Greg OMAN

UNIFYING SOME NOTIONS OF INFINITY IN ZC AND ZF

A b s t r a c t. Let $ZC - \mathcal{I}$ (respectively, $ZF - \mathcal{I}$) be the theory obtained by deleting the axiom of infinity from the usual list of axioms for Zermelo set theory with choice (respectively, the usual list of axioms for Zermelo-Fraenkel set theory). In this note, we present a collection of sentences $\exists x \varphi(x)$ for which $(ZC - \mathcal{I}) + \exists x \varphi(x)$ (respectively, $(ZF - \mathcal{I}) + \exists x \varphi(x)$) proves the existence of an infinite set.

1. Introduction

Many set-theoretic axioms postulating the existence of an infinite set have been discovered during the 20th century; Zermelo's 1908 axiom ([26]) is as follows:

Axiom of Infinity 1.1 (Zermelo)**.** *There exists a set Y with the following properties:*

1. *$\varnothing \in Y$, and*

2. *for all $y \in Y$, also $\{y\} \in Y$.*

It is easy to construct a model of Peano arithmetic (PA) via this axiom (and the remaining axioms of Zermelo set theory). To wit, suppose that Y satisfies (1) and (2) above, and let X be the intersection of all subsets of Y which contain $\varnothing$ and are closed under the function $f\colon Y \to Y$ defined by $f(y) := \{y\}$. It is routine to show that the triple $(X, f, \varnothing)$ is a Peano system[1].

Years later, von Neumann proposed a different axiom. Contemporarily, von Neumann's axiom has usurped Zermelo's in the standard axiomatization of ZFC.

Axiom of Infinity 1.2 (von Neumann)**.** *There exists a set I such that*

1. *$\varnothing \in I$, and*

2. *for every $x \in I$, also $x \cup \{x\} := x^+ \in I$.*

Assuming Axiom of Infinity 2, one obtains the least infinite ordinal ω by taking the intersection of the inductive sets (more formally, by applying separation). Moreover, it is easy to show that $(\omega, S, \varnothing)$ is a Peano system, where $S\colon \omega \to \omega$ is defined by $S(n) := n^+$.

It is clear that the sets postulated by the above axioms are infinite in some sense. Note also that, assuming either of the above axioms of infinity, one can construct an infinite set in ZF for which the defining properties (1) and (2) of the axiom fail. For example, consider a set Y as in the statement of Zermelo's axiom. Pick any $a \notin Y$, and set $Y' := \{\{y, a\}\colon y \in Y\}$. An analogous construction applies if we assume von Neumann's axiom. Thus, there is a sense in which Axioms of Infinity 1 and 2 capture something a bit more than simply "infinitude." Further, notice that the inductive property of a Peano system is, in a sense, *built into* the above axioms: by taking the

---

[1] That is, f is one-to-one, $\varnothing := e \notin f[X]$, and any set $Z \subseteq X$ which contains e and is closed under f exhausts X. This definition is due to Guiseppe Peano, who proposed it as an axiomatic foundation for number theory in 1889 ([17]).

intersection of the sets satisfying Axiom of Infinity 1 (respectively, Axiom of Infinity 2), one immediately obtains a Peano system.

Other infinity axioms been proposed which one might expect to (informally) "capture" the essence of infinity (and nothing more) within the context of set theory[2]. Possibly the two best-known are the following:

Definition 1.3 (Dedekind-Infinite Set). A set X is Dedekind-infinite if and only if there exists a bijection between X and a proper subset of itself.

Definition 1.4 (Tarski-Infinite Set). A set X is Tarski-infinite if and only if there exists a nonempty subset $\mathcal{S} \subseteq \mathcal{P}(X)$ such that for every $A \in \mathcal{S}$, there exists $B \in \mathcal{S}$ with $A \subsetneq B$.

