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COEFFICIENTS FOR NUMERICAL DIFFERENTIATION WITH CENTRAL DIFFERENCES

By Herbert E. Salzer*

The following table lists the coefficients $A^n_{2\nu-1}$ and $A^n_{2\nu}$ of mean central and central differences respectively, in the formula for the nth derivative in terms of central differences, namely,

$$\omega^n \phi_0^{(n)} = \sum_{\nu=\frac{n+1}{2}}^{\nu=k} A^n_{2\nu-1} \,\square\, \delta_0^{2\nu-1} + R_1 \text{ for } n \text{ odd, and}$$

$$\omega^n \phi_0^{(n)} = \sum_{\nu=\frac{n}{2}}^{\nu=k} A^n_{2\nu}\, \delta_0^{2\nu} + R_2 \text{ for } n \text{ even.}^1$$

Here ϕ_0 is an abbreviation for $\phi(u_0)$, the value of $\phi(u)$ at some interval point, ω is the tabular interval and $\square$ is the operator $\dfrac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}$.

Due to the fundamental importance of these coefficients in applied and pure mathematics,[2] because of their greater convenience when compared with coefficients of advancing differences, and because the exact forms of R_1 and R_2 involve coefficients of order considerably greater than that in the last coefficient used, it was the author's purpose to make this table the "ultimate" in coefficients for numerical differentiation. In brief, coefficients are given for derivatives as far as the 52nd, most of them going as far as the 42nd difference, and the rest going as far as some difference between the 42nd and 52nd. For the first 30 derivatives, exact values are given for coefficients of the first 30 differences, and also exact values are given for some coefficients of differences beyond the 30th. For all derivatives beyond the 30th, exact values are given for coefficients of differences going as far as some difference between the 41st and 52nd. Elsewhere, that is, for most of the coefficients of the 31st to 42nd differences, 18 significant figures are given, with accuracy to within 0.6 unit in the last significant figure. Wherever exact values are given, it is believed that the fractions are in lowest terms. Since the denominators are highly composite numbers whose factors consist of powers of the smallest primes, these fractions can readily be converted into decimal form with the aid of a 10- or 8-bank computing machine.

To obtain a formula somewhat similar to the Everett interpolation formula

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[1] For the forms of R_1 and R_2, see Milne-Thomson, *Calculus of Finite Differences*, p. 160, and Steffensen, *Interpolation*, pp. 66–67.

[2] For n even, $A^n_{2\nu} = [n/2\nu\,(2\nu - n)!]\, B^{(2\nu)}_{2\nu-n}(\nu)$ and for n odd $A^n_{2\nu-1} = [1/(2\nu - n - 1)!]$. $B^{(2\nu)}_{2\nu-n-1}(\nu)$ where $B^{(2\nu)}_{2\nu-n}(\nu)$ and $B^{(2\nu)}_{2\nu-n-1}(\nu)$ denote the $(2\nu - n)$th and $(2\nu - n - 1)$th Bernoulli polynomials of the (2ν)th order, for $x = \nu$. (See Milne-Thomson, pp. 127, 160.)

which employs central differences of only even order, it suffices merely to write $\frac{1}{2}(\delta_1^{2\nu-2} - \delta_{-1}^{2\nu-2})$ for $\Box\delta_0^{2\nu-1}$ and thus, for n odd,

$$\omega^n \phi_0^{(n)} = \sum_{\nu=\frac{n+1}{2}}^{\nu=k} \left(\tfrac{1}{2}A_{2\nu-1}^{n}\,\delta_1^{2\nu-2} - \tfrac{1}{2}A_{2\nu-1}^{n}\,\delta_{-1}^{2\nu-2}\right) + R_1$$

The chief use of these coefficients in computational work is to find the successive derivatives of functions that have already been tabulated to a sufficiently large number of decimal places at some suitable interval. Many calculated expressions can be made to difference well by use of extra factors and terms that make formal differentiation unfeasible or even impossible. Then numerical differentiation by means of these coefficients is the most practical procedure. For any polynomial (or function calculated from a polynomial) these coefficients give exact values for the derivatives of that polynomial (exclusive of rounding errors). Furthermore these coefficients can be employed whenever it is desired to extend the calculation of an analytic function into a neighboring two dimensional region when it has already been tabulated at regular intervals along a straight line in the region. The advantage of the large number of coefficients comes in when certain functions, such as Bessel, Legendre, error, gamma, etc., require a difference polynomial of very high degree to approximate them within a region. Since differences of the nth order are inaccurate to $2^n \times$ error in tabulated function, and also since ω^n is small for $\omega < 1$ and n large, the calculation of many functions to a number of decimal places far beyond their immediate use, is seen to be vital to their use in numerical differentiation and extension into the complex plane. Another possible use of the extremely large number of coefficients might occur when we are interested in an upper bound for the error in using the formula with a fixed number of differences. Then, if we can set an upper bound for the nth derivative of the function, the error is majorized by a known expression containing central difference coefficients of order much greater than that of the last difference employed. (See footnote 1.)

The coefficients were calculated by first obtaining the central factorial polynomials $x^{[2\nu]}$ and dividing their coefficients by the proper products of consecutive integers in accordance with (2) and (3) (See below). The polynomials $x^{[2\nu]}$ were checked by the fact that the sums of their coefficients were 0. The divisions were checked by multiplications and wherever the division involved exact cancellation, the checkback involved two exact multiplications.

In addition, all these values were checked to about 10 significant figures by a recurrence formula which was found for these coefficients. Since this formula appears to be new, its derivation is given below in detail. The formula can also be obtained from the definition of A_m^n (see footnote 2.) and the recurrence formulae for Bernoulli polynomials; but the derivation by that means is longer and more indirect than the following:

Let $\phi(u_0 + \omega x)$ be written as $\phi_{\omega x}$ or f_x, where x is the tabular interval. Then $\omega^n\phi^{(n)}(u_0 + \omega x) = \omega^n\phi_{\omega x}^{(n)} = f_x^{(n)}$; $\omega^n\phi_0^{(n)} = f_0^{(n)}$, $\phi_0 = f_0$. By Stirling's interpolation formula,

$$(1)\qquad f_x = \sum_{\nu=0}^{\nu=k}\left(\frac{x^{[2\nu]-1}}{(2\nu-1)!}\Box\delta_0^{2\nu-1} + \frac{x^{[2\nu]}}{(2\nu)!}\delta_0^{2\nu}\right) + \frac{x^{[2k+2]-1}}{(2k+1)!}f_\theta^{(2k+1)},$$

where $0 \leq \theta \leq 1$ and θ depends upon x. (See Steffensen, p. 27.) Here $x^{[2\nu]} \equiv x^2(x^2 - 1^2)(x^2 - 2^2) \cdots (x^2 - \overline{\nu - 1}^2)$, $x^{[2\nu]-1} \equiv x^{[2\nu]}/x$. Let C'^{n}_{m} denote the coefficient of x^m in $x(x^2 + 1^2)(x^2 + 2^2) \cdots (x^2 + n^2)$. Differentiating n times and letting $x = 0$, we obtain:

For n odd,

$$(2) \qquad f_0^{(n)} = \omega^n \phi_0^{(n)} = \sum_{\nu=0}^{\nu=k} (-1)^{\nu - \frac{n+1}{2}} \frac{n!\, C_n^{\nu-1}}{(2\nu - 1)!} \square \delta_0^{2\nu-1} + R_1$$

$$= \sum_{\nu=0}^{\nu=k} A^n_{2\nu-1} \square \delta_0^{2\nu-1} + R_1$$

For n even,

$$(3) \qquad f_0^{(n)} = \omega^n \phi_0^{(n)} = \sum_{\nu=0}^{\nu=k} (-1)^{\nu - \frac{n}{2}} \frac{n!}{(2\nu)!} C^{\nu-1}_{n-1} \delta_0^{2\nu} + R_2$$

$$= \sum_{\nu=0}^{\nu=k} A^n_{2\nu} \delta_0^{2\nu} + R_2$$

(See footnotes 1. and 2.)

Effectively the summation begins with $\nu = \dfrac{n+1}{2}$ for (2) and $\nu = \dfrac{n}{2}$ for (3). Obviously, from (2) and (3),

$$(4) \qquad A^{n+1}_{2\nu} = \frac{n+1}{2\nu} \cdot A^n_{2\nu-1} \text{ if } n \text{ is odd.}$$

From the evident relationship

$$(5) \qquad C_n^{\nu-1} = C^{\nu-2}_{n-2} + (\nu - 1)^2 C_n^{\nu-2},$$

for $\nu > 2$, after multiplication of both members by $(-1)^{\nu - \frac{n+1}{2}} n!/(2\nu - 1)!$ it is seen from (2) that for odd n the left member is now $A^n_{2\nu-1}$. The first term on the right is

$$\frac{n}{2\nu - 1} \frac{(n-1)!}{(2\nu - 2)!} C^{\nu-2}_{n-2} (-1)^{\nu - \frac{n}{2} - \frac{1}{2}}$$

But from (3),

$$A^{n-1}_{2\nu-2} = C'^{\nu-2}_{n-2} \frac{(n-1)!}{(2\nu - 2)!} (-1)^{\nu - 1 - \frac{n-1}{2}},$$

making the term equal to

$$\frac{n}{2\nu - 1} A^{n-1}_{2\nu-2}.$$

The second term on the right can be written as

$$(-1)^1 (\nu - 1)^2 \frac{1}{n+1} \cdot \frac{1}{2\nu - 1} \cdot \frac{(n+1)!}{(2\nu - 2)!} C_n^{\nu-2} (-1)^{\nu - 1 - \frac{n+1}{2}}.$$

But from (3),

$$A_{2\nu-2}^{n+1} = \frac{(n+1)!}{(2\nu-2)!} C_n^{\nu-2}(-1)^{\nu-1-\frac{n+1}{2}};$$

hence the term is

$$-(\nu-1)^2 \cdot \frac{1}{2\nu-1} \cdot \frac{1}{n+1} A_{2\nu-2}^{n+1}.$$

Thus the final recursion formula is

$$(6) \qquad A_{2\nu-1}^{n} = \frac{1}{2\nu-1}\left[nA_{2\nu-2}^{n-1} - (\nu-1)^2 \cdot \frac{1}{n+1} \cdot A_{2\nu-2}^{n+1}\right]$$

which is applied only when n is odd, since it is trivially true for n even. Although the foregoing proof is only for $\nu > 2$, formula (6) can be verified also for $\nu = 2$ and $\nu = 1$ provided we adopt the natural notation $A_k^0 = 0$ and $A_0^0 = 1$ respectively. Also for $n = 1$, $A_k^0 = 0$ gives the correct equation for (6) because then in (5) $C_{-1}^{\nu-2} \equiv 0$.

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Compare D2 with 2544

Please enter 4
2457 (denominators) 1, 6, 30, 140, ...
2544 (1, 12, 90, ...)
2701
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2457 cont
2544 cont
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COEFFICIENTS OF $\square\delta^n$ FOR n ODD AND OF δ^n FOR n EVEN

