# Better Approximation Schemes for Disk Graphs*

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**Abstract.** We consider Maximum Independent Set and Minimum Vertex Cover on disk graphs. We propose an asymptotic FPTAS for Minimum Vertex Cover on disk graphs of bounded ply. This scheme can be extended to an EPTAS on arbitrary disk graphs, improving on the previously known PTAS [8]. We introduce the notion of level density for disk graphs, which is a generalization of the notion of ply. We give an asymptotic FPTAS for Maximum Independent Set on disk graphs of bounded level density, which is also a PTAS on arbitrary disk graphs. The schemes are a geometric generalization of Baker's EPTASs for planar graphs [3].

## 1 Introduction

With the ever decreasing size of communication and computing devices, mobility is a key word at the start of the 21st century. Using wireless connections, mobile devices can join local or global communication networks. In trying to understand the properties of such wireless networks, the (unit) disk graph model is frequently used. A *disk graph* is the intersection graph of a set of disks in the plane. This means that each disk corresponds to a vertex of the graph and there is an edge between two vertices if the corresponding disks intersect. The set of disks is called a *disk representation* of the graph. A *unit disk graph* has a disk representation where all disks have the same radius. Besides their practical purposes, (unit) disk graphs have interesting theoretical properties as well.

In a previous paper [21], we considered unit disk graphs of bounded density, leading to new approximation schemes for several optimization problems. Here we extend these ideas to disk graphs and introduce the notion of bounded level density. We give an asymptotic FPTAS for Maximum Independent Set on disk graphs of bounded level density, which is also a PTAS on arbitrary disk graphs. Furthermore, we show there exists an EPTAS for Minimum Vertex Cover on arbitrary disk graphs, improving results of Erlebach, Jansen, and Seidel [8]. As each planar graph is a 1-ply disk graph [11], these results are a geometric generalization of the EPTASs for planar graphs by Baker [3].

## 2 Preliminaries

An *independent set* $I \subseteq V$ of a graph $G = (V, E)$ contains only non-adjacent vertices (i.e. $u, v \in I \Rightarrow (u, v) \notin E$). A set $C \subseteq V$ *covers* $V' \subseteq V$ if $u \in C$ or

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$v \in C$ for each $(u, v) \in E \cap (V' \times V')$. If $C$ covers $V$, then $S$ is a *vertex cover*. We seek independent sets of maximum size and vertex covers of minimum size.

For each instance $x$ of a maximization (minimization) problem and any $\epsilon > 0$, a *polynomial-time approximation scheme (PTAS)* delivers in time polynomial in $|x|$ (for fixed $\epsilon$) a feasible solution of value within a factor $(1 - \epsilon)$ (respectively $(1 + \epsilon)$) of the optimum. An *efficient PTAS (EPTAS)* delivers such a solution in time polynomial in $|x|$ and $f(\frac{1}{\epsilon})$ for some function $f$ only dependent on $\frac{1}{\epsilon}$, while a *fully polynomial-time approximation scheme (FPTAS)* delivers such a solution in time polynomial in $|x|$ and $\frac{1}{\epsilon}$. An *asymptotic FPTAS (FPTAS$^\omega$)* gives a feasible solution in time $|x|$ and $\frac{1}{\epsilon}$ and attains the approximation factor if $|x| > c_\epsilon$, for some constant $c_\epsilon$ only dependent on $\epsilon$.

A *path decomposition* of a graph $G = (V, E)$ is a sequence $(X_1, X_2, \ldots, X_p)$ of subsets of $V$ (called *bags*) such that 1) $\bigcup_{1 \le i \le p} X_i = V$, 2) for all $(v, w) \in E$, there is an $i$ $(1 \le i \le p)$ such that $v, w \in X_i$, and 3) $X_i \cap X_k \subseteq X_j$ for all $i, j, k$ with $1 \le i < j < k \le p$. The *width* of a path decomposition $(X_1, X_2, \ldots, X_p)$ is $\max_{1 \le i \le p} |X_i| - 1$. The *pathwidth* of a graph $G = (V, E)$ is the minimum width of any path decomposition of $G$ [18].

## 3 Previous Work

Clark, Colbourn, and Johnson [6] showed that Maximum Independent Set and Minimum Vertex Cover are NP-hard for (unit) disk graphs. The problems remain NP-hard under the assumption of bounded (level) density [20].

On unit disk graphs, Marathe et al. [13] give constant factor approximation algorithms. Different PTASs are presented by Hunt et al. [10] and Matsui [15] and PTASs exist even if no disk representation is known [17]. On $\lambda$-precision disk graphs of bounded radius ratio, Hunt et al. [10] show FPTAS$^\omega$s exist. In a $\lambda$-precision disk graph, the distance between any two disk centers is at least $\lambda$. Marx [14] gives an EPTAS for Minimum Vertex Cover on unit disk graphs.

Alber and Fiala [2] show that Maximum Independent Set is fixed-parameter tractable for $\lambda$-precision disk graphs of bounded radius ratio. Marx [14] recently showed that Maximum Independent Set is W[1]-hard for general (unit) disk graphs. However, $O(n^{O(\sqrt{k})})$-time fixed-parameter algorithms are known [1, 2].

On disk graphs, Malesińska [12] and Marathe et al. [13] give constant factor approximation algorithms. Erlebach, Jansen, and Seidel [8] give a PTAS for Maximum Independent Set and Minimum Vertex Cover. Chan [4] proposes a different PTAS for Maximum Independent Set on intersection graphs of fat objects (a set of disks is considered to be fat). Chan [5] also gives a PTAS for Maximum Independent Set on unit-height rectangle intersection graphs of bounded ply.

## 4 The Ply of Disk Graphs

Let $D = \{D_i \mid i = 1, \ldots, n\}$ be a set of disks in the plane and $G = (V, E)$ the corresponding disk graph. Scale the disks by a factor $2^w$ for some integer $w$,