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A Coupled Fixed Point Technique for Solving Coupled Systems of Functional and Nonlinear Integral Equations

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Abstract: In this paper, we obtain coupled fixed point results for F-contraction mapping satisfying a nonlinear contraction condition in the framework of complete metric space without and with a directed graph. As applications of our results, we study a problem of existence and uniqueness of solutions for a class of systems of functional equations that appears in dynamic programming and nonlinear integral equations. Finally, illustrative examples to support some our results are discussed.

Keywords: coupled common fixed point; F-contraction mapping; dynamic programming; nonlinear integral equations

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1. Introduction and Preliminaries

In the context, the symbols $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ denote the set of all real, positive real, and natural numbers, respectively.

Many authors have generalized the Banach fixed point theorem [1], which states: Let (Ω, d) be a complete metric space and Γ be a self-mapping on it, if for all $\kappa, \mu \in \Omega$ and $\xi \in [0,1)$,

$$d(\Gamma\kappa, \Gamma\mu) \leq \xi d(\kappa, \mu), \tag{1}$$

then, Γ has a unique fixed point and the sequence $\{\Gamma^n \kappa_\circ\}_{n\in\mathbb{N}}$ is convergent to the same fixed point, for all $\kappa_\circ \in \Omega$.

Some authors have worked on the right side of the inequality (1) by replacing ξ with mappings, and others, instead of the underlying space, took more general spaces (see, for example [2–5] and references therein).

In 2012, Wardowski [6] presented a new type of contraction called an F-contraction, where $F : R^+ \to R$, and showed new fixed point results related with the F-contraction. He investigated some examples to obtain a different type of contraction in the literature.

Definition 1 ([6]). *Let (Ω, d) be a metric space. A mapping $\Gamma : \Omega \longrightarrow \Omega$ is said to be an F-contraction if there exists $F \in \Sigma$ and $\tau > 0$ such that*

$$d(\Gamma\kappa, \Gamma\mu) > 0 \;\Rightarrow\; \tau + F(d(\Gamma\kappa, \Gamma\mu)) \leq F(d(\kappa, \mu)) \;\; \forall \kappa, \mu \in \Omega, \tag{2}$$

where Σ is the set of functions $F : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:
($\Im_1$) F is strictly increasing, i.e., for all $i, j \in \mathbb{R}^+$ such that $i < j$, $F(i) < F(j)$.

($\Im_2$) *For every sequence* $\{i_n\}_{n\in N}$ *of positive numbers,* $\lim_{n\to\infty} i_n = 0$ *iff* $\lim_{n\to\infty} F(i_n) = -\infty$.
($\Im_3$) *There exists* $\lambda \in (0,1)$ *such that* $\lim_{i\to 0^+} i^\lambda F(i) = 0$.

The following functions $F_i : (0,+\infty) \longrightarrow \mathbb{R}$ for $i \in \{1,2,3,4\}$ are the elements of Σ. Furthermore, substituting these functions in (2), we obtain the following contractions known in the literature, for all $\kappa, \mu \in \Omega$ with $\alpha > 0$ and $\Gamma\kappa \neq \Gamma\mu$,

$$(\mathbf{i})\ F_1(\alpha) = \ln(\alpha), \qquad d(\Gamma\kappa,\Gamma\mu) \leq e^{-\tau} d(\kappa,\mu),$$

$$(\mathbf{ii})\ F_2(\alpha) = \ln(\alpha) + \alpha, \qquad \frac{d(\Gamma\kappa,\Gamma\mu)}{d(\kappa,\mu)} e^{d(\Gamma\kappa,\Gamma\mu)-d(\kappa,\mu)} \leq e^{-\tau},$$

$$(\mathbf{iii})\ F_3(\alpha) = \frac{-1}{\sqrt{\alpha}}, \qquad d(\Gamma\kappa,\Gamma\mu) \leq \frac{1}{\left(1+\tau\sqrt{d(\kappa,\mu)}\right)^2} d(\kappa,\mu),$$

$$(\mathbf{iv})\ F_4(\alpha) = \ln(\alpha^2 + \alpha), \qquad \frac{d(\Gamma\kappa,\Gamma\mu)(1+d(\Gamma\kappa,\Gamma\mu))}{d(\kappa,\mu)(1+d(\kappa,\mu))} \leq e^{-\tau}.$$

Remark 1. *The Equation (2) implies that*

$$d(\Gamma\kappa,\Gamma\mu) < d(\kappa,\mu),$$

that is, Γ *is a contractive for all* $\kappa,\mu \in \Omega$ *such that* $\Gamma\kappa \neq \Gamma\mu$. *Hence, every F-contraction mapping is continuous.*

Remark 2 ([7]). *Consider* $F(\alpha) = \frac{-1}{\sqrt[p]{\alpha}}$, *where* $p > 1$ *and* $\alpha > 0$. *Then* $F \in \Sigma$.

By using the concept of F-contraction, Wardowski [6] established a fixed point theorem that improves the Banach contraction principle in a different way than in the known results from the literature.

Theorem 1 ([6]). *Let* Γ *be a self-mapping on a complete metric space* (Ω, d) *satisfying the condition (2). Then* Γ *has a unique fixed point* κ^*. *Moreover, for any* $\kappa_\circ \in \Omega$, *the sequence* $\{\Gamma^n\kappa_\circ\}_{n\in\mathbb{N}}$ *is convergent to* κ^*.

In 2014, Isik [8] extended this theorem for two mappings by introducing the following lemma:

Lemma 1 ([8]). *Let* (Ω, d) *be a complete metric space. Let* Υ *and* Γ *be self-mappings on it, and suppose* $F \in \Sigma$. *If there exists* $\tau > 0$ *such that*

$$\tau + F(d(\Upsilon\kappa,\Gamma\mu)) \leq F(d(\kappa,\mu)), \tag{3}$$

for all $\kappa,\mu \in \Omega$, *satisfying* $\min\{d(\Upsilon\kappa,\Gamma\mu), d(\kappa,\mu)\} > 0$. *Then the mappings* Υ *and* Γ *have a unique fixed point.*

The notion of F-contraction is generalized by Abbas et al. [9] to obtain certain fixed point results, Batra et al. [10,11] to extend it on graphs and alter distances, and Cosentino and Vetro [12] to introduce some fixed point consequences for Hardy–Rogers-type self-mappings in ordered and complete metric spaces.

Another direction, the coupled fixed point, was introduced and studied by Bhaskar and Lakhsmikantham [13]. They studied coupled fixed point results by using suitable contraction mappings and applied their results to show the existence of solutions for a periodic boundary value problem, so it has been a subject of interest by many researchers in this direction, (see, for example [14–21]).

Definition 2. *Let* Ω *be a nonempty set and* $\Upsilon,\Gamma : \Omega\times\Omega \to \Omega$ *be given mappings:*

(i) *An element* $(\kappa,\mu) \in \Omega\times\Omega$ *is said to be a coupled fixed point of the nonlinear mapping* $\Gamma : \Omega\times\Omega \to \Omega$ *if* $\Gamma(\kappa,\mu) = \kappa$ *and* $\Gamma(\mu,\kappa) = \mu$.

(ii) *An element* $(\kappa, \mu) \in \Omega \times \Omega$ *is said to be a coupled common fixed point of nonlinear mappings* Υ *and* Γ *if* $\Upsilon(\kappa, \mu) = \Gamma(\kappa, \mu) = \kappa$ *and* $\Upsilon(\mu, \kappa) = \Gamma(\mu, \kappa) = \mu$.

Consistent with Jachymski [22], let (Ω, d) be a metric space and $\Re$ the diagonal of $\Omega \times \Omega$. Let $G = (\Theta(G), \Xi(G))$ be a directed graph, where the set $\Theta(G)$ contains all vertices coinciding with Ω and the set $\Xi(G)$ contains all edges of the graph containing all loops, that is, $\Re \subseteq \Xi(G)$. In addition, we assume that the graph G has no parallel edges. We denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus,

$$\Xi(G^{-1}) = \{(\mu, \kappa) \in \Omega \times \Omega : (\kappa, \mu) \in \Xi(G)\}.$$

A set of coupled fixed points of a nonlinear mapping Γ is denoted by Δ, that is,

$$\Delta = \{(\kappa, \mu) \in \Omega \times \Omega : \Gamma(\kappa, \mu) = \kappa \text{ and } \Gamma(\mu, \kappa) = \mu\}.$$

Definition 3 ([23]). *A mapping* $\Gamma : \Omega \times \Omega \to \Omega$ *is called edge preserving if* $\big((\kappa, \theta) \in \Xi(G)$ *and* $(\mu, \varpi) \in \Xi(G^{-1})\big)$, *then,* $(\Gamma(\kappa, \mu), \Gamma(\theta, \varpi)) \in \Xi(G)$ *and* $(\Gamma(\mu, \kappa), \Gamma(\varpi, \theta)) \in \Xi(G^{-1})$.

