mathematics

MDPI

Article

Self-Dual and LCD Codes from Kneser Graphs K(n, 2) and Generalized Quadrangles

Dean Crnković * and Ana Grbac

Faculty of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia; abaric@math.uniri.hr
* Correspondence: deanc@math.uniri.hr

Abstract: In this paper, we study self-dual and LCD codes constructed from Kneser graphs K(n, 2) and collinearity graphs of generalized quadrangles using the so-called pure and bordered construction. We determine conditions under which these codes are self-dual or LCD. Further, for the codes over $\mathbb{Z}_{2k}$, we give the conditions under which they are Type II. Moreover, we study binary and ternary self-dual and LCD codes from Kneser graphs K(n, 2) and collinearity graphs of generalized quadrangles. Furthermore, from the support designs for certain weights of some of the codes, we construct strongly regular graphs and 3-designs.

Keywords: self-dual code; LCD code; strongly regular graph; block design; Kneser graph; generalized quadrangle

MSC: 94B05; 05B05; 05E30

check for updates

Citation: Crnković, D.; Grbac, A. Self-Dual and LCD Codes from Kneser Graphs K(n, 2) and Generalized Quadrangles. *Mathematics* **2024**, *12*, 3795. https://doi.org/10.3390/math12233795

Academic Editor: Abdelmejid Bayad and Marjan Mernik

Received: 16 October 2024
Revised: 19 November 2024
Accepted: 28 November 2024
Published: 30 November 2024

1. Introduction

Dougherty, Kim and Solé [1] gave the so-called pure and bordered construction of self-dual codes from two class association schemes, generalizing the Gaborit's construction of quadratic double circulant codes given in [2]. In [3], two class association schemes are used to construct linear codes with complementary duals (LCD codes). In this paper, we give conditions under which the pure and bordered constructions from Kneser graphs K(n, 2) and collinearity graphs of generalized quadrangles give self-dual or LCD binary and ternary codes. The support designs of some of the codes constructed in this paper yield strongly regular graphs and 3-designs.

A self-dual code is a code that is equal to its dual. Many of the best codes known are self-dual. For example, self-dual ternary codes include, among others, the ternary Golay code of length 12, the quadratic residue codes and the symmetry codes. Besides their well-known relation to lattices, self-dual codes are also related to some other combinatorial and algebraic structures, such as block designs and vertex operator algebras (see [4,5]). A comprehensive study of self-dual codes can be found in [6]. The study and construction of self-dual codes is a vibrant field of research, with recent results found in [7–13].

Linear codes with complementary duals, or LCD codes, are linear codes whose intersection with their duals are trivial. They were introduced in 1992 by Massey (see [14]) and have been widely applied in information protection, electronics, and cryptography. It was shown by Sendrier that LCD meets the asymptotic Gilbert–Varshamov bound (see [15]). Carlet and Guilley studied applications of binary LCD codes against side-channel attacks (SCAs) and fault injection attacks (FIAs) (see [16]). For further research on LCD codes, we refer the reader to [17–23].

The calculations in this paper are conducted using Magma V2.25-8 [24].

The paper is organized as follows. After a brief overview of linear codes, graphs, and block designs, given in Section 2, in Section 3, we determine the conditions under which the binary and ternary codes obtained from Kneser graphs K(n, 2), using the pure and the bordered construction, are self-dual or LCD. In Section 4, we study codes obtained from collinearity graphs constructed from generalized quadrangles, using the pure and the bordered construction, and determine the conditions for these codes to be self-dual or LCD. In Section 5, we construct 3-designs and strongly regular graphs from the support designs of some of the codes. Finally, in Section 6, we summarize the results.

2. Preliminaries

A graph Γ is called simple if each edge of Γ is incident with two distinct vertices and for any two distinct vertices x and y there exists, at most, one edge incident with both x and y. If there is an edge incident with the vertices x and y, then we say that x and y are adjacent. In that case, we also say that x and y are neighbors. The number of edges incident with a vertex x is called the degree of x. A graph in which all the vertices are of the same degree k is called k-regular. A simple k-regular graph with v vertices is called a strongly regular graph (SRG) with parameters (v, k, λ, μ) if the number of common neighbors of every two distinct vertices x and y is λ, if x and y are adjacent, and μ, if they are non-adjacent. Strongly regular graphs were introduced by Bose in 1963 (see [25]). For further details on strongly regular graphs, see [26,27].

