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Asymptotic Profile for Diffusion Wave Terms of the Compressible Navier–Stokes–Korteweg System

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Abstract: The asymptotic profile for diffusion wave terms of solutions to the compressible Navier–Stokes–Korteweg system is studied on $\mathbb{R}^2$. The diffusion wave with time-decay estimate was studied by Hoff and Zumbrun (1995, 1997), Kobayashi and Shibata (2002), and Kobayashi and Tsuda (2018) for compressible Navier–Stokes and compressible Navier–Stokes–Korteweg systems. Our main assertion in this paper is that, for some initial conditions given by the Hardy space, asymptotic behaviors in space–time L^2 of the diffusion wave parts are essentially different between density and the potential flow part of the momentum. Even though measuring by L^2 on space, decay of the potential flow part is slower than that of the Stokes flow part of the momentum. The proof is based on a modified version of Morawetz's energy estimate, and the Fefferman–Stein inequality on the duality between the Hardy space and functions of bounded mean oscillation.

Keywords: compressible Navier–Stokes–Korteweg system; asymptotic profile; diffusion wave; hardy space

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1. Introduction

We study the asymptotic behavior of solutions to the following compressible Navier–Stokes–Korteweg system in $\mathbb{R}^2$, called CNSK:

$$\begin{cases} \partial_t \rho + \operatorname{div} M = 0, \\ \partial_t M + \operatorname{div}\left(\frac{M \otimes M}{\rho}\right) + \nabla P(\rho) = \operatorname{div}\left(\mathcal{S}\left(\frac{M}{\rho}\right) + \mathcal{K}(\rho)\right), \\ \rho(x,0) = \rho_0, \quad M(x,0) = M_0. \end{cases} \tag{1}$$

Here, $\rho = \rho(x,t)$ and $M = (M_1(x,t), M_2(x,t))$ are unknown density and momentum, respectively, at time $t \in \mathbb{R}^+$ and position $x \in \mathbb{R}^2$; $\rho_0 = \rho_0(x)$ and $M_0 = M_0(x)$ are given initial data; $\mathcal{S}$ and $\mathcal{K}$ denote the viscous stress tensor and Korteweg stress tensor, respectively, given by

$$\begin{cases} (\mathcal{S})_{i,j}\left(\frac{M}{\rho}\right) = \left(\mu' \operatorname{div}\frac{M}{\rho}\right)\delta_{i,j} + 2\mu d_{ij}\left(\frac{M}{\rho}\right), \\ (\mathcal{K})_{i,j}(\rho) = \frac{\kappa}{2}(\Delta\rho^2 - |\nabla\rho|^2)\delta_{i,j} - \kappa \frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j}, \end{cases} \tag{2}$$

where $d_{ij}\left(\frac{M}{\rho}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_i}\left(\frac{M}{\rho}\right)_j + \frac{\partial}{\partial x_j}\left(\frac{M}{\rho}\right)_i\right)$; μ and μ' are the viscosity coefficients, supposed to be constants satisfying

$$\mu > 0, \quad \mu + \mu' \geq 0.$$

κ is the capillary constant satisfying $\kappa \geq 0$. If $\kappa = 0$ in the Korteweg tensor, the usual compressible Navier–Stokes equation (CNS) appears: $P = P(\rho)$ is pressure assumed to be a smooth function of ρ satisfying $P'(\rho_*) > 0$, where ρ_* is a given positive constant and $^\top(\rho_*, 0)$ is a given constant state for (1). We consider solutions to (1) around the constant state.

(1) is the system of equations of motion of liquid–vapor type two-phase flow with phase transition in a compressible fluid, similarly as in [1]. To describe the phase transition, this model uses the diffusive interface. Hence, the phase boundary is regarded as a narrow transition layer and change of the density prescribes fluid state. Due to the diffusive interface, it is enough to consider one set of equations and a single spatial domain and difficulty of topological change of interface do not occur. If we assume that $\kappa = 0$, the CNS that describes the motion of one-phase compressible fluid is obtained. Hence, (1) is obtained from adding higher-order derivative terms for ρ, including $\nabla \Delta \rho$ and $\nabla \rho \otimes \nabla \rho$ to CNS.

For the derivation of (1), Van der Waals [2] suggested that a phase-transition boundary be regarded as a thin transition zone, i.e., a diffusive interface caused by a steep gradient of density. On the basis of his idea, Korteweg [3] modified the stress tensor of the Navier–Stokes equation to that including term $\nabla \rho \otimes \nabla \rho$. Dunn and Serrin [4] generalized Korteweg's work and strictly provided System (1) with (2). In their recent works, Heida and Málek [5] derived (1) by the entropy production method.

We focus on the diffusion wave that stems from hyperbolic and parabolic aspects of the system. The diffusion wave is given by convolution between heat kernel and the fundamental solution to the wave equation. The importance of the diffusion wave for problems in one-dimensional cases was first recognized by Liu [6] for the study of stability of shock waves for viscous conservation laws. The multidimensional diffusion wave with a time-decay estimate of solutions was studied for CNS by Hoff and Zumbrun [7,8], and Kobayashi and Shibata [9]; for the viscoelastic equation on $\mathbb{R}^n$ ($n \geq 2$), by Shibata [10]. Let $u = {}^\top(\rho - \rho_*, M)$ be a solution to CNS and set $E := \|u_0\|_{H^{s+\ell} \cap L^1}$, where $u_0 = {}^\top(\rho_0 - \rho_*, M_0) = {}^\top(\rho(0) - \rho_*, M(0))$, s is an integer part of $n/2$ and ℓ is integer satisfying $\ell \geq 3$. Then, the authors in [7–9] showed that the linear parts decay faster than nonlinear parts do in the Duhamel formula, and the asymptotic behavior in $L^p(\mathbb{R}^n)$ ($p > 2, n \geq 2$) of solutions was presented as

$$u(t) \sim \underbrace{\begin{pmatrix} 0 \\ \mathcal{K}_\nu(t) * M_{0,in} \end{pmatrix} + \begin{pmatrix} \rho(t) - \rho_* \\ M(t) - \mathcal{K}_\nu(t) * M_{0,in} \end{pmatrix}}_{\text{solutions to linearized system}} + \underbrace{\cdots}_{\text{nonlinear parts}} \quad \text{in } L^p(\mathbb{R}^n) \quad (3)$$

as t goes to infinity. Here, notation $u(t) \sim f(t)$ in $L^p(\mathbb{R}^n)$ is defined as

$$\limsup_{t \to \infty} \|u(t) - f(t)\|_{L^p(\mathbb{R}^n)} \leq C$$

for a positive number C independent of t, similar notation is used hereafter. $\mathcal{K}_\nu = \mathcal{K}_\nu(t, x)$ is the standard heat kernel and $M_{0,in}$ is a divergence-free part of M_0, given by

$$\mathcal{K}_\nu = \mathcal{F}^{-1}(e^{-\nu|\xi|^2 t}), \quad M_{0,in} = \mathcal{F}^{-1}\left\{ \left(I_n - \frac{\xi^\top \xi}{|\xi|^2} \right) M_0 \right\}.$$

More precisely, it holds that

$$\|u(t)\|_{L^p(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})}, \quad \|u(t) - G(t) * u_0\|_{L^p(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}}$$

and

$$\left\| \begin{pmatrix} \rho(t) - \rho_* \\ M(t) - \mathcal{K}_\nu(t) * M_{0,in} \end{pmatrix} \right\|_{L^p(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})+(\frac{n-1}{4})(\frac{2}{p}-1)}$$

for $t > 0$, where $G = G(t)$ is the Green function of linearized CNS and $(\frac{n-1}{4})(\frac{2}{p} - 1) \leq 0$ when $2 \leq p \leq \infty$. $\mathcal{K}_\nu(t) * M_{0,in}$ and $M(t) - \mathcal{K}_\nu(t) * M_{0,in}$ are the Stokes flow and potential flow parts of M, respectively, in the Helmholtz decomposition. $\rho(t) - \rho_*$ and $M(t) - \mathcal{K}_\nu(t) * M_{0,in}$ are given by the Green matrix of the linearized system, which consists of the convolution with the Green functions of the diffusion and the wave equations and are called the diffusion-wave part. In addition, when $p = 2$, the behaviors of both of $\rho(t) - \rho_*$ and $M(t) - \mathcal{K}_\nu(t) * M_{0,in}$ coincide with those of $\mathcal{K}_\nu(t) * M_{0,in}$ as the parabolic-type decay rate. Kobayashi and Tsuda studied the diffusion-wave property for (1) in [11].