Definition 4 ([23]). *The mapping* $\Gamma : \Omega \times \Omega \to \Omega$ *is called G-continuous if for all* $(\kappa, \mu) \in \Omega \times \Omega$, $(\kappa^*, \mu^*) \in \Omega \times \Omega$ *and for any positive integers sequence* $\{n_i\}_{i \in \mathbb{N}}$, *with* $\Gamma^{n_i}(\kappa, \mu) \to \kappa^*$, $\Gamma^{n_i}(\mu, \kappa) \to \mu^*$, *as* $i \to \infty$ *and* $(\Gamma^{n_i}(\kappa, \mu), \Gamma^{n_i+1}(\kappa, \mu)) \in \Xi(G)$, $(\Gamma^{n_i}(\mu, \kappa), \Gamma^{n_i+1}(\mu, \kappa)) \in \Xi(G^{-1})$, *we have that*

$$\lim_{i \to \infty} \Gamma\left(\Gamma^{n_i}(\kappa, \mu), \Gamma^{n_i}(\kappa, \mu)\right) = \Gamma(\kappa^*, \mu^*) \text{ and } \lim_{i \to \infty} \Gamma\left(\Gamma^{n_i}(\mu, \kappa), \Gamma^{n_i}(\mu, \kappa)\right) = \Gamma(\mu^*, \kappa^*).$$

Definition 5 ([23]). *Let G be a directed graph and* (Ω, d) *be a complete metric space. The triple* (Ω, d, G) *satisfies the following:*

(A_1) *If for any sequence* $\{\kappa_n\}_{n \in \mathbb{N}} \subset \Omega$ *such that* $\lim_{n \to \infty} \kappa_n = \kappa$, *and* $(\kappa_n, \kappa_{n+1}) \in \Xi(G)$, *for* $n \in \mathbb{N}$, *we have* $(\kappa_n, \kappa) \in \Xi(G)$.

(A_2) *If for any sequence* $\{\kappa_n\}_{n \in \mathbb{N}} \subset \Omega$ *with* $\lim_{n \to \infty} \kappa_n = \kappa$, *and* $(\kappa_n, \kappa_{n+1}) \in \Xi(G^{-1})$, *for* $n \in \mathbb{N}$, *we have* $(\kappa_n, \kappa) \in \Xi(G^{-1})$.

As in paper [23], we define $(\Omega \times \Omega)^\Gamma$ as:

$$(\Omega \times \Omega)^\Gamma = \left[(\kappa, \mu) \in \Omega \times \Omega : (\kappa, \Gamma(\kappa, \mu)) \in \Xi(G) \text{ and } (\mu, \Gamma(\mu, \kappa)) \in \Xi(G^{-1})\right].$$

Proposition 1 ([23]). *Consider a nonlinear edge preserving mapping* $\Gamma : \Omega \times \Omega \to \Omega$, *then:*

(i) $(\kappa, \theta) \in \Xi(G)$ *and* $(\mu, \varpi) \in \Xi(G^{-1})$ *implies* $(\Gamma^n(\kappa, \mu), \Gamma^n(\theta, \varpi)) \in \Xi(G)$ *and* $(\Gamma^n(\mu, \kappa), \Gamma^n(\varpi, \theta)) \in \Xi(G^{-1})$;

(ii) $(\kappa, \mu) \in (\Omega \times \Omega)^\Gamma$ *implies* $(\Gamma^n(\kappa, \mu), \Gamma^{n+1}(\kappa, \mu)) \in \Xi(G)$ *and* $(\Gamma^n(\mu, \kappa), \Gamma^{n+1}(\mu, \kappa)) \in \Xi(G^{-1})$ *for all* $n \in \mathbb{N}$;

(iii) $(\kappa, \mu) \in (\Omega \times \Omega)^\Gamma$ *implies* $(\Gamma^n(\kappa, \mu), \Gamma^n(\mu, \kappa)) \in (\Omega \times \Omega)^\Gamma$ *for all* $n \in \mathbb{N}$.

The structure of the article is as follows. In Section 2, we obtain some coupled fixed point results under F-contraction mappings in complete metric space without and with a directed graph. In Section 3, some application to find analytical solutions for the coupled systems of functional and nonlinear integral equations are presented. In the final section, Section 4, illustrative examples are discussed to support some of our results.

2. Coupled Fixed Point Results

Now we are going to prove our main results. We begin with the following known lemma:

Lemma 2. *Let (Ω, d) be a complete metric space, $\Omega \times \Omega$ be a cartesian product, and $d_{\max}$ be defined by*

$$d_{\max}((\kappa, \mu), (\theta, \varpi)) = \max\{d(\kappa, \theta), d(\mu, \varpi)\}.$$

Then, the pair $(\Omega \times \Omega, d_{\max})$ is a complete metric space.

Here, we'll prove the first main theorem of our paper.

Theorem 2. *Let $\Upsilon, \Gamma : \Omega \times \Omega \to \Omega$ be nonlinear continuous mappings on a complete metric space (Ω, d) such that there exists $\tau > 0$ and $F \in \Sigma$ satisfying*

$$d(\Upsilon(\kappa, \mu), \Gamma(\theta, \varpi)) > 0 \Rightarrow \tau + F(d(\Upsilon(\kappa, \mu), \Gamma(\theta, \varpi))) \leq F(\max\{d(\kappa, \theta), d(\mu, \varpi)\}), \tag{4}$$

for all $\kappa, \mu \in \Omega$. Then Υ and Γ have a unique coupled common fixed point.

Proof. Consider the mappings $\hat{A}, \hat{H} : \Omega \times \Omega \to \Omega \times \Omega$ such that

$$\hat{A}(\kappa, \mu) = (\Upsilon(\kappa, \mu), \Upsilon(\mu, \kappa)) \text{ and } \hat{H}(\kappa, \mu) = (\Gamma(\kappa, \mu), \Gamma(\mu, \kappa)).$$

Next, we check if $\hat{A}$ and $\hat{H}$ satisfy the contractive condition (3) appearing in Lemma 1 on a complete metric space $\Omega \times \Omega$ (see Lemma 2). For $\kappa, \mu, \theta, \varpi \in \Omega$, suppose that

$$\begin{aligned} d_{\max}(\hat{A}(\kappa, \mu), \hat{H}(\theta, \varpi)) &= d_{\max}((\Upsilon(\kappa, \mu), \Upsilon(\mu, \kappa)), (\Gamma(\theta, \varpi), \Gamma(\varpi, \theta))) \\ &= \max\{d(\Upsilon(\kappa, \mu), \Gamma(\theta, \varpi)), d(\Upsilon(\mu, \kappa), \Gamma(\varpi, \theta))\} > 0. \end{aligned}$$

We can distinguish two cases:
Case 1. $\max\{d(\Upsilon(\kappa, \mu), \Gamma(\theta, \varpi)), d(\Upsilon(\mu, \kappa), \Gamma(\varpi, \theta))\} = d(\Upsilon(\kappa, \mu), \Gamma(\theta, \varpi))$.
Since $d(\Upsilon(\kappa, \mu), \Gamma(\theta, \varpi)) > 0$, by using the contractive condition (4), we have

$$\begin{aligned} \tau + F(d_{\max}(\hat{A}(\kappa, \mu), \hat{H}(\theta, \varpi))) &= \tau + F(d(\Upsilon(\kappa, \mu), \Gamma(\theta, \varpi))) \\ &\leq F(\max\{d(\kappa, \theta), d(\mu, \varpi)\}) \\ &= F(\max\{d(\kappa, \mu), d(\theta, \varpi)\}. \end{aligned}$$

Case 2. $\max\{d(\Upsilon(\kappa, \mu), \Gamma(\theta, \varpi)), d(\Upsilon(\mu, \kappa), \Gamma(\varpi, \theta))\} = d(\Upsilon(\mu, \kappa), \Gamma(\varpi, \theta))$.
Since $d(\Upsilon(\mu, \kappa), \Gamma(\varpi, \theta)) > 0$, by using our assumption, we get

$$\begin{aligned} \tau + F(d_{\max}(\hat{A}(\kappa, \mu), \hat{H}(\theta, \varpi))) &= \tau + F(d(\Upsilon(\mu, \kappa), \Gamma(\varpi, \theta))) \\ &\leq F(\max\{d(\mu, \varpi), d(\kappa, \theta)\}) \\ &= F(\max\{d(\kappa, \mu), d(\theta, \varpi)\}. \end{aligned}$$

Therefore, in both cases, the contractive condition (3) appearing in Lemma 1 is satisfied. So $\hat{A}$ and $\hat{H}$ have a unique common fixed point $(\kappa^*, \mu^*) \in \Omega \times \Omega$, this means that

$$\begin{aligned} (\kappa^*, \mu^*) &= \hat{A}(\kappa^*, \mu^*) = (\Upsilon(\kappa^*, \mu^*), \Upsilon(\mu^*, \kappa^*)), \\ (\kappa^*, \mu^*) &= \hat{H}(\kappa^*, \mu^*) = (\Gamma(\kappa^*, \mu^*), \Gamma(\mu^*, \kappa^*)). \end{aligned}$$

Consequently,

$$\Upsilon(\kappa^*, \mu^*) = \Gamma(\kappa^*, \mu^*) = \kappa^* \text{ and } \Upsilon(\mu^*, \kappa^*) = \Gamma(\mu^*, \kappa^*) = \mu^*.$$

Hence, (κ^*, μ^*) is a coupled common fixed point of the mappings Υ and Γ.

Suppose that there exists another coupled common fixed point $(\kappa_1, \mu_1) \in \Omega \times \Omega$ such that

$$\Upsilon(\kappa_1, \mu_1) = \Gamma(\kappa_1, \mu_1) = \kappa_1 \text{ and } \Upsilon(\mu_1, \kappa_1) = \Gamma(\mu_1, \kappa_1) = \mu_1,$$

or equivalently,

$$(\kappa_1, \mu_1) = \hat{A}(\kappa_1, \mu_1) = \hat{H}(\kappa_1, \mu_1).$$

The uniqueness of the fixed points of $\hat{A}$ and $\hat{H}$ completes the proof. □

Taking $\Upsilon = \Gamma$ in Theorem 2, we can get the pivotal following result:

Corollary 1. *Let $\Gamma : \Omega \times \Omega \to \Omega$ be continuous mappings on a complete metric space (Ω, d) such that there exist $\tau > 0$ and $F \in \Sigma$ satisfying*

$$d(\Gamma(\kappa, \mu), \Gamma(\theta, \varpi)) > 0 \ \Rightarrow \ \tau + F(d(\Gamma(\kappa, \mu), \Gamma(\theta, \varpi))) \leq F(\max\{d(\kappa, \theta), d(\mu, \varpi)\}), \tag{5}$$

for all $\kappa, \mu \in \Omega$. Then Γ has a unique coupled common fixed point.

Now, we shall define an F-G-rational contraction mapping and some related results in a directed graph.

Definition 6. *A nonlinear mapping $\Gamma : \Omega \times \Omega \to \Omega$ is said to be an F-G-rational contraction if:*

(1) Γ is edge preserving;

(2) there exists a number $\tau > 0$ such that

$$\tau + F(d(\Gamma(\kappa, \mu), \Gamma(\theta, \varpi))) \leq F\left(\frac{d(\kappa, \theta) + d(\mu, \varpi)}{2}\right),$$

for all $(\kappa, \theta) \in \Xi(G)$, $(\mu, \varpi) \in \Xi(G^{-1})$ with $d(\Gamma(\kappa, \mu), \Gamma(\theta, \varpi)) > 0$.

Lemma 3. *Let $\Gamma : \Omega \times \Omega \to \Omega$ be an F-G-rational contraction on a metric space (Ω, d) with a directed graph G. Then for all $(\kappa, \theta) \in \Xi(G)$, $(\mu, \varpi) \in \Xi(G^{-1})$, we get*

$$d(\Gamma^n(\kappa, \mu), \Gamma^n(\theta, \varpi)) \leq F\left(\frac{d(\kappa, \theta) + d(\mu, \varpi)}{2}\right) - n\tau.$$

Proof. Suppose that $(\kappa, \theta) \in \Xi(G)$, $(\mu, \varpi) \in \Xi(G^{-1})$. Since Γ is edge preserving, we can write

$$(\Gamma(\kappa, \mu), \Gamma(\theta, \varpi)) \in \Xi(G) \text{ and } (\Gamma(\mu, \kappa), \Gamma(\varpi, \theta)) \in \Xi(G^{-1}).$$

Proposition 1 (i), leads to $(\Gamma^n(\kappa, \mu), \Gamma^n(\theta, \varpi)) \in \Xi(G)$ and $(\Gamma^n(\mu, \kappa), \Gamma^n(\varpi, \theta)) \in \Xi(G^{-1})$. By definition of Γ, we get

$$\begin{aligned} F(d(\Gamma^2(\kappa, \mu), \Gamma^2(\theta, \varpi))) &= F\left(d\left(\Gamma\left[\Gamma(\kappa, \mu), \Gamma(\mu, \kappa)\right], \Gamma\left[\Gamma(\theta, \varpi), \Gamma(\varpi, \theta)\right]\right)\right) \\ &\leq F\left(\frac{d(\Gamma(\kappa, \mu), \Gamma(\theta, \varpi)) + d(\Gamma(\mu, \kappa), \Gamma(\varpi, \theta))}{2}\right) - \tau \\ &\leq F\left(\frac{d(\kappa, \theta) + d(\mu, \varpi)}{2}\right) - 2\tau. \end{aligned}$$

Therefore, by mathematical induction, we reach the conclusion. □

Lemma 4. *Let $\Gamma : \Omega \times \Omega \to \Omega$ be an F-G-rational contraction on a complete metric space (Ω, d) with a directed graph G. Then, for each $(\kappa, \mu) \in (\Omega \times \Omega)^{\Gamma}$, there exist $(\kappa^*, \mu^*) \in \Omega \times \Omega$ such that $(\Gamma^n(\kappa, \mu))_{n \in \mathbb{N}} \to \kappa^*$and $(\Gamma^n(\mu, \kappa))_{n \in \mathbb{N}} \to \mu^*$, as $n \to \infty$.*

Proof. Consider $(\kappa,\mu) \in (\Omega\times\Omega)^\Gamma$, so $(\kappa,\Gamma(\kappa,\mu)) \in \Xi(G)$ and $(\mu,\Gamma(\mu,\kappa)) \in \Xi(G^{-1})$, by Lemma 3 and putting $\theta = \Gamma(\kappa,\mu)$, $\varpi = \Gamma(\mu,\kappa)$ and $\Lambda(\kappa,\mu) = \frac{d(\kappa,\Gamma(\kappa,\mu))+d(\mu,\Gamma(\mu,\kappa))}{2}$, then we can write

$$F\left(d\left(\Gamma^n(\kappa,\mu),\Gamma^n(\Gamma(\kappa,\mu),\Gamma(\mu,\kappa))\right)\right) \le F\left(\Lambda(\kappa,\mu)\right) - n\tau,$$

that is,

$$F\left(d\left(\Gamma^n(\kappa,\mu),\Gamma^{n+1}(\kappa,\mu)\right)\right) \le F\left(\Lambda(\kappa,\mu)\right) - n\tau, \tag{6}$$

and

$$F\left(d\left(\Gamma^n(\mu,\kappa),\Gamma^{n+1}(\mu,\kappa)\right)\right) \le F\left(\Lambda(\kappa,\mu)\right) - n\tau.$$

As $n\to\infty$ in (6), we obtain that

$$\lim_{n\to\infty} F\left(d\left(\Gamma^n(\kappa,\mu),\Gamma^{n+1}(\kappa,\mu)\right)\right) = -\infty.$$

So, by $(\Im_2)$, we obtain

$$\lim_{n\to\infty} d\left(\Gamma^n(\kappa,\mu),\Gamma^{n+1}(\kappa,\mu)\right) = 0. \tag{7}$$

It follows from the axiom $(\Im_3)$ that

$$\lim_{n\to\infty}\left(d\left(\Gamma^n(\kappa,\mu),\Gamma^{n+1}(\kappa,\mu)\right)\right)^\lambda \times d\left(\Gamma^n(\kappa,\mu),\Gamma^{n+1}(\kappa,\mu)\right) = 0. \tag{8}$$

By (6), for all $n\in\mathbb{N}$, we obtain

$$\begin{aligned}&\left(d\left(\Gamma^n(\kappa,\mu),\Gamma^{n+1}(\kappa,\mu)\right)\right)^\lambda\left[F\left(d\left(\Gamma^n(\kappa,\mu),\Gamma^{n+1}(\kappa,\mu)\right)\right) - F\left(\Lambda(\kappa,\mu)\right)\right]\\ \le\ & -n\left(d\left(\Gamma^n(\kappa,\mu),\Gamma^{n+1}(\kappa,\mu)\right)\right)^\lambda \tau \le 0.\end{aligned} \tag{9}$$

Take in a count (7), (8), and passing $n\to\infty$ in (9), one observes that

$$\lim_{n\to\infty}\left(d\left(\Gamma^n(\kappa,\mu),\Gamma^{n+1}(\kappa,\mu)\right)\right)^\lambda = 0. \tag{10}$$

By (10), there exists $n_\circ\in\mathbb{N}$ such that $n\left(d\left(\Gamma^n(\kappa,\mu),\Gamma^{n+1}(\kappa,\mu)\right)\right)^\lambda \le 1$, for all $n\ge n_\circ$, or

$$d\left(\Gamma^n(\kappa,\mu),\Gamma^{n+1}(\kappa,\mu)\right) \le \frac{1}{n^{\frac{1}{\lambda}}},\ \forall n\ge n_\circ. \tag{11}$$

Using (11), for $m>n\ge n_\circ$, we have

$$\begin{aligned} d\left(\Gamma^n(\kappa,\mu),\Gamma^m(\kappa,\mu)\right) &\le d\left(\Gamma^n(\kappa,\mu),\Gamma^{n+1}(\kappa,\mu)\right) + \ldots + d\left(\Gamma^{m-1}(\kappa,\mu),\Gamma^m(\kappa,\mu)\right)\\ &\le \sum_{n\ge n_\circ}^{\infty}\frac{1}{n^{\frac{1}{\lambda}}}.\end{aligned}$$

The convergence of the series $\sum_{n\ge n_\circ}^{\infty}\frac{1}{n^{\frac{1}{\lambda}}}$ leads to $\lim_{n,m\to\infty} d\left(\Gamma^n(\kappa,\mu),\Gamma^m(\kappa,\mu)\right) = 0$. Similarly, we can write $\lim_{n,m\to\infty} d\left(\Gamma^n(\mu,\kappa),\Gamma^m(\mu,\kappa)\right) = 0$. Therefore $\left(\Gamma^n(\kappa,\mu)\right)_{n\in\mathbb{N}}$ and $\left(\Gamma^n(\mu,\kappa)\right)_{n\in\mathbb{N}}$ are Cauchy sequences in Ω. The proof is terminated by the completeness of (Ω,d). □

Theorem 3. *Let $\Gamma:\Omega\times\Omega\to\Omega$ be an F-G-rational contraction on a complete metric space (Ω,d) with a directed graph G. Consider that*

(i) *Γ is G-continuous; or*

(ii) *the triple (Ω, d, G) satisfies the conditions that (A_1), (A_2), and F are continuous. Then $\Delta \neq \emptyset$ iff $(\Omega \times \Omega)^\Gamma \neq \emptyset$;*

(iii) *if $(\kappa^*, \tau^*) \in \Lambda$ with $(\kappa^*, \tau^*) \in \Xi(G)$ and $(\mu^*, \kappa^*) \in \Xi(G^{-1})$, then $\kappa^* = \mu^*$.*

Proof. Consider $\Delta \neq \emptyset$. Then there exists $(\kappa^*, \tau^*) \in \Delta$ such that $(\kappa^*, \Gamma(\kappa^*, \mu^*)) = (\kappa^*, \kappa^*) \in \Re \subset \Xi(G)$ and $(\mu^*, \Gamma(\mu^*, \kappa^*)) = (\mu^*, \mu^*) \in \Re \subset \Xi(G^{-1})$. Hence $(\kappa^*, \Gamma(\kappa^*, \mu^*)) \in \Xi(G)$ and $(\mu^*, \Gamma(\mu^*, \kappa^*)) \in \Xi(G^{-1})$. This yields $(\Omega \times \Omega)^\Gamma \neq \emptyset$.

On the other hand, let $(\Omega \times \Omega)^\Gamma \neq \emptyset$. Then there exists $(\kappa, \mu) \in (\Omega \times \Omega)^\Gamma$; this mean that $(\kappa, \Gamma(\kappa, \mu)) \in \Xi(G)$ and $(\mu, \Gamma(\mu, \kappa)) \in \Xi(G^{-1})$.

Let a sequence of positive integers $\{n_i\}_{i \in \mathbb{N}}$. Proposition 1 (ii) gives

$$\left(\Gamma^{n_i}(\kappa, \mu), \Gamma^{n_i+1}(\kappa, \mu)\right) \in \Xi(G) \text{ and } \left(\Gamma^{n_i}(\mu, \kappa), \Gamma^{n_i+1}(\mu, \kappa)\right) \in \Xi(G^{-1}). \tag{12}$$

Applying Lemma 4 on (12), there exists $\kappa^* \in \Omega$ and $\mu^* \in \Omega$ such that

$$\lim_{i \to \infty} \Gamma^{n_i}(\kappa, \mu) = \kappa^* \text{ and } \lim_{i \to \infty} \Gamma^{n_i}(\mu, \kappa) = \mu^*. \tag{13}$$

Now, we claim that the mapping Γ has a unique coupled fixed point.

(i) Consider that Γ is G-continuous. Then we have

$$\Gamma\left(\Gamma^{n_i}(\kappa, \mu), \Gamma^{n_i}(\mu, \kappa)\right) \to \Gamma(\kappa^*, \mu^*) \text{ and } \Gamma\left(\Gamma^{n_i}(\mu, \kappa), \Gamma^{n_i}(\kappa, \mu)\right) \to \Gamma(\mu^*, \kappa^*) \text{ as } i \to \infty.$$

Applying the triangle inequality, we get

$$d(\Gamma(\kappa^*, \mu^*), \kappa^*) \leq d(\Gamma(\kappa^*, \mu^*), \Gamma^{n_i+1}(\kappa, \mu)) + d(\Gamma^{n_i+1}(\kappa, \mu), \kappa^*).$$

By the continuity of Γ and (13), one can write $d(\Gamma(\kappa^*, \mu^*), \kappa^*) = 0$, which yields $\Gamma(\kappa^*, \mu^*) = \kappa^*$. By the same manner, we can prove that $\Gamma(\mu^*, \kappa^*) = \mu^*$. Therefore a nonlinear mapping Γ has a coupled fixed point (κ^*, μ^*) and $\Delta \neq \emptyset$.

(ii) Consider the triple (Ω, d, G) satisfying the properties (A_1), (A_2). Then, we have

$$d(\Gamma^n(\kappa, \mu), \kappa^*) \in \Xi(G) \text{ and } d(\Gamma^n(\mu, \kappa), \mu^*) \in \Xi(G^{-1}).$$

Now,

$$\begin{aligned} d(\Gamma(\kappa^*, \mu^*), \kappa^*) &\leq d(\Gamma(\kappa^*, \mu^*), \Gamma^{n+1}(\kappa, \mu)) + d(\Gamma^{n+1}(\kappa, \mu), \kappa^*) \\ &\leq d(\Gamma(\kappa^*, \mu^*), \Gamma(\Gamma^n(\kappa, \mu), \Gamma^n(\mu, \kappa))) + d(\Gamma^{n+1}(\kappa, \mu), \kappa^*). \end{aligned}$$

Applying the mapping F, we obtain that

$$\begin{aligned} F\left(d(\Gamma(\kappa^*, \mu^*), \kappa^*) - d(\Gamma^{n+1}(\kappa, \mu), \kappa^*)\right) &\leq F\left(d(\Gamma(\kappa^*, \mu^*), \Gamma(\Gamma^n(\kappa, \mu), \Gamma^n(\mu, \kappa)))\right) \\ &\leq F\left(\frac{d(\kappa^*, \Gamma^n(\kappa, \mu)) + d(\mu^*, \Gamma^n(\mu, \kappa))}{2}\right) - \tau. \end{aligned} \tag{14}$$

Passing $n \to \infty$ in (14), we conclude that $F\left(d(\Gamma(\kappa^*, \mu^*), \kappa^*)\right) \leq -\infty$, that is, $d(\Gamma(\kappa^*, \mu^*), \kappa^*) = 0$, i.e., $\Gamma(\kappa^*, \mu^*) = \kappa^*$. Similarly, one can prove that $\Gamma(\mu^*, \kappa^*) = \kappa^*$. Therefore, $(\kappa^*, \mu^*) \in \Delta$.

(iii) Suppose that $\kappa^* \neq \mu^*$, since $(\kappa^*, \mu^*) \in \Xi(G)$, $(\mu^*, \kappa^*) \in \Xi(G^{-1})$, and Γ is an F-G-rational contraction mapping, we get

$$\begin{aligned} F(d(\kappa^*, \mu^*)) &= F(d(\Gamma(\kappa^*, \mu^*), \Gamma(\mu^*, \kappa^*))) \\ &\leq F\left(\frac{d(\kappa^*, \mu^*) + d(\mu^*, \kappa^*)}{2}\right) - \tau \\ &\leq F\left(d(\kappa^*, \mu^*)\right) - \tau. \end{aligned}$$

A contradiction. Hence $\kappa^* = \mu^*$. $\square$

3. Applications

Fixed point theory is one of the cornerstones in the development of mathematics since it plays a basic role in applications of many branches of mathematics, especially in differential and integral equations (see, for example [24–28]).

In this section, we study the existence of solutions for functional and nonlinear integral equations using the results proved in the previous section.

Before we present applications of our results, we need the following lemma:

Lemma 5 ([29]). *Suppose $p > 1$, $\tau > 0$ and let $\varphi_\tau^p : [0, \infty) \to [0, \infty)$ be a function defined by*

$$\varphi_\tau^p(t) = \frac{t}{(1 + \tau \sqrt[p]{t})^p}.$$

Then,

- *$\varphi_\tau^p(t)$ is strictly increasing;*
- *$\varphi_\tau^p(0) = 0$ and φ_τ^p is a concave function;*
- *for $t, s \in [0, \infty)$, $\left|\varphi_\tau^p(t) - \varphi_\tau^p(s)\right| \leq \varphi_\tau^p(|t - s|)$.*

3.1. System of Functional Equations

Dynamic programming is one of the most important tools in studying dynamic economic models, especially in optimization problems and others. One of these trends uses the fixed point technique to find solutions of a system of functional equations arising in dynamic programming (for example, see [30–37]).

