ALGEBRAIC LANGUAGES AND POLYOMINOES ENUMERATION

M. P. DELEST
Université Bordeaux II
33076 BORDEAUX Cedex, France

G. VIENNOT
Université Bordeaux I
33405 Talence, France

Abstract

The purpose of this paper is to show the use of algebraic languages theory in solving an open problem in combinatorics : give a formula for the number of convex polyominoes.

1- Introduction

Let Ω_n be a class of combinatorial objects enumerated by the integer a_n and suppose that the corresponding generating function $f(t) = \sum_{n\geq 0} a_n t^n$ is algebraic. An old idea dear to M.P. Schützenberger is to explain this algebricity by giving a bijection between Ω_n and the words L_n of length n of a certain algebraic (context free) language L defined by a non-ambiguous grammar.

Usually, an explicit formula is known for a_n or f(t) by means of classical calculus technics (recurrence relation, Lagrange inversion formula, etc...). With the non-ambiguous grammar, one can associate classically a proper algebraic system of equations (see for example Salomaa, Soittola [13]) in non-commutative power series. The unique solution contains the generating function $\underline{\underline{L}} = \sum_{w \in L} w$, of the language $L \subseteq X^*$ (free monoid generated by the alphabet X). By sending all variables $x \in X$ on t, one obtains an (ordinary) algebraic system of equations and $\underline{\underline{L}}$ becomes $f(t) = \sum_{n\geq 0} a_n t^n$. The coding with words gives more light inside the combinatorial comprehension of Ω_n.
A trivial example is with Ω_n for the set of binary trees with n vertices. Here $a_n = \frac{1}{n+1}\binom{2n}{n}$, the well known Catalan numbers, with generating function $d(t) = \frac{1-(1-4t)^{1/2}}{2t}$ satisfying the equation $d = 1 + t\, d^2$. The coding with words of the restricted Dyck language $D'^{*}_1 = D$ on two letters x, $\bar{x}$ is very classical. The non-commutative corresponding equation is $\underline{\underline{D}} = 1 + x\, \underline{\underline{D}}\, \bar{x}\, \underline{\underline{D}}$. Other examples can be found in Goldman [7] .

Deep examples are found in the work of Cori and Vauquelin [4] [5] following the numerous Tutte formulae for enumeration of planar maps (see for example [14]).

In this paper the method is reversed. Let P_n be the set of convex polyominoes (definition below) with perimeter 2n. Knuth raised the problem [9] to give some informations about the number of such polyominoes. We give an exact formula for this number p_{2n}. This formula is deduced from a coding between convex polyominoes and words of an algebraic language.

The method is in 3 steps :

(i) bijection between convex polyominoes and words. In fact three different types of polyominoes have to be considered.

(ii) solving the three corresponding algebraic systems and obtaining the generating functions f(t). These systems have about 20 to 40 equations each. We are thus forced to employ more carefully algebraic languages theory and use auxilliary languages, operators, substitutions and also multiheaded finite (or pushdown) automata The final solution has been possible with the use of the symbolic manipulation system MACSYMA from MIT.

(iii) expanding the generating function f(t) in order to obtain the formula for p_{2n}.

To the knowledge of the authors, this is the first time an open combinatoric problem is solved by using algebraic languages. These results were given in a talk at the combinatoric meeting in Oberwolfach (May 1982), and since, apparently, no "classical" analytic proof has been given.

2- CONVEX POLYOMINOES

A polyominoe is a connected, with no "cut point", union of elementary squares (or cells) of the plane $\mathbb{Z} \times \mathbb{Z}$. The polyominoe π is said to be convex if the intersection with π of any vertical or horizontal line is a connected segment (possibly empty).

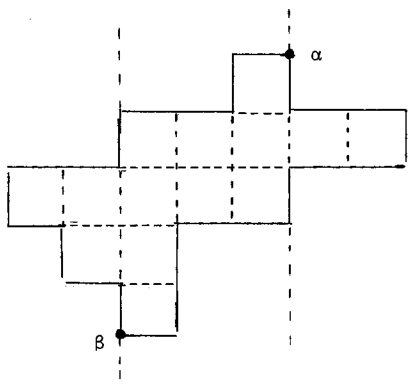

Fig. 1 - A convex polyominoe

Polyominoes are defined up to a translation. Polyominoes enumeration problems are well known in combinatorics (see for example Golomb [8]) but very few exact results are known, except for special cases or asymptotic formulae (mainly obtained by physicists, in connection with self-avoiding path problems). Some asymptotic formulae are known for the number of convex polyominoes (counted according to the area) see Bender [2] and Klarner, Rivest [9] .

Our formula has the surprising simple following form :