In this paper, we consider the linearized system for (1). Under some initial conditions given by the Hardy space $\mathcal{H}^1$(defined below), we show some space–time L^2 estimates for the density and the Stokes flow part of the momentum. The potential flow part of the momentum is also shown to grow at the rate of logarithmic order in spatial-time L^2 norm. The precise initial condition given by the Hardy space is shown below. Here, we assume a stronger initial condition by $\mathcal{H}^1$ for density than that by L^1, in contrast to [11]; thus, our results may show a gain of regularity by the Hardy space in the decay estimates. Such a gain is also obtained for heat equations (see Appendix A). Nonlinear terms are expected to decay faster than the linear terms in the Duhamel formula do, as in [7,11]. As a consequence, the leading terms of the asymptotic expansion of solution u for (1) are given by

$$u(t) \sim \underbrace{\begin{pmatrix} 0 \\ M(t) - \mathcal{K}_\nu(t) * M_{0,in} \end{pmatrix} + \begin{pmatrix} \rho(t) - \rho_* \\ \mathcal{K}_\nu(t) * M_{0,in} \end{pmatrix}}_{\text{solutions to linearized system}} + \underbrace{\cdots}_{\text{nonlinear parts}} \quad \text{in } L^2(0, \infty; \mathbb{R}^2). \quad (4)$$

Precisely, the following estimates hold true for solutions to the linearized CNSK:

$$\int_0^t \|M(\tau) - \mathcal{K}_\nu(\tau) * M_{0,in}\|_{L^2(\mathbb{R}^2)}^2 d\tau \sim \log t \quad \text{as } t \to \infty,$$

$$\|\rho - \rho_*\|_{L^2(0,\infty; L^2(\mathbb{R}^2))} + \|\mathcal{K}_\nu * M_{0,in}\|_{L^2(0,\infty; L^2(\mathbb{R}^2))} < \infty.$$

The above behaviors of the diffusion-wave parts $\rho(t) - \rho_*$ and $M(t) - \mathcal{K}_\nu(t) * M_{0,in}$ are clearly different from (3). Measuring by L^2 on space, $M(t) - \mathcal{K}_\nu(t) * M_{0,in}$ decays slower than the Stokes flow part of M does. By the dependence on κ of constants, the above estimate (4) also holds true for CNS (Theorems 2 and 3). We also obtain a decay rate of L^2 norm of density (Theorem 4). Furthermore, if $M_0 \in \mathcal{H}^1$, space–time L^2 boundedness is obtained for $M(t) - \mathcal{K}_\nu(t) * M_{0,in}$, $\rho(t) - \rho_*$ and $\mathcal{K}_\nu(t) * M_{0,in}$.

The proofs of the main results are based on Morawetz-type energy estimates for a linearized system. The diffusion-wave part of density $\rho(t) - \rho_*$ is bounded in space–time L^2. We rewrite (1) to some linear doubly dispersion equation for ρ and apply a modified version of Morawetz's energy estimate. A preliminary function is introduced in the Morawetz estimate (see (12) below), which is defined by use of a doubly Laplace-type equation. The existence of solution to the linear doubly Laplace-type equation is shown by use of the linear theory on $\mathcal{H}^1$, which may be of its own interest. Through the preliminary function, we perform Morawetz-type energy estimates utilizing the Fefferman–Stein inequality on the duality between $\mathcal{H}^1$ and the space of functions of bounded mean oscillation. Another diffusion-wave part $M(t) - \mathcal{K}_\nu(t) * M_{0,in}$ is shown to grow at the rate of order $\log t$ as t goes to infinity. Here, we use fundamental solutions for the linearized system given in [11]. Since a high-frequency part of the solutions exponentially decays, a low-frequency part only has to be estimated here. By direct computation with the explicit form of the Green matrix, we obtain the growth order for $M(t) - \mathcal{K}_\nu(t) * M_{0,in}$. For the Stokes flow part $\mathcal{K}_\nu(t) * M_{0,in}$, space–time L^2 boundedness is derived in Theorem A3 bellow. These estimates are combined for a diffusion wave, and the Stokes flow parts yields asymptotic expansion (4).

This paper is organized as follows. In Section 2 some notations and lemmas are given. In Section 3, the main results are presented. In Section 4, the proofs of the estimates for the diffusion wave parts are demonstrated.

2. Preliminaries

In this section, we introduce notations such as function spaces that are used in this paper. We also present lemmas needed in the proof of the main result.

The norm on X is denoted by $\| \cdot \|_X$ for a given Banach space X.

Let $1 \leqq p \leqq \infty$. L^p is the usual Lebesgue space of pth powered integrable and essentially bounded functions on $\mathbb{R}^2$ for a finite p and $p = \infty$, respectively. Let k be a non-negative integer. $W^{k,p}$ and H^k are the usual Sobolev spaces of order k, based on L^2 and L^p, respectively. As usual, H^0 is defined by $H^0 := L^2$.

We also use notation L^p to denote the function space of all vector fields $w = {}^\top(w_1, w_2)$ on $\mathbb{R}^2$ satisfying $w_j \in L^p$ ($j = 1, 2$), and $\| \cdot \|_{L^p}$ is norm $\| \cdot \|_{(L^p)^2}$ for brevity if no confusion occurs. Similarly, a function space X is the linear space of all vector fields $w = {}^\top(w_1, w_2)$ on $\mathbb{R}^2$ satisfying $w_j \in X$ ($j = 1, 2$), and $\| \cdot \|_X$ is norm $\| \cdot \|_{X^2}$ if no confusion occurs.

Let $u = {}^\top(\phi, m)$ with $\phi \in H^k$ and $m = {}^\top(m_1, m_2) \in H^j$. Then, norm $\|u\|_{H^k \times H^j}$ is defined as that of u on $H^k \times H^j$

$$\|u\|_{H^k \times H^j} := \left(\|\phi\|_{H^k}^2 + \|w\|_{H^j}^2 \right)^{\frac{1}{2}}.$$

In particular, if $j = k$, we put

$$H^k := H^k \times (H^k)^2, \quad \|u\|_{H^k} := \|u\|_{H^k \times (H^k)^2} \quad (u = {}^\top(\phi, m)).$$

Let X and Y be given Banach spaces. For $u = {}^\top(\phi, m) \in X \times Y$ with $m = {}^\top(m_1, m_2)$, we similarly set

$$\|u\|_{X \times Y} := \left(\|\phi\|_X^2 + \|m\|_Y^2 \right)^{\frac{1}{2}}.$$

More generally, in the case that $Y = X^2$, let

$$X := X \times X^2, \quad \|u\|_X := \|u\|_{X \times X^2}.$$

Symbols $\hat{f}$ and $\mathcal{F}[f]$ stand for the Fourier transform of f with respect to space variable x

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) := \int_{\mathbb{R}^2} f(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^2.$$

Furthermore, the inverse Fourier transform of f is defined by

$$\mathcal{F}^{-1}[f](x) := (2\pi)^{-2} \int_{\mathbb{R}^2} f(\xi) e^{i\xi \cdot x} d\xi, \quad x \in \mathbb{R}^2.$$

For a non-negative number s, $[s]$ is the Gaussian symbol that denotes the integer part of s. Symbol " $*$ " denotes the convolution on space variable x.

Now, we prepare Hardy space $\mathcal{H}^1$ and BMO space.

Definition 1. *Hardy space* $\mathcal{H}^1 = \mathcal{H}^1(\mathbb{R}^2)$ *consists of integrable functions on* $\mathbb{R}^2$, $f \in L^1(\mathbb{R}^2)$ *such that*

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \sup_{r > 0} |\phi_r * f(x)| dx$$

is finite, where $\phi_r(x) = r^{-n}\phi(r^{-1}x)$ for $r > 0$, and ϕ is a smooth function on $\mathbb{R}^2$ with compact support in an unit ball with center of the origin, $B_1(0) = \{x \in \mathbb{R}^2; |x| < 1\}$. The definition does not depend on the choice of a function ϕ.

Definition 2. *Let f be locally integrable in $\mathbb{R}^2$, $f \in L^1_{loc}(\mathbb{R}^2)$. We say that f is of bounded mean oscillation, abbreviated as BMO, if*

$$\|f\|_{\mathrm{BMO}} = \sup_{B \subset \mathbb{R}^2} \frac{1}{|B|} \int_B |f - (f)_B| dx < \infty,$$

where the supremum ranges over all finite balls $B \subset \mathbb{R}^2$, $|B|$ is the 2-dimensional Lebesgue measure of B, and $(f)_B$ denotes the integral mean of f over B, namely $(f)_B = \frac{1}{|B|} \int_B f(x)dx$.

The class of functions of BMO, modulo constants, is a Banach space with norm $\|\cdot\|_{BMO}$ defined above.

We crucially use the decisive Fefferman–Stein inequality, which means the duality between $\mathcal{H}^1(\mathbb{R}^2)$ and $BMO(\mathbb{R}^2)$, i.e., $(\mathcal{H}^1(\mathbb{R}^2))^* = BMO(\mathbb{R}^2)$. For the proof, see [12].

Lemma 1. *(Fefferman-Stein inequality) There is a positive constant C, such that, if $f \in \mathcal{H}^1(\mathbb{R}^2)$ and $g \in BMO(\mathbb{R}^2)$, then*

$$\left| \int_{\mathbb{R}^2} fg\, dx \right| \leq C\|f\|_{\mathcal{H}^1(\mathbb{R}^2)} \|g\|_{BMO(\mathbb{R}^2)}.$$

We also recall the well-known Poincaré inequality.

Lemma 2. *It holds that*
$$\|f\|_{BMO(\mathbb{R}^2)} \leq \|\nabla f\|_{L^2(\mathbb{R}^2)}$$

for $f \in H^1(\mathbb{R}^2)$.

We denote by $C^\infty_{0,\sigma}$ the set of all vector-valued functions $\phi = {}^\top(\phi_1, \phi_2)$ whose each ϕ_j ($j = 1, 2$) is C^∞ function having compact support, and satisfying that $\mathrm{div}\,\phi = 0$. For $1 \leq q < \infty$, L^q_σ is the closure of $C^\infty_{0,\sigma}$ with respect to the L^q norm.

A spatial weighted function space $W^{1,2}_w(\mathbb{R}^2)$ is defined by

$$W^{1,2}_w(\mathbb{R}^2) = \left\{ u \ : \ \frac{u}{w(x)} \in L^2(\mathbb{R}^2), \nabla u \in L^2(\mathbb{R}^2) \right\},$$

where $w(x)$ is a spatial weight defined by $w(x) = (1 + |x|)\log(2 + |x|)$.

The following Hölder type inequality was proved by Amrouche and Nguyen [13].

Lemma 3. *([13] Corollary 2.10) Let $f \in L^1_\sigma(\mathbb{R}^2)$. Then it holds true that $\int_{\mathbb{R}^2} f(x)dx = 0$ and that, for such f and any $g \in W^{1,2}_w(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$,*

$$\left| \int_{\mathbb{R}^2} fg\, dx \right| \leq C\|f\|_{L^1} \|\nabla g\|_{L^2}.$$

Since $W^{1,2}(\mathbb{R}^2) \subset W^{1,2}_w(\mathbb{R}^2)$, Lemma 3 also yields the following.

Corollary 1. *Let $f \in L^1_\sigma(\mathbb{R}^2)$. Then, there holds that $\int_{\mathbb{R}^2} f(x)dx = 0$ and, for such f and any $g \in W^{1,2}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$,*

$$\left| \int_{\mathbb{R}^2} fg\, dx \right| \leq C\|f\|_{L^1} \|\nabla g\|_{L^2}.$$

3. Main Results

In this section, we consider the linearized system corresponding to (1) and present some decay estimates for its solution. akey estimate to show (4) is space–time L^2 boundedness of the density for the linearized system. First, (1) is reformulated and linearized as follows. Hereafter, we assume that $\rho_* = 1$ without loss of generality. We also set

$$\phi = \rho - 1, \quad m = \frac{M}{\gamma}, \quad \gamma = \sqrt{P'(1)}.$$

Substituting ϕ and m into (1), we have system of equations

$$\begin{cases} \partial_t \phi + \gamma \operatorname{div} m = 0, \\ \partial_t m - \nu \Delta m - \tilde{\nu} \nabla \operatorname{div} m + \gamma \nabla \phi - \kappa_0 \nabla \Delta \phi = f(u), \\ \phi|_{t=0} = \phi_0, \quad m|_{t=0} = m_0, \end{cases} \tag{5}$$

where we use notation

$$u = {}^\top(\phi, m), \quad \phi_0 = \rho_0 - 1, \quad m_0 = \frac{M_0}{\gamma}, \quad \nu = \mu, \quad \tilde{\nu} = \mu + \mu', \quad \kappa_0 = \frac{\kappa}{\gamma}$$

and put

$$\begin{aligned} f(u) &= -\Big\{ \gamma \operatorname{div}(m \otimes m) + \gamma \operatorname{div}(P_{(1)}(\phi)\phi m \otimes m) + \frac{1}{\gamma} \nabla (P_{(2)}(\phi)\phi^2) \\ &\qquad - \nu \Delta (P_{(1)}(\phi)\phi m) - \tilde{\nu} \nabla \operatorname{div}(P_{(1)}(\phi)\phi m) - \operatorname{div} \Phi(\phi) \Big\}, \\ P_{(1)}(\phi) &= \int_0^1 f'(1 + \tau\phi)d\tau, \quad f(\tau) = \frac{1}{\tau}, \quad \tau \in \mathbb{R}, \\ P_{(2)}(\phi) &= \int_0^1 (1 - \tau)P''\Big(1 + \tau\phi\Big)d\tau, \\ \Phi(\phi) &= \kappa_0 \Big\{ \phi \Delta \phi I_2 + (\nabla \phi) \cdot (\nabla \phi) I_2 - \frac{|\nabla \phi|^2}{2} I_2 - \nabla \phi \otimes \nabla \phi \Big\}. \end{aligned}$$

Therefore, (1) is linearized as

$$\begin{cases} \partial_t \phi + \gamma \operatorname{div} m = 0, \\ \partial_t m - \nu \Delta m - \tilde{\nu} \nabla \operatorname{div} m + \gamma \nabla \phi - \kappa_0 \nabla \Delta \phi = 0, \\ \phi|_{t=0} = \phi_0, \quad m|_{t=0} = m_0. \end{cases} \tag{6}$$

By (6), ϕ satisfies the following doubly dissipative equation:

$$\begin{cases} \partial_{tt} \phi - (\nu + \tilde{\nu})\Delta \phi_t - \gamma^2 \Delta \phi + \gamma \kappa_0 \Delta^2 \phi = 0, \\ \phi(x, 0) = \phi_0, \quad \partial_t \phi(x, 0) = -\gamma \operatorname{div} m_0. \end{cases}$$

Due to the positivity of ν and $\tilde{\nu}$, we may suppose that $\nu + \tilde{\nu} = 1$ and $\gamma = 1$ without loss of generality. Then, ϕ satisfies

$$\begin{cases} \partial_{tt} \phi - \Delta \phi_t - \Delta \phi + \kappa_0 \Delta^2 \phi = 0, \\ \phi(x, 0) = \phi_0, \quad \partial_t \phi(x, 0) = -\operatorname{div} m_0. \end{cases}$$

Now, we state the existence of solutions to (7) in the energy class, defined in the following.

Definition 3. *A function ϕ defined on $(0, \infty) \times \mathbb{R}^2$ is called to be a solution to (7) if ϕ belongs to $C([0, \infty); H^2)$ with $\partial_t \phi \in C([0, \infty); L^2)$ and satisfies (7) in the distribution sense, i.e., satisfies the following conditions:*

(i) For each $T > 0$, $\varphi \in H^2$ and a.e. $0 \le t < T$,

$$(\partial_{tt}\phi, \varphi)_{L^2} + (\nabla\phi_t, \nabla\varphi)_{L^2} + (\nabla\phi, \nabla\varphi)_{L^2} + \kappa_0(\Delta\phi, \Delta\varphi)_{L^2} = 0;$$

(ii) $\phi_0(x, 0) = \phi_0$ and $\partial_t\phi(x, 0) = -\operatorname{div} m_0$.

Theorem 1. *For each $(\phi_0, m_0) \in H^2 \times H^1$, there exists a unique solution $\phi \in C([0, \infty); H^2)$ with $\partial_t\phi \in C([0, \infty); L^2)$ to (7) such that*

$$\frac{1}{2}(\|\phi_t(t_2)\|_{L^2}^2 + \|\nabla\phi(t_2)\|_{L^2}^2 + \kappa_0\|\Delta\phi(t_2)\|_{L^2}^2) + \int_{t_1}^{t_2} \|\nabla\phi_\tau(\tau)\|_{L^2}^2 d\tau$$

$$= \frac{1}{2}(\|\phi_t(t_1)\|_{L^2}^2 + \|\nabla\phi(t_1)\|_{L^2}^2 + \kappa_0\|\Delta\phi(t_1)\|_{L^2}^2) \tag{7}$$

holds for any $0 \le t_1 < t_2 < +\infty$.

Theorem 1 is valid by the standard Galerkin method based on energy inequality (7) in a similar manner to the proof of Theorem 3.1 in Huafei and Yadong [14]. In ([14] Theorem 3.1), they added $-\Delta u_{tt}$ in a linear system and considered nonlinearity. We apply a similar manner as [14] to the proof without $-\Delta u_{tt}$ and nonlinearity; thus, we do not obtain in Theorem 1 that $\partial_t\phi \in L^\infty(0, \infty; H^1)$ in contrast to ([14] Theorem 3.1); we omitted the details.

In $L^2(0, \infty; L^2)$ for a solution ϕ to (7).

Theorem 2. *Suppose that $\phi_0 \in H^2 \cap \mathcal{H}^1$, $m_0 \in H^1$ and $m_0 + \nabla\phi_0 \in \mathcal{H}^1$. Set*

$$J_0 = (\kappa_0 + \kappa_0^2)\{\|^\top(\phi_0, m_0)\|_{\mathcal{H}^1}^2 + \|\Delta\phi_0\|_{L^2}^2\} + (1 + \kappa_0)\|m_0 + \nabla\phi_0\|_{\mathcal{H}^1}^2 + \|\phi_0\|_{\mathcal{H}^1 \cap L^2}^2.$$

Let ϕ be a solution to (7). Then, it holds true that

$$\int_0^t \|\phi(\tau)\|_{L^2}^2 d\tau \le C J_0$$

for any $t > 0$, where C is a positive constant independent of t and κ_0.

In the case that $\kappa_0 = 0$, we also have the time–space L^2 estimate for linearized CNS.

Theorem 3. *Let $(\phi_0, m_0) \in H^1$, $\phi_0 \in \mathcal{H}^1$ and $m_0 + \nabla\phi_0 \in \mathcal{H}^1$. Set*

$$J_1 = \|m_0 + \nabla\phi_0\|_{\mathcal{H}^1}^2 + \|\phi_0\|_{\mathcal{H}^1 \cap L^2}^2.$$

Let ϕ be a solution to (A2) in Appendix B. Then, there holds that

$$\int_0^t \|\phi(\tau)\|_{L^2}^2 d\tau \le C J_1$$

for any $t > 0$, where C is a positive constant independent of t.

Next, we have a time-decay estimate of the solution in the energy class to (7). By Theorem 1 and the Sobolev inequality $\phi \in C([0, \infty); L^2)$. We have the following:

Theorem 4. *Under the assumption of Theorem 2, it holds that*

$$(1 + t)\|\phi(t)\|_{L^2}^2 \le C J_0$$

for any $t > 0$, where C is a positive constant independent of t.

We now recall the existence of solutions to linear system (6) in the energy class in order to consider another diffusion-wave part $m - \mathcal{K}_\nu * m_{0,in}$. System (6) is rewritten as

$$\partial_t u + Au = 0, \tag{8}$$

where

$$u = {}^\top(\phi, m), \quad A = \begin{pmatrix} 0 & \gamma \mathrm{div} \\ \gamma \nabla - \kappa_0 \nabla \Delta & -\nu \Delta - \tilde{\nu} \nabla \mathrm{div} \end{pmatrix} \tag{9}$$

Let us introduce a semigroup $S(t) = e^{-tA}$ generated by A;

$$S(t) = e^{-tA} = \mathcal{F}^{-1} e^{-t\hat{A}_\xi} \mathcal{F},$$

where

$$\hat{A}_\xi = \begin{pmatrix} 0 & i\gamma {}^\top \xi \\ i\gamma \xi + i\kappa_0 |\xi|^2 \xi & \nu |\xi|^2 I_n + \tilde{\nu} \xi^\top \xi \end{pmatrix} \quad (\xi \in \mathbb{R}^2).$$

Theorem 5. *([15] Proposition 3.3) Let s be a non-negative integer satisfying $s \geq 2$. Then, $S(t) = e^{-tA}$ is a contraction semigroup for (8) on $H^s \times H^{s-1}$. In addition, for each $u_0 = {}^\top(\phi_0, m_0) \in H^s \times H^{s-1}$ and all $T > 0$, $S(t)$ satisfies*

$$S(\cdot)u_0 \in C([0, T]; H^s \times H^{s-1}), \quad S(0)u_0 = u_0$$

and there holds the estimate

$$\|S(t)u_0\|_{H^s \times H^{s-1}} \leq \|u_0\|_{H^s \times H^{s-1}} \tag{10}$$

for $u_0 = {}^\top(\phi_0, m_0) \in H^s \times H^{s-1}$ and $t \geq 0$.

Remark 1. *Proposition 3.3 in [15] is stated on the three-dimensional case. However, the proof is based on the standard energy estimate for the resolvent problem in the Fourier space, and it can also be applied to our two-dimensional case.*

Lastly, another diffusion-wave part $m - \mathcal{K}_\nu * m_{0,in}$ is shown to grow in $L^2(0, \infty; L^2)$ at the rate of logarithmic order.

Theorem 6. *Let $u_0 = {}^\top(\phi_0, m_0) \in H^2 \times H^1$ and u be a solution of (6), $u(t) = S(t)u_0$, as in Theorem 5. Suppose that $m_0 \in L^2 \cap L^1$, $|x|m_0 \in L^1$ and $\hat{m}_0(0) \neq 0$. Then, it holds true that*

$$\int_0^t \|m(\tau) - \mathcal{K}_\nu(\tau) * m_{0,in}\|_{L^2}^2 d\tau \sim \log t \quad as \ t \to \infty,$$

precisely,

$$\limsup_{t \to \infty} \left| \int_0^t \|m(\tau) - \mathcal{K}_\nu(\tau) * m_{0,in}\|_{L^2}^2 d\tau - \log t \right| \leq C,$$

where $C = C(u_0)$ is a positive constant independent of t.

Remark 2. *In addition of the initial condition in Theorem 2, we assume that $m_0 \in \mathcal{H}^1$; then, it holds that*

$$\int_0^t \|m(\tau) - \mathcal{K}_\nu(\tau) * m_{0,in}\|_{L^2}^2 d\tau \leq C, \quad t > 0, \tag{11}$$

where $C = C(u_0)$ is a positive constant independent of t. This shows a gain of regularity by the membership in the Hardy space of data, similarly as in the decay estimates for density in

Theorems 2 and 3. a similar phenomenon was already observed in [16,17] for dissipative wave equations. The proof is given by direct computations based on the explicit form of fundamental solution (40) below and a similar argument as in Kobayashi and Misawa [16,17]. We omitted the details here.

We state the space–time L^2 boundedness for Stokes flow part $\mathcal{K}_\nu(t) * M_{0,in}$. Indeed, from (11) together with Theorem A3 in Appendix C and ([18] Chapter 3, Section 3, Theorem 3), we find that if $m_0 \in \mathcal{H}^1$ is added in the assumption of Theorem 2, space–time L^2 boundedness holds true for $M(t) - \mathcal{K}_\nu * M_{0,in}(t)$, $\rho(t) - \rho_*$ and $\mathcal{K}_\nu(t) * M_{0,in}$.

4. Proof of Main Results

4.1. Proof of Theorems 2 and 4

In this subsection, we prove Theorems 2 and 4. The proof is performed by modifying Morawetz's energy estimate. For a solution ϕ to (7), we define function w by

$$w = \int_0^t \phi(\tau)d\tau - \operatorname{div}\Phi, \tag{12}$$

where Φ is a solution to the doubly Laplace equation

$$(-\Delta + \kappa_0\Delta^2)\Phi = m_0 + \nabla\phi_0. \tag{13}$$

For the existence of a solution to (13), we have

Theorem 7. *Suppose that $\phi_0 \in H^2 \cap \mathcal{H}^1(\mathbb{R}^2)$, $m_0 \in H^1(\mathbb{R}^2)$ and $m_0 + \nabla\phi_0 \in \mathcal{H}^1(\mathbb{R}^2)$. Then, there exists a solution Φ of*

$$(-\Delta)\Phi = (I - \kappa_0\Delta)^{-1}(m_0 + \nabla\phi_0) \quad in \ \mathbb{R}^2 \tag{14}$$

such that

$$\|\nabla\Phi\|_{L^2(\mathbb{R}^2)} + \|\Delta\Phi\|_{L^2(\mathbb{R}^2)} \le C\|m_0 + \nabla\phi_0\|_{\mathcal{H}^1(\mathbb{R}^2)}, \tag{15}$$

$$\|\operatorname{div}\Delta\Phi\|_{L^2(\mathbb{R}^2)} \le C(\|\Delta\phi_0\|_{L^2(\mathbb{R}^2)} + \|^\top(\phi_0, m_0)\|_{H^1(\mathbb{R}^2)}). \tag{16}$$

The proof of Theorem 7 is in Appendix D.

By the definition of w, we derive

$$\begin{cases} w_{tt} - \Delta w_t - \Delta w + \kappa_0\Delta^2 w = 0, \\ w(0) = -\operatorname{div}\Phi, \ w_t(0) = \phi_0. \end{cases} \tag{17}$$

To estimate

$$\int_0^t \|\phi(\tau)\|_{L^2}^2 d\tau = \int_0^t \|w_\tau(\tau)\|_{L^2}^2 d\tau,$$

we take the L^2 inner product of $(17)_1$ with w; thus, we have

$$\frac{d}{dt}(w_t, w) + \frac{1}{2}\frac{d}{dt}\|\nabla w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \kappa_0\|\Delta w\|_{L^2}^2 = \|w_t\|_{L^2}^2,$$

which is integrated on time interval $(0, t)$, yielding

$$\begin{aligned} \int_0^t &\|w_\tau(\tau)\|_{L^2}^2 d\tau \\ &= \int_0^t \|\nabla w(\tau)\|_{L^2}^2 d\tau + \int_0^t \kappa_0\|\Delta w(\tau)\|_{L^2}^2 d\tau \\ &\quad + \frac{\|\nabla w(t)\|_{L^2}^2}{2} + (w_t, w) - \frac{\|\nabla w(0)\|_{L^2}^2}{2} - (w_t(0), w(0)), \end{aligned} \tag{18}$$

where $w(0) = -\mathrm{div}\,\Phi$. Now, we estimate the terms in the right-hand side of (18). Terms $\|\nabla w(0)\|_{L^2}^2$ and $(w_t(0), w(0))$ are directly estimated by (15). A test function w_t in (17), being integrated on time interval $(0, t)$ and using (15) and (16), yields estimate

$$\|w_t(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \kappa_0 \|\Delta w(t)\|_{L^2}^2 + \int_0^t \|\nabla w_\tau(\tau)\|_{L^2}^2 d\tau$$

$$\leq C(\|\phi_0\|_{L^2}^2 + \|m_0 + \nabla\phi_0\|_{\mathcal{H}^1}^2) + C\kappa_0(\|\Delta\phi_0\|_{L^2}^2 + \|^\top(\phi_0, m_0)\|_{\mathcal{H}^1}^2). \tag{19}$$

