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Comput. & Graphics Vol. 13, No. 1, pp. 59–62, 1989
Printed in Great Britain.

0097–8493/89 $3.00 + .00

SELF-SIMILAR SEQUENCES AND CHAOS FROM GAUSS SUMS

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Abstract—It is shown that the quadratic Gauss sums form self-similar sequences with well-defined scales and similarity dimensions. It is also shown that Gauss sums can be modified to yield intricate chaotic patterns.

1. FRACTAL SEQUENCES

A self-similar (fractal) sequence $\{f[n]\}$ can be defined by the power law

$$f[s \cdot n] = s^a f[n], \qquad (1)$$

with $n > 0$ ranging over all positive integers, while $f[n]$ may be real or complex. The positive integer $s > 1$ is called the *scale* of the sequence, whereas $a > 0$ is a real number which serves as the similarity (fractal) dimension of the sequence. Provided $f[n] \neq 0$, it should be noted that

$$a = \log(f[s \cdot n]/f[n])/\log(s \cdot n/n), \qquad (2)$$

which serves to validate the fractal nature[1] of the sequence by explicitly revealing the respective statuses of a and s.

The sequence $\{f[n]\}$ is completely determined by the specification of only a subset of its members. These members, countably infinite in number, are:

(i) $f[1]$ or $f[s]$,
(ii) $f[m]$; $m = 2, 3, 4, \ldots, s-2, s-1$, and
(iii) $f[m]$; $m > s$, $(m, s) = 1$.

From Eq. (1) it is readily seen that $f[1]$ and $f[s]$ are linked by the relation

$$f[s] = s^a f[1]; \qquad (3)$$

hence, the specification of either one of them is sufficient to determine the other. Furthermore, if $m < s$, then $m(\text{mod } s) = m$, or m is relatively prime to s; therefore, the specifications (ii) cannot be built up from Eq. (1) and must be independently given. Lastly, the specifications (iii) must also be made, since Eq. (1) cannot be used to generate them if the greatest common divisor of m and s is unity. But for any $n > s$, which has s as one of its divisors, $f[n]$ can be ascertained through Eq. (1). In fractal parlance, the specifications (i)–(iii) collectively are the *initiators* of the sequence, while the power law (1) serves as its *generator*[2].

A very simple example of such a sequence is the following one,

$$1, 3, 0, 9, 0, 0, 0, 27, 0, 0, 0, 0, 0, 0, 0, 81, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 243, \ldots \qquad (4a)$$

which has a scale $s = 2$ and a similarity dimension $a = \log(3)/\log(2)$. This sequence is replete with zeros, but can be easily augmented to yield a more substantial example,

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$$1, 3, 5, 9, 11, 15, 19, 27, 29, 33, 37, 45, 49, 57, 65, 81, 83, 87, 91, 99, 103, 111, 119, 135, 139, 147, 155, 171, 179, 195, 211, 243, \ldots \qquad (4b)$$

which too has a scale $s = 2$ and a similarity dimension $a = \log(3)/\log(2)$. Yet another example is furnished by the sequence

A6048

$$1, 3, 6, 8, 12, 18, 21, 27, 36, 38, 42, 48, 52, 60, 72, 78, 90, 108, 111, 117, 126, 132, 144, 162, 171, 189, 216, \ldots \qquad (5)$$

which has a scale $s = 3$ and a similarity dimension $a = \log(6)/\log(3)$.

2. FRACTAL SEQUENCES FROM GAUSS SUMS

The bi-indexed numbers defined as the quadratic Gauss sum

$$\varphi(p, q) = \sum_{r=0,1,2,\ldots,q-1} \exp[2\pi i(p/q)r^2], \qquad (6)$$

where p and q are positive integers, provide interesting examples of the fractal sequences defined above; $i = \sqrt{(-1)}$ as usual. These numbers have been much investigated in recent years, especially the celebrated Gauss sums $\varphi(1, q)$ which have a long history[3] dating back to the great Gauss himself. It has been shown by Nagell[4] that

$$\varphi(p, q) = 2\varphi(p, q/4), \qquad (7)$$

provided $p \geq 1$ is odd, and $q \geq 16$ is a power of 2. This relation forms the basis of a family of fractal sequences $\{f_p[q]\}$, $q = 1, 2, 3, \ldots$, the parameter p identifying the genus of the sequence within the family. The members of the p-th sequence thus derived are given by

$$f_p[1] = (1 + i^p)$$

$$f_p[2] = \sqrt{(2i^{p-1})} \cdot (1 + i)$$

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$$f_p[4^k] = \varphi(p, 4^k); \quad k = 1, 2, 3, \cdots$$