A t-(v, k, λ) design is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, where $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$, having $|\mathcal{P}| = v$ points, with the property that each element of $\mathcal{B}$ (called a block) is incident with exactly k points, and every t distinct point is incident with exactly λ blocks. A 2-(v, k, λ) design is called a block design. It is known that a t-(v, k, λ) design is also an s-(v, k, λ) design, for any s with $1 \leq s \leq t$. A t-design is called simple if it has no repeated blocks. If a t-design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is simple, then a block $B \in \mathcal{B}$ can be identified with the subset of the point set $\mathcal{P}$, consisting of the points incident with B. The 3-designs constructed in this paper are simple. For further details on t-designs, see [28,29].

A generalized quadrangle GQ(s, t) is an incidence structure $\mathcal{Q} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ and $\mathcal{L}$ are disjoint non-empty sets called points and lines, respectively, and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ is an incidence relation such that the following hold:

1. Each point is incident with $t+1$ lines ($t \geq 1$), and two distinct points are incident with at most one line.
2. Each line is incident with $s+1$ points ($s \geq 1$), and two distinct lines are incident with at most one point.
3. If p is a point and L is a line not incident with p, then there is a unique pair $(q, M) \in \mathcal{P} \times \mathcal{L}$, for which $p \, \mathcal{I} \, M \, \mathcal{I} \, q \, \mathcal{I} \, L$.

More information on generalized quadrangles can be found in [30].

A q-ary linear code C of length n and dimension k is a k-dimensional subspace of the vector space $\mathbb{F}_q^n$. The elements of C are called codewords. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{F}_q^n$. The Hamming distance between words x and y is the number $d(x, y) = |\{i : x_i \neq y_i\}|$. The minimum distance of the code C is defined by $d = \min\{d(x, y) : x, y \in C, \ x \neq y\}$. A q-ary linear code of length n and dimension k is called an $[n, k]_q$ code. An $[n, k]_q$ code of minimum distance d is called an $[n, k, d]_q$ code. The weight of a codeword x is $w(x) = d(x, 0) = |\{i : x_i \neq 0\}|$. For a linear code, $d = \min\{w(x) : x \in C, x \neq 0\}$. A code for which all codewords have weight divisible by four is called doubly even; singly even is for when all weights are even and there is at least one codeword x with $w(x) \equiv 2 \mod 4$. Let w_i denote the number of codewords of weight i in a code C. The weight distribution of C is the list $[\langle i, w_i \rangle : 0 \leq i \leq n]$. The support of a non-zero vector $x = (x_1, ..., x_n) \in \mathbb{F}_q^n$ is the set of indices of its non-zero coordinates, i.e., $\mathrm{supp}(x) = \{i | x_i \neq 0\}$. The support design of a code of length n for a given non-zero weight w is the design with points of n coordinate indices, and it blocks the supports of all codewords of weight w.

The dual code $C^\perp$ of a code C is the orthogonal complement under the standard inner product $\langle \cdot, \cdot \rangle$, i.e., $C^\perp = \{v \in \mathbb{F}_q^n | \langle v, c \rangle = 0 \text{ for all } c \in C\}$. A linear code C over $\mathbb{F}_q$ is called a Euclidean or classical LCD code if $C \cap C^\perp = \{0\}$. Usually, we just write LCD code in this instance. It follows that the dual of an LCD code is an LCD code. A code C

is self-orthogonal if $C \subseteq C^{\perp}$ and self-dual if equality is attained. We say the $k \times n$ matrix comprised of rows that span C generates, or is a generator matrix of a $[n,k]_q$ code C. Let G be a generator matrix of a linear code C. The code C is (Euclidean) self-orthogonal if and only if $GG^{\perp} = 0$ over the field $\mathbb{F}_q$, where 0 is the zero matrix. A self-orthogonal $[n,k]_q$ code is self-dual if and only if $k = \frac{n}{2}$. A code C is an LCD code if and only if $\det(GG^{\perp}) \neq 0$ in $\mathbb{F}_q$ (see [14]). Binary self-dual codes which are doubly even are called Type II codes; otherwise, they are called Type I. Binary Type I codes are singly even. More generally, a self-dual code C over $\mathbb{Z}_{2m}$ is called Type II if the Euclidean weight of each codeword of C is divisible by $4m$. Note that the Euclidean weight of a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_{2m}^n$ is $w_E(x) = \sum_{i=1}^{n} w_E(x_i)$, where $w_E(a) = (\min(a, 2m-a))^2$, for $a \in \mathbb{Z}_{2m}$.

In 2002, P. Gaborit introduced two constructions of self-dual codes using quadratic residues, the so-called pure and bordered construction, as a generalization of the Pless symmetry codes (see [2]). In [1], the authors further generalized this concept by studying the codes obtained by the pure and bordered constructions from two-class association schemes, namely from strongly regular graphs and doubly regular tournaments.

