Appendix

A.1 Basic Notions and Notations

Domains

$\mathbb{N} = \{0, 1, 2, \dots\}$	natural numbers including zero
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	integers, rational, real, and complex numbers
$\mathbb{K}$	an arbitrary field of characteristic zero
$\bar{\mathbb{K}}$	the algebraic closure of $\mathbb{K}$
$\mathbb{K}^n$	vector space over $\mathbb{K}$ of dimension n
$\mathbb{K}^{n\times m}$	vector space of $n \times m$ matrices over $\mathbb{K}$
R	an arbitrary commutative ring containing $\mathbb{Q}$ as subring
$R[x]$	polynomials in x with coefficients in R (p. 44)
$\mathbb{K}(x)$	rational functions in x with coefficients in $\mathbb{K}$ (p. 44)
$R[[x]]$	formal power series in x with coefficients in R (p. 18)
$\mathbb{K}((x))$	formal Laurent series in x with coefficients in $\mathbb{K}$ (p. 23)
$R^{\mathbb{N}}$	ring of sequences over R

Logic

$A \overset{!}{=} B$	desired equality: A shall be (made) equal to B
$A \overset{?}{=} B$	questioned equality: Are A and B equal?

Asymptotics

$\lim_{n\to\infty} a_n = c$	limit of a convergent sequence: $\forall\ \varepsilon > 0\ \exists\ n_0\ \forall\ n \geq n_0 : \lvert a_n - c\rvert < \varepsilon$
$a_n \sim b_n\ (n \to \infty)$	asymptotic equivalence of sequences: $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

$\lim_{z\to\zeta} a(z) = c$	limit of functions: $\forall\ \varepsilon > 0\ \exists\ \delta > 0\ \forall\ z : \lvert z-\zeta\rvert < \delta \Rightarrow \lvert a(z)-c\rvert < \varepsilon$
$a(z) \sim b(z)\ (z \to \zeta)$	asymptotic equivalence of functions: $\lim_{z\to\zeta} \frac{a(z)}{b(z)} = 1$
$a_n = \mathrm{O}(b_n)\ (n \to \infty)$	O-notation: $\exists\ c \in \mathbb{R}\ \exists\ n_0\ \forall\ n \geq n_0 : \lvert a_n\rvert \leq cb_n$
$a_n = \mathrm{o}(b_n)\ (n \to \infty)$	o-notation: $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

Sequences

$(a_n)_{n=0}^{\infty}$	the sequence $a_0, a_1, a_2, \ldots$; formally $a\colon \mathbb{N} \to \mathbb{K}$, $n \mapsto a_n$
$(a_{n,k})_{n,k=0}^{\infty}$	notation for a bivariate sequence
Δa_n	forward difference: $a_{n+1} - a_n$
$\sum_{k=u}^{v} a_k$	the sum $a_u + a_{u+1} + \cdots + a_v$. If $v < u$, the sum is defined as 0.
$\prod_{k=u}^{v} a_k$	the product $a_u a_{u+1} \cdots a_v$. If $v < u$, the product is defined as 1.
$p(x) \bullet (a_n)_{n=0}^{\infty}$	action of the polynomial $p(x) \in \mathbb{K}[x]$ on the sequence $(a_n)_{n=0}^{\infty}$ (p. 68)

Linear Algebra

$(a_i)_{i=1}^{n}$	the vector $(a_1, a_2, \ldots, a_n) \in \mathbb{K}^n$
$((a_{i,j}))_{i,j=1}^{n}$	the $n \times n$ matrix with entry $a_{i,j}$ in row i and column j.
$\det A$	determinant of the matrix $A \in \mathbb{K}^{n\times n}$
$\dim V$	dimension of the vector space V
$V \oplus W$	direct sum of the vector spaces V and W

Polynomials and formal power series

$[x^n]a(x)$	coefficient of the monomial x^n in $a(x)$
$\deg_x p(x)$, $\deg p(x)$	degree of the polynomial $p(x)$: $\max\{n \in \mathbb{N} : [x^n]p(x) \neq 0\}$
$\mathrm{lc}_x\, p(x)$, $\mathrm{lc}\, p(x)$	leading coefficient of the polynomial $p(x)$: $\mathrm{lc}\, p(x) = [x^{\deg p(x)}]p(x)$
$\mathrm{res}_x(p(x), q(x))$	resultant of the polynomials $p(x)$ and $q(x)$ (p. 124)
$\gcd(p(x), q(x))$	greatest common divisor of the polynomials $p(x)$ and $q(x)$ (p. 168)
$\mathrm{ord}\, a(x)$	order of the series $a(x)$ (p. 23)
$a(x) \odot b(x)$	Hadamard product of $a(x)$ and $b(x)$ (p. 18)
$D_x a(x)$	(formal) derivative of $a(x)$ (p. 20)
$\int_x a(x)$	(formal) integral of $a(x)$ (p. 20)
$\lim_{k\to\infty} a_k(x)$	limit of a sequence of power series (p. 24)
$a(b(x))$	composition of $a(x)$ with $b(x)$ (p. 25)
$\sum_{k=0}^{\infty} a_k(x)$, $\prod_{k=0}^{\infty} a_k(x)$	infinite sum and product of a sequence of power series (p. 28)

Common particular functions and sequences

$\lfloor x \rfloor$	floor of x: greatest $n \in \mathbb{Z}$ with $n \leq x$
$\lceil x \rceil$	ceiling of x: smallest $n \in \mathbb{Z}$ with $n \geq x$
$x^{\overline{n}}$	rising factorial: $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$
$x^{\underline{n}}$	falling factorial: $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$
$n!$	factorial: $n! = n^{\underline{n}} = 1^{\overline{n}} = 1 \cdot 2 \cdot 3 \cdots n$
$\binom{x}{k}$	binomial coefficient (p. 88)
$S_1(n,k),\ S_2(n,k)$	Stirling numbers of the first and second kind, respectively (p. 45)
B_n	Bernoulli numbers (p. 23)
$\mathbf{B}_n$	Bell numbers (p. 26)
C_n	Catalan numbers (p. 113)
p_n	partition numbers (p. 27)
F_n	Fibonacci numbers (p. 63)
H_n	harmonic numbers (p. 8)
$H_n^{(r_1,\ldots,r_m)}$	generalized harmonic numbers (p. 138)
$\Gamma(z)$	gamma function (p. 91)
$T_n(x),\ P_n(x),\ H_n(x)$	Chebyshev polynomials of the first kind (p. 54), Legendre polynomials (p. 132), Hermite polynomials (p. 161)
$J_n(x)$	Bessel functions (p. 159)

A.2 Basic Facts from Computer Algebra

Classical computer algebra focusses on algorithms for computing with polynomials. We list here only those very basic results from computer algebra which are used in some part of this book. In particular, we only state here *what* can be computed and not *how* to carry out these computations. For further background and many additional algorithmic results about polynomials, we refer to the standard text books on computer algebra [22, 63, 58].

In order to do computations in $\mathbb{K}[x]$, some technical conditions have to be imposed on the field $\mathbb{K}$. For example, there is no way to do exact computations in $\mathbb{R}$ or $\mathbb{C}$, because the elements of these fields are inherently infinite objects and cannot be faithfully represented in a (finite) computer. But computations in the field $\mathbb{Q}$ of rational numbers, in algebraic number fields such as $\mathbb{Q}(\sqrt{2})$, in rational function fields $\mathbb{Q}(x)$, or in finite fields $\mathbb{Z}_p$ can be carried out exactly. Such fields are called admissible.

Let $\mathbb{K}$ be an admissible field. Then there are algorithms

- to compute for given $a(x), b(x) \in \mathbb{K}[x]$ the coefficients of $a(x)+b(x)$, $a(x)b(x)$, $a(b(x))$, etc.;
- to compute for given $a(x), b(x) \in \mathbb{K}[x]$ the unique polynomials $q(x), r(x) \in \mathbb{K}[x]$ with $r(x) = 0$ or $\deg r(x) < \deg b(x)$ such that $a(x) = q(x)b(x) + r(x)$ (*division with remainder*);

- to compute for given $a(x), b(x) \in \mathbb{K}[x]$ the unique monic polynomial $g(x) := \gcd_x(a(x), b(x)) := \gcd(a(x), b(x)) \in \mathbb{K}[x]$ of maximal degree with $g(x) \mid a(x)$ and $g(x) \mid b(x)$ (*greatest common divisor*);
- to compute for given $a(x), b(x) \in \mathbb{K}[x]$ the unique polynomials $u(x), v(x) \in \mathbb{K}[x]$ with $g(x) := \gcd(a(x), b(x)) = u(x)a(x) + v(x)b(x)$ and $\deg u(x) < \deg b(x) - \deg g(x)$ and $\deg v(x) < \deg a(x) - \deg g(x)$ (*Bezout coefficients*);
- to compute for a given set $\{(x_0, y_0), \dots, (x_n, y_n)\} \subseteq \mathbb{K}^2$ with $x_i \neq x_j$ $(i \neq j)$ the unique polynomial $p(x) \in \mathbb{K}[x]$ of degree at most n such that $p(x_i) = y_i$ $(i = 0, \dots, n)$ (*polynomial interpolation*);
- to compute for a given $a(x) \in \mathbb{K}[x]$ a polynomial $\bar{a}(x) \in \mathbb{K}[x]$ which has the same roots as $a(x)$ in $\bar{\mathbb{K}}$ but not repeated factors (*square free part*);
- to compute for a given $a(x) \in \mathbb{K}[x]$ a representation $a(x) = a_1(x)a_2(x)^2 \cdots a_m(x)^m$ where $a_1(x), \dots, a_m(x)$ have no repeated factors and mutually disjoint sets of roots (*square free decomposition*).

Under slightly more restrictive assumptions on $\mathbb{K}$ (which are still satisfied for all fields appearing in this book), there are also algorithms

- to determine for a given polynomial $a(x) \in \mathbb{K}[x]$ the set of all $n \in \mathbb{Z}$ such that $a(n) = 0$; (*integer roots*),
- to decide for a given polynomial $a(x) \in \mathbb{K}[x]$ whether there exists $b(x) \in \mathbb{K}[x]$ with $1 < \deg b(x) < \deg a(x)$ and $b(x) \mid a(x)$ (*irreducibility*);
- to compute for a given polynomial $a(x) \in \mathbb{K}[x]$ the unique factorization $a(x) = cp_1(x)^{e_1} \cdots p_m(x)^{e_m}$ into monic irreducible polynomials (*factorization*).

All these operations can be performed efficiently, which in theory means the algorithms performing the listed operations have a runtime complexity which depends polynomially on the size of the input. In practice, using the most careful implementations of the most advanced algorithms on the most recent hardware, the manipulation of polynomials with some ten thousand terms may well be feasible.

