Tableau Calculi for the Logics of Finite k-Ary Trees

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Abstract. We present tableau calculi for the logics $\mathbf{D}_k$ ($k \geq 2$) semantically characterized by the classes of Kripke models built on finite k-ary trees. Our tableau calculi use the signs **T** and **F**, some tableau rules for Intuitionistic Logic and two rules formulated in a hypertableau fashion. We prove the Soundness and Completeness Theorems for our calculi. Finally, we use them to prove the main properties of the logics $\mathbf{D}_k$, in particular their constructivity and their decidability.

1 Introduction

In recent years there has been a growing interest (see [1,3,4,6,7,8,10,11]) in proof-theoretical characterization of *propositional intermediate logics*, that is logics laying between Intuitionistic and Classical Logic. This interest is motivated by the applications of some of these logics. As an example we recall *Dummett-Gödel Logic*, studied for its relationship with multi-valued and fuzzy logics [14]; *Jankov Logic* and *here-and-there Logic*, studied for their application to Logic Programming [15,16].

Apart from the cases of Intuitionistic and Classical Logic, the proof-theoretical characterization of Intermediate Logics given in the literature relies on variations of the standard sequent calculi or tableau calculi. As an example, the tableau calculi for the interpolable Intermediate Logics described in [1,10, 11] use new signs besides the usual signs **T** and **F** (we remark that the calculi of [10,11] give rise to space-efficient decision procedures). However, this approach seems hard to apply to several families of interesting Intermediate Logics. Another approach relies on hypersequent calculi, a natural generalization of sequent calculi; e.g., in [3] a hypersequent characterization of Dummett-Gödel Logic is presented, while in [7] the authors extend this approach to some families of Intermediate Logics with bounded Kripke models. However, also the approach based on hypersequent calculi or hypertableau calculi (the dualized version of hypersequents presented in [6]) seems to be inadequate to treat some Intermediate Logics and further variations are needed. An example is given in [6], where the notion of path-hypertableau calculus is introduced to treat the intermediate logic of finite-depth Kripke models.

Despite the wide research in this field, we remark that all the intermediate logics studied in the above mentioned papers fail to be *constructive*, where

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we call *constructive* any intermediate logic $\mathbf{L}$ satisfying the *disjunction property*: $A \vee B \in \mathbf{L}$ implies $A \in \mathbf{L}$ or $B \in \mathbf{L}$. As it is well-known, there exists a continuum of constructive intermediate logics [5,9], but, as far as we know, no proof-theoretical characterizations of constructive logics are known, apart from those given in [2]. In that paper *generalized tableau calculi* for the constructive logics $\mathbf{D}_k$ ($k \geq 2$) and for the constructive *Kreisel-Putnam Logic* are presented; however, such calculi are far from being genuine tableau calculi and are highly inefficient. Indeed, they are obtained by adding to the intuitionistic tableau calculus a special rule allowing us to introduce, at any point of the derivation, a suitable $\mathbf{T}$-signed instance of the schema characterizing the logic.

In this paper we provide tableau calculi for the intermediate constructive logics $\mathbf{D}_k$ ($k \geq 2$) of finite k-ary trees. $\mathbf{D}_k$ is the set of all the formulas valid in every Kripke model built on a finite k-ary tree. These logics have been introduced in [13], where a finite axiomatization of every $\mathbf{D}_k$ is given, and their decidability is proved. Our proof-theoretical characterization is based on a *hybrid* tableau calculus that uses the two usual signs $\mathbf{T}$ and $\mathbf{F}$, some tableau rules for Intuitionistic Logic and two rules formulated in a hypertableau fashion (a structural rule and a purely logical rule). Then we use such calculi to provide a proof of the main properties of the logics $\mathbf{D}_k$, in particular their constructivity and their decidability.

The paper is organized as follows: in Section 2 we introduce the logics $\mathbf{D}_k$ providing both the axiomatization and the semantical characterization in terms of families of Kripke models. In Section 3 we introduce the calculi $\mathbf{TD}_k$ and we prove that they characterize the logics $\mathbf{D}_k$. Finally, in Section 4 we use these calculi to prove the main properties of the logics $\mathbf{D}_k$.

2 Preliminaries

Here we consider the propositional language based on a denumerable set of atomic symbols and the logical constants $\bot, \wedge, \vee, \rightarrow$. We denote with $p, q, \ldots$, possibly with indexes, the atomic symbols and with $A, B, \ldots$, possibly with indexes, arbitrary formulas. Moreover, as usual in the setting of intermediate logics, $\neg A$ is defined as $A \rightarrow \bot$. $\mathbf{Int}$ and $\mathbf{Cl}$ denote respectively the set of intuitionistically and classically valid formulas.

An *intermediate propositional logic* (see, e.g., [5]) is any set L of formulas satisfying the following conditions: (i) L is consistent; (ii) $\mathbf{Int} \subseteq$ L; (iii) L is closed under modus ponens; (iv) L is closed under propositional substitution (where a *propositional substitution* is any function mapping every propositional variable to a formula). It is well-known that, for any intermediate logic L, L $\subseteq \mathbf{Cl}$.

Many intermediate logics can be semantically characterized by families of Kripke models. A *(propositional) Kripke model* (see, e.g., [5]) is a structure $\underline{K} = \langle P, \leq, \Vdash \rangle$, where $\langle P, \leq \rangle$ is a *poset* (partially ordered set), and $\Vdash$ (the *forcing relation*) is a binary relation between elements of P and atomic symbols such that, for every atomic symbol p, $\alpha \Vdash p$ implies $\beta \Vdash p$ for every $\beta \in P$ such that $\alpha \leq \beta$. The forcing relation is extended to arbitrary formulas as follows: