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Generalized Dimensions of Self-Affine Sets with Overlaps

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Abstract: Two decades ago, Ngai and Wang introduced a well-known finite type condition (FTC) on the self-similar iterated function system (IFS) with overlaps and used it to calculate the Hausdorff dimension of self-similar sets. In this paper, inspired by Ngai and Wang's idea, we define a new FTC on self-affine IFS and obtain an analogous formula on the generalized dimensions of self-affine sets. The generalized dimensions raised by He and Lau are used to estimate the Hausdorff dimension of self-affine sets.

Keywords: generalized dimension; self-affine set; overlap; FTC; pseudo-norm

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1. Introduction

Let $M_d(\mathbb{R})$ be the set of all $d\times d$ matrices with entries in $\mathbb{R}$, and let $A\in M_d(\mathbb{R})$ be an expanding matrix. Let $N\geq 2$ and $n_1,\ldots,n_N$ be positive integers. Write $A_j=A^{n_j}, 1\leq j\leq N$ and $\mathcal{D}=\{d_1,\ldots,d_N\}\subset\mathbb{R}^d$. Then, we can define a self-affine *iterated function system* (IFS) $\{S_j\}_{j=1}^N$ on $\mathbb{R}^d$ by

$$S_j(x)=A_j^{-1}(x+d_j),\quad j=1,\ldots,N. \tag{1}$$

According to [1], there exists a unique nonempty compact set $E:=E(\{A_j\}_{j=1}^N,\mathcal{D})$ such that

$$E=\bigcup_{j=1}^N S_j(E). \tag{2}$$

We call E a *self-affine set*. Without loss of generality, we always assume that $d_1=0$ and $n_1\geq n_2\geq\cdots\geq n_N$ throughout the paper.

The dimensional theory of self-affine IFS is a major topic in fractal geometry and dynamical systems. Many important results have been achieved in this area. In the case of self-similar IFS, i.e., A_j in (1) are similitudes as $A_j^{-1}=\rho_jR_j$, where $0<\rho_j<1$ and R_j are orthonormal matrices, the set E (2) is usually called a *self-similar set*. If the open set condition holds, the Hausdorff and box dimensions of E are clear, which satisfy $\dim_H E=\dim_B E=\alpha$, where α is the unique solution to the equation $\sum_{j=1}^N\rho_j^s=1$ (see refs. [1–3]). The IFS (1) is said to satisfy the *open set condition* (*OSC*) [1,3] if there exists a bounded nonempty open set $U\subset\mathbb{R}^d$ such that $U\supset\bigcup_{j=1}^N S_j(U)$ with disjoint union.

If the OSC does not hold, overlaps may occur; it is very challenging to obtain a simple dimensional formula. In 2001, Ngai and Wang [4] defined a *finite type condition* to deal with the self-similar IFSs with overlaps and described an algorithm for the dimension of self-similar sets. Subsequently, several other types of separation conditions were developed as well, e.g., weak separation condition, generalized finite type condition. With these conditions, it becomes easy to find the dimension of self-similar sets with overlapping structure (see refs. [5–9]). On the other hand, from the viewpoint of dynamical systems, the dimensional results of invariant measures or self-similar sets are closely related to a Ledrappier–Young type formula (see, e.g., [10] and references therein).

However, compared with self-similar sets, research on fractal dimension of self-affine sets has been progressing very slowly (see a survey paper [11]). Except for some special cases, such as the Bedford–McMullen set [12,13], it is difficult to give an exact dimensional formula (see refs. [14,15]).

Recently, by employing a pseudo-norm w rather than the Euclidean norm, He and Lau [16] introduced the generalized Hausdorff measure $\mathcal{H}^t_w$, the generalized Hausdorff dimension $\dim^w_H$ and the generalized box dimension $\dim^w_B$ (see definitions in Section 2). The w is determined by the given matrix A. The generalized dimensions are useful to estimate the exact fractal dimension of self-affine sets. Under the OSC, He and Lau obtained a formula of the generalized dimensions on a class of self-affine IFSs with equal linear parts (i.e., all $A_j \equiv A$). Later, based on He and Lau's results, Fu, Gabardo and Qiu [17] further proved that the self-affine IFS satisfies the OSC if and only if the generalized Hausdorff measure $\mathcal{H}^t_w(E) > 0$. The second author and Yang [18] also computed the generalized dimensions of the attractors of a class of self-affine graph-directed IFSs.

Motivated by the above studies, in this paper, we try to develop Ngai and Wang's idea and define a new finite type condition on the self-affine IFS (1). Then, we generalize He and Lau's dimensional result onto the overlapping situation.

Let $\Sigma = \{1, 2, \ldots, N\}, \Sigma^k = \{j_1 j_2 \ldots j_k : j_i \in \Sigma, 1 \le i \le k\}, \Sigma^* = \bigcup_{k=0}^{\infty} \Sigma^k$ with $\Sigma^0 = \{\varnothing\}$ and $\Sigma^\infty = \{j_1 j_2 \cdots : j_i \in \Sigma, i \ge 1\}$. If $I = i_1 \ldots i_h$ and $J = j_1 \ldots j_\ell$; we denote by $IJ = i_1 \ldots i_h j_1 \ldots j_\ell$ the concatenation of them. If $I = i_1 i_2 \cdots \in \Sigma^\infty$, let $I|_k = i_1 \ldots i_k$ be the restriction of the first k symbols of I. For any $J = j_1 \ldots j_k \in \Sigma^k$, we denote $|J| = k$ the length of J, and write $S_J(x) = S_{j_1} \circ \cdots \circ S_{j_k}(x)$, $A_J = \prod_{i=1}^{k} A_{j_i} = A^{j_1 + \cdots + j_k}$.

Assume $|\det A| = q$ and write $r_j = q^{-n_j/d}$, $r_J = r_{j_1} \ldots r_{j_k}$ which stand for the contraction ratios of the maps S_j and S_J under w, respectively (see Section 2). For any $k \ge 0$, we denote

$$\Lambda_k := \{j_1 \ldots j_\ell \in \Sigma^* : r_{j_1 \ldots j_\ell} \le r_1^k < r_{j_1 \ldots j_{\ell-1}}\}.$$

Let $\mathcal{V}_0 = \{(\mathrm{id}, 0)\}, \mathcal{V}_k = \{(S_J, k) : J \in \Lambda_k\}$ for $k \ge 1$ and $\mathcal{V} = \bigcup_{k \ge 0} \mathcal{V}_k$. Let U be an invariant open set of the IFS (1), i.e., U satisfies $\bigcup_{j=1}^{N} S_j(U) \subset U$. We say that $\mathbf{u} = (S_I, k)$, $\mathbf{v} = (S_J, k) \in \mathcal{V}_k$ are *neighbors* with respect to U if $S_I(U) \cap S_J(U) \ne \varnothing$. The *neighborhood* of $\mathbf{v}$ is defined as the set of all its neighbors. Two neighborhoods are said to be of the same type if they are the same in the sense of scaling and translation.

We say an IFS (1) satisfies the *finite type condition* (*FTC*) when its neighborhood types are finite. Every FTC determines an incidence matrix (see the details in Section 3). Now we turn to present the key conclusion on dimensions.

Theorem 1. *Assume the IFS (1) satisfies the FTC. Let T be the corresponding incidence matrix. Then, the self-affine set E (2) satisfies*

$$\dim^w_H E = \dim^w_B E = \frac{d \log \lambda}{n_1 \log q}$$

where λ is the spectral radius of T. Moreover, by letting $t := \dim^w_H E$, we have

$$0 < \mathcal{H}^t_w(E) < \infty.$$

From the definition, it seems that the FTC depends heavily on the invariant open set. However, the choice of open sets can be more flexible. Let V be a bounded invariant set of $\{S_j\}_{j=1}^{N}$ and define

$$\mathcal{F}_k(V) := \{S_I^{-1}S_J : \ I, J \in \Lambda_k, S_I(V) \cap S_J(V) \neq \varnothing\}, \quad \mathcal{F}(V) := \bigcup_{k=1}^{\infty} \mathcal{F}_k(V). \tag{3}$$

Theorem 2. *The IFS (1) satisfies the FTC with respect to arbitrary nonempty bounded invariant open set if and only if $\mathcal{F}(E)$ is finite.*

This paper is organized as follows. In Section 2, we give known results on generalized dimensions. In Section 3, we define the FTC of self-affine IFSs and prove Theorem 1. In Section 4, we provide some sufficient conditions for the FTC to hold, and prove Theorem 2. In Section 5, we include an illustrative example on the computation of generalized dimensions of self-affine sets.

2. Known Results on Generalized Dimensions

Following the notation in [16], let $A \in M_d(\mathbb{R})$ be an expanding matrix with $|\det A| = q$, and $B(x, r)$ the closed ball with center x and radius r. Set the region $V = A(B(0, 1)) \backslash B(0, 1)$. Take any $0 < \delta < \frac{1}{2}$ and any positive smooth even function ϕ_δ supported on $B(0, \delta)$ with $\int \phi_\delta(x)dx = 1$. We define a function w by

$$w(x) = \sum_{n \in \mathbb{Z}} q^{-n/d} \chi_V * \phi_\delta(A^n x), \ \ x \in \mathbb{R}^d \tag{4}$$

where χ_V is the characteristic function of V and $\chi_V * \phi_\delta(\cdot)$ means the convolution. The $w(\cdot)$ is called a *pseudo-norm* on $\mathbb{R}^d$.

For $F \subset \mathbb{R}^d$, $\mathrm{diam}_w F = \sup\{w(x-y) : x, y \in F\}$ stands for the w-diameter of F and $B_w(x, r) = \{y : w(x-y) \le r\}$ a w-ball.

Proposition 1 ([16])**.** *The $w(x)$ defined in (4) satisfies:*

(i) *$w(x) \ge 0$, and $w(x) = 0$ if and only if $x = 0$;*
(ii) *$w(x) = w(-x)$;*
(iii) *$w(Ax) = q^{1/d} w(x) \ge w(x)$;*
(iv) *there exists $\beta > 0$ such that $w(x+y) \le \beta \max\{w(x), w(y)\}$ for any $x, y \in \mathbb{R}^d$;*
(v) *$\mathrm{diam}_w(B_w(x, r)) \le \beta r$ for any $x \in \mathbb{R}^d$.*

It is worth mentioning that property (iii) implies that the matrix A is a similitude under w. This fact plays an important role in the present paper.

Proposition 2 ([16])**.** *Let λ_0, λ_1 be the the minimal and maximal moduli of the eigenvalues of the expanding matrix A. Then, for any $0 < \varepsilon < \lambda_0 - 1$, there exists $\sigma > 0$ such that*

$$\sigma^{-1} \| \ x \ \|^{\log q / d \log(\lambda_1 + \varepsilon)} \le w(x) \le \sigma \ \| \ x \ \|^{\log q / d \log(\lambda_0 - \varepsilon)}, \quad \text{if } \| \ x \ \| > 1,$$
$$\sigma^{-1} \| \ x \ \|^{\log q / d \log(\lambda_0 - \varepsilon)} \le w(x) \le \sigma \ \| \ x \ \|^{\log q / d \log(\lambda_1 + \varepsilon)}, \quad \text{if } \| \ x \ \| \le 1.$$

For $\alpha > 0$, the α-dimensional generalized Hausdorff measure of F with respect to w is given by

$$\mathcal{H}_w^\alpha(F) = \lim_{\delta \to 0} \mathcal{H}_{w,\delta}^\alpha(F) = \sup_{\delta > 0} \mathcal{H}_{w,\delta}^\alpha(F)$$

where

$$\mathcal{H}_{w,\delta}^\alpha(F) = \inf\left\{ \sum_{i=1}^{\infty} (\mathrm{diam}_w F_i)^\alpha : F \subset \bigcup_{i=1}^{\infty} F_i, \mathrm{diam}_w F_i \le \delta \right\}.$$

Definition 1. *The generalized Hausdorff dimension of F is defined by*

$$\dim_H^w F = \inf\{\alpha : \ \mathcal{H}_w^\alpha(F) = 0\} = \sup\{\alpha : \ \mathcal{H}_w^\alpha(F) = \infty\}.$$

Accordingly, we define the generalized box dimension. Denote by $\overline{\dim}_B^w F, \underline{\dim}_B^w F$ the upper and lower box dimensions of F under w. If the two values coincide, we say that the generalized box dimension of F exists, denoted by $\dim_B^w F$. The following consequence is trivial.

Proposition 3. *Let $F \subset \mathbb{R}^d$, and let $N_{b,\delta}^w(F)$ be the smallest number of the w-balls with radii δ needed to cover F. Then, we have*

$$\overline{\dim}_B^w F = \limsup_{\delta \to 0} \frac{\log N_{b,\delta}^w(F)}{-\log \delta} \quad \text{and} \quad \underline{\dim}_B^w F = \liminf_{\delta \to 0} \frac{\log N_{b,\delta}^w(F)}{-\log \delta}.$$

A simple relationship between the generalized dimensions is as follows.

Theorem 3 ([16]). *For any subset $F \subset \mathbb{R}^d$, we have*

$$\dim_H^w F \le \underline{\dim}_B^w F \le \overline{\dim}_B^w F$$

and

$$\frac{\log q}{d \log \lambda_1} \dim_H^w F \le \dim_H F \le \frac{\log q}{d \log \lambda_0} \dim_H^w F$$

where $q = |\det A|$ and λ_0, λ_1 are the minimal and maximal moduli of the eigenvalues of A.

3. FTC of Self-Affine IFS

We first introduce the FTC of the IFS (1), then consider the generalized dimension of the self-affine set (2).

Recall that $q = |\det A|$, $r_j = q^{-n_j/d}$ for $j = 1, \ldots, N$, and $r_1 \le \cdots \le r_N$ (as $n_1 \ge n_2 \ge \cdots \ge n_N$). Let

$$\begin{aligned} \Lambda_k &:= \{j_1 \ldots j_\ell \in \Sigma^* : \ r_{j_1 \ldots j_\ell} \le r_1^k < r_{j_1 \ldots j_{\ell-1}}\} \text{ for } k \ge 0, \\ \Lambda &:= \{\Lambda_k\}_{k \ge 0}. \end{aligned} \tag{5}$$

Define

$$\mathcal{V}_0 := \{(\mathrm{id}, 0)\}, \quad \mathcal{V}_k := \{(S_J, k) : \ J \in \Lambda_k\} \text{ for } k \ge 1, \quad \mathcal{V} := \bigcup_{k \ge 0} \mathcal{V}_k.$$

For $\mathbf{v} = (S_J, k) \in \mathcal{V}_k$, we write $S_{\mathbf{v}} = S_J$ and $r_{\mathbf{v}} = r_J$, and define a map $\pi : \Lambda \to \mathcal{V}$ by

$$\pi(J) = (S_J, k) \quad \text{for } J \in \Lambda_k.$$

Let $U \subset \mathbb{R}^d$ be a bounded invariant set under $\{S_j\}_{j=1}^N$, i.e., $\bigcup_{j=1}^N S_j(U) \subset U$. Define

$$\begin{aligned} N_U(\mathbf{v}) &:= \{\mathbf{v}' \in \mathcal{V}_k : \ S_{\mathbf{v}'}(U) \cap S_{\mathbf{v}}(U) \ne \varnothing\}, \\ \mathcal{N}_U(\mathbf{v}) &:= \{S_{\mathbf{v}'} : \ \mathbf{v}' \in N_U(\mathbf{v})\}. \end{aligned}$$

The set $N_U(\mathbf{v})$ is named as the neighborhood of $\mathbf{v}$ involving U. $\mathbf{v} \in \mathcal{V}_k$ and $\mathbf{u} \in \mathcal{V}_\ell$ are called *equivalent*, denoted by $\mathbf{v} \sim \mathbf{u}$, if $\varphi = S_{\mathbf{v}} \circ S_{\mathbf{u}}^{-1}$ is of the form $\varphi(x) = A^{-(k-\ell)n_1} x + c$ such that

$$\mathcal{N}_U(\mathbf{v}) = \varphi \mathcal{N}_U(\mathbf{u}).$$

We use $[\mathbf{v}]$ to represent the equivalence class of $\mathbf{v}$ under the relation $\sim$. Trivially, $\mathbf{v} \sim \mathbf{u}$ if and only if

$$r_{\mathbf{v}} r_1^{-k} = r_{\mathbf{u}} r_1^{-\ell} \text{ and } \{S_{\mathbf{v}}^{-1} S : S \in \mathcal{N}_U(\mathbf{v})\} = \{S_{\mathbf{u}}^{-1} S : S \in \mathcal{N}_U(\mathbf{u})\}. \tag{6}$$

Definition 2. *The IFS (1) is said to satisfy the FTC if there exists a bounded invariant open set U such that $\mathcal{V}$ contains only finitely many equivalence classes under the relation $\sim$ (i.e., $\mathcal{V}/\sim$ is finite).*

The major restriction of Ngai and Wang's FTC in [4] is that the contraction ratios of similitudes must be commensurable. In the present setting, the affine maps S_j's act as similitudes under the pseudo-norm w. More precisely, note that the linear part of S_j is $A_j\,(=A^{n_j})$. Hence, the contraction ratio of each S_j is $r_j := q^{-n_j/d}$ under the pseudo-norm w. Clearly, $r_j, j = 1, \ldots, N$, are exponentially commensurable. In this sense, the FTC is well defined.

Suppose that the IFS $\{S_j\}_{j=1}^N$ satisfies the FTC for open set U. Geometrically, iterates of U under S_I $(I \in \Lambda_k)$ generate a neighborhood system $\mathcal{N}_U(\mathbf{v})$ (where $\mathbf{v} = (S_I, k)$). FTC states that there are only finitely many distinct classes of neighborhood systems. Hence, it allows us to set up a directed graph which yields an incidence matrix T to count the number of distinct iterates. The following is a standard process to construct the directed graph and the incidence matrix.

Algorithm for constructing the directed graph:

Step I: For a vertex $\mathbf{v} = (S_I, k) \in \mathcal{V}_k$, if there exists a vertex $\mathbf{u}$ in $\mathcal{V}_{k+1}$ of the form $(S_I S_J, k+1)$ for some $J \in \Sigma^*$, we call $\mathbf{u}$ an offspring generated by $\mathbf{v}$. From $\mathbf{v}$ to $\mathbf{u}$, we label an edge: $\mathbf{v} \xrightarrow{J} \mathbf{u}$, where J is the label of the edge.

