Determining the Topology of Real Algebraic Surfaces

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Abstract. An algorithm is proposed to determine the topology of an implicit real algebraic surface in $\mathbb{R}^3$. The algorithm consists of three steps: surface projection, projection curve topology determination and surface patches composition. The algorithm provides a curvilinear wireframe of the surface and the surface patches of the surface determined by the curvilinear wireframe, which have the same topology as the surface. Most of the surface patches are curvilinear polygons. Some examples are used to show that our algorithm is effective.

1 Introduction

An implicit real algebraic surface (or curve, or hypersurface) $\mathcal{S}$ in $\mathbb{R}^u$ with degree d is defined by $f(x_1, x_2, \cdots, x_u) = 0$ where $f(x_1, x_2, \cdots, x_u) \in \mathbb{Q}[x_1, x_2, \cdots, x_u]$ is a polynomial of degree d, and $\mathbb{R}$ and $\mathbb{Q}$ are the fields of real and rational numbers, respectively. Determining the topology of an algebraic surface is not only an interesting mathematical problem, but also a key issue in computer graphics and CAGD [4, 5, 20, 22].

When $u = 1$, $\mathcal{S}$ is a set of discrete points on a line. When $u = 2$, $\mathcal{S}$ is a plane algebraic curve. Topology determination for plane algebraic curves has been studied thoroughly [1, 3, 6, 7, 9, 12, 13, 14, 16, 21]. Algorithms to determine the topology of spatial algebraic curves are also proposed in the following papers [4, 6, 9, 10]. When $u = 3$, the problem is more complex. The topology of $\mathcal{S}$ with $d = 2$ is well known. They are quadratic surfaces. But when $d \geq 3$, there are only some special surfaces whose topology can be efficiently determined [11, 12]. Fortuna et al presented an algorithm to determine the topology of non-singular, orientable real algebraic surfaces in the projective space [8]. Morse theory is used to represent an implicit algebraic surface by polyhedra in theory by Hart et al [15, 20, 22]. Theoretically, the CAD (Cylindrical Algebraic Decomposition) method proposed by Collins can be used to provide information about the topology of an algebraic surface [2, 3]. But in the general case, there exist no complete algorithms to determine the topology of an implicit algebraic surface.

In this paper, we present an algorithm to determine the topology of $\mathcal{S}$ for $u = 3, d \geq 3$. In the rest of this paper, we replace $f(x_1, x_2, x_3) = 0$ with $f(x, y, z) = 0$.

R. Martin, H. Bez, M. Sabin (Eds.): Mathematics of Surfaces 2005, LNCS 3604, pp. 121–146, 2005.

We obtain a curvilinear wireframe of the surface. The surface patches of the surface are determined by the curvilinear wireframe. Most of the surface patches are curvilinear polygons. The wireframe and the surface patches have the same topology as the surface. If needed, we can easily modify our algorithm to ensure that all the surface patches are curvilinear triangles.

The basic idea of our algorithm is as follows. We first ensure that the surface is a normal surface by performing certain transformations. We then project $\mathcal{S}$: $f(x, y, z) = 0$ to a proper plane and obtain a plane algebraic curve $\mathcal{C}$: $g(x, y) = 0$. Thirdly, we analyze the topology of $\mathcal{C}$ in a finite box, by finding its singularities, dividing the curve into plane curve segments, and dividing the box in the plane into cells. At the fourth step, we divide the spatial curve defined by $\{f(x, y, z) = 0, g(x, y) = 0\}$ into spatial curve segments and compute the number of surface patches connected with each spatial curve segment. This is the key step of the algorithm. In order to determine the number of curve segments connected with a singular point and the number of surface patches connected with a curve segment, we introduced certain minimal circles and find these numbers from the information of the intersections of the circle with the surface. The main steps of the algorithm are similar to Collins' CAD method. But, the purpose of our algorithm is different from that of the CAD method, and many aspects of the algorithm are totally new. Main parts of the algorithm are implemented in Maple and nontrivial examples are used to show that the algorithm is effective.

This paper is divided into six sections. The aim of the second section is to obtain projection curve of the surface. The third section presents an algorithm to determine the topology of the plane projection curve. Space curve segmentation, surface patch composition and the surface topology representation are discussed in the fourth section. The fifth section presents the main algorithm to obtain the topology of a given algebraic surface. Then we draw a conclusion in the last section.

2 Projection Curve of a Surface

In the following, we always assume $\mathcal{S}$ is an algebraic surface: $f(x, y, z) = 0$, where $f(x, y, z) \in \mathbb{Q}[x, y, z]$. Suppose that

$$f(x, y, z) = f_1(x, y, z)^{m_1} \cdots f_n(x, y, z)^{m_n}, \tag{1}$$

where $f_i(x, y, z) \in \mathbb{Q}[x, y, z](i = 1, \cdots, n)$ are irreducible polynomials. If a component contains variable z only, it represents some parallel planes. We can delete this kind of components before we compute the projection curve and add these planes into the topology structure and compute the intersection curve with other components after we finish the analysis. So we suppose that there does not exist this kind of components. It is clear that $f(x, y, z) = 0$ and $f_1(x, y, z) \cdots f_n(x, y, z) = 0$ have the same topology. We still denote

$$f(x, y, z) = f_1(x, y, z) \cdots f_n(x, y, z). \tag{2}$$