Barycentric Algebras and Gene Expression

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Abstract. Barycentric algebras have seen widespread application in the modeling of convex sets, semilattices, and quantum mechanics. Recently, they were developed further to encompass Boolean logic and if-then-else algebras. This paper discusses an application of barycentric algebras to systems biology. Here, they provide a calculus for the conversion from simplified Boolean models of gene transcription to fuzzy models that give a more realistic tracking of the biochemistry. Indeed, it appears that logic gates experimentally observed in cells actually follow the barycentric algebra format.

1 Introduction

Barycentric algebras (as defined in §2.3 below) are universal algebras used for modeling convex sets, semilattices, geometry, hierarchical statistical mechanics, and quantum mechanics [5,6,12,13,14,15,16,17,18]. Recently [17], they have been developed further (as *abstract barycentric algebras*) by use of the LΠ-algebras of fuzzy logic [3,10,11], incorporating B-sets [2,20,21] and if-then-else algebras [8,9]. The aim of the current paper is to show how the calculus of barycentric algebras may be used in systems biology, to provide a virtually automatic translation from simplified Boolean models of gene expression to continuous, fuzzy logic models that give a much more realistic picture of the biochemical processes involved. Experimentally observed logic gates in cells do not follow the pattern directly suggested by standard Boolean models, but their features concur exactly with the models obtained using the barycentric algebra approach [19, Fig. 3b].

The bulk of the paper comprises two parts. Section 2 gives a direct account of the algebra required. For readers who may be unfamiliar with universal algebra, §2.2 discusses concatenations of binary operations. The two key incarnations of abstract barycentric algebras, namely classic "fuzzy" barycentric algebras and their crisp Boolean counterparts, are described in §2.3.

Section 3 then focusses on the systems biology. For readers unfamiliar with molecular biology, §3.1 gives a brief account of the way cells use transcription

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factors to respond to signals and regulate gene expression. Subsequent paragraphs formulate the crisp and fuzzy models of gene regulation in the language of barycentric algebras. Once this formulation is established, Eqn. (12) provides the automatic conversion from Boolean models to fuzzy models. In §3.4, the conversion process is illustrated by the example of the AND gate. The final paragraph explains how fuzzy logic gates that have been observed experimentally in cells actually follow the barycentric algebra format.

2 Algebra

2.1 Operations on Real Numbers and Binary Digits

Although the algebra of real numbers is traditionally performed in terms of field operations such as the addition $p+q$ and product pq of real numbers p and q, the algebra discussed in this paper requires different operations, which specialize to more familiar Boolean operations on the subset $\{0,1\}$ of the reals. In fact, this specialization will also work in any field. In particular, it works if the set $\{0,1\}$ of binary digits is interpreted as the two-element (Galois) field GF(2) or field of integers modulo 2.

For a real number p, define the *complementation* $p' = 1 - p$ specializing to the Boolean $\neg p$ or NOT p on the set $\{0.1\}$ of binary digits. Note that the complementation is *involutive*: $p'' = p$. For real numbers p and q, define the *product*

$$p \cdot q = pq \tag{1}$$

specializing to the Boolean $\wedge$ or AND on $\{0,1\}$. Define the *dual product*

$$p \circ q = p + q - pq \tag{2}$$

specializing to the Boolean $\vee$ or OR on $\{0,1\}$. Note that the dual product may be defined in terms of the product and complementation using *de Morgan's law* $p \circ q = (p'q')'$ or $(p \circ q)' = p'q'$. Define the *implication*

$$p \to q = \textbf{if } (p = 0) \textbf{ then } 1 \textbf{ else } q/p \tag{3}$$

specializing to the Boolean implication $p \to q = (\neg p) \vee q$ on $\{0,1\}$. Note that the implication (3) is always defined in any field, while the division q/p is not defined for $p = 0$.

2.2 Binary Operations

If x and y are elements of a real vector space, and p is a real number, it is convenient to define

$$xy\,\underline{p} = x(1-p) + yp = xp' + yp\,, \tag{4}$$

so that $\underline{p}$ is understood as a binary operation combining the arguments x and y. Schematically, the binary operation may be understood as a circuit element or