Step II: Due to the overlaps, it is possible to have more than one $\mathbf{v} \in \mathcal{V}_k$ generating a common offspring. So we need the lexicographical order for Σ^* to obtain a reduced graph. For each vertex $\mathbf{u} \in \mathcal{V}_{k+1}$, let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be all the vertices in $\mathcal{V}_k$ that generate the offspring $\mathbf{u}$, with $\mathbf{v}_\ell \xrightarrow{J_\ell} \mathbf{u}, 1 \le \ell \le p$. If $J_1 < \cdots < J_p$ in the lexicographical order, we hold the smallest edge J_1 and eliminate all the other edges. Denote by Γ the edge set of the resulting graph. Therefore, we obtain a reduced graph $(\mathcal{V}, \Gamma)$ such that each vertex of $\mathcal{V}$ has a unique parent.

Definition 3. *Assume that $\mathcal{V}/\sim = \{[\boldsymbol{v}_1], \ldots, [\boldsymbol{v}_m]\}$. For each $1 \le i \le m$, take any representative $\boldsymbol{v} \in [\boldsymbol{v}_i]$. We define t_{ij} to be the number of offspring in $(\mathcal{V}, \Gamma)$ with type $[\boldsymbol{v}_j]$ that are generated by $\boldsymbol{v}$. Matrix $T := (t_{ij})$ is called the incidence matrix of the reduced graph $(\mathcal{V}, \Gamma)$.*

In the reduced graph $(\mathcal{V}, \Gamma)$, $\mathbf{v} \longrightarrow_{\mathcal{V}} \mathbf{u}$ means that $\mathbf{u}$ is an offspring of $\mathbf{v}$. A path in $(\mathcal{V}, \Gamma)$ is a sequence $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots)$ such that $\mathbf{v}_j \in \mathcal{V}_j$ and $\mathbf{v}_j \longrightarrow_{\mathcal{V}} \mathbf{v}_{j+1}$ for all $j \geqslant 0$, where $\mathbf{v}_0 = (\mathrm{id}, 0)$ is the root. Let $\mathcal{B}$ be the set of all paths in $(\mathcal{V}, \Gamma)$. For the given vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k$ such that $\mathbf{v}_j \longrightarrow_{\mathcal{V}} \mathbf{v}_{j+1}$, we define a *branch* as follows:

$$\mathcal{P}_{\mathbf{v}_k} := \left\{(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots) \in \mathcal{B} : \mathbf{u}_j = \mathbf{v}_j \text{ for } 0 \leqslant j \leqslant k\right\}.$$

Lemma 1. *Suppose that the IFS (1) satisfies the FTC. Let F, G be any subsets in $\mathbb{R}^d$ with $\mathrm{diam}_w G \le K_1 r_1^k$ and $\mathrm{diam}_w F \le K_2$. Then, one can find a positive integer $M = M(K_1, K_2)$ such that for all $l \ge 0$,*

$$\#\{\mathbf{v} \in \mathcal{V}_l : G \cap S_{\mathbf{v}}(F) \ne \varnothing\} \le M.$$

Proof. It is clear that there exists $a > 0$ such that if $d_w(S_{\mathbf{u}}(0), S_{\mathbf{v}}(0)) \le a r_1^k$ for any $\mathbf{u}, \mathbf{v} \in \mathcal{V}_k$, then we have

$$S_{\mathbf{u}}(U) \cap S_{\mathbf{v}}(U) \ne \varnothing. \tag{7}$$

Denote $\mathcal{C} = \{\mathbf{v} \in \mathcal{V}_k : \ G \cap S_{\mathbf{v}}(F) \neq \varnothing\}$. For $\mathbf{v} \in \mathcal{V}_k$, we see that $\text{diam}_w(S_{\mathbf{v}}(F)) \leq K_2 r_1^k$. Let $K := ((K_1 + K_2)\beta + K_2)\beta$, where β is the constant given in item (iv) of Proposition 1, then $\text{diam}_w(\bigcup_{\mathbf{v}\in\mathcal{C}} S_{\mathbf{v}}(F)) \leq K r_1^k$, which implies that for any $\mathbf{u}, \mathbf{v} \in \mathcal{C}$ we have $d_w(S_{\mathbf{u}}(0), S_{\mathbf{v}}(0)) \leq K r_1^k$.

If the conclusion were not true, with the Dirichlet's drawer principle, there exists a subset $\mathcal{C}'$ of $\mathcal{C}$ having cardinality of any size, for any $\mathbf{u}, \mathbf{v} \in \mathcal{C}'$, and one has $d_w(S_{\mathbf{u}}(0), S_{\mathbf{v}}(0)) \leq a r_1^k$. It follows from (7) that $\mathcal{C}' \subset [\mathbf{v}]$. That contradicts the definition of FTC. □

Lemma 2. *Suppose that the IFS (1) satisfies the FTC. Then, the generalized box dimension of E satisfies*

$$\liminf_{k\to\infty} \frac{\log \#\mathcal{V}_k}{-k \log r_1} \leq \dim_B^w E \leq \limsup_{k\to\infty} \frac{\log \#\mathcal{V}_k}{-k \log r_1},$$

where $r_1 = q^{-n_1/d}$.

Proof. For any sufficiently small $\delta > 0$, let $N_{b,\delta}^w(E)$ be the smallest number of w-balls with radii δ needed to cover E, and denote the w-balls by $B_1, \ldots, B_{N_{b,\delta}^w(E)}$. Set $\delta = c_1 r_1^k$ for some k with $r_1 < c_1 \leq 1$. Observe that $E = \bigcup_{\mathbf{v}\in\mathcal{V}_k} S_{\mathbf{v}}(E)$, and there exists $c_2 > 0$ such that each $S_{\mathbf{v}}(E)$ can be covered by a w-ball of radius $c_2 r_1^k$.

By Lemma 1, the cardinality of $\{\mathbf{v} \in \mathcal{V}_k : \ B_j \cap S_{\mathbf{v}}(E) \neq \varnothing\}$ is bounded by some fixed $M > 0$ for all $1 \leq j \leq N_{b,\delta}^w(E)$. Thus, $\#\mathcal{V}_k \leq M N_{b,\delta}^w(E)$. Moreover, for $c_1, c_2 > 0$, there exist two positive numbers $K^+(c_1, c_2)$ and $K^-(c_1, c_2)$ satisfying

$$K^-(c_1, c_2) N_{b,c_1 r_1^k}^w(E) \leq N_{b,c_2 r_1^k}^w(E) \leq K^+(c_1, c_2) N_{b,c_1 r_1^k}^w(E).$$

Hence,

$$\#\mathcal{V}_k \geq N_{b,c_2 r_1^k}^w(E) \geq K^-(c_1, c_2) N_{b,c_1 r_1^k}^w(E) = K^-(c_1, c_2) N_{b,\delta}^w(E). \tag{8}$$

Therefore, by Proposition 3,

$$\dim_B^w(E) \geq \liminf_{\delta\to 0} \frac{\log N_{b,\delta}^w(E)}{-\log \delta} \geq \liminf_{k\to\infty} \frac{\log \#\mathcal{V}_k/M}{-\log c_1 r_1^k} = \liminf_{k\to\infty} \frac{\log \#\mathcal{V}_k}{-k \log r_1}.$$

Similarly, by (8),

$$\dim_B^w(E) \leq \limsup_{\delta\to 0} \frac{\log N_{b,\delta}^w(E)}{-\log \delta} \leq \limsup_{k\to\infty} \frac{\log \#\mathcal{V}_k/K^-(c_1, c_2)}{-\log c_1 r_1^k} = \limsup_{k\to\infty} \frac{\log \#\mathcal{V}_k}{-k \log r_1}.$$

We complete the proof. □

Finally, we evaluate $\#\mathcal{V}_k$ by the incidence matrix T. Note that $\mathcal{V}_0 = \{(\text{id}, 0)\}$ and $\mathcal{V}/\sim = \{[\mathbf{v}_1], \ldots, [\mathbf{v}_m]\}$. We may assume that $(\text{id}, 0) \in [\mathbf{v}_1]$. According to the reduced graph $(\mathcal{V}, \Gamma)$, we have

$$\#\mathcal{V}_k = e_1^t T^k \theta \tag{9}$$

where $\theta = (1, 1, \ldots, 1)^t, \ e_1 = (1, 0, \ldots, 0)^t \in \mathbb{R}^m$.

Proof of Theorem 1. Since every vertex of the reduced graph $(\mathcal{V}, \Gamma)$ is an offspring of $(\text{id}, 0)$, all the types of neighborhood are generated from $[\mathbf{v}_1]$. Then, there exists $k_0 \geq 1$ such that $e_1^t T^{k_0} > 0$. For $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, let $\| x \| = \sum_{i=1}^m |x_i|$ be a norm on $\mathbb{R}^m$. Then,

$$\lim_{k\to\infty} (e_1^t T^k \theta)^{1/k} = \lim_{k\to\infty} (e_1^t T^{k_0} T^k \theta)^{1/k} = \lim_{k\to\infty} \| T^k \theta \|^{1/k} = \lambda.$$

So for any $\delta > 0$ and large k, we have

$$(\lambda - \delta)^k < e_1^t T^k \theta < (\lambda + \delta)^k.$$

By (9) and Lemma 2,

$$\frac{\log(\lambda-\delta)}{-\log r_1} \le \liminf_{k\to\infty} \frac{\log \#\mathcal{V}_k}{-k\log r_1} \le \dim_B^w E \le \limsup_{k\to\infty} \frac{\log \#\mathcal{V}_k}{-k\log r_1} \le \frac{\log(\lambda+\delta)}{-\log r_1}.$$

Letting $\delta \to 0$, we obtain $\dim_B^w E = \frac{d\log\lambda}{n_1 \log q}$ since $r_1 = q^{-n_1/d}$. On the other hand, $\dim_B^w E = \dim_H^w E$ always holds [19]. Hence, we prove the first part of the theorem.

For the second part, we only need to show that $\mathcal{H}_w^t(E) > 0$. Assume $\mathcal{V}/\sim = \{\mathcal{T}_1, \dots, \mathcal{T}_m\}$, where $\mathbf{v}_0 = (\mathrm{id}, 0) \in \mathcal{T}_1$. Since $\mathcal{T}_1$ generates all types, one can obtain an eigenvector $x = (b_1, \dots, b_m)^t$ of T for λ such that $b_1 > 0$ and other $b_j \geqslant 0$. Let $x^* = (a_1, \dots, a_m)^t$, $a_j = b_j/b_1$. Thus, we have $Tx^* = \lambda x^*, a_j \geqslant 0$ and $a_1 = 1$.

Recall that $\mathcal{B}$ is the set of all paths in $(\mathcal{V}, \Gamma)$. For each branch $\mathcal{P}_{\mathbf{v}_k}$ where $\mathbf{v}_k \in \mathcal{V}_k$ such that $[\mathbf{v}_k] = \mathcal{T}_i$, let

$$\mu(\mathcal{P}_{\mathbf{v}_k}) = \lambda^{-k} a_i. \tag{10}$$

In fact, μ is a measure on $\mathcal{B}$. Notice branches $\mathcal{P}_{\mathbf{v}}$ and $\mathcal{P}_{\mathbf{u}}$, with $\mathbf{v} \in \mathcal{V}_k, \mathbf{u} \in \mathcal{V}_\ell$ and $k \leqslant \ell$, intersect exactly when $\mathbf{v} = \mathbf{u}$ for $k = \ell$ or $\mathbf{u}$ is a descendant of $\mathbf{v}$ for $k < \ell$. Both have $\mathcal{P}_{\mathbf{u}} \subset \mathcal{P}_{\mathbf{v}}$. Then, it suffices to prove for each $\mathbf{v} \in \mathcal{V}$,

$$\sum_{\mathbf{u}\in\mathcal{U}} \mu(\mathcal{P}_{\mathbf{u}}) = \mu(\mathcal{P}_{\mathbf{v}}), \tag{11}$$

where $\mathcal{U}$ is the collection of offspring of $\mathbf{v}$. For any $\mathbf{v} \in \mathcal{V}_k$ and $[\mathbf{v}] = \mathcal{T}_i$, by (10), $\mu(\mathcal{P}_{\mathbf{v}}) = \lambda^{-k} a_i$, and noting that T's definition,

$$\sum_{\mathbf{u}\in\mathcal{U}} \mu(\mathcal{P}_{\mathbf{u}}) = \lambda^{-k-1}\left(\sum_{j=1}^{m} t_{ij} a_j\right) = \lambda^{-k-1}\lambda a_i = \lambda^{-k} a_i.$$

Hence, (11) holds and the claim follows from $\mu(\mathcal{B}) = \mu(\mathcal{P}_{\mathbf{v}_0}) = 1$. We define a pullback measure supported on E by μ. Observe that for all $k \geqslant 1$, we have

$$E = \bigcup_{\mathbf{v}\in\mathcal{V}_k} S_{\mathbf{v}}(E). \tag{12}$$

Since $S_{\mathbf{u}}(E) \subseteq S_{\mathbf{v}}(E)$, if $\mathbf{u}$ is an offspring of $\mathbf{v}$ in $(\mathcal{V}, \Gamma)$, each path $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots) \in \mathcal{B}$ corresponds to a unique point x in E. A point $x \in E$ has at least one path in $\mathcal{B}$ by (12). For any subset $F \subset \mathbb{R}^d$, let $\mathcal{B}(F)$ be the set of all paths in $\mathcal{B}$ that label points in $E \cap F$. Define $\mu^*(F) = \mu(\mathcal{B}(F))$. Then, $\mu^*(E) = \mu(\mathcal{B}) = 1$, and this implies that μ^* is a measure supported on E.

Finally, let $0 < \delta < r_1$. For any set $F \subset \mathbb{R}^d$ with $\mathrm{diam}_w F \leqslant \delta$, assume that $r_1^{k+1} \leqslant \mathrm{diam}_w F < r_1^k$. Lemma 1 implies that F intersects no more than M of all $S_{\mathbf{v}}(E), \mathbf{v} \in \mathcal{V}_k$. For $\ell \leqslant M$, let $\mathbf{v}_1, \dots, \mathbf{v}_\ell$ be in $\mathcal{V}_k$ such that $F \cap S_{\mathbf{v}_j}(E) \neq \varnothing$. Thus,

$$\mu^*(F \cap E) \leqslant \sum_{j=1}^{\ell} \mu(\mathcal{P}_{\mathbf{v}_k}) \leqslant M\lambda^{-k} \max_{1\leqslant i\leqslant m}\{a_i\}.$$

Notice that $\lambda^{-1} = r_1^t$. Hence,

$$\lambda^{-k} = r_1^{kt} = r_1^{-t} r_1^{(k+1)t} \leqslant r_1^{-t} (\mathrm{diam}_w F)^t.$$

Consequently, $\mu^*(F\cap E)\leqslant c(\mathrm{diam}_w F)^t$ where $c=Mr_1^{-t}\max_{1\le i\le m}\{a_i\}$. By a generalized mass distribution principle (Proposition 2.9 in [19]), we have $\mathcal{H}_w^t(E)\geqslant \mu^*(E)/C>0$. □

4. When Is FTC Fulfilled?

It seems that the definition of FTC in the previous section depends heavily on the choice of open sets. Actually, the choice can be quite flexible. In this section, we discuss this problem.

Proposition 4. *The OSC implies the FTC.*

Proof. Note that $r_j=q^{-n_j/d}$ and $n_1\ge n_2\ge\cdots\ge n_N$, then $r_1=\min_{1\le j\le N} r_j$. By (5), the definition of Λ_k, it is clear that the set $\{r_1^{-k}r_J : J\in\Lambda_k, k\ge 0\}$ is finite. Let U be the open set in the OSC, for any $k\ge 0$, all $S_J(U), J\in\Lambda_k$ are disjoint. This implies that $N_U(\mathbf{v}),\mathbf{v}\in\mathcal{V}_k$, is the single vertex $\mathbf{v}$. Then, for $\mathbf{v}\in\mathcal{V}_k$ and $\mathbf{u}\in\mathcal{V}_\ell$, $\mathbf{v}\sim\mathbf{u}$ if and only if $r_1^{-k}r_{\mathbf{v}}=r_1^{-\ell}r_{\mathbf{u}}$. Therefore, there are finitely many types of neighborhood among $[\mathbf{v}],\mathbf{v}\in\mathcal{V}$, and the FTC holds with respect to U. □

Theorem 4. *If $A\in M_d(\mathbb{Z})$ and $d_j\in\mathbb{Z}^d, j=1,\dots,N$, then the IFS (1) satisfies the FTC.*

Proof. For all $J=j_1\dots j_\ell\in\Lambda_k$, we have $A_J=\prod_{i=1}^{\ell}A_{j_i}=A^p$ for some integer $n_1k\le p<n_1(k+1)$. This implies that

$$\left\{A^{n_1k}A_J^{-1} : J=j_1\dots j_\ell\in\Lambda_k, k\ge 0\right\}\subset\{A^{-i}:0\le i<n_1\}. \tag{13}$$

Suppose that U is an invariant open set of the IFS $\{S_j\}_{j=1}^N$ in (1) with w-diameter $\mathrm{diam}_w U=C$. For $\mathbf{v}\in\mathcal{V}_k$, we denote by

$$A^{n_1k}N_U(\mathbf{v}):=\left\{\left(A^{n_1k}A_J^{-1},A^{n_1k}S_J(0)\right) : J=j_1\dots j_\ell\in\Lambda_k, \pi(J)\in N_U(\mathbf{v})\right\}.$$

By (13), we see that $A^{n_1k}A_J^{-1}$ takes no more than n_1 possible values for all $k\ge 0$. Notice that $A^{n_1k}S_J(0)\subset A^{-n_1}\mathbb{Z}^d$ for $J\in\Lambda_k$ by the assumption. Moreover, $A^{n_1k}S_J(U)$ is an open set with w-diameter $\le C$. As $A^{-n_1}\mathbb{Z}^d$ is a lattice set, it yields only several sets $\{A^{n_1k}S_J(0):\pi(J)\in N_U(\mathbf{v})\}$ which are translationally inequivalent among all $\mathbf{v}\in\mathcal{V}_k$ and all $k\ge 0$. Since $\{A^{n_1k}A_J^{-1}: J\in\Lambda_k, k\ge 0\}$ is also finte, it can be concluded that the FTC holds with respect to U. □

We modify the notion of FTC to a slightly different form which is more convenient to use. Let V be a bounded invariant set of $\{S_j\}_{j=1}^N$, and $\mathcal{F}(V)$ as in (3). We can define the relation $\sim_V$ as in (6) by replacing U with V. Then, the following result is straightforward.