A case can be made to support the claim that the existence of a Tarski-infinite set is the most natural candidate to be adopted as an infinity axiom (that is, intuitively, "Tarski-infinite" is the correct formalization of "infinite"). An informal, heuristic argument follows. If X is infinite, then there is some $x_1 \in X$. Similarly, there exists $x_2 \in X \backslash \{x_1\}$. Now choose $x_3 \in X \backslash \{x_1, x_2\}$, and so on. Then we have $\{x_1\} \subsetneq \{x_1, x_2\} \subsetneq \{x_1, x_2, x_3\} \subsetneq \cdots$, and we see that X is Tarski-infinite. Conversely, it is clear that any Tarski-infinite set is, in some sense, infinite.

Many papers have been written over the years studying various notions of finiteness and infinitude in fragments of ZFC and other set theories. For example, ZF proves that every Dedekind-infinite set is Tarski-infinite. Moreover, it can be shown in ZFC that a set is Tarski-infinite if and only if it is Dedekind-infinite. However, ZF alone is not strong enough to establish this equivalence ([18], p. 195), nor is ZF strong enough to prove that every infinite set is Tarski-infinite. In [6], the author defines and studies what he calls a T-notion of infinity, where T is a set theory (including an axiom of infinity) which is at least as strong as ZF. We refer the reader to [3]–[5], [16], [19], [21], and [22] for additional investigations of the finite and infinite in various set theories. Now consider replacing an axiom of infinity of ZFC with its negation. One then obtains so-called *finite set theory*. This theory is also well-studied in the literature; see [1], [2], [7], [11], [13], [20], and [24].

Our objective in this article is a bit different. We remove the infinity axiom from ZC (respectively, ZF) but do not replace it with its negation.

---

[2] The previous paragraph shows that neither Zermelo's axiom nor von Neumann's axiom does this, but that was not their purpose. The purpose of their axioms was simply to postulate the existence of some set that one would intuitively regard as being infinite.

Instead, we consider a large collection of sentences which, along with the other axioms of ZC minus infinity (respectively, ZF minus infinity), imply the existence of an infinite set (of course, we must make precise what we mean by "infinite set"). More precisely, we initiate an investigation of Problems 1 and 2 below.

Problem 1.5. Let $ZC - \mathcal{I}$ denote Zermelo set theory with choice minus the axiom of infinity (extensionality, pairing, separation, union, power set, and choice remain). Study the sentences $\exists x \varphi(x)$ for which $(ZC - \mathcal{I}) + \exists x \varphi(x) \models$ "There exists a Dedekind (Tarski) infinite set."

Problem 1.6. Study an analogous problem in the theory $ZF - \mathcal{I}$.

Since the axioms of choice, foundation, and replacement play a non-trivial role in this paper, we pause to state these axioms explicitly for the reader.

Axiom 1.7 (Axiom of Choice). *If $\mathcal{X}$ is a set of non-empty sets, then $\mathcal{X}$ has a choice function. Explicitly, there is a function F with domain $\mathcal{X}$ such that $F(X) \in X$ for all $X \in \mathcal{X}$.*

Axiom 1.8 (Axiom of Foundation (Regularity)). *For every non-empty set X, there exists some $m \in X$ such that $X \cap m = \varnothing$.*

Axiom 1.9 (Axiom Schema of Replacement). *Let ϕ be any formula in the language of ZFC whose free variables are among $x, y, A, w_1, \ldots, w_n$ and let B be a variable distinct from the free variables of ϕ. Then*
$$\forall A \forall w_1 \forall w_2 \ldots \forall w_n \big[\forall x (x \in A \Rightarrow \exists! y\, \phi) \Rightarrow \exists B\ \forall x \big(x \in A \Rightarrow \exists y (y \in B \wedge \phi) \big) \big].$$

We conclude the introduction by mentioning that the results of this paper are not overly technical in nature. A basic familiarity with axiomatic set theory is sufficient to digest the contents of this note.

2. Notions of Infinity in $ZC - \mathcal{I}$

We begin by reminding the reader that the theory ZC is axomatized by the usual axioms of ZFC minus the axioms of foundation and replacement. Next, we state a new definition which will play a prominent role throughout this article.