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
D1	$\frac{1}{1}$		$-\frac{1}{6}$		$\frac{1}{30}$		$-\frac{1}{140}$		$\frac{1}{630}$		$-\frac{1}{2772}$		$\frac{1}{12012}$		$-\frac{1}{51480}$		$\frac{1}{218790}$		$-\frac{1}{923780}$	
D2		$\frac{1}{1}$		$-\frac{1}{12}$		$\frac{1}{90}$		$-\frac{1}{560}$		$\frac{1}{3150}$		$-\frac{1}{16632}$		$\frac{1}{84084}$		$-\frac{1}{411840}$		$\frac{1}{1969110}$		$-\frac{1}{9237800}$
D3			1		$-\frac{1}{4}$		$\frac{7}{120}$		$-\frac{41}{3024}$		$\frac{479}{151200}$		$-\frac{59}{79200}$		$\frac{266681}{1513512000}$		$-\frac{63397}{1513512000}$		$\frac{514639}{51459408000}$	
D4				1		$-\frac{1}{6}$		$\frac{7}{240}$		$-\frac{41}{7560}$		$\frac{479}{453600}$		$-\frac{59}{277200}$		$\frac{266681}{6054048000}$		$-\frac{63397}{6810804000}$		$\frac{514639}{257297040000}$
D5					1		$-\frac{1}{3}$		$\frac{13}{144}$		$-\frac{139}{6048}$		$\frac{37}{6480}$		$-\frac{4201}{2993760}$		$\frac{3739217}{10897286400}$		$-\frac{364919}{4358914560}$	
D6						1		$-\frac{1}{4}$		$\frac{13}{240}$		$-\frac{139}{12096}$		$\frac{37}{15120}$		$-\frac{4201}{7983360}$		$\frac{3739217}{32691859200}$		$-\frac{364919}{14529715200}$
D7							1		$-\frac{5}{12}$		$\frac{31}{240}$		$-\frac{311}{8640}$		$\frac{2473}{259200}$		$-\frac{4679}{1900800}$		$\frac{5839219}{9340531200}$	
D8								1		$-\frac{1}{3}$		$\frac{31}{360}$		$-\frac{311}{15120}$		$\frac{2473}{518400}$		$-\frac{4679}{4276800}$		$\frac{5839219}{23351328000}$
D9									1		$-\frac{1}{2}$		$\frac{7}{40}$		$-\frac{67}{1260}$		$\frac{2021}{134400}$		$-\frac{21713}{5322240}$	
D10										1		$-\frac{5}{12}$		$\frac{1}{8}$		$-\frac{67}{2016}$		$\frac{2021}{241920}$		$-\frac{21713}{10644480}$
D11											1		$-\frac{7}{12}$		$\frac{41}{180}$		$-\frac{757}{10080}$		$\frac{5473}{241920}$	
D12												1		$-\frac{1}{2}$		$\frac{41}{240}$		$-\frac{757}{15120}$		$\frac{5473}{403200}$
D13													1		$-\frac{2}{3}$		$\frac{23}{80}$		$-\frac{619}{6048}$	
D14														1		$-\frac{7}{12}$		$\frac{161}{720}$		$-\frac{619}{8640}$
D15															1		$-\frac{3}{4}$		$\frac{17}{48}$	

D9									1		$-\frac{1}{2}$		$\frac{7}{40}$		$-\frac{67}{1260}$		$\frac{2021}{134400}$		$-\frac{21713}{5322240}$	
D10										1		$-\frac{5}{12}$		$\frac{1}{8}$		$-\frac{67}{2016}$		$\frac{2021}{241920}$		$-\frac{21713}{10644480}$
D11											1		$-\frac{7}{12}$		$\frac{41}{180}$		$-\frac{757}{10080}$		$\frac{5473}{241920}$	
D12												1		$-\frac{1}{2}$		$\frac{41}{240}$		$-\frac{757}{15120}$		$\frac{5473}{403200}$
D13													1		$-\frac{2}{3}$		$\frac{23}{80}$		$-\frac{619}{6048}$	
D14														1		$-\frac{7}{12}$		$\frac{161}{720}$		$-\frac{619}{8640}$
D15															1		$-\frac{3}{4}$		$\frac{17}{48}$	
D16																1		$-\frac{2}{3}$		$\frac{17}{60}$
D17																	1		$-\frac{5}{6}$	
D18																		1		$-\frac{3}{4}$
D19																			1	
D20																				1

