Automata on Gauss Words

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Abstract. In this paper we investigate the computational complexity of knot theoretic problems and show upper and lower bounds for planarity problem of signed and unsigned knot diagrams represented by Gauss words. Due to the fact the number of crossing in knots is unbounded, the Gauss words of knot diagrams are strings over infinite (unbounded) alphabet. For establishing the lower and upper bounds on recognition of knot properties we study these problems in a context of automata models over infinite alphabet.

1 Introduction

Algorithmic and computational topology is a new growing branch of modern topology. Much of the recent effort has focused on classifying the inherent complexity of topological problems. In this paper we investigate the computational complexity of knot theoretic problems and show upper and lower bounds for planarity problem of signed and unsigned knot diagrams. The main goal of proposed approach is to give a new insight on knot problems and characterise knot problems according to their computational complexity. The results presented in this paper were achieved by a combination of methods from knot theory, automata theory and computational complexity.

Knot theory is the area of topology that studies mathematical knots and links. A knot (a link) is an embedding of a circle (several circles) in 3-dimensional Euclidean space, $\mathbb{R}^3$, considered up to a smooth deformation of an ambient space. It is well established and exciting area of mathematical research with strong connections with topology, algebra and combinatorics. Examples of interactons between knot theory and computer science include works on formal language theory [1], quantum computing [2,3,4] and computational complexity [5].

Knots can be described in various ways, including various discrete representations. For example, a common method of describing a knot is a planar diagram called a knot diagram. A knot diagram is a projection of the knot onto a plane, where at each crossing we must indicate which section is "over" and which is "under", so as to be able to recreate the original knot. As an additional information we can also add a label ("'+" or "-") to each of the crossing for representing orientation of its strands.

A knot diagram can be encoded as a string of symbols O_i's (over) and U_i's (under) also known as *Gauss word*. The procedure of writing a Gauss word can be described as follows: Starting from a base point on the circle, write down the

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labels of the crossings in the counterclockwise direction, e.g. the trefoil K can be defined by a Gauss word ${U_1}^+{O_2}^+{U_3}^+{O_1}^+{U_2}^+{O_3}^+$, where indices indicate an (arbitrary) order of the crossings in the knot diagram and signs stand for orientation of each crossing. Likewise, links can be represented by *several* Gauss words - one for each component of the link.

As you can see the construction of the Gauss word is quite straightforward by reading visited crossings travelling along a circle. The inverse problem of constructing a knot from some strings of symbols from the set of O_i's and U_i's is harder and it is not always possible. It may happen that some Gauss words would not correspond to any classical (planar) diagrams. In such case we say that a Gauss word corresponds to a non-planar knot where any of its diagram should contain virtual crossings (i.e. which are not listed in the Gauss word). Most of the problems of recognising knots properties (such as virtuality, unknottedness, equivalence) are known to be decidable, with different time complexity. However their complexity in terms of computational power of devices needed to recognise the knot properties was not studied yet. In this paper we address this problem and provide first known bounds for some knot problems in this context.

The central problem which we are studying in this paper is to determine whether a given Gauss word corresponds to planar or non-planar knot. Due to the fact the number of crossing in knots is unbounded, the Gauss words of knot diagrams are strings over infinite (unbounded) alphabet. In this context we cannot estimate computational complexity in terms of classical models over finite alphabets and need to consider a new hierarchy of languages and models over infinite alphabet. Such models were recently introduced in [6,7].

In Section 2 we describe and extend the models of automata over infinite alphabet that we used for establishing the lower and upper bounds on recognition of knot properties. Then in Section 3 we show that the language of planar (non-planar) signed Gauss words can be recognised by deterministic two-way register automata by simulation of recently discovered linear time algorithm proposed in [8]. Due to the fact that the algorithm presented in [8] allows to check planarity property not only for knots but also for links we think that the proposed idea of recognising planarity by register automata can be extended for links after some minor modification. The result is final in a sense that the power of non-deterministic one-way register automata is not even enough to recognise whether an input is a Gauss word. We also conjecture that planarity problem for unsigned Gauss words is harder than the the same problem for signed Gauss words and cannot be solved by register or k-pebble automata over infinite alphabet. For the case of unsigned Gauss words we provide the upper bound by showing that planarity can be checked by deterministic linearly bounded memory automata.

2 Automata over Infinite Alphabets

Let D be an infinite set called an *alphabet*. A word, or a string over D, or shortly, D-word or D-string is a finite sequence $d_1, \dots d_n$ where $d_i \in D$, $i = 1, \dots, n$. A language over D (D-language) is a set of D-words. For a word w and a symbol