Definition 2.1. Let $\varphi(x)$ be a formula in the language of set theory (in which only x occurs free). Say that $\varphi(x)$ is K-finite (the K is for Kuratowski; see Example 2.10) provided the following is a theorem of $ZC - \mathcal{I}$:

$$\varphi(\varnothing) \wedge \forall x \forall y(\varphi(x) \Rightarrow \varphi(x \cup \{y\})).$$

Informally, the motivation for the above definition is this: $\varphi(x)$ is K-finite if $ZC - \mathcal{I} \models \varphi(s)$ for every finite set s.

There is an abundance of natural K-finite formulas; we present a sampling below. As the proofs that the formulas are K-finite are all straightforward, we omit them. The reader is referred to [14], [15], [23], and [25] for further reading on various notions of finiteness in set theory.

Example 2.2. "Every linear order on x is a well-order."

Example 2.3. "Any two well-orders on x are isomorphic."

Example 2.4. "For all y if $y \notin x$, then there is no surjection from x onto $x \cup \{y\}$."

Example 2.5. "If x has at least two elements, then there is no surjection of x onto $x \times x$."

Example 2.6. "If $x \neq \varnothing$, then x contains a maximal element with respect to every partial order on x."

Example 2.7. (Stäckel-finiteness) "x can be given a well-order such that the opposite order also well-orders x."

Example 2.8. (Dedekind-finiteness) "There is no bijection between x and a proper subset of itself."

Example 2.9. (Tarski-finiteness) "Every nonempty subset $s \subseteq \mathcal{P}(x)$ has a $\subseteq$-maximal element."

Example 2.10. (Kuratowski-finiteness) "For every $y \subseteq \mathcal{P}(x)$: if y satisfies

1. $\varnothing \in y$, and

2. for all $z \in \mathcal{P}(x)$ and $t \in x$: if $z \in y$, then $z \cup \{t\} \in y$,

then $x \in y$."

Remark 2.11. It is straightforward to check that neither choice nor foundation is needed in the proofs of the K-finiteness of the above formulas.

One of the fundamental theorems of $ZC - \mathcal{I}$ of which we shall shortly make use is the Well-Ordering Theorem. It is well-known that under ZF, the Well-Ordering Theorem is an equivalent of the axiom of choice. The standard proof of the Well-Ordering Theorem from the axiom of choice employs Hartogs' Lemma: for every set X, there is an ordinal α which cannot be mapped injectively into X. One then applies transfinite recursion to obtain a bijection between X and some ordinal $\beta \leq \alpha$; a well-order on X is easily obtained via this bijection. Now, the usual proof of Hartogs' Lemma invokes the axiom schema of replacement. In the absence of replacement, there is a work-around. The following result is known (perhaps not well-known) and is essentially Zermelo's original proof of his then-controversial Well-Ordering Theorem. We present a sketch of the proof (which does not require the axioms of infinity, foundation, or replacement) which appears on p. 13 of [10].

Lemma 2.12 (Zermelo)**.** $ZC - \mathcal{I} \models$ *every set can be well-ordered.*

Sketch of Proof.   Let X be a nonempty set, and let γ be a choice function for $\mathcal{P}(X) \backslash \{\varnothing\}$. Call a subset $Y \subseteq X$ a *γ-set* if there is a well-order $\leq_Y$ of Y such that

$$\text{for every } a \in Y : \gamma(\{z | z \notin Y \text{ or } a \leq_Y z\}) = a.$$

Observe that non-empty γ-sets exist: let $a := \gamma(X)$ and set $Y := \{a\}$. Then it is immediate that Y is a γ-set. Now, it can be shown that if Y is a γ-set with well-order $\leq_Y$ and Z is a γ-set with well-order $\leq_Z$, then $Y \subseteq Z$ and $\leq_Z$ is an extension of $\leq_Y$ or vice-versa. One then takes the union of the γ-sets and shows that this union is a γ-set which exhausts X. $\square$

We now present the main result of this section.

Theorem 2.13. *Suppose that $\varphi(x)$ is a K-finite formula. Then $(ZC - \mathcal{I}) + \exists x \neg \varphi(x) \models$ "There exists a Dedekind (Tarski) infinite set."*[3]

---

[3] Of course, there are many K-finite formulas $\varphi(x)$ for which $(ZC - \mathcal{I}) + \exists x \neg \varphi(x)$ is inconsistent (for example, take $\varphi(x)$ to be $x = x$), in which case the assertion of the theorem is trivial.

