Proof that $\underline{A334907}(n)/n! = \underline{A063079}(n+1)/\underline{A060818}(n)$ for $n \geq 0$

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Throughout this proof, we shall tacitly assume that

$$\underline{A060818}(n) = \gcd(n!, 2^n), \quad \underline{A063079}(n) = \text{numerator}\left(\binom{2n-\frac{3}{2}}{-\frac{1}{2}}\right),$$

$$\text{and} \quad \underline{A334907}(n) = \frac{\binom{4n+2}{2n+1} n!}{2^{n+1}}.$$

Before establishing the result, we state three lemmas and prove two of them.

Lemma 1. *For integer $n \geq 0$,*

$$\binom{2n+\frac{1}{2}}{-\frac{1}{2}} = \binom{2n+\frac{1}{2}}{2n+1} = \frac{\binom{4n+2}{2n+1}}{2^{4n+2}}.$$

Lemma 2. *For integer $n \geq 0$, $\gcd(n!, 2^{n+1}) = 2^{\nu_2(n!)} = \gcd(n!, 2^n)$, where $\nu_2(n!)$ is the largest power of 2 that divides $n!$.*

Proof. From Legendre's theorem, the largest power of 2 that divides $n!$ is

$$\nu_2(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor \leq \sum_{k=1}^{\infty} \frac{n}{2^k} = n.$$

This means that $\gcd(n!, 2^{n+1}) = 2^{\nu_2(n)} = \gcd(n!, 2^n)$. □

Lemma 3. *For integer $n \geq 0$,*

$$\gcd\left(\binom{4n+2,}{2n+1}, 2^{4n+2}\right) \gcd(n!, 2^n) = 2^{n+1}.$$

Proof. Throughout this proof, an empty product is by definition equal to 1. Note first that, for $n \geq 0$,

$$\begin{aligned}
\binom{4n+2}{2n+1} &= \frac{(2n+2)(2n+3)\cdots(4n+2)}{(2n+1)!} \\
&= \frac{(2n+3)(2n+5)\cdots(4n+1)2^{n+1}(n+1)(n+2)\cdots(2n+1)}{(2n+1)!} \\
&= \frac{(2n+3)(2n+5)\cdots(4n+1)2^{n+1}}{n!} \\
&= \frac{(2n+3)(2n+5)\cdots(4n+1)2^{n+1-\nu_2(n!)}}{n!/2^{\nu_2(n!)}},
\end{aligned}$$

where $\nu_2(n!)$ is defined in Lemma 2. Since $n!/2^{\nu_2(n!)}$ is odd, the largest power of 2 that divides $\binom{4n+2}{2n+1}$ is $2^{n+1-\nu_2(n!)}$. Therefore, by Lemma 2,

$$\gcd\left(\binom{4n+2,}{2n+1}, 2^{4n+2}\right)\gcd(n!, 2^n) = 2^{n+1-\nu_2(n!)} \cdot 2^{\nu_2(n!)} = 2^{n+1},$$

and this finishes the proof of the lemma. $\square$

Now we are ready to prove the main result of this note:

$$\frac{\underline{A334907}(n)}{n!} = \frac{\underline{A063079}(n+1)}{\underline{A060818}(n)} \quad \text{for integer } n \geq 0.$$

Proof. By using Lemmas 1 and 3, for integer $n \geq 0$, we have

$$\begin{aligned}
\frac{\underline{A063079}(n+1)}{\underline{A060818}(n)} &= \frac{\text{numerator}\left(\binom{2(n+1)-\frac{3}{2}}{-\frac{1}{2}}\right)}{\gcd(n!, 2^n)} \\
&= \frac{\text{numerator}\left(\frac{\binom{4n+2}{2n+1}}{2^{4n+2}}\right)}{\gcd(n!, 2^n)} \\
&= \frac{\binom{4n+2}{2n+1}}{\gcd(\binom{4n+2}{2n+1}, 2^{4n+2})\gcd(n!, 2^n)} \\
&= \frac{\binom{4n+2}{2n+1}}{2^{n+1}} = \frac{\underline{A334907}(n)}{n!}.
\end{aligned}$$

This completes the proof of the main result. $\square$