COEFFICIENTS OF $\square\delta^n$ FOR n ODD AND OF δ^n FOR n EVEN—*Continued*

$n=$	21	22	23	24	25	26	27	28	29	30
D1	$\frac{1}{3879876}$		$-\frac{1}{16224936}$		$\frac{1}{67603900}$		$-\frac{1}{280816200}$		$\frac{1}{1163381400}$	
D2		$\frac{1}{42678636}$		$-\frac{1}{194699232}$		$\frac{1}{878350700}$		$-\frac{1}{3931426800}$		$\frac{1}{17450721000}$
D3	$-\frac{178939}{74662922880}$		$\frac{10410343}{18068427336960}$		$-\frac{18500393}{133196739984000}$		$\frac{40799043101}{1215553449093984000}$		$-\frac{1411432849}{173650492727712000}$	
D4		$-\frac{178939}{410646075840}$		$\frac{10410343}{108410564021760}$		$-\frac{18500393}{865778809896000}$		$\frac{40799043101}{8508874143657888000}$		$-\frac{1411432849}{1302378695457840000}$
D5	$\frac{1473061}{72176832000}$		$\frac{7354899857}{1478325873024000}$		$\frac{1053912709}{868674391200000}$		$-\frac{593151411269}{2003659531473600000}$		$\frac{92257290523249}{1276331121548683200000}$	
D6		$\frac{1473061}{264648384000}$		$-\frac{7354899857}{5913303492096000}$		$\frac{1053912709}{3764255695200000}$		$-\frac{593151411269}{9350411146876800000}$		$\frac{92257290523249}{6381655607743416000000}$
D7	$-\frac{800993}{5094835200}$		$\frac{374322799}{9527341824000}$		$-\frac{211789363}{21660452352000}$		$\frac{1449491072231}{597364332364800000}$		$-\frac{35791253123}{59551490718720000}$	
D8		$-\frac{800993}{14010796800}$		$\frac{374322799}{28582025472000}$		$-\frac{211789363}{70396[illegible]70144000}$		$\frac{1449491072231}{2090775163276800000}$		$-\frac{35791253123}{223318090195200000}$
D9	$\frac{10684483}{9906624000}$		$-\frac{61085281}{217945728000}$		$\frac{615453101}{8550178560000}$		$-\frac{2090580163331}{114042281633280000}$		$\frac{909495616199}{196010171557200000}$	
D10		$\frac{10684483}{21794572800}$		$-\frac{61085281}{523069747200}$		$\frac{615453101}{222304641256000}$		$-\frac{2090580163331}{319318388573184000}$		$\frac{909495616199}{588030514671600000}$
D11	$-\frac{2677}{414720}$		$\frac{14091479}{7925299200}$		$-\frac{1748857}{3657830400}$		$\frac{437843543}{3464487936000}$		$-\frac{95763529279}{2902894441574400}$	
D12		$-\frac{2677}{760320}$		$\frac{14091479}{15850598400}$		$-\frac{1748857}{7925299200}$		$\frac{437843543}{8083805184000}$		$-\frac{95763529279}{7257236103936000}$
D13	$\frac{118921}{3628800}$		$-\frac{392251}{39916800}$		$\frac{189156857}{67060224000}$		$-\frac{45139487}{57480192000}$		$\frac{365667859}{1710035712000}$	
D14		$\frac{10811}{518400}$		$-\frac{392251}{68428800}$		$\frac{189156857}{124540416000}$		$-\frac{45139487}{114960384000}$		$\frac{365667859}{3664362240000}$
D15	$-\frac{1639}{12096}$		$\frac{11129}{241920}$		$-\frac{462397}{[illegible]}$		$\frac{323535929}{[illegible]}$		$\frac{74616869}{[illegible]}$	

D7	$-\frac{[illegible]}{5094835200}$		$\frac{[illegible]}{9527341824000}$		$-\frac{[illegible]}{21660452352000}$		$\frac{[illegible]}{597364332364800000}$		[illegible]	
D8		$-\frac{800993}{14010796800}$		$\frac{374322799}{28582025472000}$		$-\frac{[illegible]363}{703[illegible]0144000}$		$\frac{1449491072231}{2090775163276800000}$		$-\frac{35791253123}{223318090195200000}$
D9	$\frac{10684483}{9906624000}$		$-\frac{61085281}{217945728000}$		$\frac{615453101}{8550178560000}$		$-\frac{2090586163331}{114042281633280000}$		$\frac{909495616199}{196010171557200000}$	
D10		$\frac{10684483}{21794572800}$		$-\frac{61085281}{523069747200}$		$\frac{615453101}{22230464256000}$		$-\frac{2090586163331}{319318388573184000}$		$\frac{909495616199}{588030514671600000}$
D11	$-\frac{2677}{414720}$		$\frac{14091479}{7925299200}$		$-\frac{1748857}{3657830400}$		$\frac{437843543}{3464487936000}$		$-\frac{95763529279}{2902894441574400}$	
D12		$-\frac{2677}{760320}$		$\frac{14091479}{15850598400}$		$-\frac{1748857}{7925299200}$		$\frac{437843543}{8083805184000}$		$-\frac{95763529279}{7257236103936000}$
D13	$\frac{118921}{3628800}$		$-\frac{392251}{39916800}$		$\frac{189156857}{67060224000}$		$-\frac{45139487}{57480192000}$		$\frac{365667859}{1710035712000}$	
D14		$\frac{10811}{518400}$		$-\frac{392251}{68428800}$		$\frac{189156857}{124540416000}$		$-\frac{45139487}{114960384000}$		$\frac{365667859}{3664362240000}$
D15	$-\frac{1639}{12096}$		$\frac{11129}{241920}$		$-\frac{462397}{31933440}$		$\frac{323535929}{74724249600}$		$-\frac{74616869}{59779399680}$	
D16		$-\frac{149}{1512}$		$\frac{11129}{362880}$		$-\frac{35569}{3991680}$		$\frac{323535929}{130767436800}$		$-\frac{74616869}{112086374400}$
D17	$\frac{77}{180}$		$-\frac{331}{1890}$		$\frac{114101}{1814400}$		$-\frac{354971}{17107200}$		$\frac{844617209}{130767436800}$	
D18		$\frac{7}{20}$		$-\frac{331}{2520}$		$\frac{8777}{201600}$		$-\frac{354971}{26611200}$		$\frac{844617209}{217945728000}$
D19	$-\frac{11}{12}$		$\frac{61}{120}$		$-\frac{559}{2520}$		$\frac{7261}{86400}$		$-\frac{154477}{5322240}$	
D20		$-\frac{5}{6}$		$\frac{61}{144}$		$-\frac{43}{252}$		$\frac{7261}{120960}$		$-\frac{154477}{7983360}$
D21	1		-1		$\frac{143}{240}$		$-\frac{1193}{4320}$		$\frac{1903}{17280}$	
D22		1		$-\frac{11}{12}$		$\frac{121}{240}$		$-\frac{13123}{60480}$		$\frac{20933}{259200}$
D23			1		$-\frac{13}{12}$		$\frac{497}{720}$		$-\frac{4097}{12096}$	
D24				1		-1		$\frac{71}{120}$		$-\frac{4097}{15120}$

