Realizability Proof for Normalization of Full Differential Linear Logic

Stéphane Gimenez

Laboratoire PPS, Université Paris-Diderot – Paris 7

Abstract. Realizability methods allowed to prove normalization results on many typed calculi. *Girard* adapted these methods to systems of nets and managed to prove normalization of second order *Linear Logic* [4]. Our contribution is to provide an extension of this proof that embrace *Full Differential Linear Logic* (a logic that can describe both single-use resources and inexhaustible resources). Anchored within the realizability framework our proof is modular enough so that further extensions (to second order, to additive constructs or to any other independent feature that can be dealt with using realizability) come for free.

Keywords: Linear Logic, Proof Nets, Differential Linear Logic, Differential Interaction Nets, Realizability, Weak Normalization.

Introduction

It happens that the three differential constructs of *Differential Linear Logic* (also abbreviated as *DiLL*), namely *co-weakening*, *co-contraction* and *co-dereliction*, and the original *promotion* construct from *Linear Logic* (*LL*) can be used altogether in a system that has been called *Full Differential Linear Logic* (*Full-DiLL*). In particular, this system embeds *Differential λ-calculus*, which was the first avatar of the differential paradigm introduced by *Ehrhard* and *Regnier* [1,3]. A similar system has been studied within an intuitionist setting by *Tranquilli* [9], and a combinatorial proof of its weak normalization has already been provided by *Pagani* [7]. We provide in this paper a new proof of its weak normalization, using a reducibility technique which is of great importance because it is the only known method to prove such a result in presence of second order quantifiers.

Plan. A brief presentation of *Full-DiLL* is given in Section 1. Realizability tools used by *Girard* are introduced in Section 2 and the proof he provided for weak normalization of *LL* is recalled in Section 3. Weak normalization of *Full-DiLL* will be addressed in Section 4.

1 Full Differential Linear Logic

1.1 Syntax

In this paper, we will depict proofs of *LL* and *Full-DiLL* graphically using the *interaction nets* formalism [5], which is well suited to multiplicative constructions [6], and is the usual way to represent differential constructions, see [2].

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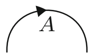
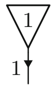
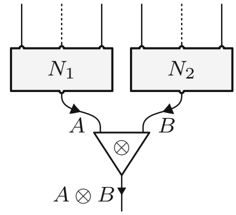
Our study will be restricted to *multiplicative* and *exponential* fragments of these logics (whose constructions are recalled bellow), because an extension of our result to *additive* and *second order* fragments can be extracted from *Girard*'s proof [4] without any significant changes, and it would otherwise be cumbersome.



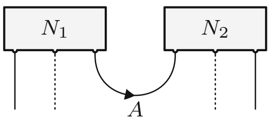
$$\frac{}{\vdash A^{\perp}, A}\ \textit{axiom}$$



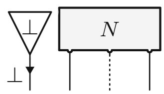
$$\frac{}{\vdash 1}\ \textit{unit}$$



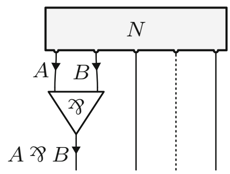
$$\frac{\vdash \Gamma_1, A \quad \vdash B, \Gamma_2}{\vdash \Gamma_1, A \otimes B, \Gamma_2}\ \textit{tensor}$$



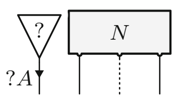
$$\frac{\vdash \Gamma_1, A \quad \vdash A^{\perp}, \Gamma_2}{\vdash \Gamma_1, \Gamma_2}\ \textit{cut}$$



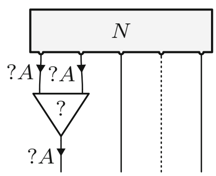
$$\frac{\vdash \Gamma}{\vdash \perp, \Gamma}\ \textit{co-unit}$$



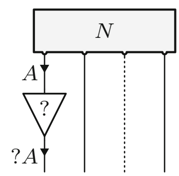
$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ \textit{par}$$



$$\frac{\vdash \Gamma}{\vdash ?A, \Gamma}\ \textit{weakening}$$



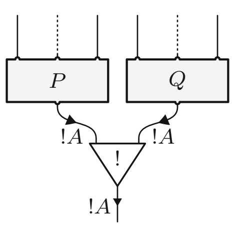
$$\frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ \textit{contraction}$$



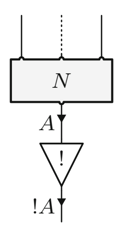
$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ \textit{dereliction}$$

Specifically in the case of *Full-DiLL*, we add the three differential constructs which are represented like this:

$$\frac{}{\vdash !A}\ \textit{co-weakening}$$



$$\frac{\Gamma_1 \vdash !A \quad \Gamma_2 \vdash !A}{\Gamma_1, \Gamma_2 \vdash !A}\ \textit{co-contraction}$$



$$\frac{\Gamma \vdash A}{\Gamma \vdash !A}\ \textit{co-dereliction}$$