Let A be an adjacency matrix of a strongly regular graph with parameters (v,k,λ,μ) and $\overline{A} = J_v - I_v - A$, where I_v and J_v are the identity and the all-one matrix of order v, respectively. For arbitrary scalars $r,s,t \in R$, where R is a finite commutative ring with identity, let $Q^R(r,s,t) = (rI_v + sA + t\overline{A})$. The matrix $P^R(r,s,t) = \left[\begin{array}{c|c} I_q & Q^R(r,s,t) \end{array}\right]$ generates a $[2v,v]$ code over R and the matrix

$$B^R(r,s,t) = \left(\begin{array}{c|c|c|c} 1 & 0\ldots0 & \alpha & \beta\ldots\beta \\ \hline 0 & & \gamma & \\ \vdots & I_v & \vdots & Q^R(r,s,t) \\ 0 & & \gamma & \end{array}\right),$$

where $\alpha, \beta, \gamma \in R$, generates a $[2v+2, v+1]$ code over R. The constructions of codes spanned by the matrices $P^R(r,s,t)$ and $B^R(r,s,t)$ are called the pure and the bordered constructions, respectively.

The following statements can be found in [1], Theorems 3.4 and 3.5.

Theorem 1. *Let Γ be a strongly regular graph with parameters (v,k,λ,μ). The code spanned by $P^R(r,s,t)$ formed from Γ is Euclidean self-dual over R if and only if*

$$\begin{aligned} r^2 + s^2k + t^2(v-k-1) &= -1, \\ 2rs - 2st + s^2\lambda + 2st(k-\lambda) + t^2(v-2k+\lambda) &= 0, \\ 2rt + s^2\mu + 2st(k-\mu) + t^2(v-2k+\mu-2) &= 0. \end{aligned}$$

The self-dual code spanned by $P^{\mathbb{Z}_{2\ell}}(r,s,t)$ is Type II if and only if

$$1 + r^2 + s^2k + t^2(v-k-1) \equiv 0 \mod 4\ell.$$

Further, the code spanned by $B^R(r,s,t)$ formed from Γ is Euclidean self-dual over R if and only if

$$\begin{aligned} r^2 + s^2k + t^2(v-k-1) &= -(1+\gamma^2), \\ 2rs - 2st + s^2\lambda + 2st(k-\lambda) + t^2(v-2k+\lambda) &= -\gamma^2, \\ 2rt + s^2\mu + 2st(k-\mu) + t^2(v-2k+\mu-2) &= -\gamma^2, \\ 1 + \alpha^2 + v\beta^2 &= 0, \\ \alpha\gamma + \beta(r + sk + t(v-k-1)) &= 0. \end{aligned}$$

The self-dual code spanned by $B_q^{\mathbb{Z}_{2\ell}}(r,s,t)$ is Type II if and only if

$$1+\gamma^2+r^2+s^2k+t^2(v-k-1)\equiv 0 \mod 4\ell$$

and

$$1+\alpha^2+v\beta^2\equiv 0 \mod 4\ell.$$

The following results are given in [3].

Theorem 2. *Let Γ be a strongly regular graph with parameters (v,k,λ,μ). If for some $x\in\mathbb{F}_q$, $x\neq 0$, it holds that*

$$\begin{aligned} r^2+s^2k-t^2-t^2k+t^2v&=x-1,\\ 2rs+s^2\lambda-2st-2st\lambda+t^2\lambda+2stk+t^2v-2t^2k&=0,\\ 2rt+s^2\mu-2st\mu+t^2\mu+2stk+t^2v-2t^2-2t^2k&=0,\end{aligned}$$

then the code spanned by $P^{\mathbb{F}_q}(r,s,t)$ formed from Γ is Euclidean LCD over $\mathbb{F}_q$. Further, if for some scalars $x,y\in\mathbb{F}_q$, $x\neq 0$, $y\neq 0$, it holds that

$$\begin{aligned} r^2+s^2k-t^2-t^2k+t^2v&=x-1-\gamma^2,\\ 2rs+s^2\lambda-2st-2st\lambda+t^2\lambda+2stk+t^2v-2t^2k&=-\gamma^2,\\ 2rt+s^2\mu-2st\mu+t^2\mu+2stk+t^2v-2t^2-2t^2k&=-\gamma^2,\\ 1+\alpha^2+v\beta^2&=y,\\ \alpha\gamma+\beta(r+sk+t(v-k-1))&=0,\end{aligned}$$

then the code spanned by $B^{\mathbb{F}_q}(r,s,t)$ formed from Γ is Euclidean LCD over $\mathbb{F}_q$.

In the sequel, we will study codes constructed by the pure and the bordered constructions from Kneser graphs K(n, 2), and collinearity graphs constructed from generalized quadrangles.

3. Codes Obtained from Kneser Graphs

The Kneser graph K(n, k) is the graph that has k-element subsets of an n-element set as the vertex set, and two vertices are adjacent when the intersection of the two corresponding subsets is empty. When $k=2$, the Kneser graph K(n, 2) is a strongly regular graph with parameters $\left(\binom{n}{2},\binom{n-2}{2},\binom{n-4}{2},\binom{n-3}{2}\right)$, which is the complement of the triangular graph $T(n)$.

The matrices $P^R(r,s,t)$ and $B^R(r,s,t)$ formed from the Kneser graph K(n, 2) will be denoted by $P_n^R(r,s,t)$ and $B_n^R(r,s,t)$, respectively.

3.1. Self-Dual Codes from Kneser Graphs

As consequences of Theorem 1, we have the following statements.

Theorem 3. *The code generated by $P_n^R(r,s,t)$ formed from an adjacency matrix of a Kneser graph $K(n,2)$ is Euclidean self-dual over R if and only if*

$$\begin{aligned} r^2+\tfrac{1}{2}(n-2)(s^2(n-3)+4t^2)&=-1\\ 4rs+8st(n-4)+s^2(n-4)(n-5)+8t^2&=0\\ 4rt+4st(n-3)+s^2(n-3)(n-4)+2t^2(n-2)&=0.\end{aligned}$$

The self-dual code generated by $P_n^{\mathbb{Z}_{2\ell}}(r,s,t)$ is Type II if and only if

$$1+r^2+\tfrac{1}{2}(n-2)(s^2(n-3)+4t^2)\equiv 0 \mod 4\ell.$$

Proof. It is stated in Theorem 1 that the code spanned by $P^R(r,s,t)$ formed from a strongly regular graph Γ with parameters (v,k,λ,μ) is Euclidean self-dual over R if and only if

$$\begin{aligned} r^2+s^2k+t^2(v-k-1)&=-1,\\ 2rs-2st+s^2\lambda+2st(k-\lambda)+t^2(v-2k+\lambda)&=0,\\ 2rt+s^2\mu+2st(k-\mu)+t^2(v-2k+\mu-2)&=0, \end{aligned}$$

and that the self-dual code spanned by $P^{\mathbb{Z}_{2\ell}}(r,s,t)$ is Type II if and only if

$$1+r^2+s^2k+t^2(v-k-1)\equiv 0 \mod 4\ell.$$

Applying these equations to the parameters $\left(\binom{n}{2},\binom{n-2}{2},\binom{n-4}{2},\binom{n-3}{2}\right)$ of the Kneser graphs $K(n,2)$, one obtains the equations from this theorem. □

Theorem 4. *The code generated by $B_n^R(r,s,t)$ formed from an adjacency matrix of a Kneser graph $K(n,2)$ is Euclidean self-dual over R if and only if*

$$\begin{aligned} r^2+\tfrac{1}{2}(n-2)(s^2(n-3)+4t^2)&=-(1+\gamma^2)\\ 4rs+8st(n-4)+s^2(n-4)(n-5)+8t^2&=-\gamma^2\\ 4rt+4st(n-3)+s^2(n-3)(n-4)+2t^2(n-2)&=-\gamma^2\\ 1+\alpha^2+\tfrac{1}{2}n(n-1)\beta^2&=0\\ \alpha\gamma+\beta(r+\tfrac{1}{2}(n-2)(s(n-3)+4t))&=0. \end{aligned}$$

The self-dual code generated by $B_n^{\mathbb{Z}_{2\ell}}(r,s,t)$ is Type II if and only if

$$1+\gamma^2+r^2+\tfrac{1}{2}(n-2)(s^2(n-3)+4t^2)\equiv 0 \mod 4\ell$$

and

$$1+\alpha^2+\tfrac{1}{2}n(n-1)\beta^2\equiv 0 \mod 4\ell.$$

Proof. As stated in Theorem 1, the code spanned by $B^R(r,s,t)$ formed from a strongly regular graph Γ with parameters (v,k,λ,μ) is Euclidean self-dual over R if and only if

$$\begin{aligned} r^2+s^2k+t^2(v-k-1)&=-(1+\gamma^2),\\ 2rs-2st+s^2\lambda+2st(k-\lambda)+t^2(v-2k+\lambda)&=-\gamma^2,\\ 2rt+s^2\mu+2st(k-\mu)+t^2(v-2k+\mu-2)&=-\gamma^2,\\ 1+\alpha^2+v\beta^2&=0,\\ \alpha\gamma+\beta(r+sk+t(v-k-1))&=0, \end{aligned}$$