Proposition 5. *$\mathcal{V}/\sim_V$ is finite if and only if $\mathcal{F}(V)$ is finite.*

Proof. By (6), the necessity is obvious. For the sufficiency, we notice that $\{r_1^{-k}r_J : J\in\Lambda_k, k\ge 0\}$ and $\mathcal{F}(V)$ are finite. □

Lemma 3. *Let $G,W\subset\mathbb{R}^d$ be two nonempty bounded invariant sets of the IFS (1). Suppose there is $k>0$ such that $S_J(W)\subset G$ for all $J\in\Lambda_k$ (in particular $W\subset G$). Then, $\mathcal{V}/_{\sim G}$ is finite, which implies that $\mathcal{V}/_{\sim W}$ is finite.*

Proof. By Proposition 5, we only need to show that $\mathcal{F}(W)$ is finite. Let

$$\widetilde{\Lambda}_k=\{J\in\Sigma^* :\ r_J\ge r_1^{k+2}\}.$$

Clearly, $\widetilde{\Lambda}_k$ is finite and

$$\Lambda_{\ell+k+1} \subset \{I_1 I_2 : \ I_1 \in \Lambda_\ell, I_2 \in \widetilde{\Lambda}_k\}.$$

Hence, any $I, J \in \Lambda_{\ell+k+1}$ can be written as $I = I_1 I_2, J = J_1 J_2$ so that $I_1, J_1 \in \Lambda_\ell, I_2, J_2 \in \widetilde{\Lambda}_k$, and $r_{I_2}, r_{J_2} \leq r_1^k$. By the assumptions on G and W, we have $S_{I_2}(W) \subset G$ and $S_{J_2}(W) \subset G$, and thus, $S_I(W) \cap S_J(W) \neq \varnothing$ implies $S_{I_1}(G) \cap S_{J_1}(G) \neq \varnothing$. Therefore,

$$\mathcal{F}(W) \subset \{S_I^{-1} f S_J : \ f \in \mathcal{F}(G), I, J \in \widetilde{\Lambda}_k\},$$

which implies that $\mathcal{F}(W)$ is finite as $\mathcal{F}(G)$ is finite. □

Proof of Theorem 2. The necessity follows from Lemma 3. For the sufficiency, suppose $\mathcal{F}(E)$ is finite, by Proposition 5; we only need to show that $\mathcal{F}(U)$ is finite for any bounded invariant open set U.

Suppose, on the contrary, that there exists a set U such that $\mathcal{F}(U)$ is infinite. Let

$$\mathcal{G}_k := \{S_I^{-1} S_J : \ r_1^k \geq r_I > r_1^{k+1}, J \in \Sigma^*\}, \quad k \geq 1.$$

Under the pseudo-norm w, the contraction ratios of the functions in $\mathcal{F}(U)$ are bounded in $[r_1, r_1^{-1}]$. Hence, for each k, $\mathcal{F}(U) \cap \mathcal{G}_k$ is finite. We can have an increasing sequence of integers $\{m_k\}_{k\geq 1}$ and words $I_k, J_k \in \Sigma^*$ such that

$$S_{I_k}^{-1} S_{J_k} \in \mathcal{F}(U) \cap \left(\mathcal{G}_{m_k} \setminus \bigcup_{i=1}^{m_k-1} \mathcal{G}_i \right), \quad k \geq 1. \tag{14}$$

So $S_{I_k}(U) \cap S_{J_k}(U) \neq \varnothing$ for all $k \geq 1$. By taking subsequences, we assume that $i_1 i_2 \ldots, j_1 j_2 \ldots \in \Sigma^\infty$ such that

$$I_k|_k = i_1 \ldots i_k, \quad J_k|_k = j_1 \ldots j_k, \quad k \geq 1. \tag{15}$$

Since U is bounded, we have $\lim_{k\to\infty} S_{i_1 \ldots i_k}(x_0) = \lim_{k\to\infty} S_{j_1\ldots j_k}(x_0) \in E$ for some $x_0 \in U$. Thus,

$$S_{i_1\ldots i_s}(E) \cap S_{j_1 \ldots j_t}(E) \neq \varnothing, \quad \text{for all } s, t \in \mathbb{N}. \tag{16}$$

For any $k \geq 1$, let $s_k, t_k \in \mathbb{N}$ such that $i_1 \ldots i_{s_k}, j_1 \ldots j_{t_k} \in \Lambda_k$. Then, (16) implies that $S_{i_1\ldots i_{s_k}}^{-1} S_{j_1\ldots j_{t_k}} \in \mathcal{F}(E)$ for all $k \geq 1$. Hence, the finiteness of $\mathcal{F}(E)$ implies that there exist $h, \ell \in \mathbb{N}$ with $h - \ell \geq 2$ such that

$$S_{i_1\ldots i_{s_h}}^{-1} S_{j_1\ldots j_{t_h}} = S_{i_1\ldots i_{s_\ell}}^{-1} S_{j_1\ldots j_{t_\ell}}. \tag{17}$$

Using (15), we can choose a sufficiently large integer k with $\sigma_k, \tau_k \in \Sigma^*$ such that $I_k = i_1 \ldots i_{s_h} \sigma_k, J_k = j_1 \ldots j_{t_h} \tau_k \in \Lambda_{m_k}$. It follows from (17) that

$$S_{I_k}^{-1} S_{J_k} = S_{i_1\ldots i_{s_\ell}\sigma_k}^{-1} S_{j_1\ldots j_{t_\ell}\tau_k}. \tag{18}$$

Since $h - \ell \geq 2$, $S_{i_1\ldots i_{s_\ell}\sigma_k}^{-1} S_{j_1\ldots j_{t_\ell}\tau_k} \in \mathcal{G}_i$ for some $i < m_k$. Hence, by (18), $S_{I_k}^{-1} S_{J_k} \in \bigcup_{i=1}^{m_k-1} \mathcal{G}_i$. This contradicts (14) and we finish the proof. □

5. An Example

We provide an example in this last section to illustrate our main results.

Example 1. *Let the IFS* $\{S_1, S_2, S_3\}$ *be as in (1) where* $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ *and* $\mathcal{D} = \{d_1 = (0,0)^t, d_2 = (8,3)^t, d_3 = (2/9, 1/4)^t\}$, $n_1 = n_2 = 2$, $n_3 = 1$.

