A Natural Deduction Calculus for Gödel-Dummett Logic Internalizing Proof-search Control Mechanisms*

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Abstract. We introduce a natural deduction calculus for the Gödel-Dummett Logic LC semantically characterized by linearly ordered Kripke models. Our calculus is inspired by an analogous calculus for Intuitionistic logic (IPL) internalizing mechanisms to reduce the proof-search space that has been used to define a goal-oriented proof-search procedure for IPL. In this paper we present the calculus for LC and we sketch its soundness and completeness.

1 Introduction

In recent years there has been a renewed interest in computational interpretations of Gödel-Dummett Logic [1,2,5], semantically characterized by linearly ordered Kripke models, also known as LC (Logic of Chains). Starting from a natural deduction calculus for LC, the mentioned papers deeply investigate the parallel interpretation proposed by Avron [4] of the characteristic rule of LC. While [2,5] are based on calculi mimicking Avron hypersequent calculus in the natural deduction setting, [1] proposes a standard natural deduction calculus extended with a rule essentially internalizing the characteristic axiom $(A \supset B) \vee (B \supset A)$ of LC. The main problem of the aforementioned calculi is that proof-search is highly inefficient. We aim at developing a natural deduction characterization of LC which allows for efficient proof-search, by applying the techniques introduced for Intuitionistic and Classical logics [8,9].

Proof-search in natural deduction calculi has been firstly investigated in [12,13], where the *intercalation calculus* has been introduced. It is based on the alternation of two phases, one applying introduction rules bottom-up, the other one applying elimination rules top-down. To provide an efficient implementation of the method for Intuitionistic Propositional Logic (IPL), in [9] we have introduced the natural deduction calculus **Nbu**, a variant of the standard natural deduction calculus for IPL [11,14]. Natural deduction derivations are represented in sequent style: a derivation with root $\Gamma \Rightarrow H$ represents a derivation of H with assumptions contained in the *context* Γ (a multiset). Following [7,10], we introduce two kinds of sequents: $\uparrow$-sequents $\Gamma \Rightarrow H \uparrow^{l}$, where the label l can be b

(blocked) or u (unblocked), and $\downarrow$-sequents $\Gamma \Rightarrow H\downarrow$. In proof-search, a derivation for an $\uparrow$-sequent σ is built by trying the bottom-up application of a rule of the calculus to σ. The label b blocks the applications of some rules, so to reduce the search-space; for instance, if $\sigma = \Gamma \Rightarrow A \vee B\uparrow^l$ and $l = u$, then we can bottom-up apply either rule $\vee I_0$, yielding $\Gamma \Rightarrow A\uparrow^u$ or $\vee I_1$, yielding $\Gamma \Rightarrow B\uparrow^u$, and continue bottom-up expansion with the obtained sequents. In contrast, if $l = b$ then the application of rules $\vee I_0$ and $\vee I_1$ is blocked, thus the sequent σ is not provable and one has to backtrack. A formula G is valid in IPL if and only if there exists a derivation $\mathcal{D}$ of the sequent $\cdot \Rightarrow G\uparrow^u$ (with empty context); a crucial point is that $\mathcal{D}$ is isomorphic to a derivation in normal form in the standard deduction calculus for IPL.

In this paper we investigate the application of the method to LC; we introduce a natural deduction calculus for LC close to **Nbu** so that proof-search can be efficiently implemented. The idea is to enhance **Nbu** by adding a rule to capture the characteristic axiom $(A \supset B) \vee (B \supset A)$ of LC. A natural candidate is the rule below, a sort of $\vee E$ elimination rule where the main premise $\Gamma \Rightarrow (A \supset B) \vee (B \supset A)\downarrow$ has been crossed out, since in LC is redundant:

$$\frac{A \supset B, \Gamma \Rightarrow C\uparrow^u \qquad B \supset A, \Gamma \Rightarrow C\uparrow^u}{\Gamma \Rightarrow C\uparrow^u}\ \mathrm{Gd}$$

Seemingly, this rule yields a non-effective proof-search strategy, since the rule must be applied bottom-up and the formulas $A \supset B$ and $B \supset A$ in the premises are arbitrary. We can bound proof-search by exploiting [3], where it is shown that to prove a goal formula G in LC we only need a finite set of instances of the axiom $(A \supset B)\vee(B \supset A)$, obtained by instantiating A and B with suitable subformulas of G. This leads to the natural deduction calculus $\mathbf{NLC}(G)$, parameterized by the goal formula G to be proved.

2 Göedel-Dummett logic LC

We consider the language based on a denumerable set of propositional variables $\mathcal{V}$, the connectives $\wedge$, $\vee$, $\supset$, and the logical constant $\bot$; $\neg A$ stands for $A \supset \bot$. By $\mathrm{Sf}(G)$ we denote the set of all subformulas of G (including G itself); $\mathrm{Sf}^{\rightarrow}(G)$ is the set of $C \in \mathrm{Sf}(G)$ such that $C \in \mathcal{V}$ or $C = A \supset B$. Let $\Gamma \vdash_L G$ denote that the formula G is provable in the logic L from the assumptions in Γ. By definition, $\vdash_{\mathrm{LC}} G$ iff $\Sigma_{\mathrm{gd}} \vdash_{\mathrm{IPL}} G$, where Σ_{gd} is the (infinite) set of all the instances of the axiom-schema $(Gd) = (A \supset B) \vee (B \supset A)$ (see e.g. [6]). In [3] it is proved that the set Σ_{gd} can be replaced by the finite set constructed from subformulas of G

$$\Sigma_{\mathrm{gd}}(G) \;=\; \{\ (A \supset B) \vee (B \supset A) \mid \{A, B\} \;\subseteq\; \mathrm{Sf}^{\rightarrow}(G)\ \}$$

Namely (a simplified proof of the theorem is discussed in Appendix A):

Theorem 1. $\vdash_{\mathrm{LC}} G$ *iff* $\Sigma_{\mathrm{gd}}(G) \vdash_{\mathrm{IPL}} G$. □

$$\dfrac{}{A, \Gamma \Rightarrow A{\downarrow}}\ \mathrm{Id} \qquad \dfrac{\Gamma \Rightarrow p{\downarrow}}{\Gamma \Rightarrow p{\uparrow}^l}\ {\downarrow\!\uparrow}\quad p \in \mathcal{V} \qquad \dfrac{\Gamma \Rightarrow \bot{\downarrow}}{\Gamma \Rightarrow F{\uparrow}^l}\ \bot E \quad F \in \mathcal{V} \cup \{\bot\}$$

$$\dfrac{\Gamma \Rightarrow A{\uparrow}^l \qquad \Gamma \Rightarrow B{\uparrow}^l}{\Gamma \Rightarrow A \wedge B{\uparrow}^l}\ \wedge I \qquad \dfrac{\Gamma \Rightarrow A_0 \wedge A_1{\downarrow}}{\Gamma \Rightarrow A_k{\downarrow}}\ \wedge E_k$$

$$\dfrac{\Gamma \Rightarrow A_k{\uparrow}^b}{\Gamma \Rightarrow A_0 \vee A_1{\uparrow}^l}\ \vee I_k \qquad \dfrac{\Gamma \Rightarrow A \supset B{\downarrow} \quad \Gamma \Rightarrow A{\uparrow}^b}{\Gamma \Rightarrow B{\downarrow}}\ \supset E$$

$$\dfrac{\Gamma \Rightarrow A \vee B{\downarrow} \qquad \Gamma \Rightarrow B \supset A{\downarrow}}{\Gamma \Rightarrow A{\downarrow}}\ \vee E_0 \qquad \dfrac{\Gamma \Rightarrow A \vee B{\downarrow} \qquad \Gamma \Rightarrow A \supset B{\downarrow}}{\Gamma \Rightarrow B{\downarrow}}\ \vee E_1$$

$$\dfrac{\Gamma \Rightarrow B{\uparrow}^l}{\Gamma \Rightarrow A \supset B{\uparrow}^l}\ \supset I_1 \quad A \in \Gamma \qquad \dfrac{A, \Gamma \Rightarrow B{\uparrow}^u}{\Gamma \Rightarrow A \supset B{\uparrow}^l}\ \supset I_2 \quad A \notin \Gamma$$

$$\dfrac{A \supset B, \Gamma \Rightarrow D{\uparrow}^u \quad B \supset A, \Gamma \Rightarrow D{\uparrow}^u}{\Gamma \Rightarrow D{\uparrow}^u}\ \mathrm{Gd} \qquad \begin{array}{l} D \in \mathcal{V} \cup \{\bot\} \text{ or } D = D_0 \vee D_1 \\ \{A, B\} \subseteq \mathrm{Sf}^{\rightarrow}(G) \text{ or } A \vee B \in \mathrm{Sf}(G) \\ A \supset B \notin \Gamma \text{ and } B \supset A \notin \Gamma \end{array}$$

Fig. 1. The natural deduction calculus **NLC**(G) ($l \in \{b, u\}$, $k \in \{0, 1\}$).

3 The calculus NLC(G)

The natural deduction calculi we consider in this section act on sequents of the form $\Gamma \Rightarrow \delta$ where Γ, the *context*, is a finite, possibly empty, multiset of formulas and δ has the form $H{\downarrow}$ or $H{\uparrow}^l$, with H any formula and $l \in \{b, u\}$. For calculi and derivations we use the definitions and notations of [14]. In particular, applications of rules of a calculus **C** are depicted as trees with sequents as nodes. A *derivation* of **C** is a tree where every leaf is an *axiom sequent*, i.e., a sequent obtained by instantiating a zero-premise rule of **C**. When needed, we write **C**-*tree* and **C**-*derivation* to emphasize the calculus at hand. A sequent σ is *provable* in **C**, and we write $\mathbf{C} \vdash \sigma$, if there exists a **C**-derivation with root sequent σ.

The calculus **NLC**(G) is displayed in Fig. 1. It consists of introduction (I) and elimination (E) rules for every logical connective plus the *coercion rule* (${\downarrow\!\uparrow}$), the $\bot$ elimination rule ($\bot E$), and assumption introduction rule (Id). The *major formula* of an elimination rule is the formula with the characteristic connective and the *major premise* is the sequent having the major formula in the right-hand side; e.g., in the rule $\supset E$ the major formula is $A \supset B$ and the major premise is $\Gamma \Rightarrow A \supset B{\downarrow}$. The calculus **NLC**$(G)$, parametrized by the goal formula G to be proved, is a variant of the natural deduction calculus **Nbu** for IPL presented in [9] (see Fig. 2). In **NLC**(G), the standard three-premise $\vee E$ rule, which is harmful for proof-search, is replaced by the two-premise rules $\vee E_0$ and $\vee E_1$, where both the conclusion and the premises are ${\downarrow}$-sequents. To grasp the computational meaning of these rules, one has to consider that in LC the formula $A \vee B$ is equivalent to $((A \supset B) \supset B) \wedge ((B \supset A) \supset A)$ (see Ex. 1 below) [3]. The characteristic rule of LC is Gd. To get a feasible proof-search, formulas $A \supset B$ and $B \supset A$ are not arbitrary, but are chosen from a finite set of formulas which depends on the goal formula G (this is the reason why the calculus is parametrized by the goal formula G). Moreover, to narrow the proof-search space, we require that the label of the conclusion has the form $\Gamma \Rightarrow D{\uparrow}^u$ (with label u), where D has a special form. It is easy to check that $\mathbf{NLC}(G)$ satisfies the following property (soundness):

Theorem 2. $\mathbf{NLC}(G) \vdash \cdot \Rightarrow G{\uparrow}^u$ *implies* $\vdash_{\mathrm{LC}} G$. □

In the rest of this section we sketch completeness of $\mathbf{NLC}(G)$, i.e., the converse of Th. 2. We provide an example of derivation (another example is at the end of Appendix B).

Example 1. Let G be the formula $(p \vee q) \leftrightarrow ((p \supset q) \supset q) \wedge ((q \supset p) \supset p)$ (with $\leftrightarrow$ defined as usual). We prove that G is valid in LC by showing an $\mathbf{NLC}(G)$-derivation of $\cdot \Rightarrow G{\uparrow}^u$.

$$A \;=\; (p \supset q) \supset q \qquad B \;=\; (q \supset p) \supset p \qquad C \;=\; A \wedge B \qquad D \;=\; p \vee q$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ D \Rightarrow A{\uparrow}^u\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ D \Rightarrow B{\uparrow}^u\end{array}}{D \Rightarrow C{\uparrow}^u}\,\wedge I}{\cdot \Rightarrow D \supset C{\uparrow}^u}\,\supset I_2 \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_3\\ p \supset q, C \Rightarrow D{\uparrow}^u\end{array} \quad \begin{array}{c}\mathcal{D}_4\\ q \supset p, C \Rightarrow D{\uparrow}^u\end{array}}{C \Rightarrow D{\uparrow}^u}\,\mathrm{Gd}}{\cdot \Rightarrow C \supset D{\uparrow}^u}\,\supset I_2}{\cdot \Rightarrow D \leftrightarrow C{\uparrow}^u}\,\wedge I$$

Below we detail the $\mathbf{NLC}(G)$-derivations $\mathcal{D}_1$ and $\mathcal{D}_3$ ($\mathcal{D}_2$ and $\mathcal{D}_4$ are similar).

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \supset q, D \Rightarrow D{\downarrow}}\,\mathrm{Id} \qquad \dfrac{}{p \supset q, D \Rightarrow p \supset q{\downarrow}}\,\mathrm{Id}}{p \supset q, D \Rightarrow q{\downarrow}}\,\vee E_1}{p \supset q, D \Rightarrow q{\uparrow}^u}\,{\downarrow\uparrow}}{D \Rightarrow A{\uparrow}^u}\,\supset I_2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \supset q, C \Rightarrow C{\downarrow}}\,\mathrm{Id}}{p \supset q, C \Rightarrow (p \supset q) \supset q{\downarrow}}\,\wedge E_0 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{p, p \supset q, C \Rightarrow p \supset q{\downarrow}}\,\mathrm{Id} \qquad \dfrac{\dfrac{}{p, p \supset q, C \Rightarrow p{\downarrow}}\,\mathrm{Id}}{p, p \supset q, C \Rightarrow p{\uparrow}^b}\,{\downarrow\uparrow}}{p, p \supset q, C \Rightarrow q{\downarrow}}}{p, p \supset q, C \Rightarrow q{\uparrow}^b}\,{\downarrow\uparrow}}{p \supset q, C \Rightarrow p \supset q{\uparrow}^b}\,\supset I_2}{p \supset q, C \Rightarrow q{\downarrow}}\,\supset E}{p \supset q, C \Rightarrow q{\uparrow}^b}\,{\downarrow\uparrow}}{p \supset q, C \Rightarrow D{\uparrow}^u}\,\vee I_1$$

◊

[3] Actually, one could introduce $\vee$ as a defined operator and drop out its I/E-rules.

$$\mathbf{Nbu} \; = \; \mathbf{NLC}(G) \setminus \{\mathrm{Gd}, \; \vee E_0, \; \vee E_1\} \cup \{\vee E\}$$

$$\mathbf{NLC}^*(G) \; = \; \mathbf{NLC}(G) \setminus \{\mathrm{Gd}\} \cup \{\mathrm{Gd}^*\} \qquad \mathbf{NLC}^*_\vee(G) \; = \; \mathbf{NLC}^*(G) \cup \{\vee E\}$$

$$\frac{\Gamma \Rightarrow A \vee B{\downarrow} \quad A, \Gamma \Rightarrow D{\uparrow^u} \quad B, \Gamma \Rightarrow D{\uparrow^u}}{\Gamma \Rightarrow D{\uparrow^u}} \vee E \qquad \begin{array}{l} D \in \mathcal{V} \cup \{\bot\} \text{ or } D = D_0 \vee D_1 \\ A \notin \Gamma \text{ and } B \notin \Gamma \end{array}$$

$$\frac{A \supset B, \Gamma \Rightarrow D{\uparrow^u} \quad B \supset A, \Gamma \Rightarrow D{\uparrow^u}}{\Gamma \Rightarrow D{\uparrow^l}} \mathrm{Gd}^* \qquad \begin{array}{l} l \in \{b, u\} \\ D \in \mathcal{V} \cup \{\bot\} \text{ or } D = D_0 \vee D_1 \\ \{A, B\} \subseteq \mathrm{Sf}^{\rightarrow}(G) \text{ or } A \vee B \in \mathrm{Sf}(G) \\ A \supset B \notin \Gamma \text{ and } B \supset A \notin \Gamma \end{array}$$

Fig. 2. The calculi **Nbu**, $\mathbf{NLC}^*(G)$, $\mathbf{NLC}^*_\vee(G)$.

In [9] it is proved that **Nbu** is a sound and complete calculus for IPL:

Theorem 3. *$\Gamma \vdash_{\mathrm{IPL}} G$ iff* $\mathbf{Nbu} \vdash \Gamma \Rightarrow G{\uparrow^u}$. □

Let us assume $\vdash_{\mathrm{LC}} G$. By Th. 1, it holds that $\Sigma_{\mathrm{gd}}(G) \vdash_{\mathrm{IPL}} G$; by Th. 3, there exists an **Nbu**-derivation $\mathcal{D}$ of the sequent $\Sigma_{\mathrm{gd}}(G) \Rightarrow G{\uparrow^u}$. We show that $\mathcal{D}$ can be turned into an $\mathbf{NLC}(G)$-derivation of $\cdot \Rightarrow G{\uparrow^u}$. The translation requires some non-trivial steps; to handily describe them, we introduce the auxiliary calculi $\mathbf{NLC}^*(G)$ and $\mathbf{NLC}^*_\vee(G)$ (see Fig. 2). We point out that $\mathbf{NLC}^*(G)$ is obtained by replacing in $\mathbf{NLC}(G)$ the rule Gd with the more liberal rule Gd^*; $\mathbf{NLC}^*_\vee(G)$ is obtained by adding the rule $\vee E$ to $\mathbf{NLC}^*(G)$. Let

$$\mathrm{Sf}^*(G) \; = \; \mathrm{Sf}(G) \cup \{\, A \supset B \mid \{A, B\} \subseteq \mathrm{Sf}^{\rightarrow}(G) \text{ or } A \vee B \in \mathrm{Sf}(G) \,\}$$

$\mathbf{NLC}^*_\vee(G)$ enjoys the following extended version of subformula property:

Theorem 4. *Let $\mathcal{D}$ be an $\mathbf{NLC}^*_\vee(G)$-derivation of $\cdot \Rightarrow G{\uparrow^u}$ and let $\Gamma \Rightarrow \delta$ be a sequent occurring in $\mathcal{D}$. Then:*

(i) $\Gamma \subseteq \mathrm{Sf}^*(G)$.
(ii) $\delta = H{\uparrow^l}$ *implies* $H \in \mathrm{Sf}(G)$.
(iii) $\delta = H{\downarrow}$ *implies* $H \in \mathrm{Sf}^*(G)$. □

Given $h \in \{b, u\}$, by Ψ^h we denote the partial function mapping an $\mathbf{NLC}^*_\vee(G)$-tree $\mathcal{T}$ with root sequent $\Gamma \Rightarrow C{\uparrow^l}$ ($l \in \{b, u\}$) to an $\mathbf{NLC}^*_\vee(G)$-tree with root sequent $\Gamma \Rightarrow C{\uparrow^h}$. When defined, $\Psi^h(\mathcal{T})$ is obtained by possibly changing some of the labels l in the bottom part of $\mathcal{T}$ with h. The definition of $\Psi^h(\mathcal{T})$ is displayed in Fig. 3. The following properties can be easily checked:

(P1) if $\mathcal{D}$ is an $\mathbf{NLC}^*_\vee(G)$-derivation of $\Gamma \Rightarrow C{\uparrow^l}$, then $\Psi^u(\mathcal{D})$ is an $\mathbf{NLC}^*_\vee(G)$-derivation of $\Gamma \Rightarrow C{\uparrow^u}$.
(P2) if $\mathcal{D}$ is an $\mathbf{NLC}^*(G)$-derivation of $\Gamma \Rightarrow C{\uparrow^l}$ and $h \in \{b, u\}$, then $\Psi^h(\mathcal{D})$ is an $\mathbf{NLC}^*(G)$-derivation of $\Gamma \Rightarrow C{\uparrow^h}$.

$$\Psi^h\left(\dfrac{\begin{matrix}\mathcal{T}_1\\ \Gamma' \Rightarrow \delta\end{matrix}}{\Gamma \Rightarrow C{\uparrow}^l}\,\mathcal{R}\right) = \dfrac{\begin{matrix}\mathcal{T}_1\\ \Gamma' \Rightarrow \delta\end{matrix}}{\Gamma \Rightarrow C{\uparrow}^h}\,\mathcal{R} \qquad \mathcal{R} \in \{\downarrow\uparrow,\ \bot E,\ \vee I_k,\ \supset I_2\}$$

$$\Psi^h\left(\dfrac{\begin{matrix}\mathcal{T}_1\\ \Gamma \Rightarrow A{\uparrow}^l\end{matrix} \qquad \begin{matrix}\mathcal{T}_2\\ \Gamma \Rightarrow B{\uparrow}^l\end{matrix}}{\Gamma \Rightarrow A \wedge B{\uparrow}^l}\,\wedge I\right) = \dfrac{\begin{matrix}\Psi^h(\mathcal{T}_1)\\ \Gamma \Rightarrow A{\uparrow}^h\end{matrix} \qquad \begin{matrix}\Psi^h(\mathcal{T}_2)\\ \Gamma \Rightarrow B{\uparrow}^h\end{matrix}}{\Gamma \Rightarrow A \wedge B{\uparrow}^h}\,\wedge I$$

$$\Psi^h\left(\dfrac{\begin{matrix}\mathcal{T}_1\\ \Gamma \Rightarrow B{\uparrow}^l\end{matrix}}{\Gamma \Rightarrow A \supset B{\uparrow}^l}\,\supset I_1\right) = \dfrac{\begin{matrix}\Psi^h(\mathcal{T}_1)\\ \Gamma \Rightarrow B{\uparrow}^h\end{matrix}}{\Gamma \Rightarrow A \supset B{\uparrow}^h}\,\supset I_1$$

$$\Psi^h\left(\dfrac{\begin{matrix}\mathcal{T}_1\\ A \supset B, \Gamma \Rightarrow D{\uparrow}^u\end{matrix} \quad \begin{matrix}\mathcal{T}_2\\ B \supset A, \Gamma \Rightarrow D{\uparrow}^u\end{matrix}}{\Gamma \Rightarrow D{\uparrow}^l}\,\mathrm{Gd}^*\right) = \dfrac{\begin{matrix}\mathcal{T}_1\\ A \supset B, \Gamma \Rightarrow D{\uparrow}^u\end{matrix} \quad \begin{matrix}\mathcal{T}_2\\ B \supset A, \Gamma \Rightarrow D{\uparrow}^u\end{matrix}}{\Gamma \Rightarrow D{\uparrow}^h}\,\mathrm{Gd}^*$$

Let $\mathcal{R} = \vee E$ (hence $l = u$). Then $\Psi^u(\mathcal{T}) = \mathcal{T}$, $\Psi^b(\mathcal{T})$ is not defined

Fig. 3. Definition of $\Psi^h(\mathcal{T})$ ($\mathcal{R}$ is the root rule of $\mathcal{T}$)

Let $\mathcal{T}$ be an $\mathbf{NLC}^*_\vee(G)$-tree with root sequent $C, \Gamma \Rightarrow \delta$. By $\Phi^-_C(\mathcal{T})$ we denote the $\mathbf{NLC}^*_\vee(G)$-tree obtained by deleting an occurrence of C from the left contexts of the sequents in $\mathcal{T}$, with some caution with rule $\supset I_1$. Formally:

$$\Phi^-_C\left(\dfrac{\begin{matrix}\mathcal{T}_1\\ C, \Gamma \Rightarrow B{\uparrow}^l\end{matrix}}{C, \Gamma \Rightarrow C \supset B{\uparrow}^l}\,\supset I_1\right) = \dfrac{\begin{matrix}\Psi^u(\mathcal{T}_1)\\ C, \Gamma \Rightarrow B{\uparrow}^u\end{matrix}}{\Gamma \Rightarrow C \supset B{\uparrow}^l}\,\supset I_1 \qquad \text{if } C \notin \Gamma$$

$$\Phi^-_C\left(\dfrac{\mathcal{T}_1 \quad \cdots \quad \mathcal{T}_n}{C, \Gamma \Rightarrow \delta}\,\mathcal{R}\right) = \dfrac{\Phi^-_C(\mathcal{T}_1) \quad \cdots \quad \Phi^-_C(\mathcal{T}_n)}{\Gamma \Rightarrow \delta}\,\mathcal{R} \qquad \text{in all the other cases}$$

One can easily check that the map Φ^-_C is well-defined (note that in its definition the map Ψ^h is only used with h set to u). We remark that, if $\mathcal{D}$ is an $\mathbf{NLC}^*_\vee(G)$-derivation, the tree $\Phi^-_C(\mathcal{D})$ might not be an $\mathbf{NLC}^*_\vee(G)$-derivation. For instance, let us assume that $\mathcal{D}$ contains a leaf $\sigma_C = C, \Gamma \Rightarrow C{\downarrow}$, with $C \notin \Gamma$; then, σ_C is replaced by the open leaf $\Gamma \Rightarrow C{\downarrow}$. We can prove that:

Lemma 1. $\mathbf{Nbu} \vdash \Sigma_{\mathrm{gd}}(G) \Rightarrow G{\uparrow}^u$ *implies* $\mathbf{NLC}^*_\vee(G) \vdash \cdot \Rightarrow G{\uparrow}^u$. □

Let $\mathcal{T}$ be an $\mathbf{NLC}^*(G)$-tree with root sequent $\Gamma \Rightarrow \delta$ and C a formula. By $\Phi^+_C(\mathcal{D})$ we denote the $\mathbf{NLC}^*(G)$-tree with root sequent $C, \Gamma \Rightarrow \delta$ obtained by adding an occurrence of C to the left contexts of $\mathcal{T}$, with some caution with rule $\supset I_2$, namely:

$$\Phi_C^+\left(\dfrac{\begin{array}{c}\mathcal{T}_1\\ A,\Gamma \Rightarrow B{\uparrow}^u\end{array}}{\Gamma \Rightarrow A \supset B{\uparrow}^l}\supset I_2\right) = \dfrac{\begin{array}{c}\Psi^l(\mathcal{T}_1)\\ C,\Gamma \Rightarrow B{\uparrow}^l\end{array}}{C,\Gamma \Rightarrow A \supset B{\uparrow}^l}\supset I_1 \qquad \text{if } C = A$$

$$\Phi_C^+\left(\dfrac{\begin{array}{c}\mathcal{T}_1\\ A\supset B,\Gamma \Rightarrow D{\uparrow}^u\end{array}\quad \begin{array}{c}\mathcal{T}_2\\ B\supset A,\Gamma \Rightarrow D{\uparrow}^u\end{array}}{\Gamma \Rightarrow D{\uparrow}^l}\text{Gd}^*\right) = \begin{array}{c}\Psi^l(\mathcal{T}_k)\\ C,\Gamma \Rightarrow D{\uparrow}^l\end{array} \qquad \begin{array}{l} C \in \{A\supset B, B\supset A\}\\ k = \begin{cases}1 & \text{if } C = A\supset B\\ 2 & \text{otherwise}\end{cases}\end{array}$$

$$\Phi_C^+\left(\dfrac{\mathcal{T}_1 \quad \cdots \quad \mathcal{T}_n}{\Gamma \Rightarrow \delta{\uparrow}^l}\mathcal{R}\right) = \dfrac{\Phi_C^+(\mathcal{T}_1) \quad \cdots \quad \Phi_C^+(\mathcal{T}_n)}{C,\Gamma \Rightarrow \delta{\uparrow}^l}\mathcal{R} \qquad \text{in all the other cases}$$

It is easy to check that if $\mathcal{D}$ is an $\mathbf{NLC}^*(G)$-derivation of $\Gamma \Rightarrow \delta$, then $\Phi_C^+(\mathcal{D})$ is an $\mathbf{NLC}^*(G)$-derivation of $C,\Gamma \Rightarrow \delta$ (see (P2)). For $\Delta = \{C_1,\dots,C_n\}$, Φ_Δ^+ is the composite map $\Phi_{C_1}^+ \circ \cdots \circ \Phi_{C_n}^+$ (namely, we add all the formulas in Δ).

Lemma 2. $\mathbf{NLC}_\vee^*(G) \vdash \cdot \Rightarrow H{\uparrow}^u$ *implies* $\mathbf{NLC}^*(G) \vdash \cdot \Rightarrow H{\uparrow}^u$.

Proof. Let $\mathcal{D}$ be an $\mathbf{NLC}_\vee^*(G)$-derivation of $\cdot \Rightarrow H{\uparrow}^u$. We show that we can eliminate all the applications of rule $\vee E$. Let us consider a subderivation $\mathcal{D}'$ of $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow A\vee B{\downarrow}\end{array}\quad \begin{array}{c}\mathcal{D}_1\\ A,\Gamma \Rightarrow D{\uparrow}^u\end{array}\quad \begin{array}{c}\mathcal{D}_2\\ B,\Gamma \Rightarrow D{\uparrow}^u\end{array}}{\Gamma \Rightarrow D{\uparrow}^u}\vee E \qquad A\notin\Gamma,\ B\notin\Gamma$$

where $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_2$ do not contain applications of $\vee E$, hence $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_2$ are $\mathbf{NLC}^*(G)$-derivations. We show that we can replace $\mathcal{D}'$ with an $\mathbf{NLC}^*(G)$-derivation $\mathcal{G}$ of $\Gamma \Rightarrow D{\uparrow}^u$. Let us assume $B\supset A \in \Gamma$ and let us consider the $\mathbf{NLC}^*(G)$-tree $\Phi_A^-(\mathcal{D}_1)$ having root sequent $\Gamma \Rightarrow D{\uparrow}^u$. Note that $\Phi_A^-(\mathcal{D}_1)$ might contain open leaves of the form $\sigma_A = \Gamma,\Delta \Rightarrow A{\downarrow}$. We can replace σ_A with the $\mathbf{NLC}^*(G)$-derivation

$$\dfrac{\begin{array}{c}\Phi_\Delta^+(\mathcal{D}_0)\\ \Gamma,\Delta \Rightarrow A\vee B{\downarrow}\end{array}\quad \dfrac{}{\Gamma,\Delta \Rightarrow B\supset A{\downarrow}}\text{Id}}{\sigma_A \;=\; \Gamma,\Delta \Rightarrow A{\downarrow}}\vee E_0$$

After such replacements, we get an $\mathbf{NLC}^*(G)$-derivation $\mathcal{G}$ of $\Gamma \Rightarrow D{\uparrow}^u$. The case $A\supset B \in \Gamma$ is symmetric. Finally, let us assume $A\supset B \notin \Gamma$ and $B\supset A \notin \Gamma$. Proceeding as in the previous cases, starting from the $\mathbf{NLC}^*(G)$-tree $\Phi_{A\supset B}^+(\Phi_B^-(\mathcal{D}_2))$ we can build an $\mathbf{NLC}^*(G)$-derivation $\mathcal{E}_1$ of $A\supset B,\Gamma \Rightarrow D{\uparrow}^u$; similarly, from $\Phi_{B\supset A}^+(\Phi_A^-(\mathcal{D}_1))$ we can build $\mathbf{NLC}^*(G)$-derivation $\mathcal{E}_2$ of $B\supset A,\Gamma \Rightarrow D{\uparrow}^u$. We can replace $\mathcal{D}'$ with the $\mathbf{NLC}^*(G)$-derivation

$$\dfrac{\begin{array}{c}\mathcal{E}_1\\ A\supset B,\Gamma \Rightarrow D{\uparrow}^u\end{array}\quad \begin{array}{c}\mathcal{E}_2\\ B\supset A,\Gamma \Rightarrow D{\uparrow}^u\end{array}}{\Gamma \Rightarrow D{\uparrow}^u}\text{Gd}^*$$

Note that the displayed application of rule Gd* is sound since $\Gamma \Rightarrow A \vee B \downarrow$ occurs in $\mathcal{D}$ hence, by Theorem 4(iii), $A \vee B \in \mathrm{Sf}(G)$. By repeatedly applying such replacements, we cross out all the applications of $\vee E$ and we eventually get an $\mathbf{NLC}^*(G)$-derivation of $\cdot \Rightarrow H{\uparrow}^u$. □

By applying standard permutation steps, consisting in moving down the applications of Gd*, so that the conclusion can be labelled by u, we get:

Lemma 3. $\mathbf{NLC}^*(G) \vdash \cdot \Rightarrow G{\uparrow}^u$ *implies* $\mathbf{NLC}(G) \vdash \cdot \Rightarrow G{\uparrow}^u$. □

Putting things together we get the completeness of $\mathbf{NLC}(G)$:

Theorem 5. $\vdash_{\mathrm{LC}} G$ *implies* $\mathbf{NLC}(G) \vdash \cdot \Rightarrow G{\uparrow}^u$. □

To conclude, in this short paper we present the calculus $\mathbf{NLC}(G)$ and we show that it is sound and complete for LC. We plan to design a proof-search strategy for $\mathbf{NLC}(G)$, similar to the one developed for **Nbu** in [9], and to investigate the computational interpretation of $\mathbf{NLC}(G)$ and the relationship with the calculi in [1,2,5].

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A Kripke semantics and proof of Theorem 1

In this section we provide a semantic proof of Theorem 1 inspired from [3]. An IPL-model is a Kripke model [6] $\mathcal{K} = \langle W, \leq, \rho, V \rangle$, where $\langle W, \leq \rangle$ is a poset over the set of worlds W with minimum ρ and $V : W \to 2^{\mathcal{V}}$ is a function such that $\alpha \leq \beta$ implies $V(\alpha) \subseteq V(\beta)$. If $\leq$ is a linear order over W, we say that $\mathcal{K}$ is a linear model. The *forcing relation* $\Vdash$ is defined as follows:

- $\mathcal{K}, \alpha \nVdash \bot$ and, for every $p \in \mathcal{V}$, $\mathcal{K}, \alpha \Vdash p$ iff $p \in V(\alpha)$;
- $\mathcal{K}, \alpha \Vdash A \wedge B$ iff $\mathcal{K}, \alpha \Vdash A$ and $\mathcal{K}, \alpha \Vdash B$;
- $\mathcal{K}, \alpha \Vdash A \vee B$ iff $\mathcal{K}, \alpha \Vdash A$ or $\mathcal{K}, \alpha \Vdash B$;
- $\mathcal{K}, \alpha \Vdash A \supset B$ iff, for every $\beta \in W$ such that $\alpha \leq \beta$, $\mathcal{K}, \beta \nVdash A$ or $\mathcal{K}, \beta \Vdash B$.

Let Γ be a set of formulas. By $\mathcal{K}, \alpha \Vdash \Gamma$ we mean that $\mathcal{K}, \alpha \Vdash A$, for every $A \in \Gamma$. A formula A is valid in a model $\mathcal{K}$ iff $\mathcal{K}, \alpha \Vdash A$ for every $\alpha \in W$. It is well-known that (see e.g. [6]):

- IPL is the set of the formulas valid in all Kripke models;
- LC is the set of the formulas valid in all linear Kripke models.

We introduce a filtration technique on Kripke models based on quotientation [6]. Let $\mathcal{K} = \langle W, \leq, \rho, V \rangle$, let G be a formula and α, β two worlds of $\mathcal{K}$. We set:

- $\alpha \subseteq_G \beta$ iff, for every $A \in \mathrm{Sf}^{\to}(G)$, $\mathcal{K}, \alpha \Vdash A$ implies $\mathcal{K}, \beta \Vdash A$;
- $\alpha \equiv_G \beta$ iff $\alpha \subseteq_G \beta$ and $\beta \subseteq_G \alpha$.

Note that $\alpha \leq \beta$ implies $\alpha \subseteq_G \beta$, while the converse might not hold. We point out that $\equiv_G$ is an equivalence relation on W; by $[\alpha]$ we denote the $\equiv_G$-equivalence class containing α. The *quotient* model $\mathcal{K}_{/G} = \langle W', \leq', \rho', V' \rangle$ is defined as follows:

- W' is the set of equivalence classes generated by $\equiv_G$ and $\rho' = [\rho]$;
- $[\alpha] \leq' [\beta]$ iff $\alpha \subseteq_G \beta$;
- $V'([\alpha]) = V(\alpha) \cap \mathrm{Sf}^{\to}(G)$.

One can easily check that $\mathcal{K}_{/G}$ is a well-defined Kripke model. In particular: ρ' is the minimum element of $\mathcal{K}_{/G}$ and $[\alpha] \leq' [\beta]$ implies $V([\alpha]) \subseteq V([\beta])$. In the next lemma we prove that quotientation with respect to G preserves the forcing of subformulas of G; moreover if all the formulas in $\Sigma_{\mathrm{gd}}(G)$ are valid in $\mathcal{K}$, then $\mathcal{K}_{/G}$ is linear.

Lemma 4. *Let $\mathcal{K} = \langle W, \leq, \rho, V \rangle$ be a Kripke model and let G be a formula.*

(i) For every $\alpha \in W$ and $C \in \mathrm{Sf}(G)$, $\mathcal{K}, \alpha \Vdash C$ iff $\mathcal{K}_{/G}, [\alpha] \Vdash C$.
(ii) If $\mathcal{K}, \rho \Vdash \Sigma_{\mathrm{gd}}(G)$, then $\mathcal{K}_{/G}$ is a linear model.

Proof. Let $\mathcal{K}_{/G} = \langle W', \leq', \rho', V' \rangle$. The proof of (i) goes by induction on the structure of C. We only discuss the case $C = A \supset B$ the other being trivial. Let $\mathcal{K}, \alpha \nVdash A \supset B$. Then, there exists $\beta \in W$ such that $\alpha \leq \beta$, $\mathcal{K}, \beta \Vdash A$ and $\mathcal{K}, \beta \nVdash B$. By induction hypothesis, $\mathcal{K}', [\beta] \Vdash A$ and $\mathcal{K}', [\beta] \nVdash B$. Since

$\alpha \leq \beta$, we get $[\alpha] \leq' [\beta]$, and this proves that $\mathcal{K}', [\alpha] \nVdash A \supset B$. Conversely, let $\mathcal{K}', [\alpha] \nVdash A \supset B$. Then, there exists $\beta \in W$ such that $[\alpha] \leq' [\beta]$, $\mathcal{K}', [\beta] \Vdash A$ and $\mathcal{K}', [\beta] \nVdash B$. By induction hypothesis, we get $\mathcal{K}, \beta \Vdash A$ and $\mathcal{K}, \beta \nVdash B$, which implies $\mathcal{K}, \beta \nVdash A \supset B$. Since $A \supset B \in \mathrm{Sf}^{\rightarrow}(G)$ and $\alpha \subseteq_G \beta$, we conclude $\mathcal{K}, \alpha \nVdash A \supset B$.

(ii) Let $\mathcal{K}, \rho \Vdash \Sigma_{\mathrm{gd}}(G)$ and let us assume, by absurd, that $\mathcal{K}'$ is not a linear model. Then, there exist two worlds α and β of $\mathcal{K}$ such that $[\alpha] \not\leq' [\beta]$ and $[\beta] \not\leq' [\alpha]$, namely $\alpha \not\subseteq_G \beta$ and $\beta \not\subseteq_G \alpha$. This implies that there is $A \in \mathrm{Sf}^{\rightarrow}(G)$ such that $\mathcal{K}, \alpha \Vdash A$ and $\mathcal{K}, \beta \nVdash A$ and $B \in \mathrm{Sf}^{\rightarrow}(G)$ such that $\mathcal{K}, \beta \Vdash B$ and $\mathcal{K}, \alpha \nVdash B$. It follows that $\mathcal{K}, \rho \nVdash D$, where $D = (A \supset B) \vee (B \supset A)$. This yields a contradiction, since $D \in \Sigma_{\mathrm{gd}}(G)$ and $\mathcal{K}, \rho \Vdash \Sigma_{\mathrm{gd}}(G)$; we conclude that $\mathcal{K}'$ is linear. □

Given a logic L, by $\Gamma \models_L A$ we mean that A is a *logical consequence* of Γ in L, namely: for every L-model $\mathcal{K}$ and every world α of $\mathcal{K}$, if $\mathcal{K}, \alpha \Vdash \Gamma$ then $\mathcal{K}, \alpha \Vdash A$.