and the self-dual code spanned by $B_q^{\mathbb{Z}_{2\ell}}(r,s,t)$ is Type II if and only if

$$1+\gamma^2+r^2+s^2k+t^2(v-k-1)\equiv 0 \mod 4\ell$$

and

$$1+\alpha^2+v\beta^2\equiv 0 \mod 4\ell.$$

Applying the above given equations to the parameters $\left(\binom{n}{2},\binom{n-2}{2},\binom{n-4}{2},\binom{n-3}{2}\right)$ of the Kneser graphs $K(n,2)$, one obtains the equations from this theorem. □

Below, we study binary and ternary self-dual codes constructed from Kneser graphs K$(n,2)$ in more detail.

3.1.1. Self-Dual Binary Codes from Kneser Graphs $\mathrm{K}(n,2)$

In Tables 1 and 2, we give the conditions under which the constructions $P_n^{\mathbb{F}_2}(r,s,t)$ and $B_n^{\mathbb{F}_2}(r,s,t)$ from Kneser graphs K(n, 2) yield self-dual binary codes. Further, we give conditions under which the self-dual binary codes are Type II.

Table 1. Binary codes from Kneser graphs K(n, 2), pure construction.

r	s	t	$P_n^{\mathbb{F}_2}(r,s,t)$	Type II
0	0	1	Never	-
0	1	0	$n \equiv 0 \mod 4$	$n \equiv 0 \mod 8$
0	1	1	$n \equiv 0, 1 \mod 4$	$n \equiv 0 \mod 8$
1	0	0	Always	Never
1	0	1	$n \equiv 0 \mod 2$	Never
1	1	0	Never	-
1	1	1	$n \equiv 2, 3 \mod 4$	$n \equiv 3, 6 \mod 8$

Table 2. Binary codes from Kneser graphs K(n, 2), bordered construction.

r	s	t	$B_n^{\mathbb{F}_2}(r,s,t)$	Type II
0	0	1	Never	-
0	1	0	$n \equiv 0 \mod 4\,; \alpha = 1, \gamma = 0$	Never
			$n \equiv 2 \mod 4\,; \alpha = 0, \beta = \gamma = 1$	$n \equiv 6 \mod 8$
0	1	1	$n \equiv 0, 1 \mod 4\,; \alpha = 1, \beta = \gamma = 0$	Never
			$n \equiv 2, 3 \mod 4\,; \alpha = 0, \beta = \gamma = 1$	$n \equiv 3, 6 \mod 8$
1	0	0	$\alpha = 1, \beta = \gamma = 0$	Never
1	0	1	$n \equiv 0 \mod 2\,; \alpha = 1, \beta = \gamma = 0$	Never
1	1	0	Never	-
1	1	1	Never	-

3.1.2. Self-Dual Ternary Codes from Kneser Graphs $\mathrm{K}(n,2)$

After examining all cases for parameters r, s, t, we obtain that there are no cases under which the constructions from Kneser graphs K(n, 2) yield self-dual ternary codes, using pure construction. Further, it turns out that the only case in which we can obtain self-dual ternary codes using bordered construction is the following:

- $B_n^{\mathbb{F}_3}(r,s,t), r \neq 0, s \neq 0, t \neq 0$ and $n \equiv 2 \mod 3\,; \alpha\gamma + \beta r = 0, 1 + \alpha^2 + \beta^2 = 0, st = 1.$

3.2. LCD Codes Obtained from Kneser Graphs

Theorems 5 and 6 can be proven by applying Theorem 2 to the parameters of Kneser graphs $K(n,2)$, similar to the way Theorems 3 and 4 are proven by applying Theorem 1.

Theorem 5. *Let Γ be a Kneser graph $K(n,2)$. If for some $x \in \mathbb{F}_q$, $x \neq 0$, it holds that*

$$\begin{aligned} r^2 + \tfrac{1}{2}(n-2)(s^2(n-3) + 4t^2) &= x - 1, \\ 4rs + 8st(n-4) + s^2(n-4)(n-5) + 8t^2 &= 0, \\ 4rt + 4st(n-3) + s^2(n-3)(n-4) + 2t^2(n-2) &= 0, \end{aligned}$$

then the code generated by $P_n^{\mathbb{F}_q}(r,s,t)$ formed from an adjacency matrix of Γ is Euclidean LCD over $\mathbb{F}_q$.