On the other hand, from taking the L^2 inner product of $-\kappa_0\Delta w$ with (17) and integrating on $(0, t)$, we obtain that

$$\frac{1}{2}\kappa_0\|\Delta w(t)\|_{L^2}^2 + \kappa_0 \int_0^t \|\Delta w(\tau)\|_{L^2}^2 d\tau + \kappa_0(\nabla w_t, \nabla w) + \kappa_0^2 \int_0^t \|\nabla\Delta w(\tau)\|_{L^2}^2 d\tau$$

$$= \frac{1}{2}\kappa_0\|\Delta w(0)\|_{L^2}^2 + \kappa_0 \int_0^t \|\nabla w_\tau(\tau)\|_{L^2}^2 d\tau + \kappa_0(\nabla w_t(0), \nabla w(0))$$

and thus,

$$\frac{1}{2}\kappa_0\|\Delta w(t)\|_{L^2}^2 + \kappa_0 \int_0^t \|\Delta w(\tau)\|_{L^2}^2 d\tau + \kappa_0^2 \int_0^t \|\nabla\Delta w(\tau)\|_{L^2}^2 d\tau$$

$$\leq C\kappa_0\{\|\nabla\phi_0\|_{L^2}^2 + \|\nabla\mathrm{div}\,\Phi\|_{L^2}^2 + \|\Delta\mathrm{div}\,\Phi\|_{L^2}^2 + \int_0^t \|\nabla w_\tau(\tau)\|_{L^2}^2 d\tau\}$$

$$+ C\kappa_0(\|\nabla w(t)\|_{L^2}^2 + \|\nabla w_t(t)\|_{L^2}^2). \tag{20}$$

Terms having Φ in (20) are estimated by (15) and (16). The fourth and fifth terms on the right-hand side of (20) are also estimated by (19). For term $\|\nabla w_t(t)\|_{L^2}^2 = \|\nabla\phi(t)\|_{L^2}^2$, we apply the standard energy estimate obtained from (7) with a test function ϕ_t

$$\frac{1}{2}(\|\phi_t(t)\|_{L^2}^2 + \|\nabla\phi(t)\|_{L^2}^2 + \kappa_0\|\Delta\phi(t)\|_{L^2}^2) + \int_0^t \|\nabla\phi_\tau(\tau)\|_{L^2}^2 d\tau$$

$$= \frac{1}{2}(\|\phi_t(0)\|_{L^2}^2 + \|\nabla\phi(0)\|_{L^2}^2 + \kappa_0\|\Delta\phi(0)\|_{L^2}^2)$$

$$\leq C(\|\mathrm{div}\,m_0\|_{L^2}^2 + \|\nabla\phi_0\|_{L^2}^2) + C\kappa_0\|\Delta\phi_0\|_{L^2}^2$$

and thus,

$$\kappa_0 \int_0^t \|\Delta w(\tau)\|_{L^2}^2 d\tau \leq C(\kappa_0 + \kappa_0^2)\{\|^\top(\phi_0, m_0)\|_{\mathcal{H}^1}^2 + \|\Delta\phi_0\|_{L^2}^2\}$$

$$+ C\kappa_0\|m_0 + \nabla\phi_0\|_{\mathcal{H}^1}^2. \tag{21}$$

For the estimation of the L^2-norm on a space of w and the one on the time–space of its spatial gradient, let

$$v = \int_0^t w(s)ds \tag{22}$$

and proceed to the energy estimates of v. From the direct calculation, v satisfies

$$\begin{cases} v_{tt} - \Delta v - \Delta v_t + \kappa_0\Delta^2 v = \phi_0 + \Delta\mathrm{div}\Phi, \\ v(0) = 0, \qquad v_t(0) = -\mathrm{div}\Phi. \end{cases} \tag{23}$$

A test function v_t in (23) gives

$$\frac{1}{2}\frac{d}{dt}\|v_t(t)\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|_{L^2}^2 + \|\nabla v_t(t)\|_{L^2}^2 + \frac{1}{2}\kappa_0\frac{d}{dt}\|\Delta v(t)\|_{L^2}^2$$
$$= (\phi_0 + \Delta\mathrm{div}\Phi, v_t(t))$$
$$= \frac{d}{dt}(\phi_0 + \Delta\mathrm{div}\Phi, v(t)), \tag{24}$$

being integrated on $(0, t)$ and yielding

$$\frac{1}{2}\|v_t(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla v(t)\|^2 + \int_0^t \|\nabla v_s(s)\|_{L^2}^2 ds + \frac{1}{2}\kappa_0\|\Delta v(s)\|_{L^2}^2$$
$$= (\phi_0 + \Delta\mathrm{div}\Phi, v(t)) + \frac{1}{2}\|w(0)\|_{L^2}^2. \tag{25}$$

Now, the first term of the right-hand side of (25) is

$$(\phi_0 + \Delta\mathrm{div}\Phi, v(t)) = (\phi_0, v(t)) + (\Delta\mathrm{div}\Phi, v(t)) \tag{26}$$

of which the first term is evaluated by the use of Lemmas 1 and 2 and Young's inequality as

$$|(\phi_0, v(t))| \leq C\|\phi_0\|_{\mathcal{H}^1}\|v(t)\|_{\mathrm{BMO}} \quad \leq \quad C\|\phi_0\|_{\mathcal{H}^1}\|\nabla v(t)\|_{L^2}$$
$$\leq \quad \frac{1}{8}\|\nabla v(t)\|_{L^2}^2 + C\|\phi_0\|_{\mathcal{H}^1}^2$$

and, the second term is controlled by (15) as

$$|(\Delta\mathrm{div}\Phi, v(t))| \quad = \quad |(\Delta\Phi, \nabla v(t))|$$
$$\leq \quad \|m_0 + \nabla\phi_0\|_{\mathcal{H}^1}\|\nabla v(t)\|_{L^2}$$
$$\leq \quad \frac{1}{8}\|\nabla v(t)\|_{L^2}^2 + C\|m_0 + \nabla\phi_0\|_{\mathcal{H}^1}^2.$$

Since $w(0) = -\mathrm{div}\Phi$, the right-hand side of (25) is bounded by

$$(\phi_0 + \Delta\mathrm{div}\Phi, v(t)) + \frac{1}{2}\|w(0)\|_{L^2}^2$$
$$\leq \frac{1}{4}\|\nabla v(t)\|_{L^2}^2 + C\|\phi_0\|_{\mathcal{H}^1}^2 + C\|m_0 + \nabla\phi_0\|_{\mathcal{H}^1}^2.$$

Thus, it follows that

$$\frac{1}{2}\|v_t(t)\|^2 + \frac{1}{4}\|\nabla v(t)\|^2 + \int_0^t \|\nabla v_s(s)\|^2 ds + \frac{1}{2}\kappa_0\|\Delta v(s)\|_{L^2}^2$$
$$\leq C\|\phi_0\|_{\mathcal{H}^1}^2 + C\|m_0 + \nabla\phi_0\|_{\mathcal{H}^1}^2. \tag{27}$$

Gathering (18), (19), (21), and (27), we obtain Theorem 2.

On the basis of Theorems 2 and 4 is proved as follows. We set a total energy of w as

$$E(w(t)) = \|w_t(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \kappa_0\|\Delta w(t)\|_{L^2}^2.$$

By the proof of Theorem 2 and integration by parts, we find that

$$CJ_0 \quad \geq \quad \int_0^t E(w(s))ds$$
$$= \quad tE(w(t)) - \int_0^t s\frac{d}{ds}E(w(s))ds.$$

Since, by integration by parts again,

$$
\begin{aligned}
\frac{d}{dt}E(w(t)) &= \frac{d}{dt}\left\{\|\nabla w(t)\|_{L^2}^2 + \|w_t(t)\|_{L^2}^2 + \kappa_0\|\Delta w(t)\|_{L^2}^2\right\} \\
&= 2(\nabla w_t(t), \nabla w(t)) + 2(w_{tt}(t), w_t(t)) + 2(\kappa_0\Delta w, \Delta w_t) \\
&= -2(w_t(t), \Delta w(t)) + 2(w_{tt}(t), w_t(t)) + 2(\kappa_0\Delta^2 w, w_t) \\
&= 2(w_{tt}(t) - \Delta w(t) + \kappa_0\Delta^2 w(t), w_t(t)) \\
&= 2(\Delta w_t(t), w_t(t)) \\
&= -2\|\nabla w_t(t)\|_{L^2}^2 \\
&= -2\|\nabla \phi(t)\|_{L^2}^2,
\end{aligned}
$$

we have

$$
CJ_0 \geq tE(w(t)) + 2\int_0^t s\|\nabla \phi(s)\|^2 ds. \tag{28}
$$

This, together with (19), gives the assertion of Theorem 4. The proof is completed. □

4.2. Proof of Theorem 6

In this section, we show the validity of Theorem 6. By taking the Fourier transform of (6) with respect to space variable x, we have the following ordinary differential equation with a parameter ξ.

$$
\begin{cases}
\partial_t\hat{\phi}(t,\xi) + i\gamma\xi \cdot \hat{m}(t,\xi) = 0, \\
\partial_t\hat{m}(t,\xi) + \nu|\xi|^2\hat{m}(t,\xi) + \tilde{\nu}\xi(\xi \cdot \hat{m}(t,\xi)) + i\gamma\xi\hat{\phi}(t,\xi) + i\xi\kappa_0|\xi|^2\hat{\phi}(t,\xi) = 0, \\
\hat{\phi}(0,\xi) = \hat{\phi}_0, \quad \hat{m}(0,\xi) = \hat{m}_0.
\end{cases} \tag{29}
$$

Therefore, the solutions of (6) are given by the following formulas by [11]. Let $K = \dfrac{2\sqrt{\kappa_0\gamma}}{\nu + \tilde{\nu}}$ and $B = \dfrac{2\gamma}{\nu + \tilde{\nu}}$. For $|\xi| \neq 0, B/\sqrt{1 - K^2}$ when $0 < K < 1$ and $|\xi| \neq 0$ when $K \geq 1$, the Fourier transforms of ϕ and m are given explicitly by formulas

$$
\begin{aligned}
\hat{\phi} &= \frac{\lambda_+(\xi)e^{\lambda_-(\xi)t} - \lambda_-(\xi)e^{\lambda_+(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)}\hat{\phi}_0 - i\gamma\frac{e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)}\xi \cdot \hat{m}_0, \\
\hat{m} &= e^{-\nu|\xi|^2 t}\hat{m}_0 - i\xi(\gamma + \kappa_0|\xi|^2)\left(\frac{e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)}\right)\hat{\phi}_0 \\
&\quad + \left(\frac{\lambda_+(\xi)e^{\lambda_+(\xi)t} - \lambda_-(\xi)e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} - e^{-\nu|\xi|^2 t}\right)\frac{\xi(\xi \cdot \hat{m}_0)}{|\xi|^2},
\end{aligned} \tag{30}
$$

where $\lambda_\pm$ are given by

$$
\lambda_\pm(\xi) = -A(|\xi|^2 \pm \sqrt{|\xi|^4 - B^2|\xi|^2 - K^2|\xi|^4}) \tag{31}
$$

with a positive constant $A = \dfrac{\nu + \tilde{\nu}}{2}$ and stand for roots of the characteristic equation of (29). If $0 < K < 1$ and $\min\left\{\frac{1}{2}, \frac{B}{2\sqrt{1-K^2}}\right\} \leq |\xi| \leq 2\frac{B}{\sqrt{1-K^2}}$, $\hat{\phi}$ and $\hat{m}$ are represented as

$$
\begin{aligned}
\hat{\phi} &= \frac{1}{2\pi i}\oint_\Gamma \frac{(z + A|\xi|^2)e^{zt}}{z^2 + (\nu+\tilde{\nu})|\xi|^2 z + \kappa_0\gamma|\xi|^4 + \gamma^2|\xi|^2}dz\hat{\phi}_0 \\
&\quad - \frac{\gamma}{2\pi}\oint_\Gamma \frac{e^{zt}}{z^2 + (\nu+\tilde{\nu})|\xi|^2 z + \kappa_0\gamma|\xi|^4 + \gamma^2|\xi|^2}dz\xi\cdot\hat{m}_0,
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\hat{m} &= e^{-\nu|\xi|^2 t}\hat{m}_0 - \frac{\gamma\xi}{2\pi}\oint_\Gamma \frac{e^{zt}}{z^2 + (\nu+\tilde{\nu})|\xi|^2 z + \kappa_0\gamma|\xi|^4 + \gamma^2|\xi|^2}dz\hat{\phi}_0 \\
&\quad + \left(\frac{1}{2\pi i}\oint_\Gamma \frac{ze^{zt}}{z^2 + (\nu+\tilde{\nu})|\xi|^2 z + \kappa_0\gamma|\xi|^4 + \gamma^2|\xi|^2}dz\right)\frac{\xi(\xi\cdot\hat{m}_0)}{|\xi|^2},
\end{aligned}
\tag{33}
$$

where Γ is a closed pass surrounding $\lambda_\pm$ and included in set $\{z \in \mathbb{C}\,|\,\mathrm{Re}\,z \leq -c_0\}$, and c_0 is a positive number satisfying

$$
\max_{\min\left\{\frac{1}{2}, \frac{B}{2\sqrt{1-K^2}}\right\}\leq|\xi|\leq 2\frac{B}{\sqrt{1-K^2}}} \mathrm{Re}\,\lambda_\pm \leq -2c_0.
$$

Cut-off functions φ_1, φ_∞ and φ_M in $C^\infty(\mathbb{R}^2)$ are defined by [11] as follows: in the case such that $K \neq 1$, φ_1 is given by

$$
\varphi_1(\xi) = \left\{\begin{array}{l} 1 \ \text{for} \ |\xi| \leq \frac{1}{2} \\ 0 \ \text{for} \ |\xi| \geq 1 \end{array}\right. \quad \text{if} \ \frac{B}{2\sqrt{|1-K^2|}} > 1;
$$

$$
\varphi_1(\xi) = \left\{\begin{array}{l} 1 \ \text{for} \ |\xi| \leq \dfrac{B}{2\sqrt{|1-K^2|}} \\ 0 \ \text{for} \ |\xi| \geq 1 \end{array}\right. \quad \text{if} \ \frac{B}{2\sqrt{|1-K^2|}} \leq 1 < \frac{B}{\sqrt{2|1-K^2|}};
$$

$$
\varphi_1(\xi) = \left\{\begin{array}{l} 1 \ \text{for} \ |\xi| \leq \dfrac{B}{2\sqrt{|1-K^2|}} \\ 0 \ \text{for} \ |\xi| \geq \dfrac{B}{\sqrt{2|1-K^2|}} \end{array}\right. \quad \text{if} \ \frac{B}{\sqrt{2|1-K^2|}} \leq 1.
$$

φ_∞ and φ_M are

$$
\varphi_\infty(\xi) = \left\{\begin{array}{l} 1 \ \text{for} \ |\xi| \geq \dfrac{2B}{\sqrt{|1-K^2|}} \\ 0 \ \text{for} \ |\xi| \leq \dfrac{\sqrt{2}B}{\sqrt{|1-K^2|}}, \end{array}\right. \quad \varphi_M(\xi) = 1 - \varphi_1(\xi) - \varphi_\infty(\xi).
$$

In the case that $K = 1$, φ_1 and φ_∞ are

$$
\varphi_1(\xi) = \left\{\begin{array}{l} 1 \ \text{for} \ |\xi| \leq \frac{1}{2} \\ 0 \ \text{for} \ |\xi| \geq 1, \end{array}\right. \quad \varphi_\infty(\xi) = \left\{\begin{array}{l} 0 \ \text{for} \ |\xi| \leq \frac{1}{2} \\ 1 \ \text{for} \ |\xi| \geq 1, \end{array}\right.
$$

$$
\varphi_1(\xi) + \varphi_\infty(\xi) = 1.
$$

We define solution operators E_1 and E_∞ on a low- and high-frequency part of (6), respectively, as follows:

$$
\begin{aligned}
E_1(t) &= (E_{1,\phi}(t), E_{1,m}(t)), \\
E_\infty(t) &= (E_{\infty,\phi}(t), E_{\infty,m}(t)),
\end{aligned}
\tag{34}
$$

where

$$E_{1,\phi}(t)(\phi_0, m_0)(x) = \mathcal{F}^{-1}[\varphi_1(\xi)\hat{\phi}(t,\xi)](x), \tag{35}$$

$$E_{1,m}(t)(\phi_0, m_0)(x) = \mathcal{F}^{-1}[\varphi_1(\xi)\hat{m}(t,\xi)](x), \tag{36}$$

$$E_{\infty,\phi}(t)(\phi_0, m_0)(x) = \mathcal{F}^{-1}[(\varphi_M(\xi) + \varphi_\infty(\xi))\hat{\phi}(t,\xi)](x), \tag{37}$$

$$E_{\infty,m}(t)(\phi_0, m_0)(x) = \mathcal{F}^{-1}[(\varphi_M(\xi) + \varphi_\infty(\xi))\hat{m}(t,\xi)](x). \tag{38}$$

In [11], the solution operator is shown to have an exponential decay in time on the high-frequency part (34). In fact we have

Theorem 8. *([11] Theorem 3.2) Let $1 \le p \le \infty$. Then, it holds that*

$$\|\partial_t^k \partial_x^\alpha E_\infty(t)(\phi_0, m_0)\|_{L^p}$$
$$\le C_{k,\alpha} e^{-ct} \left\{ (1 + t^{-\delta_1 - \frac{|\alpha|}{2} - k}) \left[\|\phi_0\|_{L^p} + \|m_0\|_{L^p} \right] + t^{-\delta_2 - \frac{|\alpha|}{2} - k} \|\phi_0\|_{L^p} \right\} \tag{39}$$

for $t > 0$, $k \ge 0$ and $|\alpha| \ge 0$, where $(\delta_1, \delta_2) = (1/2, 1)$ and $(1, 3/2)$ for $K \ne 1$ and $K = 1$, respectively.

Therefore, in order to show Theorem 6, it is enough to consider the low-frequency part. We estimate the Green function. We put

$$L_{11,j}(t,x) = \mathcal{F}^{-1}\left\{ \frac{\lambda_+(\xi)e^{\lambda_-(\xi)t} - \lambda_-(\xi)e^{\lambda_+(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \varphi_j(\xi) \right\}(x),$$

$$L_{12,j}(t,x) = \mathcal{F}^{-1}(-i\gamma^\top \xi \hat{L}_j),$$

$$\hat{L}_j(t,\xi) = \frac{e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \varphi_j(\xi),$$

$$L_{21,j}(t,x) = \mathcal{F}^{-1}\left\{ -\xi(i\gamma + k_0|\xi|^2)\hat{L}_j \right\},$$

$$L_{22,j}(t,x) = K_{1,j}(t,x) + K_{2,j}(t,x) - K_{3,j}(t,x),$$

$$K_{1,j}(t,x) = \mathcal{F}^{-1}\left[e^{-\nu|\xi|^2 t}\varphi_j(\xi) \right](x)I_n,$$

$$K_{2,j}(t,x) = \mathcal{F}^{-1}\left\{ \frac{\lambda_+(\xi)e^{\lambda_+(\xi)t} - \lambda_-(\xi)e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \frac{\xi^\top \xi}{|\xi|^2} \varphi_j(\xi) \right\}(x),$$

$$K_{3,j}(t,x) = \mathcal{F}^{-1}\left[e^{-\nu|\xi|^2 t} \frac{\xi^\top \xi}{|\xi|^2} \varphi_j(\xi) \right](x)$$

for $j = 1, \infty$. We see from (30) that

$$E_j(t)(\phi_0, m_0) = \begin{pmatrix} L_{11,j}(t, \cdot) & L_{12,j}(t, \cdot) \\ L_{21,j}(t, \cdot) & L_{22,j}(t, \cdot) \end{pmatrix} * \begin{pmatrix} \phi_0 \\ m_0 \end{pmatrix} \tag{40}$$

for $j = 1, \infty$.

We set

$$K_1 m_0 = \mathcal{F}^{-1}\left[\frac{\lambda_+(\xi)e^{\lambda_+(\xi)t} - \lambda_-(\xi)e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \frac{\xi^\top \xi}{|\xi|^2} \varphi_1(\xi)\hat{m}_0 \right].$$

$K_1 m_0$ is a part of the Green matrix and corresponds to the diffusion-wave part $m - \mathcal{K}_\nu * m_{0,in}$. Our claim is the following estimate.

Proposition 1. *Let $m_0 \in L^2 \cap L^1$, $|x|m_0 \in L^1$ and $\hat{m}_0 \ne 0$. Then, it holds that*

$$\limsup_{t \to \infty} \left| \int_0^t \|K_1 m_0(\tau)\|_{L^2}^2 d\tau - \log t \right| \le C_1,$$

where $C_1 = C_1(u_0)$ is a positive constant independent of t.

Proof. By the Plancherel theorem and (31), we see that there exists a positive constant C, such that

$$C\|e^{-|\xi|^2 t}\hat{m}_0(\xi)\|_{L^2(|\xi|\le c_1)}^2 \le \|K_1 m_0(t)\|_{L^2}^2 \le \frac{1}{C}\|e^{-|\xi|^2 t}\hat{m}_0(\xi)\|_{L^2(|\xi|\le c_1)}^2,$$

where $c_1 = 1$ when $\frac{B}{2\sqrt{|1-K^2|}} > 1$ or $\frac{B}{2\sqrt{|1-K^2|}} \le 1 < \frac{B}{\sqrt{2|1-K^2|}}$ and $c_1 = \frac{B}{\sqrt{2|1-K^2|}}$ when $\frac{B}{\sqrt{2|1-K^2|}} \le 1$. Hence, we have to estimate $\|e^{-|\xi|^2 t}\hat{m}_0(\xi)\|_{L^2(|\xi|\le c_1)}^2$. It follows from the polar coordinates that

$$\|e^{-|\xi|^2 t}\hat{m}_0(\xi)\|_{L^2(|\xi|\le c_1)}^2 = C_\omega \int_0^{c_1} |e^{-r^2 t}\hat{m}_0(r\omega)|^2 r\,dr,$$

where $r = |\xi|$, $\omega = \xi/|\xi|$, and C_ω is some positive constant that appears in the polar coordinates. Changing variables $r\sqrt{t} = s$, we have

$$\int_0^{c_1} |e^{-r^2 t}\hat{m}_0(\omega r)|^2 r\,dr = t^{-1}\int_0^{c_1\sqrt{t}} e^{-2s^2}|\hat{m}_0(s\omega/\sqrt{t})|^2 s\,ds.$$

This, together with the fundamental theorem of calculus for $\hat{m}_0$, implies that

$$\left|\|e^{-|\xi|^2 t}\hat{m}_0(\xi)\|_{L^2(|\xi|\le c_1)}^2 - C_\omega t^{-1}|\hat{m}_0(0)|^2\int_0^{c_1\sqrt{t}} e^{-2s^2} s\,ds\right|$$

$$= C_\omega\left|t^{-1}\int_0^{c_1\sqrt{t}}(|\hat{m}_0(s\omega/\sqrt{t})|^2 - |\hat{m}_0(0)|^2)e^{-2s^2} s\,ds\right|$$

$$\le Ct^{-1}\int_0^{c_1\sqrt{t}} |\hat{m}_0(s\omega/\sqrt{t}) - \hat{m}_0(0)|^2 e^{-2s^2} s\,ds$$

$$= Ct^{-1}\int_0^{c_1\sqrt{t}}\left|\int_0^1 \nabla_\xi\hat{m}_0\left(\theta\frac{s\omega}{\sqrt{t}}\right)d\theta\frac{s\omega}{\sqrt{t}}\right|^2 e^{-2s^2} s\,ds$$

$$\le Ct^{-1}\int_0^{c_1\sqrt{t}}\|\nabla_\xi\hat{m}_0\|_{L^\infty}^2\left|\frac{s\omega}{\sqrt{t}}\right|^2 e^{-2s^2} s\,ds$$

$$\le C\||x|m_0\|_{L^1}^2 t^{-2}\int_0^{c_1\sqrt{t}} e^{-2s^2} s^3\,ds$$

$$\le C_2 t^{-2} \tag{41}$$

for a positive constant $C_2 = C_2(u_0)$ independent of t. $\hat{m}_0(0) \ne 0$ by our assumption. Since

$$\int_0^1 \|e^{-|\xi|^2\tau}\hat{m}_0(\xi)\|_{L^2}^2 d\tau \le C\|m_0\|_{L^2}^2,$$

we estimate $\int_1^t \|e^{-|\xi|^2\tau}\hat{m}_0(\xi)\|_{L^2}^2 d\tau$. We set

$$I_1(t) := C_\omega|\hat{m}_0(0)|^2 t^{-1}\int_0^{c_1\sqrt{t}} e^{-2s^2} s\,ds.$$

Applying (41) yields that, for $t \ge 1$,

$$\left|\int_1^t\left\{\|e^{-|\xi|^2\tau}\hat{m}_0(\xi)\|_{L^2(|\xi|\le c_1)}^2 - I_1(\tau)\right\}d\tau\right| \le C_2\int_1^t \tau^{-2}d\tau \le C_2. \tag{42}$$

Let positive constants Q_1 and Q_2 be defined by

$$Q_1 := \int_0^{c_1} e^{-2s^2} s^3 ds, \quad Q_2 := \int_0^{\infty} e^{-2s^2} s^3 ds.$$

It obviously follows that, for $t \geq 1$,

$$Q_1 \leq \int_0^{c_1\sqrt{t}} e^{-2s^2} s^3 ds \leq Q_2.$$

This implies that, for $t \geq 1$,

$$\int_1^t C_\omega |\hat{m}_0(0)|^2 Q_1 \tau^{-1} d\tau \leq \int_1^t I_1(\tau) d\tau \leq \int_1^t C_\omega |\hat{m}_0(0)|^2 Q_2 \tau^{-1} d\tau$$

and thus,

$$C_\omega |\hat{m}_0(0)|^2 Q_1 \log t \leq \int_1^t I_1(\tau) d\tau \leq C_\omega |\hat{m}_0(0)|^2 Q_2 \log t. \tag{43}$$

(42) and (43) yield that

$$\limsup_{t\to\infty} \left| \int_1^t \|e^{-|\xi|^2 \tau} \hat{m}_0(\xi)\|_{L^2(|\xi|\leq c_1)}^2 d\tau - \log t \right| \leq CC_2.$$

Therefore, there exists a positive constant $C_1 = C_1(u_0)$ independent of t, such that

$$\limsup_{t\to\infty} \left| \int_0^t \|K_1 m_0(\tau)\|_{L^2}^2 d\tau - \log t \right| \leq C_1.$$

Since other parts of diffusion wave $m - \mathcal{K}_\nu * m_{0,in}$ that appear in the Green matrix on the low-frequency part are estimated similarly to Proposition 1, we obtain the estimation in Theorem 6. The proof is completed.   □

5. Conclusions

We studied the asymptotic behavior of solutions to the compressible Navier–Stokes–Korteweg system in $\mathbb{R}^2$. Concerning the linearized system for (1), under some initial conditions given by Hardy space $\mathcal{H}^1$, we showed some space–time L^2 estimates for the density and Stokes flow parts of the momentum. The potential flow part of the momentum was also shown to grow at the rate of logarithmic order in space–time L^2 norm. The asymptotic behaviors in space–time L^2 of the diffusion-wave parts were shown to be essentially different between density and the potential flow part of the momentum. Nonlinear terms are expected to decay faster than the linear terms in the Duhamel formula do, as in [7,11]. As a consequence, the leading terms of the asymptotic expansion of solution u for (1) were given by

$$u(t) \sim \underbrace{\begin{pmatrix} 0 \\ M(t) - \mathcal{K}_\nu(t) * M_{0,in} \end{pmatrix} + \begin{pmatrix} \rho(t) - \rho_* \\ \mathcal{K}_\nu(t) * M_{0,in} \end{pmatrix}}_{\text{solutions to linearized system}} + \underbrace{\cdots}_{\text{nonlinear parts}} \quad \text{in } L^2(0, \infty; \mathbb{R}^2). \tag{44}$$

Analysis for asymptotic behavior in the two-dimensional case is difficult because the time decay of solutions to the linear system in two-dimensional cases is slower than that in higher-dimensional cases. To overcome this difficulty, we used a gain of regularity by the Hardy space; by the L^2 energy estimate of the Morawetz type, we succeeded to derive the asymptotic behavior (44).