Consider the following system of functional equations:

$$\begin{cases} \omega(\kappa) = \sup_{\mu \in D} \left\{\sigma(\kappa, \mu) + \Pi\left(\kappa, \mu, \omega(\eta(\kappa, \mu)), \rho(\eta(\kappa, \mu))\right)\right\} \\ \rho(\kappa) = \sup_{\mu \in D} \left\{\sigma(\kappa, \mu) + \Pi\left(\kappa, \mu, \rho(\eta(\kappa, \mu)), \omega(\eta(\kappa, \mu))\right)\right\} \end{cases}, \tag{15}$$

appearing in the study of dynamic programming [38–41], where $\kappa \in S$ and S is a state space, D is a decision space, $\eta : S \times D \to S$, $\sigma : S \times D \to \mathbb{R}$, and $\Pi : S \times D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Let $\phi(S)$ denote the set of all bounded real-valued functions on a nonempty set S, and for any $\theta \in \phi(S)$, define

$$\|\theta\| = \sup_{\kappa \in S} |\theta(\kappa)|.$$

It is well known that $\phi(S)$ endowed with the sup metric

$$d(\ell, \xi) = \sup_{\kappa \in S} |\ell(\kappa) - \xi(\kappa)|,$$

for all $\ell, \xi \in \phi(S)$, is a complete metric space.

System (15) will be considered under the following conditions:

(i) $\sigma : S \times D \to \mathbb{R}$ and $\Pi : S \times D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are bounded functions;

(ii) for arbitrary points $\kappa \in S$, $\mu \in D$ and $\theta, \varpi, \theta_1, \varpi_1 \in \mathbb{R}$ such that

$$|\Pi(\kappa,\mu,\theta,\varpi) - \Pi(\kappa,\mu,\theta_1,\varpi_1)| \leq \frac{\max\{|\theta-\theta_1|, |\varpi-\varpi_1|\}}{\left(1+\tau\sqrt[p]{\max|\theta-\theta_1|, |\varpi-\varpi_1|\}}\right)^p}.$$

Theorem 4. *Under assumptions (i) and (ii), system (15) has a unique bounded common solution in $\phi(S) \times \phi(S)$.*

Proof. Define the operator Γ on $\phi(S)$ as

$$\Gamma(\omega,\rho)(\kappa,\mu) = \sup_{\mu\in D}\{\sigma(\kappa,\mu) + \Pi(\kappa,\mu,\omega(\eta(\kappa,\mu)),\rho(\eta(\kappa,\mu)))\},$$

for all $(\omega,\rho) \in \phi(S)$ and $\kappa \in S$. Since functions σ and Π are bounded, then Γ is well-defined.

Now, we will show that Γ satisfies the condition (5) appearing in Corollary 1 with the sup metric d. Let $(\omega,\rho), (\omega_1,\rho_1) \in \phi(S) \times \phi(S)$. Then, by (ii), we get

$$\begin{aligned}
& d(\Gamma(\omega,\rho),\Gamma(\omega_1,\rho_1)) \\
= {} & \sup_{\kappa\in S}|\Gamma(\omega,\rho)(\kappa) - \Gamma(\omega_1,\rho_1)(\kappa)| \\
= {} & \sup_{\kappa\in S}\Big|\sup_{\mu\in D}\{\sigma(\kappa,\mu)+\Pi(\kappa,\mu,\omega(\eta(\kappa,\mu)),\rho(\eta(\kappa,\mu)))\} \\
& - \sup_{\mu\in D}\{\sigma(\kappa,\mu)+\Pi(\kappa,\mu,\omega_1(\eta(\kappa,\mu)),\rho_1(\eta(\kappa,\mu)))\}\Big| \\
= {} & \sup_{\kappa\in S}\left\{\sup_{\mu\in D}|\Pi(\kappa,\mu,\omega(\eta(\kappa,\mu)),\rho(\eta(\kappa,\mu))) - \Pi(\kappa,\mu,\omega_1(\eta(\kappa,\mu)),\rho_1(\eta(\kappa,\mu)))|\right\} \\
\leq {} & \sup_{\kappa\in S}\left\{\sup_{\mu\in D}\left(\frac{\max\{|\omega(\eta(\kappa,\mu))-\omega_1(\eta(\kappa,\mu))|, |\rho(\eta(\kappa,\mu))-\rho_1(\eta(\kappa,\mu))|\}}{\left(1+\tau\sqrt[p]{\max\{|\omega(\eta(\kappa,\mu))-\omega_1(\eta(\kappa,\mu))|, |\rho(\eta(\kappa,\mu))-\rho_1(\eta(\kappa,\mu))|\}}\right)^p}\right)\right\} \\
\leq {} & \sup_{\kappa\in S}\left(\frac{\max\{\|\omega(\eta(\kappa,\mu))-\omega_1(\eta(\kappa,\mu))\|, \|\rho(\eta(\kappa,\mu))-\rho_1(\eta(\kappa,\mu))\|\}}{\left(1+\tau\sqrt[p]{\max\{\|\omega(\eta(\kappa,\mu))-\omega_1(\eta(\kappa,\mu))\|, \|\rho(\eta(\kappa,\mu))-\rho_1(\eta(\kappa,\mu))\|\}}\right)^p}\right) \\
\leq {} & \frac{\max\{d(\omega,\omega_1), d(\rho,\rho_1)\}}{\left(1+\tau\sqrt[p]{\max\{d(\omega,\omega_1), d(\rho,\rho_1)\}}\right)^p},
\end{aligned}$$

where we have used the nondecreasing character of φ_τ^p (Lemma 5). Therefore,

$$d(\Gamma(\omega,\rho),\Gamma(\omega_1,\rho_1)) \leq \frac{\max\{d(\omega,\omega_1), d(\rho,\rho_1)\}}{\left(1+\tau\sqrt[p]{\max\{d(\omega,\omega_1), d(\rho,\rho_1)\}}\right)^p}.$$

This yields that

$$\sqrt[p]{d(\Gamma(\omega,\rho),\Gamma(\omega_1,\rho_1))} \leq \frac{\sqrt[p]{\max\{d(\omega,\omega_1), d(\rho,\rho_1)\}}}{1+\tau\sqrt[p]{\max\{d(\omega,\omega_1), d(\rho,\rho_1)\}}},$$

or equivalently,

$$\frac{1+\tau\sqrt[p]{\max\{d(\omega,\omega_1), d(\rho,\rho_1)\}}}{\sqrt[p]{\max\{d(\omega,\omega_1), d(\rho,\rho_1)\}}} \leq \frac{1}{\sqrt[p]{d(\Gamma(\omega,\rho),\Gamma(\omega_1,\rho_1))}}.$$

This leads to

$$\frac{1}{\sqrt[p]{\max\{d(\omega,\omega_1),d(\rho,\rho_1)\}}}+\tau\leq\frac{1}{\sqrt[p]{d(\Gamma(\omega,\rho),\Gamma(\omega_1,\rho_1))}},$$

and it follows that

$$\tau-\frac{1}{\sqrt[p]{d(\Gamma(\omega,\rho),\Gamma(\omega_1,\rho_1))}}\leq-\frac{1}{\sqrt[p]{\max\{d(\omega,\omega_1),d(\rho,\rho_1)\}}}.$$

This tells us that Γ satisfies the contractive condition (5) with $F(\alpha)=\frac{-1}{\sqrt[p]{\alpha}}\in\Sigma$ (Remark 2). By Corollary 1, there exists a unique coupled fixed point of Γ. That is, the Equations (15) have a unique bounded solution in $\phi(S)\times\phi(S)$. $\square$

3.2. Nonlinear Integral Equations

In this part, we deal with Volterra-type integral equations. This part is divided into two parts: in the first part, we will apply the result of Corollary 1 to prove the existence and uniqueness of solutions of a system of nonlinear integral equations in complete metric space. In the second part, we will discuss at least one solution for a different system of nonlinear integral equations by using the results of Theorem 3 in a directed graph under the same space.

Now, consider the following first system of nonlinear integral equations:

$$\begin{cases}\kappa(t)=\hbar(t)+\int\limits_0^1\chi(t,s,\kappa(s),\mu(s))ds\\ \mu(t)=\hbar(t)+\int\limits_0^1\chi(t,s,\mu(s),\kappa(s))ds\end{cases},\tag{16}$$

where $\kappa(t)$ and $\mu(t)$ are unknown variables and $\hbar(t)$ is the deterministic free term defined for all $t\in[0,1]$.

System (16) will be considered under the following conditions:

(1) $\hbar(t)\in C[0,1]$, where $C[0,1]$ is the space of all continuous functions on $[0,1]$.
(2) $\chi:[0,1]\times[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying

$$|\chi(t,s,\theta,\varpi)-\chi(t,s,\theta_1,\varpi_1)|\leq\frac{\max\{|\theta-\theta_1|,|\varpi-\varpi_1|\}}{\left(1+\tau\sqrt[p]{\max\{|\theta-\theta_1|,|\varpi-\varpi_1|\}}\right)^p},$$

for any $t,s\in[0,1]$ and $\theta,\varpi,\theta_1,\varpi_1\in\mathbb{R}$, where $\tau>0$ and $p>1$.