Proof. (1) We assume that $\varphi(x)$ is K-finite and consider the theory $(ZC - \mathcal{I}) + \exists x \neg \varphi(x)$. Let Y be a set such that $\neg \varphi(Y)$, and let $\leq$ be a well-order on Y. Observe that by definition of "K-finite formula," Y is not empty. For $y \in Y$, we let $seg(y) := \{y' \in Y : y' < y\}$ and define $S \colon Y \to Y$ as follows:

$$S(y) = \begin{cases} \min(\{y' : y < y'\}) & \text{if } \{y' : y < y'\} \neq \varnothing, \text{and} \\ y & \text{otherwise.} \end{cases}$$

We consider two cases:

Case 1. $\varphi(seg(y))$ for every $y \in Y$. In this case, we claim that Y does not possess a maximal element with respect to $\leq$. Indeed, suppose such a y_0 exists. Then $Y = seg(y_0) \cup \{y_0\}$. But since $\varphi(seg(y_0))$ and because $\varphi(x)$ is K-finite, it follows that $\varphi(seg(y_0) \cup \{y_0\})$, that is, $\varphi(Y)$. This contradicts $\neg \varphi(Y)$. Now let e be the $\leq$-minimal element of Y. It is clear that $S \colon Y \to Y \backslash \{e\}$ is an injection. Thus Y is Dedekind-infinite. Moreover, $\{seg(y) : y \in Y\} \subseteq \mathcal{P}(Y)$ shows that Y is Tarski-infinite as well.

Case 2. $\neg \varphi(seg(y))$ holds for some $y \in Y$. Let $y_0 \in Y$ be least such that $\neg \varphi(seg(y_0))$. Because $\varphi(\varnothing)$, y_0 is not the $\leq$-least element of Y. We claim that if $y < y_0$, then there exists z such that $y < z < y_0$. Indeed, suppose there is $y \in Y$ with $y < y_0$, yet there is no z with $y < z < y_0$. Then $seg(y_0) = seg(y) \cup \{y\}$. By minimality of y_0, we see that $\varphi(seg(y))$. As $\varphi(x)$ is K-finite, also $\varphi(seg(y) \cup \{y\})$. But then $\varphi(seg(y_0))$, and we have reached a contradiction. Now the argument presented in Case 1 shows that $seg(y_0)$ is Dedekind and Tarski-infinite. The proof is complete. $\square$

Corollary 2.14. *If $\varphi(x)$ is K-finite, then $(ZC-\mathcal{I})+\exists x \neg \varphi(x) \models$ "There exists a Peano system."*

Proof. By Theorem 2.13, it suffices to prove that $(ZC - \mathcal{I})+$"There exists a Dedekind-infinite set"$\models$"There exists a Peano system." Toward this end, suppose that D is Dedekind-infinite, and let $f \colon D \to D'$ be a bijection of D with a proper subset D' of D. Now choose any $e \in D \backslash D'$, and let $X := \bigcap \{Y \subseteq D : e \in Y \text{ and } Y \text{ is closed under } f\}$. It is patent that (X, f, e) is a Peano system. $\square$

3.   Notions of Infinity in $ZF - \mathcal{I}$

We continue the investigation initiated in the previous section, but we drop the axiom of choice. In its place, we add the axioms of replacement and foundation (regularity), that is, we consider the theory $ZF - \mathcal{I}$ of Zermelo-Fraenkel set theory without choice or infinity.

   We begin with a couple of observations. First, the standard proof of the Transfinite Recursion Theorem invokes neither choice nor infinity (see the text [8] for the precise formulation of the Transfinite Recursion Theorem in ZFC and its proof). Thus:

Lemma 3.1. $ZF - \mathcal{I} \models$ *"the Transfinite Recursion Theorem."*

Armed with the Transfinite Recursion Theorem, we may define the sets V_α of the cumulative hierarchy as usual. Further, foundation and replacement yield

Lemma 3.2. $ZF - \mathcal{I} \models$ *"Every set is a member of V_α for some ordinal α."*

   Recall from Example 2.9 that a set x is Tarski-finite (henceforth abbreviated T-finite) if and only if every nonempty subset of $\mathcal{P}(x)$ has a $\subseteq$-maximal element. Let us agree to call an ordinal α a *natural number* if and only if α is a T-finite set[4].