COEFFICIENTS OF $\square\delta^n$ FOR n ODD AND OF δ^n FOR n EVEN—*Continued*

$n =$	21	22	23	24	25	26	27	28	29	30
D25					1		$-\frac{7}{6}$		$\frac{19}{24}$	
D26						1		$-\frac{13}{12}$		$\frac{247}{360}$
D27							1		$-\frac{5}{4}$	
D28								1		$-\frac{7}{6}$
D29									1	
D30										1

COEFFICIENTS OF $\square\delta^n$ FOR n ODD AND OF δ^n FOR n EVEN—*Continued*

$n =$	31	32	33	34
D1	−.(9) 2079[illegible]872190124186		.(10) 504142720460907118	
D2		−.(10) 129974295118827616		.(11) 296554541447592422
D3	.(8) 1971[illegible]947345435825		−.(9) 479242062914258009	
D4		.(9) 246499934181794782		−.(10) 563814191663832952
D5	−.(7) 1766[illegible]655051227041		.(8) 431920677914820020	
D6		−.(8) 331174978221050702		.(9) 762212961026152977
D7	.(6) 1486[illegible]9074889066 07		−.(7) 367439244965674668	
D8		.(7) 371677268722266517		−.(8) 864562929330999218
D9	−.(5) 1169[illegible]111983929049		.(6) 293559954562556477	
D10		−.(6) 365350349949778277		.(7) 863411631066342581
D11	.(5) 853[illegible]734105294141		−.(5) 218966886599609371	
D12		.(5) 319876025289485303		−.(6) 772824305645680132
D13	−.(4) 5726[illegible]53630974676		.(4) 151433286724973585	
D14		−.(4) 250548573463551421		.(5) 623548827691067703
D15	.(3) 35027[illegible]06592425417		−.(4) 963026094828706203	
D16		.(3) 175135203296212709		−.(4) 453188750507626448
D17	−.(2) 19277[illegible]44231482150		.(3) 557544666501642063	
D18		−.(2) 108433656130208710		.(3) 295170705794986974
D19	.(2) 9397[illegible]921530556451		−.(2) 290246025415709609	
[illegible]		.(2) 587334325956597782		−.(2) 1707[illegible]
D21	$-\frac{1889}{47520}$		.(1) 133743538322531378	
D22		$-\frac{1889}{69120}$		.(2) 865399365616379505
D23	$\frac{128699}{907200}$		$-\frac{12798671}{239500800}$	