The following results (see [25]) are important in the sequel.

Suppose that $\kappa\in C[0,1]$ and let G_κ be a function such that $G_\kappa\in C[0,1]$ and for $\kappa,\mu\in C[0,1]$,

$$\begin{aligned}d(G_\kappa,G_\mu)&=\sup\{|G_\kappa(t)-G_\mu(t)|:t\in[0,1]\}\\&\leq d(\kappa,\mu)=\sup\{|\kappa(t)-\mu(t)|:t\in[0,1]\}.\end{aligned}$$

Theorem 5. *Under conditions* (1) *and* (2), *system (16) has a unique solution* $(\kappa^*,\mu^*)\in C[0,1]\times C[0,1]$.

Proof. For $\kappa,\mu\in C[0,1]$ and $t\in[0,1]$, we define $\Gamma(\kappa,\mu)$ by

$$\Gamma(\kappa,\mu)(t)=\hbar(t)+\int\limits_0^1\chi(t,s,\kappa(s),\mu(s))ds.$$

In virtue of (1) and (2) and the fact that the operator G is continuous on $C[0,1]$, it is clear that if $\kappa, \mu \in C[0,1]$, then $\Gamma(\kappa, \mu) \in C[0,1]$. Therefore,

$$\Gamma : C[0,1] \times C[0,1] \to C[0,1].$$

Next, we check that Γ satisfies the contractive condition (5) appearing in Corollary 1.

In fact, suppose that $d(\Gamma(\kappa, \mu), \Gamma(\theta, \varpi)) > 0$, then for all $t \in [0,1]$, one can write

$$\begin{aligned}
& |\Gamma(\kappa,\mu)(t) - \Gamma(\theta,\varpi)(t)| \\
= & \left| \int_0^1 \chi(t,s,\kappa(s),\mu(s))ds - \int_0^1 \chi(t,s,\theta(s),\varpi(s))ds \right| \\
= & \left| \int_0^1 \chi(t,s,G_\kappa(s),\mu(s))ds - \int_0^1 \chi(t,s,G_\theta(s),\varpi(s))ds \right| \\
\leq & \int_0^1 |\chi(t,s,G_\kappa(s),\mu(s)) - \chi(t,s,G_\theta(s),\varpi(s))| \, ds \\
\leq & \int_0^1 \frac{\max\{|G_\kappa(s) - G_\theta(s)|, |\mu(s) - \varpi(s)|\}}{\left(1 + \tau \sqrt[p]{\max\{|G_\kappa(s) - G_\theta(s)|, |\mu(s) - \varpi(s)|\}}\right)^p} ds \\
\leq & \int_0^1 \frac{\max\{d(G_\kappa(s), G_\theta(s)), d(\mu(s), \varpi(s))\}}{\left(1 + \tau \sqrt[p]{\max\{d(G_\kappa(s), G_\theta(s)), d(\mu(s), \varpi(s))\}}\right)^p} ds \\
\leq & \int_0^1 \frac{\max\{d(\kappa,\theta), d(\mu(s), \varpi(s))\}}{\left(1 + \tau \sqrt[p]{\max\{d(\kappa,\theta), d(\mu(s), \varpi(s))\}}\right)^p} ds \\
= & \frac{\max\{d(\kappa,\theta), d(\mu,\varpi)\}}{\left(1 + \tau \sqrt[p]{\max\{d(\kappa,\theta), d(\mu,\varpi)\}}\right)^p},
\end{aligned}$$

where we have used the nondecreasing character of φ_τ^p (Lemma 5) and the fact that $d(G_\kappa, G_\mu) \leq d(\kappa, \mu)$. Therefore,

$$d(\Gamma(\kappa,\mu), \Gamma(\theta,\varpi)) \leq \frac{\max\{d(\kappa,\theta), d(\mu,\varpi)\}}{\left(1 + \tau \sqrt[p]{\max\{d(\kappa,\theta), d(\mu,\varpi)\}}\right)^p}.$$

This yields that

$$\sqrt[p]{d(\Gamma(\kappa,\mu), \Gamma(\theta,\varpi))} \leq \frac{\sqrt[p]{\max\{d(\kappa,\theta), d(\mu,\varpi)\}}}{1 + \tau \sqrt[p]{\max\{d(\kappa,\theta), d(\mu,\varpi)\}}},$$

or equivalently,

$$\frac{1 + \tau \sqrt[p]{\max\{d(\kappa,\theta), d(\mu,\varpi)\}}}{\sqrt[p]{\max\{d(\kappa,\theta), d(\mu,\varpi)\}}} \leq \frac{1}{\sqrt[p]{d(\Gamma(\kappa,\mu), \Gamma(\theta,\varpi))}},$$

or

$$\frac{1}{\sqrt[p]{\max\{d(\kappa,\theta), d(\mu,\varpi)\}}} + \tau \leq \frac{1}{\sqrt[p]{d(\Gamma(\kappa,\mu), \Gamma(\theta,\varpi))}},$$

or

$$\tau - \frac{1}{\sqrt[p]{d(\Gamma(\kappa,\mu), \Gamma(\theta,\varpi))}} \leq -\frac{1}{\sqrt[p]{\max\{d(\kappa,\theta), d(\mu,\varpi)\}}}.$$

This tells us that the contractive condition is satisfied with $F(\alpha) = \frac{-1}{\sqrt[p]{\alpha}} \in \Sigma$ (Remark 2).

By Corollary 1, there exists a unique coupled fixed point of a mapping Γ, i.e., there exists a unique $(\kappa^*, \mu^*) \in C[0,1] \times C[0,1]$ such that for any $t \in [0,1]$,

$$\begin{cases} \kappa^*(t) = \Gamma(\kappa_\circ, \mu_\circ)(t) = \hbar(t) + \int\limits_0^1 \chi(t, s, \kappa_\circ^*(s), \mu_\circ^*(s))ds \\ \mu^*(t) = \Gamma(\mu_\circ, \kappa_\circ)(t) = \hbar(t) + \int\limits_0^1 \chi(t, s, \mu_\circ^*(s), \kappa_\circ^*(s))ds \end{cases},$$

and this completes the proof. □

Next, consider the second nonlinear system of integral equations as the form:

$$\begin{cases} \kappa(t) = \int_0^B \aleph(t,s)\chi(t,s,\kappa(s),\mu(s))ds \\ \mu(t) = \int_0^B \aleph(t,s)\chi(t,s,\mu(s),\kappa(s))ds \end{cases}, \tag{17}$$

where $t, s \in [0, B]$ with $B > 0$.

Let $\Omega = C([0, B], \mathbb{R}^n)$ with the norm, $\|\kappa\| = \max_{t\in[0,B]} |\kappa(t)|$ for all $\kappa \in C([0, B], \mathbb{R}^n)$. Suppose that G is a directed graph defined by the following:

$$\kappa, \mu \in \Omega,\ \kappa \le \mu \Leftrightarrow \kappa(t) \le \mu(t) \text{ for every } t \in [0, B].$$

Then $(\Omega, \|.\|)$ is a complete metric space equipped with G.

If we take $\Xi(G) = \{(\kappa, \mu) \in \Omega \times \Omega : \kappa \le \mu\}$, then the diagonal $\Re$ of $\Omega \times \Omega$ is included in $\Xi(G)$. On the other hand, $\Xi(G^{-1}) = \{(\kappa, \mu) \in \Omega \times \Omega : \mu \le \kappa\}$. Moreover, $(\Omega, \|.\|, G)$ has the properties (A_1) and (A_2). In this case $(\Omega \times \Omega)^\Gamma = \{(\kappa, \mu) \in \Omega \times \Omega : \kappa \le \Gamma(\kappa, \mu)$ and $\Gamma(\mu, \kappa) \le \mu\}$.

Theorem 6. *The system (17) has at least one solution as long as the following conditions are satisfied:*

(i) $\chi : [0, B] \times [0, B] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $\aleph : [0, B] \times [0, B] \to \mathbb{R}^n$ *are continuous functions such that*

$$\int_0^B \aleph(t,s)ds \le \frac{B}{\tau};$$

(ii) *for all* $\kappa, \mu, \theta, \varpi \in \mathbb{R}^n$ *with* $\kappa \le \theta$, $\mu \le \varpi$, *we have* $\chi(t, s, \kappa, \mu) \le \chi(t, s, \theta, \varpi)$ $\forall\, t, s \in [0, B]$;

(iii) *there exists* $\tau > 0$ *such that*

$$|\aleph(t,s,\kappa,\mu) - \aleph(t,s,\theta,\varpi)| \le \frac{\tau}{B} \frac{\frac{1}{2}\max\{|\kappa-\theta|, |\mu-\varpi|\}}{\left(1 + \tau \sqrt[p]{\frac{1}{2}\max\{|\kappa-\theta|, |\mu-\varpi|\}}\right)^p},$$

for any $t, s \in [0, B]$, $\kappa, \mu, \theta, \varpi \in \mathbb{R}^n$ *and* $p > 1$;

(iv) *there exists* $(\kappa_\circ, \mu_\circ) \in \Omega \times \Omega$ *such that*

$$\begin{cases} \kappa_\circ(t) = \int_0^B \aleph(t,s)\chi(t,s,\kappa_\circ(s),\mu_\circ(s))ds \\ \mu_\circ(t) = \int_0^B \aleph(t,s)\chi(t,s,\mu_\circ(s),\kappa_\circ(s))ds \end{cases},$$

where $t \in [0, B]$.