   We now prove a trivial but useful lemma (the lemma is well-known in ZFC, but it holds even if there is no set containing all natural numbers; this is known also). We include the easy proof for completeness.

Lemma 3.3. $ZF - \mathcal{I}$ *proves the following:*

  1. $0 := \varnothing$ is a natural number.

  2. If n is a natural number, so is $n + 1 := n \cup \{n\}$.

  3. If n is a natural number and $x < n$, then x is a natural number.

---

[4] This is a common definition of "natural number" in the absence of an infinity axiom. Another popular definition is as follows: an ordinal number α is a natural number if and only if for every ordinal $\beta \leq \alpha$, either $\beta = \varnothing$ or β is a successor ordinal. It is not hard to show that these definitions are equivalent in $ZF - \mathcal{I}$; see p. 80 of [12].

4. (*Induction Principle*) *Suppose that $\varphi(x)$ is a formula such that $\varphi(0)$ and for every natural number n: if $\varphi(n)$, then $\varphi(n+1)$. Then $\varphi(m)$ for every natural number m.*

Proof. Items (1) and (2) follow immediately by definition of "natural number," Example 2.9, and Remark 2.11. Assertion (3) is easily deduced from the fact that every ordinal is transitive and any subset of a T-finite set is T-finite (this is trivial to verify). Finally, suppose that $\varphi(x)$ is a formula such that $\varphi(0)$ and for every natural number n: if $\varphi(n)$, then $\varphi(n+1)$. Suppose by way of contradiction that there is a natural number m such that $\neg\varphi(m)$. We may assume that m is the least such natural number. Since $\varphi(0)$, we have $m \neq 0$. Let $n < m$ be arbitrary. Then (3) implies that n is a natural number. The minimality of m yields $\varphi(n)$. But then also $\varphi(n+1)$, by assumption. Since $\neg\varphi(m)$, we conclude that $n+1 < m$. Now observe that m is a nonempty subset of $\mathcal{P}(m)$, and for every $i \in m$, both $i+1 \in m$ and $i \subsetneq i+1$. But then m is not T-finite, a contradiction. This contradiction completes the argument. $\square$

Next, we present another new definition.

Definition 3.4. Let $\varphi(x)$ be a formula. Say that $\varphi(x)$ is a V_ω-formula provided the following is a theorem of $ZF - \mathcal{I}$:

$$\forall x \forall y((y \text{ is a natural number} \wedge x \in V_y) \Rightarrow \varphi(x)).$$

In what follows, "ω exists" abbreviates the sentence, "There exists a set to which every natural number belongs." From such a set, one can extract the usual ordinal ω by separation.

Proposition 3.5. *Let $\varphi(x)$ be a V_ω-formula. Then*

$$(ZF - \mathcal{I}) + \exists x \neg \varphi(x) \models \text{``}\omega \text{ exists.''}$$

Proof. We let $\varphi(x)$ be a V_ω-formula. Recall from Lemma 3.2 that $ZF - \mathcal{I} \models$"Every set is a member of V_α for some ordinal α." As $\varphi(x)$ is a V_ω-formula, we see that $(ZF - \mathcal{I}) + \exists x \neg \varphi(x) \models$"There exists an ordinal α which is not a natural number." The class of ordinal numbers is linearly ordered (which can be proved in $ZF - \mathcal{I}$; cf. [9]). This fact along with (3) of Lemma 3.3 implies that every natural number belongs to α. $\square$

Ultimately, we shall establish an analog of Theorem 2.13 in the theory $ZF - \mathcal{I}$. First we prove a lemma.