D14		−.(4) 250548573463551421		.(5) 623548827691067703
D15	.(3) 350270406592425417		−.(4) 963026094828706203	
D16		.(3) 175135203296212709		−.(4) 453188750507626448
D17	−.(2) 192770944231482150		.(3) 557544666501642063	
D18		−.(2) 108433656130208710		.(3) 295170705794986974
D19	.(2) 939734921530556451		−.(2) 290246025415709609	
D20		.(2) 587334325956597782		−.(2) 170732956126888005
D21	$-\frac{1889}{47520}$		.(1) 133743538322531378	
D22		$-\frac{1889}{69120}$		.(2) 865399365616379505
D23	$\frac{128699}{907200}$		$-\frac{12798671}{239500800}$	
D24		$\frac{128699}{1209600}$		$-\frac{752863}{19958400}$
D25	$-\frac{155}{378}$		$\frac{26129}{145152}$	
D26		$-\frac{2015}{6048}$		$\frac{19981}{145152}$
D27	$\frac{9}{10}$		$-\frac{1649}{3360}$	
D28		$\frac{63}{80}$		$-\frac{97}{240}$
D29	$-\frac{4}{3}$		$\frac{731}{720}$	
D30		$-\frac{5}{4}$		$\frac{43}{48}$
D31	1		$-\frac{17}{12}$	
D32		1		$-\frac{4}{3}$
D33			1	
D34				1

NOTE: The numbers in parentheses indicate the number of zeros following the decimal po nt.

COEFFICIENTS OF $\Box\delta^n$ FOR n ODD AND OF δ^n FOR n EVEN—*Continued*

n =	35	36	37	38
D1	−.(10) 12243466[illegible]3363157		.(11) 297814039500072545	
D2		−.(12) 680192559352017540		.(12) 156744231315827655
D3	.(9) 116641547[illegible]3274881		−.(10) 284274191180413510	
D4		.(10) 129601719714749868		−.(11) 299235990716224747
D5	−.(8) 10570047[illegible]4547481		.(9) 258860627461260762	
D6		−.(9) 176167451040912468		.(10) 408727306517780150
D7	.(8) 907596711[illegible]0018681		−.(8) 224099665331048834	
D8		.(8) 201688158062226374		−.(9) 471788769117997545
D9	−.(7) 735162936[illegible]6148568		.(7) 183729345132738942	
D10		−.(7) 204211926779485713		.(8) 483498276665102478
D11	.(6) 558912518[illegible]8279239		−.(6) 142022859133025606	
D12		.(6) 186304172906093080		−.(7) 448493239367449283
D13	−.(5) 396471456[illegible]6061111		.(5) 102984825164391108	
D14		−.(5) 154183344099579321		.(6) 379417776921440925
D15	.(4) 26060128[illegible]3731551		−.(5) 696401783733230524	
D16		.(4) 115822794254991800		−.(5) 293221803677149694
D17	−.(3) 15741572[illegible]5483500		.(4) 436119004851847989	
D18		−.(4) 787078648732417500		.(4) 206582686508770100
D19	.(3) 86511873[illegible]4001979		−.(3) 250851838049935434	
D20		.(3) 480621516991112211		−.(3) 132027283184176544
D21	−.(2) 42724550[illegible]2280436		.(2) 131203101948698907	
D22		−.(2) 261094478090282488		.(3) 759596906018783146
D23	.(1) 186649405[illegible]11632694		−.(2) 616314040327660533	
D24		.(1) 124432937007755129		−.(2) 389250972838522442
D25	$-\frac{10745}{152064}$		.(1) 255954508010972958	
D26		$\frac{139685}{2737152}$		.(1) 175126768639086761
D27	10097 [illegible]			