Proof. Define the mapping $\Gamma : \Omega \times \Omega \to \Omega$ as

$$\Gamma(\kappa, \mu)(t) = \int_0^B \aleph(t,s)\chi(t,s,\kappa(s),\mu(s))ds,\ t \in [0, B].$$

Now we prove that Γ is G-edge preserving. Let $\kappa, \mu, \theta, \varpi \in \Omega$ with $\kappa \leq \theta$, $\mu \leq \varpi$. Then we have

$$\begin{aligned} \Gamma(\kappa, \mu)(t) &= \int_0^B \aleph(t,s)\chi(t,s,\kappa(s),\mu(s))ds \\ &\leq \int_0^B \aleph(t,s)\chi(t,s,\theta(s),\varpi(s))ds = \Gamma(\theta,\varpi)(t) \text{ for all } t \in [0,B]. \end{aligned}$$

Similarly, $\Gamma(\mu,\kappa)(t) \leq \Gamma(\varpi,\theta)(t)$.
Next, the condition (iv) follows that

$$(\Omega \times \Omega)^{\Gamma} = \{(\kappa, \mu) \in \Omega \times \Omega : \kappa \leq \Gamma(\kappa, \mu) \text{ and } \Gamma(\mu, \kappa) \leq \mu\} \neq \emptyset.$$

Finally,

$$\begin{aligned} |\Gamma(\kappa,\mu)(t) - \Gamma(\theta,\varpi)(t)| &\leq \int_0^B \aleph(t,s)\left|\chi(t,s,\kappa(s),\mu(s)) - \chi(t,s,\theta(s),\varpi(s))\right| ds \\ &\leq \int_0^B \aleph(t,s)\left(\frac{\tau}{B}\frac{\frac{1}{2}\max\{|\kappa-\theta|,|\mu-\varpi|\}}{\left(1+\tau\sqrt[p]{\frac{1}{2}\max\{|\kappa-\theta|,|\mu-\varpi|\}}\right)^p}\right)ds \\ &\leq \frac{\frac{1}{2}\max\{|\kappa-\theta|,|\mu-\varpi|\}}{\left(1+\tau\sqrt[p]{\frac{1}{2}\max\{|\kappa-\theta|,|\mu-\varpi|\}}\right)^p} \\ &\leq \frac{\left(\frac{\|\kappa-\theta\|+\|\mu-\varpi\|}{2}\right)}{\left(1+\tau\sqrt[p]{\frac{\|\kappa-\theta\|+\|\mu-\varpi\|}{2}}\right)^p}, \text{ since } \max\{a,b\} \leq a+b \\ &= \frac{\left(\frac{d(\kappa,\theta)+d(\mu,\varpi)}{2}\right)}{\left(1+\tau\sqrt[p]{\frac{d(\kappa,\theta)+d(\mu,\varpi)}{2}}\right)^p}, \end{aligned}$$

where we have used the nondecreasing character of φ_τ^p (Lemma 5). Therefore,

$$d(\Gamma(\kappa,\mu),\Gamma(\theta,\varpi)) \leq \frac{\left(\frac{d(\kappa,\theta)+d(\mu,\varpi)}{2}\right)}{\left(1+\tau\sqrt[p]{\frac{d(\kappa,\theta)+d(\mu,\varpi)}{2}}\right)^p}.$$

This yields that

$$\sqrt[p]{d(\Gamma(\kappa,\mu),\Gamma(\theta,\varpi))} \leq \frac{\sqrt[p]{\frac{d(\kappa,\theta)+d(\mu,\varpi)}{2}}}{1+\tau\sqrt[p]{\frac{d(\kappa,\theta)+d(\mu,\varpi)}{2}}},$$

or equivalently,

$$\frac{1+\tau\sqrt[p]{\frac{d(\kappa,\theta)+d(\mu,\varpi)}{2}}}{\sqrt[p]{\frac{d(\kappa,\theta)+d(\mu,\varpi)}{2}}} \leq \frac{1}{\sqrt[p]{d(\Gamma(\kappa,\mu),\Gamma(\theta,\varpi))}},$$

or

$$\frac{1}{\sqrt[p]{\frac{d(\kappa,\theta)+d(\mu,\varpi)}{2}}} + \tau \leq \frac{1}{\sqrt[p]{d(\Gamma(\kappa,\mu),\Gamma(\theta,\varpi))}}.$$

This implies that

$$\tau - \frac{1}{\sqrt[p]{d(\Gamma(\kappa,\mu),\Gamma(\theta,\varpi))}} \leq -\frac{1}{\sqrt[p]{\frac{d(\kappa,\theta)+d(\mu,\varpi)}{2}}}.$$

This says us that Γ is an F-G-rational contraction with $F(\alpha) = \frac{-1}{\sqrt[p]{\alpha}} \in \Sigma$ (Remark 2). By Theorem 3, there exists at least the coupled fixed point of a mapping Γ that is the solution of the integral system (17). □

4. Illustrative Examples

In this section, we present some numerical examples to justify the requirements of some results.

Example 1. *Let $\Omega = [0,\infty)$ and $d(\kappa,\mu) = |\kappa-\mu|$. It is clear that (Ω,d) is a complete metric space. Define $\Upsilon,\Gamma : \Omega\times\Omega \to \Omega$ as follows:*

$$\Upsilon(\kappa,\mu) = \begin{cases} \frac{\kappa-4\mu}{5} & \text{if} \quad \kappa \geq 4\mu \\ 0 & \text{if} \quad \kappa < \mu \end{cases} \quad \text{and} \quad \Gamma(\kappa,\mu) = \begin{cases} \frac{\kappa-\mu}{5} & \text{if} \quad \kappa \geq \mu \\ 0 & \text{if} \quad \kappa < \mu \end{cases}.$$

It is obvious that Υ and Γ are continuous. Define a function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\alpha) = \ln(\alpha)$ for $\alpha > 0$. To prove the condition (4) of Theorem 2, we discuss the following cases:

(a) *If $\kappa \geq 4\mu$ and $\theta \geq \varpi$, we have $\Upsilon(\kappa,\mu) = \frac{\kappa-4\mu}{5}$ and $\Gamma(\theta,\varpi) = \frac{\theta-\varpi}{5}$. Then*

$$\begin{aligned} d(\Upsilon(\kappa,\mu),\Gamma(\theta,\varpi)) &= \left|\frac{\kappa-4\mu}{5} - \frac{\theta-\varpi}{5}\right| = \left|\frac{\kappa-\theta}{5} + \frac{\varpi-4\mu}{5}\right| \\ &\leq \left|\frac{\kappa-\theta}{5}\right| + \left|\frac{4\mu-\varpi}{5}\right| \leq \left|\frac{\kappa-\theta}{5}\right| + \left|\frac{\kappa-\theta}{5}\right| \\ &= \frac{2}{5}|\kappa-\theta| \\ &\leq \frac{2}{5}\max\{|\kappa-\theta|,|\mu-\varpi|\}. \end{aligned}$$

Applying F, we can write

$$\ln(d(\Upsilon(\kappa,\mu),\Gamma(\theta,\varpi))) \leq \ln(\frac{2}{5}\max\{|\kappa-\theta|,|\mu-\varpi|\}) = \ln(\frac{2}{5}) + \ln(\max\{|\kappa-\theta|,|\mu-\varpi|\}),$$

or equivalently,

$$\ln(\frac{5}{2}) + \ln(d(\Upsilon(\kappa,\mu),\Gamma(\theta,\varpi))) \leq \ln(\max\{|\kappa-\theta|,|\mu-\varpi|\}),$$

which implies that

$$\tau + F(d(\Upsilon(\kappa,\mu),\Gamma(\theta,\varpi))) \leq F(\max\{d(\kappa,\theta),d(\mu,\varpi)\}). \tag{18}$$

(b) *If $\kappa \geq 4\mu$ and $\theta < \varpi$, we get $\Upsilon(\kappa,\mu) = \frac{1}{5}(\kappa-\mu)$ and $\Gamma(\theta,\varpi) = 0$. Then*

$$\begin{aligned} d(\Upsilon(\kappa,\mu),\Gamma(\theta,\varpi)) &= \left|\frac{\kappa-\mu}{5} - 0\right| = \left|\frac{(\kappa-\theta)+(\varpi-\mu)+(\theta-\varpi)}{5}\right| \\ &\leq \frac{|\kappa-\theta|}{5} + \frac{|\varpi-\mu|}{5}. \end{aligned}$$

Since $a+b \leq 2\max\{a,b\}$, we get

$$d(\Upsilon(\kappa,\mu),\Gamma(\theta,\varpi)) \leq \frac{2}{5}\max\{|\kappa-\theta|,|\varpi-\mu|\},$$

and in the same way as in (a), we have (18).