Theorem 6. *Let Γ be a Kneser graph $K(n,2)$. If for some scalars $x, y \in \mathbb{F}_q$, $x \neq 0$, $y \neq 0$, it holds that*

$$\begin{aligned} r^2 + \tfrac{1}{2}(n-2)(s^2(n-3)+4t^2) &= x - 1 - \gamma^2, \\ 4rs + 8st(n-4) + s^2(n-4)(n-5) + 8t^2 &= -\gamma^2, \\ 4rt + 4st(n-3) + s^2(n-3)(n-4) + 2t^2(n-2) &= -\gamma^2, \\ 1 + \alpha^2 + \tfrac{1}{2}n(n-1)\beta^2 &= y, \\ \alpha\gamma + \beta(r + \tfrac{1}{2}(n-2)(s(n-3)+4t)) &= 0, \end{aligned}$$

then the code generated by $B_n^{\mathbb{F}_q}(r,s,t)$ formed from an adjacency matrix of Γ is Euclidean LCD over $\mathbb{F}_q$.

Below, we study LCD codes spanned by $P_n^{\mathbb{F}_2}(r,s,t)$, $B_n^{\mathbb{F}_2}(r,s,t)$, $P_n^{\mathbb{F}_3}(r,s,t)$ and $B_n^{\mathbb{F}_3}(r,s,t)$ constructed from Kneser graphs K(n, 2) in more detail.

3.2.1. LCD Binary Codes from Kneser Graphs K$(n,2)$

In Table 3, we present conditions under which the constructions from Kneser graphs K(n, 2) produce LCD binary codes.

Table 3. LCD binary codes from Kneser graphs K(n, 2).

r	s	t	$P_n^{\mathbb{F}_2}(r,s,t)$	$B_n^{\mathbb{F}_2}(r,s,t)$
0	0	1	$n \equiv 0 \mod 2$	$n \equiv 0 \mod 2\,;\, \gamma = 0,$ $\alpha + \frac{1}{2}n(n-1)\beta = 0$
0	1	0	Never	Never
0	1	1	Never	Never
1	0	0	Never	Never
1	0	1	Never	Never
1	1	0	$n \equiv 0 \mod 4$	$n \equiv 0 \mod 4\,;\, \alpha = \gamma = 0$ $n \equiv 2 \mod 4\,;\, \gamma = 1, \alpha + \beta = 0$
1	1	1	$n \equiv 0,\ 1 \mod 4$	$n \equiv 0,\ 1 \mod 4\,;\, \alpha = \gamma = 0$ $n \equiv 2,\ 3 \mod 4\,;\, \gamma = 1, \alpha + \beta = 0$

3.2.2. LCD Ternary Codes from Kneser Graphs K$(n,2)$

In Table 4, we present conditions under which the constructions from Kneser graphs K(n, 2) produce LCD ternary codes. All congruences are given modulo 3.

Table 4. LCD ternary codes from Kneser graphs K(n, 2).

r	s	t	$P_n^{\mathbb{F}_3}(r,s,t)$	$B_n^{\mathbb{F}_3}(r,s,t)$
$\neq 0$	0	0	Always	$\beta = \gamma = 0$
$\neq 0$	$\neq 0$	0	$n \equiv 0\,;\, rs = 1$	$n \equiv 0\,;\, \beta = \gamma = 0, rs = 1$
$\neq 0$	0	$\neq 0$	Never	Never
0	$\neq 0$	0	$n \equiv 1$	$\beta = \gamma = 0$
0	0	$\neq 0$	Never	Never
0	$\neq 0$	$\neq 0$	$n \equiv 2\,;\, st = 2$	$n \equiv 2\,;\, \gamma = 0, 1 + \alpha^2 + \beta^2 \neq 0, st = 2$ $n \equiv 2\,;\, \alpha = 0, \gamma^2 = 1, st = 1$
$\neq 0$	$\neq 0$	$\neq 0$	$n \equiv 0,\ 1\,;\, rs = 1, st = 1$ $n \equiv 2\,;\, rs = 2, st = 1$	$n \equiv 0\,;\, \gamma = 0, rs = 1, st = 1$ $n \equiv 0\,;\, \alpha\gamma + \beta(r + 2t) = 0, \gamma^2 = 1, rt = 2$ $n \equiv 1\,;\, \alpha\gamma + \beta t = 0, \gamma^2 = 1, rs = 2$ $n \equiv 2\,;\, \beta = \gamma = 0, rs = 2, st = 1$

4. Codes Obtained from Collinearity Graphs of Generalized Quadrangles

The collinearity graph of a generalized quadrangle GQ(n,m) is the graph whose vertices are the points of the generalized quadrangle, and two vertices are adjacent if and only if the corresponding points are collinear. This graph is a strongly regular graph with parameters $((n+1)(nm+1), n(m+1), n-1, m+1)$.