Some of the items in the list above generalize from univariate polynomials to polynomials in several variables. In particular it is possible to compute the greatest common divisor of multivariate polynomials. However, for multivariate polynomials there is in general no Bezout representation of the greatest common divisor, because $\mathbb{K}[x_1, \dots, x_n]$ is not a Euclidean domain. It is a unique factorization domain, though, and for many types of fields $\mathbb{K}$ there are algorithms that split a given multivariate polynomial into irreducible factors.

A.3 A Collection of Formal Power Series Identities

We list here the most frequently needed identities related to quantities appearing in the text. Additional series expansions can be found in the book of Wilf [62]. Further identities for special functions can be found in tables like the classical volume of Abramowitz and Stegun [5] or its recent successor project [19]. A version of the latter is also available electronically at http://dlmf.nist.gov (Digital Library of Mathematical Functions)

$$\log(1-x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n \qquad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad \sum_{n=0}^{\infty} n^{\overline{k}} x^n = \frac{k!}{(1-x)^{k+1}}$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \qquad \sum_{n=0}^{\infty} H_n x^n = \frac{1}{1-x} \log \frac{1}{1-x}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \sum_{n,k=0}^{\infty} \binom{n}{k} x^n y^k = \frac{1}{1-x-xy}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad \sum_{n=0}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}$$

$$(1+x)^{\lambda} = \sum_{n=0}^{\infty} \binom{\lambda}{n} x^n \qquad \sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$$

$$x^n = \sum_{k=0}^{n} S_2(n,k) x^{\underline{k}} \qquad \sum_{k=0}^{n} S_1(n,k) x^k = x^{\underline{n}}$$

$$\frac{x}{\exp(x)-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \text{ (Bernoulli numbers)} \qquad \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$$

$$\exp(\mathrm{e}^x - 1) = \sum_{n=0}^{\infty} \frac{\mathbf{B}_n}{n!} x^n \text{ (Bell numbers)} \qquad \sum_{n=0}^{\infty} T_n(x) y^n = \frac{1-xy}{1-2xy+y^2}$$

$$P_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{2^n} \binom{n}{k} \binom{2n-2k}{k} x^{n-2k} \qquad \sum_{n=0}^{\infty} P_n(x) y^n = \frac{1}{\sqrt{1-2xy+y^2}}$$

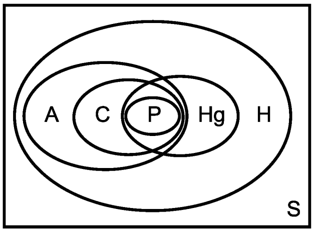
$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k} \qquad \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^n = \exp(2xy - y^2)$$

$${}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| x\right) = \sum_{k=0}^{\infty} \frac{a_1^{\overline{k}} a_2^{\overline{k}} \cdots a_p^{\overline{k}}}{b_1^{\overline{k}} \cdots b_q^{\overline{k}}} \frac{x^k}{k!}$$

A.4 Closure Properties at One Glance

We summarize the closure properties of the classes of formal power series discussed in this book. Not all of the facts stated in the tables below are explicitly mentioned in the text. The following abbreviations are used:

F polynomial generating function / finite sequence
P generating function of the form $p(x)/(1-x)^d$ / sequences which from some index on agree with a polynomial sequence
C rational generating function / C-finite sequence
Hg hypergeometric series
A algebraic series
H holonomic series
S formal power series



Unary operations

For $1/a(x)$ it is assumed that $a(0) \neq 0$.

$a(x)$	$\frac{1}{a(x)}$	$\frac{a(x)-a(0)}{x}$	$\frac{1}{1-x}a(x)$	$D_x a(x)$	$\int_x a(x)$
F	C	F	P	F	F
P	C	P	P	P	H
C	C	C	C	C	H
Hg	S	Hg	H	Hg	H
A	A	A	A	A	H
H	S	H	H	H	H
S	S	S	S	S	S

Binary operations

For $a(b(x))$ it is assumed that $b(0) = 0$.

$a(x)$	$b(x)$	$a(x)+b(x)$	$a(x)b(x)$	$a(x) \odot b(x)$	$a(b(x))$
F	F	F	F	F	F
F	P	P	P	F	P
P	F	P	P	F	C
P	P	P	P	P	C
C	C	C	C	C	C
Hg	Hg	H	H	Hg	S
A	A	A	A	H	A
H	F	H	H	F	H
H	A	H	H	H	H
H	H	H	H	H	S
S	S	S	S	S	S

A.5 Software

The algorithms described in this book as well as many other algorithms which are useful for solving problems related to the Concrete Tetrahedron have been implemented and are available in computer algebra systems. We give here a brief overview of some of the most basic maneuvers in Maple and Mathematica. Details can be found in the documentation of the respective pieces of software.

Maple

Gosper's and Zeilberger's algorithm are part of the built-in *sum* command:

$sum(k\,k!, k = 0..n);$

$$(n+1)! - 1 \tag{1}$$

$sum(\mathrm{binomial}(n,k), k = 0..n);$

$$2^n \tag{2}$$

For definite sums, the *sum* command applies first Zeilberger's algorithm to find a recurrence for the sum and then uses Petkovšek's algorithm to solve this recurrence. If no closed form exists, some standardized format of the sum is returned:

$sum(\mathrm{binomial}(n,k)^2\,\mathrm{binomial}(n+k,k)^2, k = 0..n);$

$$\mathrm{hypergeom}([-n,-n,n+1,n+1],[1,1,1],1) \tag{3}$$

For getting the recurrence of a sum, there is the command *Zeilberger* in the *Hypergeometric* section of the package *SumTools*:

$SumTools[Hypergeometric][Zeilberger](\mathrm{binomial}(n,k)^2\,\mathrm{binomial}(n+k,k)^2, n, k, N);$

$$\Big[(6n^2+12n+8+n^3)N^2+(-34n^3-153n^2-231n-117)N+3n+1+n^3+3n^2, \tag{4}$$
$$\frac{(-4-2n^2-6n-\frac{3}{2}k+k^2)\,k^4\,\mathrm{binomial}(n,k)^2\,\mathrm{binomial}(n+k,k)^2\,(16n+24)}{(-n-1+k)^2\,(-n-2+k)^2}\Big]$$

Hypergeometric solutions of holonomic recurrence equations can be found via the built-in *rsolve* command.

$rsolve(\{2\,(n^2+2n+2)\,a(n)-(n^3+5n^2+4n+4)\,a(n+1)+(n^2+1)\,(n+2)\,a(n+2)=0, a(0)=1, a(1)=1\}, a(n));$

$$-n+\frac{2^n}{\Gamma(n+1)} \tag{5}$$

A description of Maple's procedures for hypergeometric summation is given in [3].

Functions for executing closure properties for holonomic sequences and power series are available in the package *gfun* by Salvy and Zimmermann [49]. This package allows, for example, to compute a recurrence for $(a_n+b_n)_{n=0}^{\infty}$ or $(a_nb_n)_{n=0}^{\infty}$ given recurrences for $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$. Note that both input recurrences must be represented with the same function symbol, and that this symbol is also used in the output recurrence:

$gfun[`rec+rec`](\{f(n+2)=f(n+1)+f(n), f(0)=0, f(1)=1\},$
$\{f(n+1)=\frac{n+1}{n+2}f(n), f(0)=1\}, f(n))$

$$\{(n^3+9n^2+22n+14)\,f(n)+(-n^2-9n-14)\,f(n+1)+(-19n^2-53n -42-2n^3)\,f(n+2)+(n^3+10n^2+31n+28)\,f(n+3), f(0)=1, f(1)=\frac{3}{2}, f(2)=\frac{4}{3}\} \quad (6)$$

$gfun[`rec*rec`](\{f(n+2)=f(n+1)+f(n), f(0)=0, f(1)=1\},$
$\{f(n+1)=\frac{n+1}{n+2}f(n), f(0)=1\}, f(n))$

$$\{(-n-1)\,f(n)+(-n-2)\,f(n+1)+(n+3)\,f(n+2), f(0)=0, f(1)=\frac{1}{2}\} \quad (7)$$

The package also provides guessing facilities:

$gfun[listtorec]([1,1,2,5,14,42,132,429,1430,4862,16796], f(n));$

$$[\{(-4n-2)\,f(n)+(n+2)\,f(n+1), f(0)=1\}, ogf] \quad (8)$$

$gfun[listtoalgeq]([1,1,2,5,14,42,132,429,1430,4862,16796], f(x));$

$$[-1+f(x)-x\,f(x)^2, ogf] \quad (9)$$

Conversely, the *series* command determines the first terms of a series

$series((1-sqrt(1-4x))/(2x), x=0,10);$

$$1+x+2x^2+5x^3+14x^4+42x^5+132x^6+429x^7+1430x^8+\mathrm{O}(x^9) \quad (10)$$

Also Puiseux expansions can be computed with this command, for instance the expansion of the generating function for Catalan numbers at its singularity $x=1/4$:

$series((1-sqrt(1-4x))/(2x), x=1/4,3);$

$$4-4\mathrm{i}(x-\tfrac{1}{4})^{1/2}-8x+16\mathrm{i}(x-\tfrac{1}{4})^{3/2}+32(x-\tfrac{1}{4})^2-64\mathrm{i}(x-\tfrac{1}{4})^{5/2}+\mathrm{O}((x-\tfrac{1}{4})^3) \quad (11)$$

Mathematica

Like its cousin in Maple, the **Sum** command in Mathematica resorts to the algorithms of Gosper and Zeilberger for simplifying hypergeometric sums. As an alternative to this built-in command, there are also special purpose add-on packages which provide functionality for doing summation and related calculations. Some packages are available at

http://www.risc.jku.at/research/combinat/software/

There is a package of Paule and Schorn [42] for hypergeometric summation, a package by Mallinger [39] for holonomic closure properties and guessing, and several other useful packages with sophisticated algorithms which are not discussed in this book, such as Schneider's package Sigma for simplification of nested sums [50], Kauers's multivariate guessing library [31], Wegschaider's package for multivariate hypergeometric summation [61], or Koutschan's package for multivariate holonomic series [36]. Of these, we give here just some examples for the Paule-Schorn package and the Mallinger package:

In[1]:= **<< zb.m**
Fast Zeilberger Package by Peter Paule and Markus Schorn (enhanced by Axel Riese – © RISC Linz – V 3.54 (02/23/05)

In[2]:= **Gosper[$k\,k!$, k]**

Out[2]= $\{k k! = \Delta_k k!\}$

In[3]:= **Gosper[$k\,k!$, {k, 0, n}]**
If `n' is a natural number, then:

Out[3]= $\{\sum_{k=0}^{n} kk! == -1+(n+1)n!\}$

In[4]:= **Zb[Binomial[n,k]^2 Binomial[$n+k,k$]^2, k, n]**
If `n' is a natural number, then:

Out[4]= $\{(1+n)^2 F[k,n]-(3+2n)(39+51n+17n^2)F[k,n+1]+(2+n)^3 F[k,n+2] == \Delta_k F[k,n]R[k,n]\}$

In[5]:= **Show[F]**

Out[5]= $\binom{n}{k}^2\binom{k+n}{n}^2$

In[6]:= **Show[R]**

Out[6]= $\dfrac{k^4(-12k(3+2n)+8k^2(3+2n)-16(3+2n)(2+3n+n^2))}{(1-k+n)^2(2-k+n)^2}$

In[7]:= **Zb[Binomial[n,k]^2 Binomial[$n+k,k$]^2, {k, 0, n}, n]**
If `n' is a natural number, then:

Out[7]= $\{(1+n)^2\mathrm{SUM}[n]-(3+2n)(39+51n+17n^2)\mathrm{SUM}[n+1]+(2+n)^3\mathrm{SUM}[n+2] == 0\}$

In[8]:= **<< GeneratingFunctions.m**
GeneratingFunctions Package by Christian Mallinger – © RISC Linz – V 0.68 (07/17/03)

In[9]:= **REPlus[{$f[n+2] == f[n]+f[n+1], f[0] == 0, f[1] == 1$}, {$f[n+1] == \frac{n+1}{n+2}f[n], f[0] == 1$}, $f[n]$]**

Out[9]= $\{(1+n)(14+8n+n^2)f[n]-(2+n)(7+n)f[1+n]-(3+n)(14+13n+2n^2)f[2+n]+(4+n)(7+6n+n^2)f[3+n] == 0, f[0] == 1, f[1] == \frac{3}{2}, f[2] == \frac{4}{3}\}$

In[10]:= **REHadamard[{$f[n+2] == f[n]+f[n+1], f[0] == 0, f[1] == 1$}, {$f[n+1] == \frac{n+1}{n+2}f[n]$, $f[0] == 1$}, $f[n]$]**

Out[10]= $\{(-1-n)f[n]+(-2-n)f[1+n]+(3+n)f[2+n] == 0, f[0] == 0, f[1] == \frac{1}{2}\}$

In[11]:= **GuessRE[{1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796}, $f[n]$]**

Out[11]= $\{\{-2(1+2n)f[n]+(2+n)f[1+n] == 0, f[0] == 1\}, \mathrm{ogf}\}$

Mathematica's built-in **Series** command is useful for going the other direction:

In[12]:= **Series[(1 – Sqrt[1 – 4x])/(2x), {x, 0, 5}]**

Out[12]= $1+x+2x^2+5x^3+14x^4+42x^5+\mathrm{O}[x]^6$

In[13]:= **Series[(1 – Sqrt[1 – 4x])/(2x), {x, 1/4, 3}]**

Out[13]= $2-(4\mathrm{i})\sqrt{-\frac{1}{4}+x}-8(-\frac{1}{4}+x)+(16\mathrm{i})(-\frac{1}{4}+x)^{3/2}+32(-\frac{1}{4}+x)^2-(64\mathrm{i})(-\frac{1}{4}+x)^{5/2}-128(-\frac{1}{4}+x)^3+\mathrm{O}[-\frac{1}{4}+x]^{7/2}$

A Mathematica implementation of Petkovšek's algorithm by Petkovšek himself is available at

http://www.fmf.uni-lj.si/~petkovsek/software.html

This implementation returns the shift quotients of hypergeometric solutions as output:

```
In[14]:= << Hyper.m
In[15]:= Hyper[2(n^2 + 2n + 2)a[n] - (n^3 + 5n^2 + 4n + 4)a[n + 1] + (n^2 + 1)(n + 2)a[n + 2], a[n],
         Solutions → All]
```

Out[15]= $\left\{\frac{2}{n+1}, \frac{1+n}{n}\right\}$

A.6 Solutions to Selected Problems

Problem 1.1 Induction on n.

Problem 1.2 1. $(n+1)^2$; 2. $\frac{1}{6}n(n+1)(2n+1)$; 3. $\frac{1}{4}n^2(n+1)^2$.

Problem 1.3 $\frac{n+1}{2n}$.

Problem 1.4 From the definition of the Riemann integral we obtain

$$\log(n) = \int_1^n \frac{1}{x}\,dx \le \sum_{k=1}^{n-1} 1 \times \max_{x\in[k,k+1]} \frac{1}{x} = \sum_{k=1}^{n-1} \frac{1}{k} \le H_n,$$

so $H_n - \log(n) \ge 0$ $(n \ge 1)$. Secondly, from

$$\left(1+\frac{1}{n}\right)^{n+1} \ge \mathrm{e} \qquad (n \ge 1)$$

we obtain

$$(n+1)\big(\log(n+1) - \log(n)\big) \ge 1 \qquad (n \ge 1)$$

by taking logarithm on both sides. Dividing both sides by $n+1$ and using $H_{n+1} - H_n = \frac{1}{n+1}$ gives

$$H_n - \log(n) \ge H_{n+1} - \log(n+1) \qquad (n \ge 1),$$

so $H_n - \log(n)$ is decreasing. The claim follows.

Problem 1.5 1. $\frac{x}{(1-x)^2}$; 2. $-\frac{x(x+1)}{(1-x)^3}$; 3. $-\frac{1}{x}\log(1-x)$.

Problem 1.6 $a(2x)$.

Problem 1.7 Use the relation $\log(-z) = \mathrm{i}\pi + \log(z)$.

Problem 2.2 Use the order as degree function. Then the greatest common divisor of two power series $a(x), b(x) \in \mathbb{K}[[x]]$ is the series of lower order, for whenever $\operatorname{ord} a(x) < \operatorname{ord} b(x)$, then $a(x) \mid b(x)$.

Problem 2.3 Using the definition, the Cauchy product formula, and the binomial theorem, we can calculate

$$\exp(ax)\exp(bx) = \left(\sum_{n=0}^{\infty} a^n \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} b^n \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\right)\frac{x^n}{n!}$$
$$= \sum_{n=0}^{\infty}(a+b)^n \frac{x^n}{n!} = \exp((a+b)x).$$

Problem 2.4 Hopefully none.

Problem 2.7 If $D(a) = 0$ and $D(b) = 0$, then $D(a+b) = D(a) + D(b) = 0$ and $D(ab) = D(a)b + aD(b) = 0$.

Problem 2.8 1. $(n+1)2^n$; 2. $\frac{1}{2}(n+1)(n+2)$; 3. $\frac{1}{2^n n!}$.

Problem 2.9 "⇒" If $(a_n(x))_{n=0}^{\infty}$ is a Cauchy sequence then for every fixed $n \in \mathbb{N}$ there is some k_0 such that for $k, l \geq k_0$ the first n terms of $a_k(x)$ and $a_l(x)$ agree. Hence if we set $a_n := [x^n]a_k(x)$ for some $k \geq k_0$ then this definition will not depend on the choice of k. Now set $a(x) := \sum_{n=0}^{\infty} a_n x^n$ with a_n defined in this way. Then we have $\operatorname{ord}(a(x) - a_k(x)) > n$ for all $k > k_0$, hence $(a_n(x))_{n=0}^{\infty}$ converges to $a(x)$.

"⇐" Suppose that $(a_n(x))_{n=0}^{\infty}$ converges to $a(x)$. Let $n \in \mathbb{N}$ and $k_0 \in \mathbb{N}$ be such that for all $k \geq k_0$ we have $\operatorname{ord}(a(x) - a_k(x)) > n$. Then for $k, l \geq k_0$ we have $\operatorname{ord}(a(x) - a_k(x)) > n$ and $\operatorname{ord}(a(x) - a_l(x)) > n$. Now $\operatorname{ord}(a_k(x) - a_l(x)) = \operatorname{ord}((a(x) - a_l(x)) - (a(x) - a_k(x))) \geq \max(\operatorname{ord}(a(x) - a_l(x)), \operatorname{ord}(a(x) - a_k(x))) > n$, as desired.

Problem 2.10 Let $(K_n)_{n=0}^{\infty}$ be the sequence in question. We show by induction that $\operatorname{ord}(K_{n+1} - K_n) > n$ for all $n \in \mathbb{N}$. The claim then follows.

For $n = 0$ the claim is true because $\operatorname{ord}(\frac{x}{1-x} - x) = 2$. Assume now that it is true for some $n \in \mathbb{N}$. Then

$$\operatorname{ord}(K_{n+1} - K_n) > n \Longrightarrow \operatorname{ord}((1 - K_{n+1}) - (1 - K_n)) > n$$
$$\Longrightarrow \operatorname{ord}\left(\frac{1}{1-K_{n+1}} - \frac{1}{1-K_n}\right) > n \Longrightarrow \operatorname{ord}\left(\frac{x}{1-K_{n+1}} - \frac{x}{1-K_n}\right) > n+1$$
$$\Longrightarrow \operatorname{ord}(K_{n+2} - K_{n+1}) > n+1,$$

as desired.

Problem 2.12 Consider $B(x) := \frac{x}{\exp(x)-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$. We clearly have

$$[x^n](\exp(x) - 1)B(x) = [x^n]x = 0 \qquad (n \geq 2).$$

Now observe that

$$\big(\exp(x) - 1\big)B(x) = \left(\sum_{n=1}^{\infty} \frac{1}{n!}x^n\right)\left(\sum_{n=0}^{\infty} \frac{B_n}{n!}x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n-1} \frac{1}{k!(n-k)!}B_k\right)x^n.$$

Taking the coefficient of n and multiplying by $n!$ gives

$$[x^n]\big(\exp(x)-1\big)B(x) = \sum_{k=0}^{n-1}\binom{n}{k}B_k \qquad (n \geq 0).$$

The claim follows.

Problem 2.14 From $D_x \exp(e^x-1) = e^x \exp(e^x-1)$ we obtain

$$\sum_{n=0}^{\infty}(n+1)\frac{\mathbf{B}_{n+1}}{(n+1)!}x^n = \left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right)\left(\sum_{n=0}^{\infty}\frac{\mathbf{B}_n}{n!}x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\mathbf{B}_k\right)x^n.$$

Comparing coefficients of x^n and multiplying by $n!$ on both sides gives the claim.

Problem 2.15 Writing $a(x) = \sum_{k=0}^{\infty} a_k x^k$, we have

$$[x^n]\frac{1}{1-x}a\Big(\frac{x}{x-1}\Big) = [x^n]\sum_{k=0}^{\infty}a_k(-1)^k\frac{x^k}{(1-x)^{k+1}} = \sum_{k=0}^{\infty}a_k(-1)^k\binom{n}{k}.$$

Now use that $\binom{n}{k} = 0$ for $k > n$.