(c) *If $\kappa < 4\mu$, $\theta \geq \varpi$, $\Upsilon(\kappa,\mu) = 0$ and $\Gamma(\theta,\varpi) = \frac{1}{5}(\theta - \varpi)$, we obtain the inequality (18).*

(d) *If $\kappa < \mu$, $\theta < \varpi$, $\Upsilon(\kappa,\mu) = 0$ and $\Gamma(\theta,\varpi) = 0$, it is trivial.*

According to the four cases, the mappings Υ and Γ satisfy the contractive condition (4) of Theorem 2 with $\tau = \ln(\frac{5}{2}) > 0$. Thus, $(0,0) \in \Omega \times \Omega$ is a unique coupled common fixed point of Υ and Γ.

Example 2. *Consider the following coupled system of functional equations:*

$$\begin{cases} \omega(\kappa) = \sup_{\mu\in\mathbb{R}}\left\{\arctan(\kappa + 3\,|\mu|) + \left(\frac{1}{1+\kappa^2} + \frac{1}{1+e^{\mu}} + \frac{1}{2}\frac{|\omega(\eta)|}{\left(1+8\sqrt[3]{|\omega(\eta)|}\right)^3} + \frac{1}{2}\frac{|\rho(\eta)|}{\left(1+5\sqrt[3]{|\rho(\eta)|}\right)^3}\right)\right\} \\ \rho(\kappa) = \sup_{\mu\in\mathbb{R}}\left\{\arctan(\kappa + 3\,|\mu|) + \left(\frac{1}{1+\kappa^2} + \frac{1}{1+e^{\mu}} + \frac{1}{2}\frac{|\rho(\eta)|}{\left(1+8\sqrt[3]{|\rho(\eta)|}\right)^3} + \frac{1}{2}\frac{|\omega(\eta)|}{\left(1+5\sqrt[3]{|\omega(\eta)|}\right)^3}\right)\right\} \end{cases}, \tag{19}$$

for $\kappa \in [0,1]$.

System (19) is a particular case of system (15), where $S = [0,1]$ and $D = \mathbb{R}$. It is clear that the assumption (i) of Theorem 4 is satisfied. For (ii), we have

$$\begin{aligned} &|\Pi\,(\kappa,\mu,\omega(\eta(\kappa,\mu)),\rho(\eta(\kappa,\mu))) - \Pi\,(\kappa,\mu,\omega_1(\eta(\kappa,\mu)),\rho_1(\eta(\kappa,\mu))))| \\ \leq\;& \frac{1}{2}\left|\frac{|\omega(\eta)|}{\left(1+8\sqrt[3]{|\omega(\eta)|}\right)^3} - \frac{|\omega_1(\eta)|}{\left(1+8\sqrt[3]{|\omega_1(\eta)|}\right)^3}\right| + \frac{1}{2}\left|\frac{|\rho(\eta)|}{\left(1+5\sqrt[3]{|\rho(\eta)|}\right)^3} - \frac{|\rho_1(\eta)|}{\left(1+5\sqrt[3]{|\rho_1(\eta)|}\right)^3}\right| \\ =\;& \frac{1}{2}\left|\varphi_8^3(|\omega(\eta)|) - \varphi_8^3(|\omega_1(\eta)|)\right| + \frac{1}{2}\left|\varphi_5^3(|\rho(\eta)|) - \varphi_5^3(|\rho_1(\eta)|)\right| \\ \leq\;& \frac{1}{2}\varphi_8^3(||\omega(\eta)| - |\omega_1(\eta)||) + \frac{1}{2}\varphi_5^3(||\rho(\eta)| - |\rho_1(\eta)||) \\ \leq\;& \frac{1}{2}\varphi_8^3(|\omega(\eta) - \omega_1(\eta)|) + \frac{1}{2}\varphi_5^3(|\rho(\eta) - \rho_1(\eta)|) \\ \leq\;& \frac{1}{2}\varphi_8^3(\max\{|\omega-\omega_1|,|\rho-\rho_1|\}) + \frac{1}{2}\varphi_5^3(\max\{|\omega-\omega_1|,|\rho-\rho_1|\}) \\ \leq\;& 2\times\frac{1}{2}\varphi_5^3(\max\{|\omega-\omega_1|,|\rho-\rho_1|\}) = \frac{\max\{|\omega-\omega_1|,|\rho-\rho_1|\}}{(1+5\sqrt[3]{\max\{|\omega-\omega_1|,|\rho-\rho_1|\}})^3}, \end{aligned}$$

where we used Lemma 5. Therefore (ii) holds with $\tau = 5$ and $p = 3$. By Theorem 4, system (19) has a unique solution in $\phi(S) \times \phi(S)$.

Example 3. *Consider another coupled system of nonlinear integral equations as follows:*

$$\begin{cases} \kappa(t) = e^t + \int\limits_0^1 \left(t^2 + \frac{s}{1+s} + \frac{1}{2}\frac{|\kappa(s)|}{\left(1+10\sqrt[5]{\kappa(s)}\right)^5} + \frac{1}{2}\frac{|\mu(s)|}{\left(1+7\sqrt[5]{|\mu(s)|}\right)^5}\right) ds \\ \mu(t) = e^t + \int\limits_0^1 \left(t^2 + \frac{s}{1+s} + \frac{1}{2}\frac{|\mu(s)|}{\left(1+10\sqrt[5]{\mu(s)}\right)^5} + \frac{1}{2}\frac{|\kappa(s)|}{\left(1+7\sqrt[5]{|\kappa(s)|}\right)^5}\right) ds \end{cases}, \tag{20}$$

for $t \in [0,1]$.

System (20) is a particular case of system (16), where $\hbar(t) = e^t$ and

$$\chi(t,s,\theta(s),\varpi(s)) = t^2 + \frac{s}{1+s} + \frac{1}{2}\frac{|\theta(s)|}{\left(1+10\sqrt[5]{|\theta(s)|}\right)^5} + \frac{1}{2}\frac{|\varpi(s)|}{\left(1+7\sqrt[5]{|\varpi(s)|}\right)^5}.$$

It is clear that condition (1) *of Theorem 5 is satisfied. For condition* (2), *we have*

$$
\begin{aligned}
& |\chi(t,s,\theta(s),\varpi(s)) - \chi(t,s,\theta_1(s),\varpi_1(s))| \\
\leq\ & \frac{1}{2}\left|\frac{|\theta(s)|}{\left(1+10\sqrt[5]{|\theta(s)|}\right)^5} - \frac{|\theta_1(s)|}{\left(1+10\sqrt[5]{|\theta_1(s)|}\right)^5}\right| + \frac{1}{2}\left|\frac{|\varpi(s)|}{\left(1+7\sqrt[5]{|\varpi(s)|}\right)^5} - \frac{|\varpi_1(s)|}{\left(1+7\sqrt[5]{|\varpi_1(s)|}\right)^5}\right| \\
=\ & \frac{1}{2}\left|\varphi_{10}^5(|\theta(s)|) - \varphi_{10}^5(|\theta_1(s)|)\right| + \frac{1}{2}\left|\varphi_7^5(|\varpi(s)|) - \varphi_7^5(|\varpi_1(s)|)\right| \\
\leq\ & \frac{1}{2}\varphi_{10}^5(||\theta(s)| - |\theta_1(s)||) + \frac{1}{2}\varphi_7^5(||\varpi(s)| - |\varpi_1(s)||) \\
\leq\ & \frac{1}{2}\varphi_{10}^5(|\theta(s) - \theta_1(s)|) + \frac{1}{2}\varphi_7^5(|\varpi(s) - \varpi_1(s)|) \\
\leq\ & \frac{1}{2}\varphi_{10}^5(\max\{|\theta - \theta_1|, |\varpi - \varpi_1|\}) + \frac{1}{2}\varphi_7^5(\max\{|\theta - \theta_1|, |\varpi - \varpi_1|\}) \\
\leq\ & 2 \times \frac{1}{2}\varphi_7^5(\max\{|\theta - \theta_1|, |\varpi - \varpi_1|\}) = \frac{\max\{|\theta - \theta_1|, |\varpi - \varpi_1|\}}{(1+7\sqrt[5]{\max\{|\theta - \theta_1|, |\varpi - \varpi_1|\}})^5},
\end{aligned}
$$

where we have used Lemma 5. Therefore, condition (2) *holds with* $\tau = 7$ *and* $p = 5$. *By Corollary 1, system (20) has a unique solution* $(\kappa^*, \mu^*) \in C[0,1] \times C[0,1]$.

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