Conditional Semi-Thue Systems for Presenting Monoids

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Abstract

There are well known examples of monoids in literature which do not admit a finite and canonical presentation by a semi-Thue system over a fixed alphabet, not even over an arbitrary alphabet. We introduce conditional Thue and semi-Thue systems similar to conditional term rewriting systems as defined by Kaplan. Using these conditional semi-Thue systems we give finite and canonical presentations of the examples mentioned above. Furthermore we show, that every finitely generated monoid with decidable word problem is embeddable in a monoid which has a finite canonical conditional presentation.

1 Introduction

Thue and semi-Thue systems [Boo85, Jan88] can be used to examine questions concerning monoids and groups. A Thue system R over an alphabet Σ induces a congruence on Σ^*, the congruence classes form the monoid M_R. A monoid M is finitely presented by (Σ, R) if M is isomorphic to M_R and both Σ and R are finite, it is finitely generated, if only Σ is finite. If R viewed as a semi-Thue system induces a canonical, i.e. confluent and noetherian, relation, it can be used to decide the word problem of M: Two strings u and v are congruent if and only if u and v have the same common irreducible descendant.

Therefore no monoid with an undecidable word problem admits a finite and canonical presentation. It has been shown by Narendran and Squier that there exist finitely presented monoids with decidable word problem which do not have a finite and canonical presentation using a fixed alphabet, see e.g. [KN85], resp. using an arbitrary but finite alphabet [Squ87].

To overcome this gap between decidability of the word problem and the existence of finite and canonical presentations we introduce in this paper conditional Thue and semi-Thue systems. They are defined similar to conditional term rewriting systems, see e.g. [Kap84, Kap87, JW86, Gan87]. We show, that using conditional semi-Thue systems we get finite and canonical presentations for the examples of Narendran and Squier. Furthermore we are able to strengthen a result of Bauer [Bau81]: Each finitely generated monoid with decidable word problem can be embedded in a monoid, presented by a finite, canonical, and conditional semi-Thue system.

Different conditional string rewriting systems have been used already by Siekmann and Szabo [SS82] to give a finite and canonical presentation of idempotent monoids. They use variables within their rules and a different system to evaluate the premises of a conditional rule and therefore are not a conditional semi-Thue system according to our definition.

2 Conditional Semi-Thue Systems

The form of the conditional rules we use follows that of conditional term rewriting systems as defined e.g. by Kaplan [Kap84]. Therefore the induced congruences are more difficult to handle as for unconditional systems. For example, the congruences are not decidable in general. These problems can be solved by introducing reductive systems analogously to simplifying and reductive conditional term rewriting systems [Kap87, JW86].

A *conditional Thue system* R is a set of conditional equations. Each equation consists of a *conclusion* $u_0 = v_0$ and a finite set $\{u_i = v_i | 1 \leq i \leq n\}$ of *premises*, all u_i, v_i are strings over an alphabet Σ. We write such a conditional equation as

$$\forall_{i=1}^{n} u_i = v_i :: u_0 = v_0$$

The relation $\Leftrightarrow_R$ is defined as follows: $u \Leftrightarrow_R v$ if and only if there exist $x, y \in \Sigma^*$ and an equation $\forall_{i=1}^{n} u_i = v_i :: u_0 = v_0$ in R such that $u = xu_0y$ and $v = xv_0y$ or $u = xv_0y$ and $v = xu_0y$ and for all $1 \leq i \leq n$ we have $xu_iy \Leftrightarrow_R^* xv_iy$. $\Leftrightarrow_R^*$ is the *Thue congruence* induced by R. x and y are the left resp. right context of the occurrence of u_0 resp. v_0 in u. In this case R is called a *left-right* conditional Thue system. If only the right context y is used in evaluating the premises, i.e. $u_iy \Leftrightarrow_R^* v_iy$, we call R a *right* conditional system.

To define conditional *semi-Thue* systems we restrict the application of the equations. Let $\forall_{i=1}^{n} u_i = v_i :: u_0 \rightarrow v_0$ be a rule of a conditional semi-Thue system R. u_0 is the left-hand side of this rule, v_0 is the right-hand side. Now $u \rightarrow_R v$ if and only if $u = xu_0y$ and $v = xv_0y$, where $x, y \in \Sigma^*$, and xu_iy and xv_iy have each a common descendant modulo $\rightarrow_R$ for $1 \leq i \leq n$. As for conditional Thue systems we distinguish left-right and right conditional semi-Thue systems.

Notice that the premises must have a common descendant in the context of rule application instead of being congruent as it was in the definition of conditional Thue systems. This causes the first difference to unconditional systems: The Thue congruence and the symmetric, transitive, and reflexive closure of the reduction relation may not coincide anymore. To recover this property we need in addition confluence of R.

Lemma 1 cf. [Kap84, theorem 3.2.]
a) There exists a conditional Thue system R with $\Leftrightarrow_R^* \neq \leftrightarrow_R^*$.
b) If R is confluent, we have $\Leftrightarrow_R^* = \leftrightarrow_R^*$.

Both $\Leftrightarrow_R^*$ and $\leftrightarrow_R^*$ are compatible with concatenation, hence the congruence classes modulo $\Leftrightarrow_R^*$ resp. $\leftrightarrow_R^*$ form a monoid. As a direct consequence of the lemma above these monoids are the same if R is confluent.

For a finite unconditional system R the relations $\Leftrightarrow_R$ and $\rightarrow_R$ are decidable. This changes, too, when considering conditional systems, cf. [Kap84, theorem 3.3.].

Lemma 2 *There exists a finite conditional Thue system R such that $\Leftrightarrow_R$ and $\rightarrow_R$ are undecidable.*

Similar to the case of conditional term rewriting systems there is a sufficient criterion such that the reduction relation becomes decidable: We call a conditional semi-Thue system *reductive* if for all rules in R the strings in the premises and the right-hand side are smaller than the left-hand side wrt. to a wellfounded ordering which is compatible with concatenation. In analogy to [Kap87, theorem 1.6.] we have

Lemma 3 *Let R be a finite reductive conditional semi-Thue system, then $\rightarrow_R$ is noetherian and decidable.*

The results of the lemmata 1 and 3 can be combined: If R is finite, confluent, and reductive, then we have $\Leftrightarrow_R^* = \leftrightarrow_R^*$ and $\Leftrightarrow_R^*$ resp. $\leftrightarrow_R^*$ are decidable. Hence R can be used to decide the word problem by means of string rewriting.