Problem 2.16 The product rule implies $D_x b(x)^n = nb(x)^{n-1}b'(x)$ for all $n \geq 0$. Therefore, if $a(x) = \sum_{n=0}^{\infty} a_n x^n$ then

$$D_x a(b(x)) = \sum_{n=0}^{\infty} a_n D_x b(x)^n = \sum_{n=0}^{\infty} a_n n b(x)^{n-1} b'(x) = a'(b(x))b'(x),$$

the first step being justified by reference to the hint.

Problem 2.17 $1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6-\frac{5}{128}x^8+\frac{7}{256}x^{10}+\cdots$

Problem 2.19 First set $q_n^{(1)} := 2q_{2n}-q_n$ to get rid of the term α_1/n. Next, because of $q_n^{(1)} = \beta/n^2+\cdots$ we have $q_{2n}^{(1)} = \beta/4n^2+\cdots$, so the quadratic term can be eliminated by taking $4q_{2n}^{(1)}-q_n^{(1)}$. Since this converges to three times the original limit, we set

$$q_n^{(2)} := \tfrac{1}{3}\left(4q_{2n}^{(1)}-q_n^{(1)}\right) = \tfrac{1}{3}(8q_{4n}-6q_{2n}+q_n).$$

For $q_n = C_{n+1}/C_n$ and $n=7$ this yields the estimate $\frac{719}{180} \approx 3.99444$ for the limit. The general formula for eliminating the first k terms in the asymptotic expansion is $q_n^{(k)} := (2^k q_{2n}^{(k-1)} - q_n^{(k-1)})/(2^k-1)$.

Problem 2.20 1. $(1-x-x^2)f(x) = x$; 2. $f(x)^2 = 1-x-x^2$; 3. $(1-x)^2 f'(x) - (2x^2-4x+1)f(x) = 0$.

Problem 3.1 Yes. The high order coefficients of a truncated power series are considered unknown whereas the high order coefficients of a polynomial are zero. For example, if $a(x) = 1+x+x^2+\mathrm{O}(x^3)$ is a truncated power series then $a(x)^2 = 1+$

$2x+3x^2+\mathrm{O}(x^3)$ and we know nothing about the coefficient of x^3 in $a(x)^2$. On the other hand, if $a(x)=1+x+x^2$ is a polynomial, then $a(x)^2=1+2x+3x^2+2x^3+x^4$.

Problem 3.2 Yes. The object $\sum_{n=0}^{\infty}(1+x^n)y^n$ belongs to $\mathbb{K}[x][[y]]$ but not to $\mathbb{K}[[y]][x]$.

Problem 3.3 Since we assume throughout the book that $\mathbb{K}$ is a field of characteristic zero, $\mathbb{K}[x]$ and $\mathrm{Pol}(\mathbb{K})$ are isomorphic. For fields of positive characteristic, this is not the case.

Problem 3.5 $x^{\overline{n}}=\sum_{k=0}^{n}(-1)^{n+k}S_1(n,k)x^k$; $x^n=\sum_{k=0}^{n}(-1)^{n+k}S_2(n,k)x^{\overline{k}}$.

Problem 3.6 1. $x^{\underline{3}}+5x^{\underline{2}}+5x^{\underline{1}}+1$; 2. $2x^{\underline{3}}+11x^{\underline{2}}+3x^{\underline{1}}+2$; 3. $5x^{\underline{3}}+17x^{\underline{2}}+7x^{\underline{1}}+3$.

Problem 3.8 Let $a_k(x):=\sum_{n=k}^{\infty}\frac{S_2(n,k)}{n!}x^n$. We start with

$$S_2(n+1,k+1)=(k+1)S_2(n,k+1)+S_2(n,k)\qquad(n,k\geq 0).$$

Multiplying both sides by $x^n/n!$ and summing from k to ∞ gives

$$D_x a_{k+1}(x)=(k+1)a_{k+1}(x)+a_k(x)\qquad(k\geq 0).$$

For $k=0$ we have $S_2(n,k)=\delta_{n,0}$, so $a_0(x)=1$. Furthermore, we have $a_k(0)=0$ for all $k>0$. Together with this initial conditions, the differential equation determines each $a_k(x)$ uniquely. For $k=1,2,3,\ldots$ we find

$$a_1(x)=\mathrm{e}^x-1,\quad a_2(x)=\tfrac{1}{2}(\mathrm{e}^x-1)^2,\quad a_3(x)=\tfrac{1}{6}(\mathrm{e}^x-1)^3,\ldots$$

and the general form $a_k(x)=\frac{1}{k!}(\mathrm{e}^x-1)^k$ is easily confirmed by induction on k.

For the result about the Bell numbers, observe that

$$\sum_{n=0}^{\infty}\frac{\mathbf{B}_n}{n!}x^n=\mathrm{e}^{\mathrm{e}^x-1}=\sum_{k=0}^{\infty}\frac{1}{k!}(\mathrm{e}^x-1)^k=\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{S_2(n,k)}{n!}x^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{S_2(n,k)}{n!}x^n,$$

where in the last step we exploited that $S_2(n,k)=0$ for $k<0$ or $k>n$. The desired identity now follows by comparing coefficients and multiplying by $n!$.

Problem 3.9 1. The first identity is clear by $\exp(kx)=\sum_{d=0}^{\infty}k^d\frac{y^d}{d!}$. Next, we have

$$\sum_{k=0}^{n}\exp(ky)=\sum_{k=0}^{n}\exp(y)^k=\frac{\exp(y)^{n+1}-1}{\exp(y)-1}.$$

Multiplying by $y/(d+1)!$ and taking the coefficient of y^{d+1} on both sides leads to the desired identity.

2. The first identity is obtained from

$$\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}y^n=\frac{y\exp(xy)}{\exp(y)-1}=\frac{y}{\exp(y)-1}\sum_{n=0}^{\infty}\frac{x^n}{n!}y^k=\left(\sum_{n=0}^{\infty}\frac{B_n}{n!}y^n\right)\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}y^k\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{B_k x^{n-k}}{k!(n-k)!}\right)y^k$$

by comparing coefficients of y^k and multiplying by $n!$. As a consequence, we get

$$\begin{aligned} B_{d+1}(n+1) - B_{d+1} &= \sum_{k=0}^{d+1} B_k \binom{d+1}{k} (n+1)^{d+1-k} - B_{d+1} \\ &= \sum_{k=0}^{d} B_k \binom{d+1}{k} (n+1)^{d-k+1}. \end{aligned}$$

Problem 3.10 If $(H_n)_{n=0}^{\infty}$ were a polynomial sequence, then also $\Delta(H_n)_{n=0}^{\infty} = (\frac{1}{n+1})_{n=0}^{\infty}$.

Problem 3.11 In Mathematica:

```
genfun[poly_,x_] := Module[{c,n,k},
  c = Table[(-1)^(n+k) StirlingS2[n,k], {k,0,Length[c] - 1}, {n,0,Length[c] - 1}];
  c = c . CoefficientList[poly /. x -> x - 1,x];
  Together[Sum[c[[k]](k - 1)!/(1 - x)^k, {k,1,Length[c]}]]]
```

Problem 3.13 1. $2x^4 + 10x^3 + 16x^2 + 11x + 6$; 2. no solution; 3. $x - 1$.

Problem 3.14 A similar reasoning as for the recurrence case gives the following case distinction: If $\deg q(x) + 1 \neq \deg r(x)$ then

$$\deg a(x) = \deg p(x) - \max(\deg q(x), \deg r(x)).$$

Otherwise, if $\mathrm{lc}\, r(x)/\mathrm{lc}\, q(x)$ is not an integer then $\deg a(x) = \deg p(x) - \deg r(x)$. Otherwise $\deg a(x) \leq \max(\deg p(x) - \deg r(x), \mathrm{lc}\, r(x)/\mathrm{lc}\, q(x))$.

Problem 3.15 A general bound is given in Sect. 7.4.

Problem 3.16 The chromatic polynomial is the same as for the graph in Fig. 3.3: $k^4 - 5k^3 + 8k^2 - 4k$. Setting $k = 1000$ yields 995007996000 colorings.

Problem 4.1 Use repeated squaring to compute $\phi^{2^{1000}}$ with a decent approximation of ϕ (a few hundred digits, say). Depending on the computer algebra system, it might be necessary to divide by some power of 10 from time to time. The result is $F_{2^{1000}} = 419087604\ldots$.

Problem 4.2 Use the logarithmic computation scheme for Fibonacci numbers and keep intermediate results reduced modulo 10^{10}, i.e., do the computations in the residue class ring $\mathbb{Z}_{10^{10}}$. The result is $F_{2^{1000}} = \ldots 48059253307$.

Problem 4.3 $F_{2n} + F_{2n+1} - F_{n+1}$. More generally, $\sum_{k=0}^{n} F_{m+k} = F_{n+m} + F_{n+m+1} - F_{m+1}$ for every $m \in \mathbb{N}$.

Problem 4.6 The finite continued fraction equals F_{n+2}/F_{n+1}. The infinite one is therefore ϕ.

Problem 4.8 If $p(x)$ is the characteristic polynomial of a C-finite recurrence satisfied by $(a_n)_{n=0}^{\infty}$, then letting this polynomial act as an operator on both sides of the inhomogeneous equation shows that $(u_n)_{n=0}^{\infty}$ satisfies the C-finite recurrence whose characteristic polynomial is $p(x)(x^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0)$.

Problem 4.9 If $(H_n)_{n=0}^{\infty}$ is C-finite, then by Theorem 4.2 also $(H_{n+1}-H_n)_{n=0}^{\infty}$ is C-finite, so it suffices to show that $(1/(n+1))_{n=0}^{\infty}$ is not C-finite. Assume it were. Then

$$c_0\frac{1}{n}+c_1\frac{1}{n+1}+\cdots+c_r\frac{1}{n+r}=0 \qquad (n\geq 1)$$

for some constants $c_0,\dots,c_r$ with $c_r\neq 0$. Multiply by $n(n+1)\cdots(n+r)$ to obtain a polynomial relation of degree at most $r-1$. This relation, being valid for all integers $n\geq 1$, actually holds for all $n\in\mathbb{C}$. Setting $n=-r$ implies $c_r=0$, a contradiction.

Problem 4.10 Induction on k. For $k=1$ we have $S_2(n,1)=1$ $(n\geq 0)$, which is clearly C-finite. If $(S_2(n,k))_{n=0}^{\infty}$ is C-finite for some k, then the recurrence

$$S_2(n+1,k+1)-(k+1)S_2(n,k+1)=S_2(n,k) \qquad (n,k\geq 0),$$

in combination with the result of Problem 4.8, implies that $(S_2(n,k+1))_{n=0}^{\infty}$ is C-finite.

Problem 4.11 1. The identities are immediate consequences of Cassini's identity. It follows that $u(F_n,F_{n+1})^2-1=0$ for all $n\in\mathbb{N}$, and consequently

$$a(F_n,F_{n+1})(u(F_n,F_{n+1})^2-1)=0 \quad (n\in\mathbb{N})$$

for any $a(x,y)\in\mathbb{Q}[x,y]$.

2. Write $p(x,y)=p_0(x)+p_1(x)y+\cdots+p_d(x)y^d$ for $p_i(x)\in\mathbb{Q}[x]$. Proceed by induction on d. For $d<2$ there is nothing to show. For $d\geq 2$ consider

$$\bar{p}(x,y)=p(x,y)-p_d(x)y^{d-2}(u(x,y)-1).$$

We have $\deg_y\bar{p}(x,y)<2$, and by the induction hypothesis there is some $\bar{a}(x,y)$ with $\bar{p}(x,y)=q(x,y)+\bar{a}(x,y)(u(x,y)-1)$ and $q(x,y)$ at most linear in y. Setting $a(x,y)=\bar{a}(x,y)-p_d(x)y^{d-2}$ completes the induction step. The conclusion follows because $u(F_{2n},F_{2n+1})-1=0$ for all $n\in\mathbb{N}$.

3. If at least one of $q_0(x)$ and $q_1(x)$ is nonzero then $d:=\max(\deg q_0(x),1+\deg q_1(x))$ is a nonnegative integer. In this case, we have

$$\lim_{n\to\infty}\frac{q_0(F_{2n})+q_1(F_{2n})F_{2n+1}}{F_{2n}^d}=[x^d]q_0(x)+\phi\,[x^{d-1}]q_1(x)\neq 0,$$

so we cannot have $q_0(F_{2n})+q_1(F_{2n})F_{2n+1}=0$ for all $n\in\mathbb{N}$ then. The conclusion follows by combining this result with the previous step.

4. The argument for (F_{2n+1},F_{2n}) in place of (F_{2n},F_{2n+1}) is fully analogous. (In the asymptotics argument, divide by F_{2n+1}^d and find $1/\phi$ instead of ϕ in the limit expression.) To get the final conclusion, observe that if $p(F_n,F_{n+1})=0$ for all $n\in\mathbb{N}$ then in particular for all even n, implying that $p(x,y)$ is a multiple of $u(x,y)-1$, as well as for all odd n, implying that $p(x,y)$ is a multiple of $u(x,y)+1$. Putting things together, $p(x,y)$ must be a multiple of $u(x,y)^2-1$, as claimed.

In general, if $(a_n^{(1)})_{n=0}^{\infty}, \dots, (a_n^{(m)})_{n=0}^{\infty}$ are some sequences in a field $\mathbb{K}$, the set of all polynomials $p \in \mathbb{K}[x_1,\dots,x_m]$ with $p(a_n^{(1)},\dots,a_n^{(m)}) = 0$ for all $n \in \mathbb{N}$ forms an ideal in the ring $\mathbb{K}[x_1,\dots,x_m]$. If the $(a_n^{(i)})_{n=0}^{\infty}$ are C-finite in $\mathbb{Q}$, then a basis of this ideal can be computed by an algorithm [32].

Problem 4.12 No. For example, the solution of the recurrence $a_{n+2} = a_n$ is not uniquely determined by requesting that $a_0 = a_2 = 1$, because these conditions are satisfied by the two distinct solutions $(1)_{n=0}^{\infty}$ and $((-1)^n)_{n=0}^{\infty}$.

Problem 4.13 Because of $D_x^k \bar{a}(x) = \sum_{n=0}^{\infty} a_{n+k} \frac{x^n}{n!}$ for every k, there is a one-to-one correspondence between differential equations with constant coefficients for $\bar{a}(x)$ and recurrence equations with constant coefficients for $(a_n)_{n=0}^{\infty}$.

Problem 4.14 The characteristic polynomial of the recurrence is also the characteristic polynomial of the matrix M. Its roots $u_1,\dots,u_r$ are therefore the eigenvalues of M. As they are distinct, it follows that M is equivalent to a diagonal matrix with $u_1,\dots,u_r$ on the diagonal. A direct calculation confirms that $(1,u_i,\dots,u_i^{r-1})$ is an eigenvector for u_i, and hence the representation $M = TDT^{-1}$ is established.

If $(a_n)_{n=0}^{\infty}$ is a solution of the recurrence, then

$$(a_n, a_{n+1}, \dots, a_{n+r-1}) = M^n (a_0, a_1, \dots, a_{r-1}) \quad (n \in \mathbb{N}).$$

Because of $M^n = (TDT^{-1})^n = TD^nT^{-1}$, this implies that $(a_n)_{n=0}^{\infty}$ can be written as a linear combination of $(u_i^n)_{n=0}^{\infty}$ $(i = 1,\dots,r)$. Conversely, every vector $(a_0, a_1, \dots, a_{r-1})$ of initial values gives rise to a solution $(a_n)_{n=0}^{\infty}$. This implies Theorem 4.1 for the present situation.

Problem 5.1 For instance via $1 \Rightarrow 4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$.

Problem 5.2 Write $p(x,y) = p_0(x) + p_1(x)y + p_2(x)y + \cdots + p_d(x)y^d$ for $p_i(x) \in \mathbb{K}[x]$. Suppose one of the $p_i(x)$ is not the zero polynomial. Then there would be some n_0 with $p_i(n_0) \neq 0$. Then the univariate polynomial $p(n_0,y) \in \mathbb{K}[y]$ would not be the zero polynomial, although $p(n_0,m) = 0$ for all $m \in \mathbb{N}$. Contradiction.

Problem 5.3 $(p(n)a^n)_{n=0}^{\infty}$ where $p(x) \in \mathbb{K}[x]$ and $a \in \mathbb{K} \setminus \{0\}$.

Problem 5.4 $\frac{6}{\sqrt{5}\pi} n^{-2} 2^{5n} 3^{6n} 5^{-5n}$.

Problem 5.5 Substitute $f(x) = \sum_{n=0}^{\infty} \frac{a^{\overline{n}} b^{\overline{n}}}{c^{\overline{n}} n!} x^n$, $f'(x) = \sum_{n=0}^{\infty} \frac{(a+1)^{\overline{n}} (b+1)^{\overline{n}}}{(c+1)^{\overline{n}} n!} x^n$, and $f''(x) = \sum_{n=0}^{\infty} \frac{(a+2)^{\overline{n}} (b+2)^{\overline{n}}}{(c+2)^{\overline{n}} n!} x^n$ into the left hand side of the differential equation, and check that the coefficient of x^n simplifies to zero for every $n \in \mathbb{N}$.

Problem 5.6 1. After multiplying the equation by $(1-x)^a$, compare coefficients on both sides. The coefficient of x^0 is 1 on both sides. For $n > 0$, the coefficient of

x^n on the left is

$$[x^n](1-x)^a {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| x\right) = [x^n]\sum_{k=0}^{\infty}\binom{a}{k}(-1)^k x^k \sum_{k=0}^{\infty}\frac{a^{\overline{k}}b^{\overline{k}}}{c^{\overline{k}}k!}x^k$$
$$= \sum_{k=0}^{n}\binom{a}{n-k}(-1)^{n-k}\frac{a^{\overline{k}}b^{\overline{k}}}{c^{\overline{k}}k!}$$

and on the right hand side we have

$$[x^n]{}_2F_1\left(\begin{matrix}a,c-b\\c\end{matrix}\middle| \frac{x}{x-1}\right) = \sum_{k=0}^{n}\frac{a^{\overline{k}}(c-b)^{\overline{k}}}{c^{\overline{k}}k!}\underbrace{[x^n]\frac{x^k}{(x-1)^k}}_{=(-1)^k\binom{n-1}{k-1}}.$$

It is therefore enough to prove the summation identity

$$\sum_{k=0}^{n}(-1)^{n-k}\frac{a^{\overline{k}}b^{\overline{k}}}{c^{\overline{k}}k!}\binom{a}{n-k} = \sum_{k=0}^{n}(-1)^k\frac{a^{\overline{k}}(c-b)^{\overline{k}}}{c^{\overline{k}}k!}\binom{n-1}{k-1} \qquad (n\geq 1).$$

This can be done with Zeilberger's algorithm. It will find that both sides satisfy the recurrence equation

$$(a-b-n)nS_n + (a(b-c-n-1)+(n+1)(b+c+2n+1))S_{n+1}$$
$$-(n+2)(c+n+1)S_{n+2} = 0 \qquad (n\geq 1).$$

After checking the identity for $n=1,2$, it follows for all $n\geq 0$ by induction.

2. ${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| x\right) = (1-x)^{-a}{}_2F_1\left(\begin{matrix}a,c-b\\c\end{matrix}\middle| \frac{x}{x-1}\right) = (1-x)^{-a}{}_2F_1\left(\begin{matrix}c-b,a\\c\end{matrix}\middle| \frac{x}{x-1}\right)$
$= (1-x)^{-a}\left(1-\frac{x}{x-1}\right)^{-(c-b)}{}_2F_1\left(\begin{matrix}c-b,c-a\\c\end{matrix}\middle| x\right) = (1-x)^{c-a-b}{}_2F_1\left(\begin{matrix}c-b,c-a\\c\end{matrix}\middle| x\right).$

3. Multiply the Euler transform by $(1-x)^{a+b-c}$ and compare coefficients of x^n. Then on the right hand side there is just $\frac{(c-a)^{\overline{n}}(c-b)^{\overline{n}}}{c^{\overline{n}}n!}$ while on the left hand side we get

$$(1-x)^{a+b-c}{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| x\right) = \sum_{n=0}^{\infty}\binom{a+b-c}{n}(-1)^n x^n \sum_{n=0}^{\infty}\frac{a^{\overline{n}}b^{\overline{n}}}{c^{\overline{n}}n!}x^n$$
$$= \sum_{n=0}^{\infty}\underbrace{\left(\sum_{k=0}^{n}\frac{a^{\overline{k}}b^{\overline{k}}}{c^{\overline{k}}k!}\underbrace{\binom{a+b-c}{n-k}(-1)^{n-k}}_{=\frac{(c-a-b)^{\overline{n}}(-n)^{\overline{k}}}{n!(1+a+b-c-n)^{\overline{k}}}}\right)}_{=\frac{(c-a-b)^{\overline{n}}}{n!}{}_3F_2\left(\begin{matrix}-n,a,b\\c,1+a+b-c-n\end{matrix}\middle| 1\right)}x^n.$$

The identity follows.

Problem 5.7 Compare coefficients of x^n on both sides.

Problem 5.8 1. For every i with $\gcd(u_1(x), u_2(x+i)) \neq 0$ there must exist a point $(\xi, \zeta) \in \bar{\mathbb{K}}^2$ where ξ is a root of $u_1(x)$ and ζ is a root of $u_2(x)$ and $\xi = \zeta - i$. Since $u_1(x), u_2(x)$ each have only finitely many roots, there can be only finitely many pairs (ξ, ζ). Therefore, there can also be only finitely many such i.

2. First, we have

$$\frac{p(x+1)}{p(x)} \frac{\bar{u}_1(x)}{\bar{u}_2(x)} = \frac{g(x)g(x-1)\cdots g(x-i+1)}{g(x-1)g(x-2)\cdots g(x-i)} \frac{\bar{u}_1(x)}{\bar{u}_2(x)} = \frac{g(x)\bar{u}_1(x)}{g(x-i)\bar{u}_2(x)} = \frac{u_1(x)}{u_2(x)}.$$

Secondly, the choice of $g(x)$ implies $\gcd(\bar{u}_1(x), \bar{u}_2(x+i)) = 1$. Finally, it is clear that the set of all j with $\gcd(\bar{u}_1(x), \bar{u}_2(x+j)) \neq 1$ is contained in the set of all j with $\gcd(\bar{u}_1(x), \bar{u}_2(x+j)) \neq 1$ because $\bar{u}_1(x) \mid u_1(x)$ and $\bar{u}_2(x) \mid u_2(x)$.

3. The desired $i \in \mathbb{N}$ are precisely the positive integer roots of the univariate polynomial $\operatorname{res}_x(p(x), q(x+t)) \in \mathbb{K}[t]$.

4./5. In Maple:

```
gosperForm := proc(u,x)
  local p,q,r,i,j,k,g;
  p := 1; q := numer(u); r := denom(u);
  j := max(select(is, {solve(resultant(q, subs(x = x+i, r), x), i)}, integer));
  while j >= 0 do
    g := gcd(q, subs(x = x+j, r));
    q := q/g; r := r/subs(x = x-j, g);
    p := p * product(subs(x = x-k, g), k = 1..j);
    j := max(select(is, {solve(resultant(q, subs(x = x+i, r), x), i)}, integer));
  od;
  return([p, q, subs(x = x-1, r)]);
end;
```

Problem 5.9 For $p(x), q(x), r(x) \in \mathbb{K}[x]$ with $\gcd(q(x), r(x+i)) = 1$ for all $i \in \mathbb{N} \setminus \{0\}$ consider the equation

$$p(x) = q(x)y(x+1) - r(x)y(x).$$

If there are two different solutions $y_1(x), y_2(x) \in \mathbb{K}[x]$, then their difference $y_h(x) := y_1(x) - y_2(x)$ satisfies $q(x)y_h(x+1) = r(x)y_h(x)$. In this case we have

$$\frac{p(x+1)}{p(x)} \frac{q(x)}{r(x+1)} = \frac{p(x+1)/\big(y_h(x+1)r(x+1)\big)}{p(x)/\big(y_h(x)r(x)\big)},$$

so the summand sequence $(a_n)_{n=0}^{\infty}$ in question is essentially $a_n = \frac{p(n)}{y_h(n)r(n)}$.

Writing the general solution of the Gosper equation as $y(x) = y_1(x) + c y_h(x)$ with c a constant, Gosper's algorithm returns $w(n)a_n = \frac{r(n)}{p(n)}(y_1(n) + c y_h(n)) \frac{p(n)}{y_h(n)r(n)} = \frac{y_1(n)}{y_h(n)} + c$. The choice of a solution of the Gosper equation therefore corresponds to a choice of the additive constant in the solution of the telescoping equation.

When $(a_n)_{n=0}^{\infty}$ is hypergeometric but not rational, then there is no such choice because then $(a_n)_{n=0}^{\infty}$ and the constant sequence $(1)_{n=0}^{\infty}$ are not similar.

Problem 5.10 Let $u_n = \prod_{k=0}^{n-1}\left(-\frac{c_0(k)}{c_1(k)}\right)$. Then $(u_n)_{n=0}^{\infty}$ is hypergeometric and $u_n \neq 0$ for all $n \in \mathbb{N}$. Substituting $s_n = u_n \bar{s}_n$ into the equation and dividing on both sides by $-c_0(n)u_n$ yields the new equation

$$\bar{s}_{n+1} - \bar{s}_n = -\frac{a_n}{c_0(n)u_n}.$$

This is now a telescoping equation with a hypergeometric sequence on the right hand side, so Gosper's algorithm can be used for finding its hypergeometric solutions. Every solution $(\bar{s}_n)_{n=0}^{\infty}$ gives rise to a solution $(u_n\bar{s}_n)_{n=0}^{\infty}$ of the original equation.

Problem 5.11 1. $\frac{x+1+n}{x+1}\binom{n+x}{n}$; 2. $(2n+1)4^{-n}\binom{2n}{n}$; 3. $(n+1)\big(m(m^2-7m+3)\sqrt{5}-(3m^3-7m^2+19m-6)\big)/6(2m^3\sqrt{5}+(m^4+5m^2-1))$; 4. $2-\frac{n!}{(2n+1)!}$; 5. $\frac{2}{3}+\frac{4(n-1)}{3(n+2)}4^n$; 6. $16^{-n}\binom{2n}{n}^2$.

Problem 5.12 Applying Gosper's algorithm to $a_k = \frac{1}{k}$ leads to the Gosper equation $1 = xy(x+1) - xy(x)$ which has obviously no polynomial solution $y(x)$.

Problem 5.13 Like in Zeilberger's algorithm, apply Gosper's algorithm to $(c_0 + c_1k + \cdots + c_dk^d)/k!$ for a priori undetermined $c_0,\ldots,c_d$ and find suitable values for the c_i during the computation. The smallest d where a nontrivial solution can be found is $d = 1$. Here we get $c_0 = 1, c_1 = -1$, thus $p(x) = x - 1$.

Problem 5.14 1. $(2n-1)4^{n-1}$ $(n \geq 1)$; 2. $(2n+1)(-1)^n$ $(n \geq 0)$; 3. $2^{-n}\binom{2n}{n}$ $(n \geq 0)$; 4. 2^n $(n \geq 0)$; 5. $4^{-n}\binom{2n}{n}$ $(n \geq 0)$; 6. 0 $(n \geq 1)$.

Problem 5.15 $(n+2)^3s_{n+2} - (2n+3)(17n^2+51n+39)s_{n+1} + (n+1)^3s_n = 0$ $(n \geq 0)$.

Problem 5.16 With $c_0(t) = 2t+1$ and $c_1(t) = -2(t+1)$ we have

$$\int \left(c_0(n)\cos(\phi)^{2n} + c_1(n)\cos(\phi)^{2n+2}\right)d\phi = -\sin(\phi)\cos(\phi)^{2n+1}.$$

For the specific boundaries 0 and $\pi/2$ the right hand side becomes zero. Therefore the definite integral $I(n)$ on the right hand side of Wallis identity satisfies the recurrence equation $c_0(n)I(n) + c_1(n)I(n+1) = 0$ $(n \geq 0)$. Since the right hand side satisfies the same recurrence and both sides trivially agree for $n = 0$, the identity follows by induction.

Problem 6.1 $\frac{1+\sqrt{1-4x}}{2x} = \frac{1}{x} - 1 - x - 2x^2 + \cdots$ is not a power series.

Problem 6.2 $(4x^2 - x)a''(x) + (10x-2)a'(x) + 2a(x) = 0$.

Problem 6.3 The recursive structure of the continued fraction implies that $K(x) = x/(1-K(x))$. Clearing denominators gives the algebraic equation $K(x)^2 - K(x) + x = 0$. This equation has a unique formal power series solution whose constant term is zero. Since $xC(x)$ satisfies the same equation, we must have $K(x) = xC(x)$.

Problem 6.4 The vector space $V = \mathbb{K}(x) \oplus a(x)\mathbb{K}(x) \oplus \cdots \oplus a(x)^{d-1}\mathbb{K}(x)$ contains 1 and $a(x), a'(x), a''(x), \ldots$. Because of $\dim V = d$, any d elements are linearly dependent over $\mathbb{K}(x)$. In particular, $1, a(x), a'(x), \ldots, a^{(d-1)}(x)$ are linearly dependent. The dependence gives the desired equation.

Problem 6.5 $1 + \frac{4}{9}x + \frac{8}{243}x^2 + \frac{64}{6561}x^3 + \cdots$; $-\frac{1}{2} - 4\sqrt{3}x^{1/2} - \frac{2}{9}x + \frac{2}{27\sqrt{3}}x^{3/2} + \cdots$; $-\frac{1}{2} + 4\sqrt{3}x^{1/2} - \frac{2}{9}x - \frac{2}{27\sqrt{3}}x^{3/2} + \cdots$.

Problem 6.6 Observe that $\sum_{n=0}^{\infty} c_n x^n = a(x^2) + xb(x^2)$ and resort to Theorem 6.1.

Problem 6.7 $(x^2+x-1)^2xy^2 + (x^2-x+1)(x^2+x-1)y + (x^4+2x^3-2x^2-x+1)$.

Problem 6.8 $\frac{1}{2}\sqrt{5/6\pi}(4/5)^n n^{-3/2}$.

Problem 6.9 $4y^3 - (24x+3)y + 8x^2 + 20x - 1$ can be discovered by automated guessing as explained in Sect. 2.6. To prove that this "guessed" equation is correct, plug the series into the polynomial and simplify to zero (using summation algorithms whenever appropriate). Alternatively, convert the minimal polynomial into a differential equation (via Theorem 6.1), check compatibility of the series with this equation, and compare a suitable number of initial terms.

$\sum_{n=0}^{\infty} \binom{1/2}{n}^2 x^n$ is hypergeometric but not algebraic. $1/(1-x-x^2)$ is algebraic but not hypergeometric.

Problem 6.10 1. $a_{n,k} = \sum_j (-2)^{n-j}\binom{k+j}{j}\binom{k}{n-j}$ $(n, k \geq 0)$.

2. The recurrence is $a_{n+2,n+2} = -4\,\frac{n+1}{n+2}\,a_n$ $(n \geq 0)$. Together with the initial values $a_{0,0} = 1$ and $a_{1,1} = 0$ implies $a_{2n,2n} = (-1)^n\binom{2n}{n}$ and $a_{2n+1,2n+1} = 0$ $(n \geq 0)$.

3. $(4x^2+1)y^2 - 1 = 0$.

4. Plug $\sum_{n=0}^{\infty}\binom{2n}{n}(-x^2)^n$ into the left hand side of the equation and simplify the resulting expression to zero.

A proof for the general statement can be found in [54, Theorem 6.3.3].

Problem 6.12 Suppose $m(x,y) \in \mathbb{K}[x,y]$ is such that $m(x, a(x)) = 0$. Substituting $x \mapsto b(x)$ gives $m(b(x), a(b(x))) = 0$, so $m(b(x), x) = 0$. The claim follows.

Problem 6.13 1. We have $1/n! \sim cn^{-n+\frac{1}{2}}e^n$ for some constant c, whereas the coefficient sequences of algebraic power series grow like $n^{\alpha}d^n$ for some constants α, d.

2. Suppose $p(x,y) = p_0(x) + p_1(x)y + \cdots + p_d(x)y^d$ is an annihilating polynomial of $\exp(x)$. We may assume that d is minimal and that among all annihilating polynomials of degree d, the choice is made such that $\deg_x p_d(x)$ is as small as can be. Then $d \geq 1$ and $p_0(x)$ is not the zero polynomial. Differentiating $p(x, \exp(x)) = 0$ with respect to x implies that

$$q(x,y) := p_0'(x) + (p_1'(x) + p_1(x))y + \cdots + (p_d'(x) + dp_d(x))y^d$$

is another annihilating polynomial for $\exp(x)$. Now consider $q(x,y) - dp(x,y)$. This cannot be the zero polynomial because $d \neq 0$ and $\deg_x p_0'(x) < \deg_x p_0(x)$ implies that $[y^0](q(x,y) - dp(x,y)) = p_0'(x) - dp_0(x)$ is not the zero polynomial. But

$[y^d](q(x,y) - dp(x,y)) = p_d'(x)$. This is in contradiction to the minimality assumptions because either $p_d'(x) = 0$, then d was not minimal, or otherwise $\deg_x p_d'(x) < \deg_x p_d(x)$ and the degree of $p_d(x)$ was not minimal.

3. Suppose $p(x,y) = p_0(x) + p_1(x)y + \cdots + p_d(x)y^d$ is an annihilating polynomial of $\exp(x)$. We may assume that $p_d(x)$ is not the zero polynomial. Set $u = -\deg_x p_d(x)$ and $v = -d$. Then $\lim_{z\to\infty} p(z,\mathrm{e}^z)z^u\mathrm{e}^{vz} = \operatorname{lc} p_d(x) \neq 0$ while $p(z,\mathrm{e}^z) = 0$ for all $z \in \mathbb{R}$. Contradiction.

4. If $\exp(x)$ is algebraic then so are $\exp(\mathrm{i}x)$ and $\exp(-\mathrm{i}x)$. Hence, also $\sin(x) = \frac{1}{2\mathrm{i}}(\exp(\mathrm{i}x) - \exp(-ix))$, and hence $x/\sin(x)$. But the latter power series, regarded as an analytic function, has a pole at $k\pi$ for every $k \in \mathbb{Z} \setminus \{0\}$. These are infinitely many. But the singularities of an algebraic function are roots of a univariate polynomial and therefore there are at most finitely many of them. Therefore, $x/\sin(x)$ cannot be algebraic, and therefore $\exp(x)$ cannot be algebraic either.

5. Suppose there is a nontrivial relation of the proposed form. Then $p_0(x)$ cannot be the zero polynomial, for otherwise multiplying by $n!$ on both sides would yield a contradiction to the linear independence statement of Theorem 4.1. Assume again that d is minimal and derive a smaller relation subtracting $dp_d(n+1)$ times the original relation from $(n+1)p_d(n)$ times the relation obtained from the original relation by shifting $n \mapsto n+1$. The first term in the resulting relation is the $dp_0(n)p_d(n+1) - (n+1)p_0(n+1)p_d(n)$ which cannot be identically zero because the $p_0(x)$ is not the zero polynomial and the two terms have different degree in n. This is the desired contradiction. To see that $\exp(x)$ is not algebraic, observe that comparing coefficients of x^n in a relation $p(x, \exp(x)) = 0$ would give rise to a nontrivial relation of the form we just proved to be impossible.

Problem 6.14 If $\log(1+x)$ was algebraic, then, since $\exp(\log(1+x)) = 1+x$, also $\exp(x)$ would be algebraic by Problem 6.12. This is not the case by Problem 6.13, so $\log(1+x)$ is not algebraic either. Now $\sum_{n=0}^{\infty} H_n x^n = -\frac{\log(1-x)}{1-x}$ cannot be algebraic either, for if it was, then multiplying by $x-1$ and substituting $x \mapsto -x$ would yield again an algebraic series, while $\log(1+x)$ was shown to be not algebraic.

Problem 6.15 1. Counterexample: $\sum_{n=0}^{\infty} 2^n x^n = 1/(1-2x)$ is algebraic as series, but 2^n grows too quickly to be algebraic as a sequence.

2. Counterexample: $a_n = 1/(n+1)$ is algebraic as a sequence, because $(n+1)a_n - 1 = 0$ $(n \geq 0)$, but $\sum_{n=0}^{\infty} a_n x^n = \log(1-x)$ is not algebraic as a series.

Problem 6.16 C_n.

Problem 6.17 $\frac{1-\sqrt{1-4x}}{2x\sqrt{1-4x}} = 1 + 3x + 10x^2 + 35x^3 + \cdots$.

Problem 7.2 1. $\frac{1}{4}n((1-n) + 2(n+1)H_n)$; 2. $2n - (2n+1)H_n + (n+1)H_n^2$; 3. $(n+1)H_n^{(2)} - H_n$.

Problem 7.3 1. n; 2. 2^n; 3. $2^n + 3^n$; 4. $n!$; 5. $\binom{2n}{n}$; 6. $2^n + \binom{2n}{n}$; 7. $n!$; 8. $n! + (2n)!$; 9. $1/(n! + (2n)!)$.

Problem 7.4 Let $c_n = b_n - a_n$. Then $c_n \neq 0$ for finitely many $n \in \mathbb{N}$ only. Therefore $c(x) := \sum_{n=0}^{\infty} c_n x^n$ is a polynomial, and thus holonomic. (It satisfies, for instance, the differential equation $c'(x)a(x) - c(x)a'(x) = 0$.) Since $(a_n)_{n=0}^{\infty}$ is holonomic and also $(c_n)_{n=0}^{\infty}$ and $b_n = a_n + c_n$ ($n \in \mathbb{N}$), it follows that $(b_n)_{n=0}^{\infty}$ is holonomic.

The corresponding statement is true for C-finite sequences and for coefficient sequences of algebraic power series, but not for hypergeometric sequences.

Problem 7.7 All.

Problem 7.8 Let $a(x)$ be the series in question. Then for every $k \in \mathbb{N}$ there is some polynomial $q_k(x) \in \mathbb{Q}[x]$ of degree k such that $D_x^k a(x) = q_k(\exp(x))a(x)$. This follows directly from a repeated application of the chain rule. Now if there was an equation

$$p_0(x)a(x) + p_1(x)a'(x) + \cdots + p_r(x)D_x^r a(x) = 0$$

then this would imply

$$\big(p_0(x) + p_1(x)q_1(\exp(x)) + \cdots + p_r(x)q_r(\exp(x))\big)a(x) = 0.$$

Dividing by $a(x)$ gives a polynomial equation for $\exp(x)$ which is nontrivial because $\deg q_k(x) = k$ for every k. We have reached a contradiction to the result of Problem 6.13 and therefore a holonomic differential equation for $a(x)$ cannot exist.

Problem 7.9 $a_k := (\frac{1}{2})^k \binom{2k}{k}$ is clearly holonomic as a sequence. Hence $a(x) := \sum_{k=0}^{\infty} a_k x^k$ is holonomic as a power series. Hence $\frac{1}{1-x} a(\frac{x}{x-1})$ is holonomic as a power series. Hence $[x^n] \frac{1}{1-x} a(\frac{x}{x-1})$ is holonomic as a sequence. By the Euler transform, the latter is equal to s_n.

Problem 7.10 Starting from the obvious equations satisfied by $1/\sqrt{1-4t^2}$, $\exp(t)$, and $4t(xy - t(x^2+y^2))/(1-4t^2)$, construct a differential equation for the right hand side. This gives

$$(1-4t^2)^2 a'(t) + 4((2x^2+2y^2-1)t - 4xyt^2 - xy + 4t^3)a(t) = 0.$$

This differential equation translates into the recurrence equation

$$16(n+1)a_n - 16xya_{n+1} - 4(2n - 2x^2 - 2y^2 + 5)a_{n+2} - 4xya_{n+3} + (n+4)a_{n+4} = 0$$

valid for $n \geq 0$. Now verify that $H_n(x)H_n(y)/n!$ satisfies this recurrence as well and compare four initial values.

Problem 7.11 Start with the differential equation from Problem 5.5 and the obvious equations satisfied by $4x/(1+x)^2$, $(1+x)^{2a}$ and x^2, and construct differential equations for the series on the left hand side and the right hand side of the equation. It turns out that both sides satisfy

$$4ab(x-1)f(x) + 2(x+1)(-2a\,x + bx^2 + b - x^2)f'(x) - (x-1)x(x+1)^2 f''(x) = 0.$$

Finally, compare a suitable number of initial values.

Problem 7.12 The recurrence corresponding to the differential equation is

$$52a_n+(2705-104n)a_{n+1}+2(26n^2-1353n-54)a_{n+2}+53(n-50)(n+3)a_{n+3}=0.$$

It has a singularity at $n=50$, so fixing a_0,a_1,a_2 does not determine the value a_{53}.

Problem 7.13 See Sect. 8.3 of [44].

Problem 7.15 $h_n=n!$. To obtain the other solution, plug the proposed form of u_n into the equation. Using that h_n is a solution, the equation can be simplified to a first order equation for $\bar{h}_n$. Its solution is $\bar{h}_n=(-1)^n/n!$.

More generally, whenever $(h_n)_{n=0}^{\infty}$ is some solution to a recurrence of order r, there is a second solution $(u_n)_{n=0}^{\infty}$ with $u_n=h_n\sum_{k=0}^{n-1}\bar{h}_k$ where $(\bar{h}_n)_{n=0}^{\infty}$ satisfies a recurrence of order $r-1$.

Problem 7.16 $\xi=1$ or $\xi=-1$.

Problem 7.17 1. If $(h_n)_{n=0}^{\infty}$ is such that $u(n)h_{n+1}+v(n)h_n$ $(n\geq 0)$ for some polynomials $u(x),v(x)\in\mathbb{K}[x]$, then every solution $(a_n)_{n=0}^{\infty}$ of the given inhomogeneous equation will also satisfy the homogeneous equation

$$\begin{aligned}&v(n)p_0(n)a_n+(u(n)p_0(n+1)+v(n)p_1(n))a_{n+1}+\cdots\\&+(u(n)p_{r-1}(n+1)+v(n)p_r(n))a_{n+r}+u(n)p_r(n+1)a_{n+r+1}=0\quad(n\geq 0).\end{aligned}$$

Use Petkovšek's algorithm to determine the hypergeometric solutions of this equation, and then check which of these also satisfies the original equation. Return those as answer and discard the others.

2. Petkovšek's algorithm returns the shift quotients $w(x)\in\mathbb{K}(x)$ of all the hypergeometric solutions. We need to detect for a given $w(x)\in\mathbb{K}(x)$ whether it is actually the shift quotient of a rational function, i.e., whether there exists $u(x)\in\mathbb{K}(x)$ such that $u(x+1)/u(x)=w(x)$. This can be done by computing a Gosper form of $w(x)$: If $w(x)=p(x+1)q(x)/p(x)/r(x+1)$ is a Gosper form of $w(x)$, then $w(x)$ is the shift quotient of a rational function if and only if $q(x)=r(x)$. In this case $u(x)=p(x)/q(x)$.

It should be remarked that this is a rather brutal way of finding rational solutions of holonomic recurrence equations. Better algorithms can be found in the literature [2].

Problem 7.18 Induction on k. For $k=0$ we have $S_1(n,0)=0$ $(n\geq 1)$, which is clearly holonomic. If $(S_1(n,k))_{n=0}^{\infty}$ is holonomic for some $k\geq 0$, then the general recurrence

$$S_1(n,k)+(n-1)S_1(n-1,k)=S_1(n-1,k-1)\qquad(n,k>0)$$

for Stirling numbers of the first kind (Ex. 3.7) implies

$$S_1(n,k+1)=(-1)^n(n-1)!\sum_{i=0}^{n-1}\frac{S_1(i,k)}{(-1)^i i!}\qquad(n>0),$$

so $(S_1(n,k+1))_{n=0}^{\infty}$ is holonomic as well.

The claimed harmonic number representations can be checked by explicitly computing recurrence equations for the $(S_1(n,k))_{n=0}^{\infty}$ ($k = 1,2,3,4$), checking that the claimed expressions satisfy these recurrence equations, and comparing a suitable number of initial values.

Problem 7.19 1. The worst case situation is exactly the same as for Quicksort.
2. $c_{n,i} = (n-1) + \frac{1}{n}\left(\sum_{k=1}^{i-1} c_{n-k,i-k} + \sum_{k=i+1}^{n} c_{k-1,i}\right)$ and $c_{n,1} = c_{n,n} = n-1$.
Elimination of the summation signs leads to the recurrence

$$c_{n,i} - c_{n+1,i} - c_{n+1,i+1} + c_{n+2,i+1} = \frac{2}{n+2} \qquad (n \geq 1, 1 < i < n).$$

3. The terms for $n = 1,2,\ldots,30$ read 0, 1, $\frac{8}{3}$, $\frac{53}{12}$, $\frac{197}{30}$, $\frac{87}{10}$, $\frac{467}{42}$, $\frac{1133}{84}$, $\frac{1013}{63}$, $\frac{23447}{1260}$, $\frac{49249}{2310}$, $\frac{664327}{27720}$, $\frac{4822549}{180180}$, $\frac{118115}{4004}$, $\frac{1166365}{36036}$, $\frac{1810561}{51480}$, $\frac{58338047}{1531530}$, $\frac{27874461}{680680}$, $\frac{511162633}{11639628}$, $\frac{5449841867}{116396280}$, $\frac{161087299}{3233230}$, $\frac{6141864151}{116396280}$, $\frac{74689129967}{1338557220}$, $\frac{10489933451}{178474296}$, $\frac{29559627277}{478056150}$, $\frac{867902775947}{13385572200}$, $\frac{1363691268007}{20078358300}$, $\frac{5698145622329}{80313433200}$, $\frac{5748829529089}{77636318760}$, $\frac{89786686813897}{1164544781400}$.

4. We found the equation

$$\begin{aligned}
&x(x+1)(x-1)^3(2x^6+2x^5-18x^4-17x^3-9x^2-30x-30)a^{(3)}(x)\\
&\quad + 2(x-1)^2(9x^8+15x^7-94x^6-148x^5-83x^4-207x^3-282x^2+90)a''(x)\\
&\quad + 2(x-1)(15x^8+24x^7-175x^6-274x^5-320x^4-582x^3\\
&\qquad\qquad -528x^2+480x+360)a'(x)\\
&\quad + 4(3x^7+11x^6-40x^5-227x^4-195x^3-42x^2+300x+90)a(x) = 0.
\end{aligned}$$

5. We found $c_1(x) = \frac{x^2-3}{(x-1)^2x^4}$, $c_2(x) = \frac{1}{(x-1)^2}$, and

$$c_3(x) = \frac{x^6 - 18 + 2(x^2-3)\log(x+1) + 2(2x^4+x^2-3)\log(1-x)}{(x-1)^2x^4}.$$

6. For the above choice of $c_1(x), c_2(x), c_3(x)$ we get $\alpha_1 = 3, \alpha_2 = -\frac{1}{2}, \alpha_3 = -\frac{1}{2}$.
7. On $U := \{z \in \mathbb{C} : |z| < 1\}$ we can define the analytic function

$$c\colon U \to \mathbb{C}, \quad c(z) = \alpha_1 c_1(z) + \alpha_2 c_1(z) + \alpha_3 c_3(z).$$

It has singularities at $z = \pm 1$ and there we have

$$c(z) \sim \tfrac{1}{2}(1+\log(1+z)) \quad (z \to -1)$$
$$c(z) \sim \frac{2(1+\log(2))}{(1-z)^2} \qquad (z \to 1).$$

The growth implied by the second estimate dominates the growth implied by the first. Therefore

$$[x^n]c(x) \sim 2(1+\log(2))\,n \quad (n \to \infty).$$

8. $\frac{2n^5+28n^4+123n^3+166n^2-81n-210}{(n+1)(n+2)(n+3)(n+4)} + \frac{n^4+15n^3+85n^2+209n+198}{(n+1)(n+2)(n+3)(n+4)}(-1)^n - 2(n+6)\sum_{k=1}^{n}\frac{(-1)^k}{k} - 10H_n$.

A.7 Bibliographic Remarks

Chapter 1

Hoare's original paper on Quicksort is [29]; the complexity analysis given here already appears there and is used as an example for the analysis of algorithms in many introductory textbooks. Further examples for analysis of algorithms can be found in the seminal volumes of Knuth [34, 33]. The book Concrete Mathematics [24] arose as an offspring of Knuth's ground breaking work in this area.

Havil [27] gives a fine and very readable account on the mysteries surrounding Euler's magic constant γ.

Chapter 2

Wilf's introduction to generating functions [62] contains further information and additional examples for the usage of formal power series in combinatorics and other branches of mathematics. Ongoing research on formal power series is presented at the annual meetings of the international conference series FPSAC ("Formal Power Series and Algebraic Combinatorics").

Sokal [52] gives a "ridiculously simple" version of the implicit function theorem for analytic functions as well as for formal power series.

Sloane's collection of integer sequences has originally appeared as a book [51]. One of the first uses of computers for detecting possible equations among sequences or power series was by Pivar and Finkelstein [45], the idea has subsequently been adapted to various different types of equations [49, 39, 37, 31, 28].

Richardson's convergence acceleration technique dates back to 1910 [47], a survey of more recent developments can be found in [13].

Chapter 3

Most of the material in this chapter is part of the mathematical folklore.

We follow Concrete Mathematics for the notation of rising and falling factorials; many other notations are used in the literature. In particular, the symbol $(x)_n$ may refer to $x^{\overline{n}}$ as well as to $x^{\underline{n}}$.

For a collection of additional facts on Stirling numbers, see Sect. 6.1 of Concrete Mathematics. Also Stirling numbers appear in a variety of different notations.

The algorithm for solving first order inhomogeneous linear recurrence equations with polynomial coefficients appears in Gosper's paper [23].

The theory behind the examples for partition analysis was developed in the late 19th century by MacMahon [38] and revived and brought to the computer by Andrews, Paule and Riese in a series of articles. The examples given here are taken from [9]. For more recent work, see [8] and the references given there.

Chapter 4

The term C-finite was coined by Zeilberger in [65], he also calls holonomic recurrences *P-finite* because they have polynomial coefficients where C-finite recurrences have constant coefficients.

Stanley discusses rational generating functions in Chap. 4 of [53]. Combinatorial examples as well as pointers to the literature can be found there. For number theoretic aspects of C-finite sequences we refer to the comprehensive text [20].

A general summation algorithm for sums over C-finite sequences is due to Greene and Wilf [25]. Their algorithm includes as a special case the one we describe.

For general aspects of the theory of orthogonal polynomials, see Chap. 6 in the book of Temme [55] and the relevant chapters of [7].

Chapter 5

Andrews, Askey and Roy [7] is excellent reference for classical aspects of the theory of hypergeometric series.

Gosper's algorithm originally appeared in [23], Paule [41] provides an algebraic explanation of it. The original article of Zeilberger's algorithm is [64]. There are also textbooks solely devoted to hypergeometric summation [44, 35], and further references to the literature are given there.

The elliptic arc length example is taken from [6].

Monthly problems which can be solved with the help of summation algorithms are collected in [40]. Also the examples we have given are taken from there.

Chapter 6

The kernel method first appeared as a solution to Exercise 2.2.1-4 in [33], see [10, 12] and the references given there for generalizations.

The solution of algebraic equations in terms of Puiseux series was already proposed almost two centuries before Puiseux by Isaac Newton.

Buchberger's theory of Gröbner bases [17] provides a computational alternative to resultant computations.

More on the connection of context free languages to algebraic power series can be found in [21]. Also detailed background information on asymptotic techniques can be found there.

Facts about Legendre polynomials are collected in [18].

Chapter 7

Concrete Mathematics [24] discusses harmonic numbers and their summation in Sects. 6.3 and 6.4. For algebraic relations among generalized harmonic numbers, their automated computation, and their relevance in particle physics, we refer to [1] and the references give there. Apery's proof of the irrationality of $\zeta(3)$ can be found in [56].

Zeilberger promoted holonomic sequences and power series in [65], there he discusses a more general definition applicable to sequences and series in several variables. Closure properties can be proven and computed also in this case. Stanley treats holonomic objects in Sect. 6.4 of [54] under the names D-finite (for power series) and P-recursive (for sequences). Summation algorithms for general multivariate holonomic sequences and functions are described in [15] and [14].

The solution of linear differential equations in terms of generalized series goes back to the work of Frobenius at the end of the 19th century.

The criterion that $a(x)$ and $1/a(x)$ are holonomic if and only if $a'(x)/a(x)$ is algebraic can be found as Exercise 1.39 in [57]. The original source is [26].

Petkovšek's algorithm first appeared in [43] and is also described in [44, 35]. Abramov and van Hoeij present their summation algorithm in [4].

Proofs for the claimed permutation statistics are given in [62]. For techniques for high performance computations of constants, see the book [11]. Bessel functions are described at length in the classical book [60].

The fast median search algorithm discussed in Problem 7.19 is described in Chap. 9 of [16].

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