Lemma 3.6. $ZF - \mathcal{I}$ *proves each of the following:*

1. *Every subset of a natural number is equinumerous with a natural number.*

2. *For all natural numbers n: if x and y are sets both of which are equinumerous with n, then $x \cup y$ is equinumerous with a natural number.*

3. *For all natural numbers n: if x is a set equinumerous with n, then $\mathcal{P}(x)$ is equinumerous with a natural number.*

Sketch of proof. We work in the theory $ZF - \mathcal{I}$.

(1) Easy induction.

(2) We induct on n, the case $n = 0$ being trivial. Assume the claim holds for some natural number n, and suppose that x and y are sets which are both equinumerous with $n+1$. Henceforth we use the familiar notation $a \sim b$ to denote the assertion that a and b are equinumerous sets. Now, there exists $x_0 \in x$ ($y_0 \in y$) with $x \backslash \{x_0\} \sim n$ ($y \backslash \{y_0\} \sim n$). By the inductive hypothesis, $(x \backslash \{x_0\}) \cup (y \backslash \{y_0\}) \sim m$ for some natural number m. It is easy to see that $x \cup y$ is equinumerous with m, $m+1$, or $(m+1)+1$.

(3) As above, we proceed by induction on n, the case $n = 0$ being trivial. Suppose the assertion holds for the natural number n, and let $x \sim n + 1$. Then there exists $x_0 \in x$ such that $x \backslash \{x_0\} \sim n$. Observe that $\mathcal{P}(x) = \mathcal{P}(x \backslash \{x_0\}) \cup \{y \cup \{x_0\} \colon y \in \mathcal{P}(x \backslash \{x_0\})\}$. We now invoke (2) and the inductive hypothesis to finish the proof. $\square$

We give two final definitions and examples, then we establish the main result of this section.

Definition 3.7. Let $\varphi(x)$ be a formula. Say that $\varphi(x)$ is card-invariant if and only if $ZF - \mathcal{I}$ proves the following:

$$\forall x \forall y (x \sim y \Rightarrow (\varphi(x) \Leftrightarrow \varphi(y))).$$

Before presenting our next example, we remind the reader (Example 2.10) that a set x is Kuratowski-finite provided for every $y \subseteq \mathcal{P}(x)$: if y satisfies

1. $\varnothing \in y$, and

2. for all $z \in \mathcal{P}(x)$ and $t \in x$: if $z \in y$, then $z \cup \{t\} \in y$,

then $x \in y$.

Example 3.8. $ZF - \mathcal{I}$ proves: if x and x' are sets such that $x \sim x'$ and x is Kuratowski-finite, then x' is Kuratowski-finite as well. Therefore, $\varphi(x) :=$ "x is Kuratowski-finite" is a card-invariant formula.

Proof. Let x and x' be sets and $f \colon x \to x'$ be a bijection. Assume that x is Kuratowski-finite. Clearly $\overline{f} \colon \mathcal{P}(x) \to \mathcal{P}(x')$ defined by $\overline{f}(\alpha) := \{f(\beta) \colon \beta \in \alpha\}$ is a bijection as well. Let $y' \subseteq \mathcal{P}(x')$ be arbitrary, and assume that

$$\varnothing \in y', \text{ and} \tag{3.1}$$

$$\text{for all } z' \in \mathcal{P}(x') \text{ and } t' \in x' : \text{ if } z' \in y', \text{ then } z' \cup \{t'\} \in y'. \tag{3.2}$$

We must show that $x' \in y'$. Toward this end, observe that

$$y' = \overline{f}[y] \text{ for some } y \subseteq \mathcal{P}(x).^5 \tag{3.3}$$

By (3.1) above, we see that $\varnothing \in \overline{f}[y]$; thus $\overline{f}(i) = \varnothing$ for some $i \in y$. Hence $i = \varnothing$, and

$$\varnothing \in y. \tag{3.4}$$

Next, let $z \in \mathcal{P}(x)$ and $t \in x$ be arbitrary, and assume that $z \in y$. We shall prove that

$$z \cup \{t\} \in y. \tag{3.5}$$

First, $\overline{f}(z) \in \mathcal{P}(x')$ and $f(t) \in x'$. As $z \in y$, also $\overline{f}(z) \in \overline{f}[y] = y'$. Invoking (3.2) and (3.3),

$$\overline{f}(z) \cup \{f(t)\} \in \overline{f}[y]. \tag{3.6}$$

It follows immediately by definition of $\overline{f}$ that $\overline{f}(\{t\}) = \{f(t)\}$. We now have

$$\overline{f}(z \cup \{t\}) = \overline{f}(z) \cup \overline{f}(\{t\}) = \overline{f}(z) \cup \{f(t)\} \in \overline{f}[y]. \tag{3.7}$$