$$f_p[4^{k+1/2}] = \varphi(p, 4^{k+1/2}); \quad k = 1, 2, 3, \cdots$$

with $f_p[q] = 0$ otherwise. Note that $\varphi(p, 1) = 1$, and $\varphi(p, 2) = 0$ for odd p; it would have been nice to be able to define $f_p[1]$ and $f_p[2]$ in terms of the sums (6), but even so. The scale s for these sequences can be immediately seen to be equal to 4, while the dimension $a = 0.5$. Furthermore, $|f_p[q]|$ is seen to be independent of (odd) p.

Similar fractal sequences $\{f_N[q]\}$ can also be constructed by utilizing another identity given in[4],

$$\varphi(1, 2^q N) = (1 + i)\sqrt{(2^q N)}, \tag{8}$$

which holds for all integer $N > 0$ and $q \geq 2$. Such sequences would have a scale $s = 2$, while their similarity dimension would still be $a = 0.5$: verification of this statement is given by the ratio

$$\log(\varphi(1, 2^q N)/\varphi(1, 2^{q-1} N))/\log(2^q N/2^{q-1} N) = \log(\sqrt{2})/\log(2) = 0.5. \tag{9}$$

3. EXTENSION TO THE COMPLEX PLANE

The fractal sequences given in Section 2 are comprised of either zeros or complex numbers, but p and q are positive integers. Thus, complex patterns cannot be obtained from them, unlike the case of the Collatz sequence $(3n + 1)$[5-7]. Recently, Berry and Goldberg[8] have studied the quadratic sum

$$S_L(\tau) = \sum_{r=1,2,\ldots,L} \exp[i\pi\tau r^2], \tag{10a}$$

which can specialize to

$$S_q(2/q) = \varphi(1, q). \tag{10b}$$

The parameter τ is always real in their study, and they have worked out a renormalization transform to explore $S_L(\tau)$ in the complex domain.

An extension of the quadratic Gauss sums can, however, be fruitfully utilized to yield complicated patterns provided p can be made complex. Accordingly, we can define the function

$$F(z; q, c) = c + \sum_{r=0,1,2,\ldots,q-1} \exp[2\pi i z q^{-1} r^2], \tag{11}$$

where z and c are complex. In general, the Julia sets of the mapping $z \Rightarrow F(z; q, c)$ yield very intricate patterns. For $q = 1$, $F(z; q, c) = c + 1$, which is not very interesting. When $q = 2$, then $F(z; q, c) = c + 1 + \exp[2\pi i z]$. The Julia set for the resulting process can be equivalently expressed as $z \Rightarrow \exp[2\pi i(c + 1)]\exp[2\pi i z]$, which is a take-off on the mapping $z \Rightarrow \lambda \exp[z]$ due to Misiurewicz[9] and Devaney[10]. When viewed on a monochrome monitor using the program given here, the most distinguishing feature of such sets is a multitude of swirling speckled bands of varying sizes. For higher q, the detail is even more spectacular, chiefly consisting of gouged-out bulbs of different sizes. Color pictures are highly intricate; as examples, see Figs. 1 and 2 for $F(z; 2, 0)$ and $F(z; 3, 0)$, respectively.

The function $F(z; q, c)$ can be made even more interesting by using higher order Gauss sums, or even negative order sums, e.g., consider the function

$$F(z; q, n, c) = c + \sum_{r=0,1,2,\ldots,q-1} \exp[2\pi i z q^{-1} r^n], \tag{12}$$

in which the integer n can be zero, negative or positive. Yet another generalization, reminiscent of the Cornu spirals[6], is the function

$$F(z; \alpha, n, c) = c + \int_0^\alpha dr \exp[2\pi i z(\alpha + 1)^{-1} r^n], \tag{13}$$

where $\alpha > 0$ is positive real.

Despite Gauss' assertion that "the theory of numbers is the Queen of mathematics" because queens do not dirty their hands with grime, this branch of mathematics is increasingly being applied. Thus, the quadratic Gauss sums $\varphi(1, q)$ have been utilized in what may be termed quadratic-residue phase gratings, the ultimate in that application being the frosted glass and the sound diffusors in concert halls[11]. And here, these same sums have been linked to the eminently utilitarian concepts of fractals and chaos.

Acknowledgement—We thank Dr. C. A. Pickover of IBM Watson Research Center for generously creating Figs. 1 and 2 for this communication.

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