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Sueli I.R. Costa • Frédérique Oggier Antonio Campello • Jean-Claude Belfiore Emanuele Viterbo

# Lattices Applied to Coding for Reliable and Secure Communications



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