---

[5] The set $\overline{f}[y]$ denotes the image of y under $\overline{f}$.

Therefore, $\overline{f}(z \cup \{t\}) = \overline{f}(\alpha)$ for some $\alpha \in y$. Since $\overline{f}$ is one-to-one, (3.5) follows. Because x is Kuratowski-finite, (3.4) and (3.5) yield that $x \in y$. We deduce that $\overline{f}(x) \in \overline{f}[y]$. By definition of $\overline{f}$, we have $\overline{f}(x) = x'$. Recall from (3.3) that $y' = \overline{f}[y]$. Hence $x' \in y'$, which was to be shown.   □

Definition 3.9. Let $\varphi(x)$ be a formula. Call $\varphi(x)$ an ω-formula if and only if $ZF - \mathcal{I}$ proves:

$$\forall x(x \text{ is a natural number} \Rightarrow \varphi(x)).$$

Example 3.10. Let $\varphi(x)$ be "x is transitive". Then it is easy to see that $\varphi(x)$ is an ω-formula.

We now present our second theorem.

Theorem 3.11. *Suppose that $\varphi(x)$ is a card-invariant ω-formula. Then $(ZF - \mathcal{I}) + \exists x \neg \varphi(x) \models$ "ω exists."*

Proof. Let $\varphi(x)$ be an ω-formula. Recall that $V_0 = \varnothing$ and for every natural number n, we have $V_{n+1} = \mathcal{P}(V_n)$. Induction and (3) of Lemma 3.6 imply that V_n is equinumerous with a natural number for every natural number n. Next, let n be a natural number and let $y \in V_n$ be arbitrary. Since V_n is transitive, $y \subseteq V_n$. Because V_n is equinumerous with a natural number, we deduce from (1) of Lemma 3.6 that y is also equinumerous with a natural number. As $\varphi(x)$ is a card-invariant ω-formula, we see that $(ZF - \mathcal{I}) \models \varphi(y)$. We have shown that $\varphi(x)$ is a V_ω-formula. Invoking Proposition 3.5 completes the argument.   □

We end the article with the following corollary and subsequent remark.

Corollary 3.12. *$(ZF - \mathcal{I}) + \exists x \neg \varphi(x) \models$ "ω exists," where $\varphi(x)$ is any of the formulas presented in Examples 1–9.*

Proof. Let $\varphi(x)$ be such a formula. Recall that $\varphi(x)$ is K-finite. Moreover, this can be established without the axiom of choice (Remark 2.11). Thus $ZF - \mathcal{I} \models$ "$\varphi(x)$ is K-finite." It is easy to see that $\varphi(x)$ is card-invariant (we proved this explicitly in Example 3.8 for $\varphi(x) := $ "x is Kuratowski-finite"). Moreover, the K-finiteness of $\varphi(x)$ implies immediately (via induction) that $\varphi(x)$ is an ω-formula. We are now done by Theorem 3.11.   □

Remark 3.13. As stated in the introduction, ZF is not strong enough to prove that a set is Dedekind-infinite if and only if it is Tarski-infinite. However, Corollary 3.12 shows that $ZF-\mathcal{I} \models (\exists x(x \text{ is Dedekind-infinite})) \Leftrightarrow (\exists x(x \text{ is Tarski-infinite}))$.

Acknowledgments

References

[1] S. Baratella, R. Ferro, A theory of sets with the negation of the axiom of infinity, *Math. Logic Quart.* **39**:3 (1993), 338–352.