D15	.(4) 260601287073731551		−.(5) 696401783733230524	
D16		.(4) 115822794254991[illegible]		−.(5) 293221803677149694
D17	−.(3) 157415729746483500		.(4) 436119004851847989	
D18		−.(4) 787078648732417500		.(4) 206582686508770100
D19	.(3) 86[illegible]118730584001979		−.(3) 250851838049935434	
D20		.(3) 480621516991112211		−.(3) 132027283184176544
D21	−.(2) 42[illegible]245509602280436		.(2) 131203101948698907	
D22		−.(2) 261094478090282488		.(3) 759596906018783146
D23	.(1) 186649405511632694		−.(2) 616314040327660533	
D24		.(1) 124432937007755129		−.(2) 380250972838522442
D25	$-\frac{10745}{152064}$		.(1) 255954508010972958	
D26		$-\frac{139685}{2737152}$		.(1) 175126768639086761
D27	$\frac{10097}{44800}$		$-\frac{1633259}{17740800}$	
D28		$\frac{10097}{57600}$		$-\frac{85961}{1267200}$
D29	$-\frac{5861}{10080}$		$\frac{112423}{403200}$	
D30		$-\frac{5861}{12096}$		$\frac{5917}{26880}$
D31	$\frac{91}{80}$		$-\frac{41287}{60480}$	
D32		$\frac{91}{90}$		$-\frac{2173}{3780}$
D33	$-\frac{3}{2}$		$\frac{19}{15}$	
D34		$-\frac{17}{12}$		$\frac{17}{15}$
D35	1		$-\frac{19}{12}$	
D36		1		$-\frac{3}{2}$
D37			1	
D38				1

COEFFICIENTS OF $\Box\delta^n$ FOR n ODD AND OF δ^n FOR n EVEN—*Continued*

$n =$	43	44	45	46	47	48	49	50	51	52
D33	$-\frac{342773}{1814400}$									
D34		$-\frac{5827141}{39916800}$								
D35	$\frac{12881}{25920}$		$-\frac{804103}{3421440}$							
D36		$\frac{1171}{2880}$		$-\frac{34961}{190080}$						
D37	$-\frac{10637}{10080}$		$\frac{179101}{302400}$		$-\frac{240923}{831600}$					
D38		$-\frac{18373}{20160}$		$\frac{147953}{302400}$		$-\frac{4577537}{19958400}$				
D39	$\frac{407}{240}$		$-\frac{72841}{60480}$		$\frac{423767}{604800}$		$-\frac{13463}{38016}$			
D40		$\frac{37}{24}$		$-\frac{3167}{3024}$		$\frac{423767}{725760}$		$-\frac{13463}{47520}$		
D41	$-\frac{11}{6}$		$\frac{667}{360}$		$-\frac{5167}{3780}$		$\frac{119507}{145152}$		$-\frac{20575789}{47900160}$	
D42		$-\frac{7}{4}$		$\frac{203}{120}$		$-\frac{5167}{4320}$		$\frac{119507}{172800}$		$-\frac{1582753}{4561920}$
D43	1		$-\frac{23}{12}$		$\frac{121}{60}$		$-\frac{9335}{6048}$		$\frac{3488771}{3628800}$	
D44		1		$-\frac{11}{6}$		$\frac{1331}{720}$		$-\frac{20537}{15120}$		$\frac{2952037}{3628800}$
D45			1		-2		$\frac{35}{16}$		$-\frac{3497}{2016}$	

D39	$\frac{407}{240}$		$-\frac{[illegible]}{60480}$		$\frac{[illegible]}{604800}$		$\frac{[illegible]}{38016}$			
D40		$\frac{37}{24}$		$-\frac{3167}{3024}$		$\frac{423767}{725760}$		$-\frac{13463}{47520}$		
D41	$-\frac{11}{6}$		$\frac{667}{360}$		$-\frac{5167}{3780}$		$\frac{119507}{145152}$		$-\frac{20575789}{47900160}$	
D42		$-\frac{7}{4}$		$\frac{203}{120}$		$-\frac{5167}{4320}$		$\frac{119507}{172800}$		$-\frac{1582753}{4561920}$

D43	1		$-\frac{23}{12}$		$\frac{121}{60}$		$-\frac{9335}{6048}$		$\frac{3488771}{3628800}$	
D44		1		$-\frac{11}{6}$		$\frac{1331}{720}$		$-\frac{20537}{15120}$		$\frac{2952037}{3628800}$
D45			1		-2		$\frac{35}{16}$		$-\frac{3497}{2016}$	
D46				1		$-\frac{23}{12}$		$\frac{161}{80}$		$-\frac{6187}{4032}$
D47					1		$-\frac{25}{12}$		$\frac{1703}{720}$	
D48						1		-2		$\frac{131}{60}$
D49							1		$-\frac{13}{6}$	
D50								1		$-\frac{25}{12}$
D51									1	
D52										1