The matrices $P^R(r,s,t)$ and $B^R(r,s,t)$ formed from the collinearity graph of the generalized quadrangle GQ(n,m) are denoted by $P^R_{n,m}(r,s,t)$ and $B^R_{n,m}(r,s,t)$, respectively.

4.1. Self-Dual Codes from Collinearity Graphs of GQ(n,m)

The following theorems follow from Theorem 1, in the same manner as Theorems 3 and 4.

Theorem 7. *The code generated by $P^R_{n,m}(r,s,t)$ formed from an adjacency matrix of a collinearity graph obtained from a generalized quadrangle GQ(n,m) is Euclidean self-dual over R if and only if*

$$r^2 + s^2 n(m+1) + t^2 n^2 m = -1,$$
$$2rs + 2stnm + (n-1)(s^2 + t^2 nm) = 0,$$
$$2rt + 2st(n-1)(m+1) + (m+1)(s^2 - t^2 n) + t^2 m(n^2+1) = 0.$$

The self-dual code generated by $P^{\mathbb{Z}_{2\ell}}_{n,m}(r,s,t)$ is Type II if and only if

$$1 + r^2 + s^2 n(m+1) + t^2 n^2 m \equiv 0 \mod 4\ell.$$

Theorem 8. *The code generated by $B^R_{n,m}(r,s,t)$ formed from an adjacency matrix of a collinearity graph obtained from a generalized quadrangle GQ(n,m) is Euclidean self-dual over R if and only if*

$$r^2 + s^2 n(m+1) + t^2 n^2 m = -(1+\gamma^2),$$
$$2rs + 2stnm + (n-1)(s^2 + t^2 nm) = -\gamma^2,$$
$$2rt + 2st(n-1)(m+1) + (m+1)(s^2 - t^2 n) + t^2 m(n^2+1) = -\gamma^2,$$
$$1 + \alpha^2 + (n+1)(nm+1)\beta^2 = 0,$$
$$\alpha\gamma + \beta(r + sn(m+1) + tn^2 m) = 0.$$

The self-dual code generated by $B^{\mathbb{Z}_{2\ell}}_{n,m}(r,s,t)$ is Type II if and only if

$$1 + \gamma^2 + r^2 + s^2 n(m+1) + t^2 n^2 m \equiv 0 \mod 4\ell$$

and

$$1 + \alpha^2 + (n+1)(nm+1)\beta^2 \equiv 0 \mod 4\ell.$$

In the following section, we study binary and ternary self-dual codes spanned by $P^R_{n,m}(r,s,t)$ and $B^R_{n,m}(r,s,t)$.

4.1.1. Self-Dual Binary Codes from Collinearity Graphs of GQ(n,m)

In Tables 5 and 6, we give conditions under which the constructions from collinearity graphs of GQ(n,m) yield self-dual binary codes, and determine which of the self-dual codes are Type II. Congruences for Type II codes are given modulo 4.

Table 5. Binary codes from collinearity graphs of GQ(n,m), pure construction.

r	s	t	$P^{\mathbb{F}_2}_{n,m}(r,s,t)$	Type II
0	0	1	$n,m \equiv 1 \mod 2$	$n^2 m \equiv 3$
0	1	0	Never	-
0	1	1	$n \equiv 1 \mod 2$, $m \equiv 0, 1 \mod 2$	$n(nm+m+1) \equiv 3$
1	0	0	Always	Never
1	0	1	$n,m \equiv 0 \mod 2$	$n^2 m \equiv 2$

Table 5. *Cont.*

r	s	t	$P^{\mathbb{F}_2}_{n,m}(r,s,t)$	Type II
1	1	0	$n, m \equiv 1 \mod 2$	$n(m+1) \equiv 2$
1	1	1	$n \equiv 0 \mod 2, m \equiv 0, 1 \mod 2$	$n(nm+m+1) \equiv 2$

Table 6. Ternary codes from collinearity graphs of GQ(n, m), bordered construction.