[2] N. V. Belyakin, S.P. Odintsov, Nonstandard analysis and the axiom of determinacy, *Algebra i Logika* **32**:6 (1993), 607–617, 711; translation in *Algebra and Logic* **32**:6 (1994), 328–333.

[3] P. Bernays, A system of axiomatic set theory. III. Infinity and enumerability. Analysis, *J. Symbolic Logic* **7** (1942), 65–89.

[4] E. W. Beth, Axiomatique de la théorie des ensembles sans axiome de l'infini, *Bull. Soc. Math. Belg.* **16** (1964), 127–136.

[5] J. Drabbe, Les axiomes de l'infini dans al théorie des ensembles sans axiome de substitution, *C.R. Acad. Sci. Paris Sér.* A-B 268 (1969), A137–A138.

[6] J. Degen, Some aspects and examples of infinity notions, *Math. Logic. Quart.* **40**:1 (1994), 111–124.

[7] A. Enayat, J. Schmerl, A. Visser, ω-models of finite set theory, in: Set theory, arithmetic, and foundations of mathematics: theorems, philosophies, pp. 43–65, Lect. Notes Log., 36, Assoc. Symbol. Logic, La Jolla, CA, 2011.

[8] H. Enderton, Elements of set theory, Academic Press [Harcourt Brace Jovanovish, Publishers], New York-London, 1977.

[9] T. Jech, Set theory. The third millennium edition, revised and expanded, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[10] A. Kanamori, D. Gabbay, J. Woods (editors), Handbook of the history of logic: sets and extensions in the twentieth century, Elsevier, 2012.

[11] R. Kay, T. Wong, On interpretations of arithmetic and set theory, *Notre Dame J. Formal Logic* **48**:4 (2007), 497–510.

[12] G. B. Keene, Abstract sets and finite ordinals. An introduction to the study of set theory, International Series of Monographs on Pure and Applied Mathematics, Vol. 23 Pergamon Press, New York-Oxford-London-Paris, 1961.

[13] L. Kirby, Substandard models of finite set theory, *Math. Log. Q.* **56**:6 (2010), 631–642.

[14] A. Levi, The independence of various definitions of finiteness, *Fundamenta Mathematicae* **46** (1958), 1–13.

[15] A. Mostowski, On the independence of definitions of finiteness in a system of logic, *Ann. Soc. Math. Polon.* Ser. II (1938), 1–54.

[16] C. Parsons, Developing arithmetic in set theory without infinity: some historical remarks, *Hist. Philos. Logic* **8**:2 (1987), 201–213.

[17] G. Peano, Arithmetices principia, nova methodo exposita, 1889, 83–97.

[18] L. Polkowski, Rough sets. Mathematical foundations. Advances in Soft Computing. Physica-Verlag, Heidelberg, 2002.

[19] K. Sato, The strength of extensionality II - weak set theories without infinity, *Ann. Pure Appl. Logic* **162**:8 (2011), 579–646.

[20] A. Sochor, The alternative set theory. Set theory and hierarchy theory, Proc. Second Conf., Bierutowice, 1975), pp. 259–271. Lecture Notes in Math., Vol. 537, Springer, Berlin, 1976.

[21] L. Spilsiak, P. Vojtas, Dependences between definitions of finiteness, *Czech. Math. J.* **38** (1988), 389–397.

[22] A. Tarski, On finite sets, *Fundamenta Mathematicae* Volume 6H (1924), 45–95.

[23] P. Vejjajiva, S. Panasawatwong, A note on weakly Dedekind finite sets, *Notre Dame J. Form. Log.* **55**:3 (2014), 413–417.

[24] P. Vopenka, Axiome der Theorie endlichter Mengen, *Casopis Pest. Mat.* **89** (1964), 312–317.

[25] A. C. Walczak-Typke, The first-order theory of weakly Dedekind-finite sets, *J. Symbolic Logic* **70**:4 (2005), 1161–1170.

[26] E. Zermelo, Untersuchungen ber die Grundlagen der Mengenlehre I, *Mathematische Annalen* **65**:2 (1908), 261–281.

Department of Mathematics
The University of Colorado
Colorado Springs, CO 80918
USA

`goman@uccs.edu`