Concerning future works, it is important to consider how the pressure term has an effect on the asymptotic behavior of (1). As shown in our paper [19], since (1) governs the

motion of two-phase fluids, pressure is a nonmonotone function. When pressure decreases, solutions are expected to be unstable due to positive eigenvalues in linear systems. Hence, we will study the asymptotic behavior of solutions with relation to a critical value, such that $P'(\rho) = 0$ holds or an initial condition of pressure. Furthermore, from the point of view of engineering, analysis of two-phase fluids in bounded or unbounded domains with boundaries is more important. On the basis of our analysis of the Caucy problem, we will study the asymptotic behavior of solutions to CNSK under boundary conditions.

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Appendix A

We consider the following Cauchy problem:

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } (0, \infty) \times \mathbb{R}^2, \\ u = u_0 & \text{on } \{t = 0\} \times \mathbb{R}^2. \end{cases} \tag{A1}$$

Solution u to (A1) satisfies estimate

$$\int_0^\infty \|u(t)\|_{L^2(\mathbb{R}^2)}^2 dt \le C \|u_0\|_{\mathcal{H}^1}^2$$

for $u_0 \in \mathcal{H}^1$, while $\mathcal{H}^1 \subset L^1$ and for $u_0 \in L^1$, estimate

$$\int_0^\infty \|u(t)\|_{L^2(\mathbb{R}^2)}^2 dt \le C \|u_0\|_{L^1}^2$$

generally does not hold. This shows a subtle gain of regularity by the Hardy space.

Appendix B

We treat the linearized CNS, that is, (7) with zero capillary constant $\kappa_0 = 0$.

$$\begin{cases} \partial_{tt}\phi - \Delta\phi_t - \Delta\phi = 0, \\ \phi(x, 0) = \phi_0, \quad \partial_t\phi(x, 0) = -\operatorname{div} m_0. \end{cases} \tag{A2}$$

Definition A1. *A function ϕ defined on $(0, \infty) \times \mathbb{R}^2$ is called to be a solution to (A2) if ϕ belongs to $C([0, \infty); H^1)$ with $\partial_t\phi \in C([0, \infty); L^2)$ and satisfies (A2) in the distribution sense, i.e., satisfies the following conditions:*
(i) For each $T > 0$, $\varphi \in H^1$ and a.e. $0 \le t < T$,

$$(\partial_{tt}\phi, \varphi)_{L^2} + (\nabla\phi_t, \nabla\varphi)_{L^2} + (\nabla\phi, \nabla\varphi)_{L^2} = 0;$$

(ii) $\phi_0(x, 0) = \phi_0$ and $\partial_t\phi(x, 0) = -\operatorname{div} m_0$.

The existence of a unique solution to (A2) is well-known, as follows. For the proof, we can refer to Proposition 2.1 in Ikehata, Todorova, and Yordanov [20] using the Lumer–Phillips theorem.

Theorem A1. *For each $(\phi_0, m_0) \in H^1$ there exists a unique solution $\phi \in C([0, \infty); H^1)$ with $\partial_t \phi \in C([0, \infty); L^2)$ to (A2), such that*

$$\frac{1}{2}(\|\phi_t(t_2)\|_{L^2}^2 + \|\nabla\phi(t_2)\|_{L^2}^2) + \int_{t_1}^{t_2} \|\nabla\phi_\tau(\tau)\|_{L^2}^2 d\tau$$
$$= \frac{1}{2}(\|\phi_t(t_1)\|_{L^2}^2 + \|\nabla\phi(t_1)\|_{L^2}^2)$$

holds for any $0 \le t_1 < t_2 < +\infty$.

Appendix C

Let us introduce the following incompressible Stokes system:

$$\begin{cases} \partial_t u - \nu\Delta u + \nabla P = 0 & \text{in } (0, \infty) \times \mathbb{R}^2, \\ \text{div } u = 0 & \text{in } (0, \infty) \times \mathbb{R}^2, \\ u(0, x) = M_{0,in} & \text{in } \mathbb{R}^2. \end{cases} \tag{A3}$$

The following Helmholtz decomposition is well-known (cf., Simader and Sohr [21])

$$L^q(\mathbb{R}^2) = L_\sigma^q(\mathbb{R}^2) \oplus G^q(\mathbb{R}^2), \quad (1 < q < \infty),$$

where $G^q(\mathbb{R}^2)$ denotes the set of all functions of the potential flow part, defined by $G^q = \{\nabla p \in L^q(\mathbb{R}^2); p \in L_{\text{loc}}^q(\mathbb{R}^2)\}$. Here, we denote by P_q the projection operator from $L^q(\mathbb{R}^2)$ to $L_\sigma^q(\mathbb{R}^2)$. On the whole, space P_q is given by the Riesz operator

$$P_q f = \mathcal{F}^{-1}\left\{\left(I_2 - \frac{\xi^\top \xi}{|\xi|^2}\right)\hat{f}\right\}.$$

Applying the Helmholtz projection to Stokes Equation (A3) derives the following system.

$$\begin{cases} \partial_t u - \nu P_q\Delta u = 0 & \text{in } (0, \infty) \times \mathbb{R}^2, \\ \text{div } u = 0 & \text{in } (0, \infty) \times \mathbb{R}^2, \\ u(0, x) = u_0 := M_{0,in} & \text{in } \mathbb{R}^2. \end{cases} \tag{A4}$$

We define Stokes operator A_q on L_σ^q by $A_q = -P_q\Delta$ with domain $D(A_q) = W^{2,q}(\mathbb{R}^2) \cap L_\sigma^q(\mathbb{R}^2)$. Concerning the existence of solutions to (A4), we have

Theorem A2. *(Giga and Sohr [22]) $-A_q$ generates a uniformly bounded holomorphic semigroup $\{e^{-tA_q}\}_{t\ge 0}$ of class C^0 in $L_\sigma^q(\mathbb{R}^2)$.*

The solution to (A4) satisfies $u(t) = \mathcal{K}_\nu(t) * M_{0,in}$; we can thus estimate solution u to (A4) as follows.

Theorem A3. *Let $M_{0,in} \in L^2 \cap L^1$ and u be a solution to (A4). Then, u satisfies estimate*

$$\int_0^t \|u(s)\|_{L^2}^2 ds \le C\|M_{0,in}\|_{L^1}^2$$

uniformly for t.

We give the proof of Theorem A3 here.

Proof of Theorem A3. Put $v(t, x) := \int_0^t u(s, x)ds$. Then, v satisfies

$$\begin{cases} \partial_t v - \nu \Delta v = u_0 & \text{in } (0, \infty) \times \mathbb{R}^2, \\ v(0, x) = 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{A5}$$

Here, we used $P_2(\Delta u) = \Delta(P_2 u) = u$ in (A4). A test function $\partial_t w$ in (A5), being integrated on time interval $(0, t)$, and $\partial_t w = u$ yields estimate

$$\int_0^t \|u(s)\|_{L^2}^2 ds + \frac{\nu}{2}\|\nabla v(t)\|_{L^2}^2 = (u_0, v(t)) \tag{A6}$$

The first term of the right-hand side in (A6) is estimated by Corollary 1 as follows.

$$|(u_0, v(t))| \le C\|u_0\|_{L^1}\|\nabla v(t)\|_{L^2} \le C_1\|u_0\|_{L^1}^2 + \frac{\nu}{4}\|\nabla v(t)\|_{L^2}^2. \tag{A7}$$

(A6) and (A7) derive the desired estimate. The proof is completed. □

Appendix D

Here, we demonstrate the proof of Theorem 7.

Proof of Theorem 7. Now, we define operator T for $f \in \mathcal{H}^1$ by

$$Tf = \mathcal{F}^{-1}((1 + \kappa_0|\xi|^2)^{-1}\hat{f}) := K * f.$$

From direct computation, we see that

$$|\partial_\xi^\alpha \hat{K}(\xi)| \le C|\xi|^{-|\alpha|}, \quad \text{for any } \xi \neq 0 \text{ and } |\alpha| \ge 0$$

for a positive constant C independent of κ_0. Then, it follows from Shimizu and Shibata ([23] Theorem 2.3) that

$$|\partial_\xi^\alpha K(x)| \le C_3|x|^{-2-|\alpha|} \quad (x \neq 0)$$

holds true for a positive constant C_3 independent of κ_0. By this fact and the multiplier type theorem on the Hardy space as in Stein ([18] Chapter 3, Section 3.2, Theorem 4), we find that T is a bounded operator on $\mathcal{H}^1$ and

$$\|Tf\|_{\mathcal{H}^1} \le C\|f\|_{\mathcal{H}^1},$$

where C is independent of κ_0. Therefore, $(I - \kappa_0\Delta)^{-1}$ is bounded on $\mathcal{H}^1$ and thus, $(I - \kappa_0\Delta)^{-1}(m_0 + \nabla\phi_0) \in \mathcal{H}^1$ for $m_0 + \nabla\phi_0 \in \mathcal{H}^1$. Then, from [24] we obtain the existence of a solution to (13) satisfying (15). Furthermore, since $(I - \kappa_0\Delta)^{-1}$ is bounded on L^2, we have estimation

$$\begin{aligned} \|\text{div } \Delta\Phi\|_{L^2} &= \|\text{div }(I - \kappa_0\Delta)^{-1}(m_0 + \nabla\phi_0)\|_{L^2} \\ &\le C\|\text{div }(m_0 + \nabla\phi_0)\|_{L^2} \\ &\le C(\|\Delta\phi_0\|_{L^2} + \|^\top(\phi_0, m_0)\|_{H^1}), \end{aligned}$$

from which (16) is obtained. The proof is completed. □

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