Lemma 5. *$\models_{\mathrm{LC}} G$ iff $\Sigma_{\mathrm{gd}}(G) \models_{\mathrm{IPL}} G$.*

Proof. Let us assume $\not\models_{\mathrm{LC}} G$. Then, there exists a linear Kripke model $\mathcal{K}$ and a world α in $\mathcal{K}$ such that $\mathcal{K}, \alpha \nVdash G$. Since all the formulas in $\Sigma_{\mathrm{gd}}(G)$ are valid in $\mathcal{K}$, we conclude $\Sigma_{\mathrm{gd}}(G) \not\models_{\mathrm{IPL}} G$. Conversely, let us assume, $\Sigma_{\mathrm{gd}}(G) \not\models_{\mathrm{IPL}} G$. Then, there exists a Kripke model $\mathcal{K}$ and a world α in $\mathcal{K}$ such that $\mathcal{K}, \alpha \Vdash \Sigma_{\mathrm{gd}}(G)$ and $\mathcal{K}, \alpha \nVdash G$. By Lemma 4, $\mathcal{K}_{/G}$ is an LC-model (Point (ii)) such that $\mathcal{K}_{/G}, [\alpha] \nVdash G$ (Point (i)), and this implies $\not\models_{\mathrm{LC}} G$. □

By Lemma 5 and the fact that logical consequence relation $\models_L$ and provability relations $\vdash_L$ coincide, with $L \in \{\mathrm{IPL}, \mathrm{LC}\}$, Theorem 1 follows.

B Proof of the main results of Section 3

Given a formula G, the set $\mathrm{Sf}^+(G)$ of *strictly positive subformulas* of G (see [14]) is the smallest set of formulas such that:

- $G \in \mathrm{Sf}^+(G)$;
- $A \odot B \in \mathrm{Sf}^+(G)$, with $\odot \in \{\wedge, \vee\}$, implies $A \in \mathrm{Sf}^+(G)$ and $B \in \mathrm{Sf}^+(G)$;
- $A \supset B \in \mathrm{Sf}^+(G)$ implies $B \in \mathrm{Sf}^+(G)$.

Given a multiset of formulas Γ, the set $\mathrm{Sf}^+(\Gamma)$ is the union of the sets $\mathrm{Sf}^+(G)$, for every $G \in \Gamma$. By induction on the depth of $\mathbf{NLC}^*_{\vee}(G)$-derivations, one can easily prove that:

Lemma 6. *$\mathbf{NLC}^*_{\vee}(G) \vdash \Gamma \Rightarrow C{\downarrow}$ implies $C \in \mathrm{Sf}^+(\Gamma)$.* □

Theorem 4. *Let $\mathcal{D}$ be an $\mathbf{NLC}^*_{\vee}(G)$-derivation of $\cdot \Rightarrow G{\uparrow}^u$ and let $\Gamma \Rightarrow \delta$ be a sequent occurring in $\mathcal{D}$. Then:*

(i) $\Gamma \subseteq \mathrm{Sf}^*(G)$.
(ii) $\delta = C{\uparrow}^l$ *implies* $C \in \mathrm{Sf}(G)$.
(iii) $\delta = C{\downarrow}$ *implies* $C \in \mathrm{Sf}^*(G)$.

Proof. By induction on the depth of $\sigma = \Gamma \Rightarrow \delta$, namely the distance between σ and the root sequent $\sigma_r = \cdot \Rightarrow G{\uparrow}^u$ of $\mathcal{D}$. In the base case $\sigma = \sigma_r$ and points (i) and (ii) trivially hold. Let $\sigma \neq \sigma_r$. Then, σ is the assumption of the application of a rule $\mathcal{R}$ of $\mathbf{NLC}^*_\vee(G)$; by σ_0 we denote the conclusion of $\mathcal{R}$. By IH, we can assume that σ_0 satisfies points (i)–(iii). We proceed by a case analysis on $\mathcal{R}$, only detailing some significant cases.

$$\mathcal{R} = {\downarrow\!\uparrow} \qquad \frac{\sigma \ = \ \Gamma \Rightarrow p{\downarrow}}{\sigma_0 \ = \ \Gamma \Rightarrow p{\uparrow}^l}\ {\downarrow\!\uparrow}$$

By IH, $\Gamma \subseteq \mathrm{Sf}^*(G)$ and $p \in \mathrm{Sf}(G)$, hence (i) and (iii) hold.

$$\mathcal{R} \ = \ \bot E \qquad \frac{\sigma \ = \ \Gamma \Rightarrow \bot{\downarrow}}{\sigma_0 \ = \ \Gamma \Rightarrow F{\uparrow}^l}\ \bot E$$

By IH, $\Gamma \subseteq \mathrm{Sf}^*(G)$, hence (i) holds. Since $\mathbf{NLC}^*_\vee(G) \vdash \sigma$, by Lemma 6 we get $\bot \in \mathrm{Sf}^+(\Gamma)$. It follows that $\bot \in \mathrm{Sf}^*(G)$, and this proves (iii).

$$\mathcal{R} \ = \ \wedge I \qquad \frac{\sigma \ = \ \Gamma \Rightarrow A{\uparrow}^l \qquad \Gamma \Rightarrow B{\uparrow}^l}{\sigma_0 \ = \ \Gamma \Rightarrow A \wedge B{\uparrow}^l}\ \wedge I$$

By IH, $\Gamma \subseteq \mathrm{Sf}^*(G)$ and $A \wedge B \in \mathrm{Sf}(G)$, which implies $A \in \mathrm{Sf}(G)$, hence (i) and (ii) hold.

$$\mathcal{R} \ = \ \wedge E_k \qquad \frac{\sigma \ = \ \Gamma \Rightarrow A_0 \wedge A_1{\downarrow}}{\sigma_0 \ = \ \Gamma \Rightarrow A_k{\downarrow}}\ \wedge E_k$$

By IH, $\Gamma \subseteq \mathrm{Sf}^*(G)$, hence (i) holds. Since $\mathbf{NLC}^*_\vee(G) \vdash \sigma$, by Lemma 6 we get $A_0 \wedge A_1 \in \mathrm{Sf}^+(\Gamma)$. It follows that $A_0 \wedge A_1 \in \mathrm{Sf}^*(G)$, hence (iii) holds.

$$\mathcal{R} \ = \supset E \qquad \frac{\sigma_1 \ = \ \Gamma \Rightarrow A \supset B{\downarrow} \qquad \sigma \ = \ \Gamma \Rightarrow A{\uparrow}^b}{\sigma_0 \ = \ \Gamma \Rightarrow B{\downarrow}}\ \supset E$$

By IH, $\Gamma \subseteq \mathrm{Sf}^*(G)$, hence (i) holds. Since $\mathbf{NLC}^*_\vee(G) \vdash \sigma_1$, by Lemma 6 we get $A \supset B \in \mathrm{Sf}^+(\Gamma)$, hence $A \supset B \in \mathrm{Sf}^*(G)$. This implies that $A \supset B \in \mathrm{Sf}(G)$ or $\{A, B\} \subseteq \mathrm{Sf}^\rightarrow(G)$ or $A \vee B \in \mathrm{Sf}(G)$. In either case $A \in \mathrm{Sf}(G)$, hence (ii) holds.

$$\mathcal{R} \ = \supset I_2 \qquad \frac{\sigma \ = \ A, \Gamma \Rightarrow B{\uparrow}^u}{\sigma_0 \ = \ \Gamma \Rightarrow A \supset B{\uparrow}^l}\ \supset I_2$$

By IH, $\Gamma \subseteq \mathrm{Sf}^*(G)$ and $A \supset B \in \mathrm{Sf}(G)$, which implies $\{A, B\} \subseteq \mathrm{Sf}(G)$, hence (i) and (ii) hold.

$$\mathcal{R} \ = \ \mathrm{Gd}^* \qquad \frac{\sigma \ = \ A \supset B, \Gamma \Rightarrow D{\uparrow}^u \quad B \supset A, \Gamma \Rightarrow D{\uparrow}^u}{\sigma_0 \ = \ \Gamma \Rightarrow D{\uparrow}^l}\ \mathrm{Gd}^*$$

By IH, $\Gamma \subseteq \mathrm{Sf}^*(G)$ and $D \in \mathrm{Sf}(G)$. By the side conditions of application of rule Gd^*, $A \supset B \in \mathrm{Sf}^*(G)$, hence (i) and (ii) hold. □

The following lemma states a sufficient condition to guarantee that $\Phi^-_C(\mathcal{D})$ is an $\mathbf{NLC}^*_\vee(G)$-derivation:

Lemma A.1. *Let $\mathcal{D}$ be an $\mathbf{NLC}^*_\vee(G)$-derivation of $C, \Gamma \Rightarrow \delta$ and let us assume that $\mathcal{D}$ does not contain applications of rule* Id *with main formula C. Then, $\Phi^-_C(\mathcal{D})$ is an $\mathbf{NLC}^*_\vee(G)$-derivation of $\Gamma \Rightarrow \delta$.*

Proof. By induction on the depth of $\mathcal{D}$. Note that, in the base case, $\mathcal{D}$ only consists of the sequent $C, \Gamma \Rightarrow A{\downarrow}$ and, by hypothesis, $A \neq C$, hence $A \in \Gamma$. This implies that $\Phi^-_C(\mathcal{D}) = \Gamma \Rightarrow A{\downarrow}$ is an $\mathbf{NLC}^*_\vee(G)$-derivation. □

Lemma 1. $\mathbf{Nbu} \vdash \Sigma_{\mathrm{gd}}(G) \Rightarrow G{\uparrow}^u$ *implies* $\mathbf{NLC}^*_\vee(G) \vdash \cdot \Rightarrow G{\uparrow}^u$.

Proof. Let $\mathcal{D}$ be an **Nbu**-derivation of $\Sigma_{\mathrm{gd}}(G) \Rightarrow G{\uparrow}^u$; note that $\mathcal{D}$ is an $\mathbf{NLC}^*_\vee(G)$-derivation as well. We can eliminate from $\mathcal{D}$ all the applications of rule Id having main formulas in $\Sigma_{\mathrm{gd}}(G)$ as follows. Let $\sigma = \Gamma \Rightarrow C{\downarrow}$ be the conclusion of an application of rule Id with $C \in \Sigma_{\mathrm{gd}}(G)$, and let us assume $C = (A \supset B) \vee (B \supset A)$ where $\{A, B\} \subseteq \mathrm{Sf}^{\rightarrow}(G)$. Then, σ must be the major premise of an application of rule $\vee E$ with conclusion $\sigma_0 = \Gamma \Rightarrow D{\uparrow}^u$. Let $\mathcal{D}'$ be the subderivation of $\mathcal{D}$ with root sequent σ_0 defined as:

$$\dfrac{\dfrac{}{\sigma \;=\; \Gamma \Rightarrow C{\downarrow}}\,\mathrm{Id} \qquad \begin{array}{c}\mathcal{D}_1\\ A \supset B, \Gamma \Rightarrow D{\uparrow}^u\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ B \supset A, \Gamma \Rightarrow D{\uparrow}^u\end{array}}{\sigma_0 \;=\; \Gamma \Rightarrow D{\uparrow}^u}\,\vee E \qquad \begin{array}{c} A \supset B \notin \Gamma\\ B \supset A \notin \Gamma\end{array}$$

We can replace $\mathcal{D}'$ with the $\mathbf{NLC}^*_\vee(G)$-derivation

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ A \supset B, \Gamma \Rightarrow D{\uparrow}^u\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ B \supset A, \Gamma \Rightarrow D{\uparrow}^u\end{array}}{\Gamma \Rightarrow D{\uparrow}^u}\,\mathrm{Gd}^*$$

By repeatedly applying such replacements, we eventually get an $\mathbf{NLC}^*_\vee(G)$-derivation of $\Sigma_{\mathrm{gd}}(G) \Rightarrow G{\uparrow}^u$ such that the main formulas of applications of rule Id do not belong to $\Sigma_{\mathrm{gd}}(G)$. By Lemma A.1, we conclude $\mathbf{NLC}^*_\vee(G) \vdash \cdot \Rightarrow G{\uparrow}^u$. □

We conclude this section with a further example of $\mathbf{NLC}(G)$-derivation.

Example 2. Let $G = \neg p \vee \neg\neg p$; an $\mathbf{NLC}(G)$-derivation of G is (recall that $\neg A$ stands for $A \supset \bot$):

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ p, \neg\neg p \supset \neg p \Rightarrow \bot{\downarrow}\end{array}}{p, \neg\neg p \supset \neg p \Rightarrow \bot{\uparrow}^b}\,\bot E}{\neg\neg p \supset \neg p \Rightarrow \neg p{\uparrow}^b}\,{\supset} I_2}{\neg\neg p \supset \neg p \Rightarrow \neg p \vee \neg\neg p{\uparrow}^u}\,\vee I_0 \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2\\ \neg p, \neg p \supset \neg\neg p \Rightarrow \bot{\downarrow}\end{array}}{\neg p, \neg p \supset \neg\neg p \Rightarrow \bot{\uparrow}^b}\,\bot E}{\neg p \supset \neg\neg p \Rightarrow \neg\neg p{\uparrow}^b}\,{\supset} I_2}{\neg p \supset \neg\neg p \Rightarrow \neg p \vee \neg\neg p{\uparrow}^u}\,\vee I_1}{\cdot \Rightarrow \neg p \vee \neg\neg p{\uparrow}^u}\,\mathrm{Gd}$$

where $\mathcal{D}_1$ is $(A = \neg\neg p \supset \neg p)$

$$\dfrac{\dfrac{\dfrac{}{p, A \Rightarrow A{\downarrow}}\,\text{Id} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg p, p, A \Rightarrow \neg p{\downarrow}}\,\text{Id} \qquad \dfrac{\dfrac{}{\neg p, p, A \Rightarrow p{\downarrow}}\,\text{Id}}{\neg p, p, A \Rightarrow p{\uparrow}^b}\supset E}{\neg p, p, A \Rightarrow \bot{\downarrow}}\supset E}{\neg p, p, A \Rightarrow \bot{\uparrow}^u}\bot E}{p, A \Rightarrow \neg\neg p{\uparrow}^b}\supset I_2}{p, A \Rightarrow \neg p{\downarrow}}\supset E \qquad \dfrac{\dfrac{}{p, A \Rightarrow p{\downarrow}}\,\text{Id}}{p, A \Rightarrow p{\uparrow}^b}\,{\downarrow}{\uparrow}}{p, A \Rightarrow \bot{\downarrow}}\supset E$$

and $\mathcal{D}_2$ is similar. $\Diamond$