Fix $U = (0,1) \times (0,1)$ as an invariant open set, and let $\mathcal{T}_1$ be the equivalence type of $\mathbf{v}_0 = (\mathrm{id}, 0)$. The iterates of U under S_1, S_2, S_3 are as shown in Figure 1. It is easy to check that

$$\Lambda_1 = \{(1), (2), (31), (32), (33)\}$$

and

$$\mathcal{V}_1 = \{\mathbf{v}_1 = (S_1, 1), \mathbf{v}_2 = (S_2, 1), \mathbf{v}_{31} = (S_{31}, 1), \mathbf{v}_{32} = (S_{32}, 1), \mathbf{v}_{33} = (S_{33}, 1)\}$$

where $S_1 = (x/9, y/4)^t$, $S_2 = (x/9 + 8/9, y/4 + 3/4)^t$, $S_{31} = (x/27 + 2/27, y/8 + 1/8)^t$, $S_{32} = (x/27 + 10/27, y/8 + 1/2)^t$, $S_{33} = (x/9 + 8/81, y/4 + 3/16)^t$.

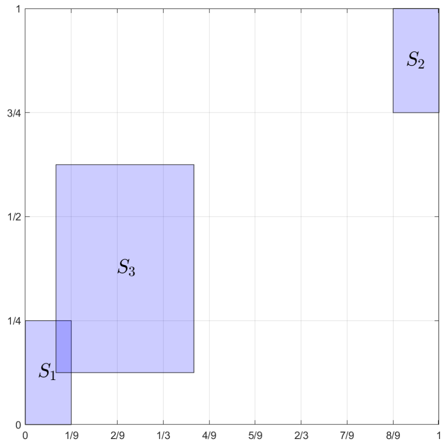
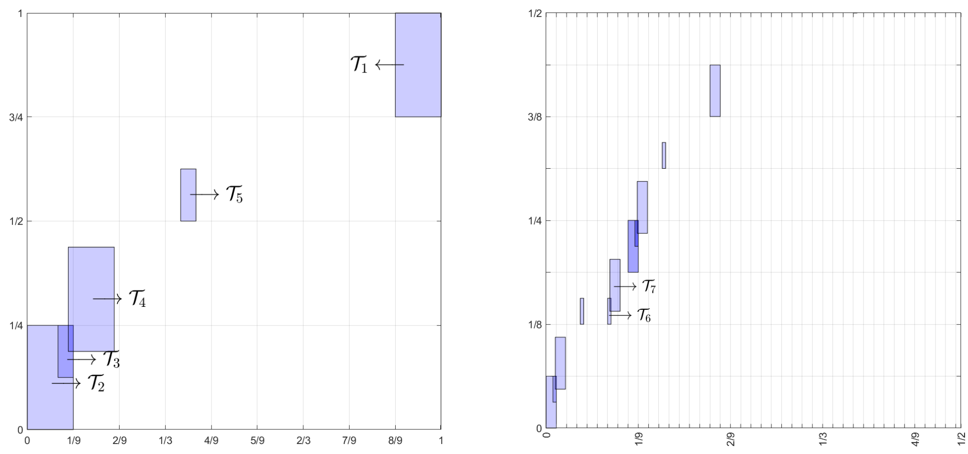


Figure 1. The iterates of U under $\{S_1, S_2, S_3\}$.

Step I: Denote $[\mathbf{v}_1] := \mathcal{T}_2$, $[\mathbf{v}_{31}] := \mathcal{T}_3$, $[\mathbf{v}_{33}] := \mathcal{T}_4$, $[\mathbf{v}_{32}] := \mathcal{T}_5$, $[\mathbf{v}_2] := \mathcal{T}_1$. Then, a type $\mathcal{T}_1$ vertex yields one offspring of each of the types $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5$ (see Figure 2a). All of these offspring are also in $(\mathcal{V}, \Gamma)$, and therefore,

$$\mathcal{T}_1 \longrightarrow \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5.$$



(**a**) (**b**)

Figure 2. The iterates of U under S_I's.

Step II: Since $\mathbf{v}_1$ is of type $\mathcal{T}_2$, it generates five offspring with vertices

$$(11), (12), (131), (132), (133) \in \Lambda_2,$$

which are of types $\mathcal{T}_2, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_5, \mathcal{T}_4$, respectively. Note that $S_{12} = S_{331}$, and $(1) < (2)$ in the lexicographical order. The edge (2) connecting $\mathbf{v}_1$ to $\mathbf{v}_{12}$ is cancelled in the reduced graph. Notice that $\mathbf{v}_{31}$ is of type $\mathcal{T}_3$; it gives rise to three offspring with vertices

$$(311), (312), (313) \in \Lambda_2.$$

It is easy to check that the types of $\mathbf{v}_{311}, \mathbf{v}_{313}$ are different from the neighborhood types above and $[\mathbf{v}_{312}] = \mathcal{T}_3$. Denote $[\mathbf{v}_{311}] := \mathcal{T}_6$, $[\mathbf{v}_{313}] := \mathcal{T}_7$ (see Figure 2b). Since $S_{312} = S_{3331}$ and $(2) < (31)$, the edge (31) connecting $\mathbf{v}_{33}$ to $\mathbf{v}_{3331}$ is removed. Hence,

$$\begin{aligned} \mathcal{T}_2 &\longrightarrow \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5, \\ \mathcal{T}_3 &\longrightarrow \mathcal{T}_3 + \mathcal{T}_6 + \mathcal{T}_7. \end{aligned}$$

Using **Step I** and **Step II** again, we have

$$\begin{aligned} \mathcal{T}_4 &\longrightarrow \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_4 + \mathcal{T}_5, \\ \mathcal{T}_5 &\longrightarrow \mathcal{T}_5 + \mathcal{T}_6 + \mathcal{T}_7, \\ \mathcal{T}_6 &\longrightarrow \mathcal{T}_3 + \mathcal{T}_6 + \mathcal{T}_7, \\ \mathcal{T}_7 &\longrightarrow \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_4 + \mathcal{T}_5. \end{aligned}$$

These are all the equivalences and the incidence matrix is

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

By Theorem 1, the generalized dimensions of the self-affine set E:

$$\dim_H^w E = \dim_B^w E = \frac{2 \log \lambda}{2 \log 6} \approx 0.72$$

where $\lambda \approx 3.63$. Furthermore, Theorem 3 states that the Hausdorff dimension $\dim_H E$ lies in $[0.59, 0.93]$.

6. Conclusions

This paper defines a new finite type condition on self-affine iterated function systems and obtains a dimensional formula on the generalized dimensions of self-affine sets, while the generalized dimensions are useful to estimate the classical Hausdorff dimension of self-affine sets. Specific conditions for the finite type condition to hold are also discussed.

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