r	s	t	$B^{\mathbb{F}_2}_{n,m}(r,s,t)$	Type II
0	0	1	$n, m \equiv 1 \mod 2$; $\alpha = 1, \beta = \gamma = 0$	Never
0	1	0	$n, m \equiv 0 \mod 2$; $\alpha = 0, \beta = \gamma = 1$	$n(m+1) \equiv 2, n^2m \equiv 0$
0	1	1	$n \equiv 0 \mod 2, m \equiv 0, 1 \mod 2$; $\alpha = 0, \beta = \gamma = 1$	$n(nm+m+1) \equiv 2$
			$n \equiv 1 \mod 2, m \equiv 0, 1 \mod 2$; $\alpha = 1, \beta = \gamma = 0$	Never
1	0	0	$\alpha = 1, \beta = \gamma = 0$	Never
1	0	1	$n, m \equiv 0 \mod 2$; $\alpha = 1, \beta = \gamma = 0$	Never
1	1	0	$n, m \equiv 1 \mod 2$; $\alpha = 1, \beta = \gamma = 0$	Never
1	1	1	Never	-

4.1.2. Self-Dual Ternary Codes Obtained from Collinearity Graphs of GQ(n, m) Graphs

After examining all cases for parameters r, s, t, we obtain that the only case under which the constructions from collinearity graphs of GQ(n, m) yield self-dual ternary codes using the pure construction is in the following:

- $P^{\mathbb{F}_3}_{n,m}(r,s,t)$, $r = s = 0, t \neq 0$ and $n \equiv 1 \mod 3, m \equiv 2 \mod 3$.

Further, it turns out that we cannot obtain self-dual codes by using the bordered construction.

4.2. LCD Codes from Collinearity Graphs of GQ(n, m)

The following theorems can be proven by applying Theorem 2 to the parameters of collinearity graphs of GQ(n, m).

Theorem 9. *Let Γ be a collinearity graph of GQ(n, m). If for some $x \in \mathbb{F}_q$, $x \neq 0$, it holds that*

$$\begin{aligned} r^2 + s^2n(m+1) + t^2n^2m &= x - 1, \\ 2rs + 2stnm + (n-1)(s^2 + t^2nm) &= 0, \\ 2rt + 2st(n-1)(m+1) + (m+1)(s^2 - t^2n) + t^2m(n^2+1) &= 0, \end{aligned}$$

then the code generated by $P^{\mathbb{F}_q}_{n,m}(r,s,t)$ formed from an adjacency matrix of Γ is Euclidean LCD over $\mathbb{F}_q$.

Theorem 10. *Let Γ be a collinearity graph of GQ(n, m). If for some scalars $x, y \in \mathbb{F}_q$, $x \neq 0$, $y \neq 0$, it holds that*

$$\begin{aligned} r^2 + s^2n(m+1) + t^2n^2m &= x - 1 - \gamma^2, \\ 2rs + 2stnm + (n-1)(s^2 + t^2nm) &= -\gamma^2, \\ 2rt + 2st(n-1)(m+1) + (m+1)(s^2 - t^2n) + t^2m(n^2+1) &= -\gamma^2, \\ 1 + \alpha^2 + (n+1)(nm+1)\beta^2 &= y, \\ \alpha\gamma + \beta(r + sn(m+1) + tn^2m) &= 0, \end{aligned}$$

then the code generated by $B^{\mathbb{F}_q}_{n,m}(r,s,t)$ formed from an adjacency matrix of Γ is Euclidean LCD over $\mathbb{F}_q$.

Further, we study more closely binary and ternary LCD codes constructed from the collinearity graphs of generalized quadrangles, by the pure and the bordered construction.

4.2.1. LCD Binary Codes Obtained from Collinearity Graphs of GQ(n, m)

In Table 7, we present conditions under which the constructions from collinearity graphs of GQ(n, m) produce LCD binary codes.

Table 7. LCD binary codes from collinearity graphs of GQ(n, m).

r	s	t	$P^{\mathbb{F}_2}_{n,m}(r,s,t)$	$B^{\mathbb{F}_2}_{n,m}(r,s,t)$
0	0	1	$n, m \equiv 0 \mod 2$	$n, m \equiv 0 \mod 2\,; \gamma = 0, \alpha + \beta = 0$
				$n, m \equiv 1 \mod 2\,; \alpha = \beta = 0, \gamma = 1$
0	1	0	$n, m \equiv 1 \mod 2$	$n, m \equiv 1 \mod 2\,; \alpha = \gamma = 0$
0	1	1	Never	Never
1	0	0	Never	Never
1	0	1	$n, m \equiv 1 \mod 2$	$n, m \equiv 1 \mod 2\,; \alpha = \gamma = 0$
1	1	0	Never	$n, m \equiv 0 \mod 2\,; \gamma = 1, \alpha + \beta = 0$
1	1	1	$n \equiv 1 \mod 2, m \equiv 0, 1 \mod 2$	$n \equiv 0 \mod 2, m \equiv 0, 1 \mod 2\,; \gamma = 1, \alpha + \beta = 0$
				$n \equiv 1 \mod 2, m \equiv 0, 1 \mod 2\,; \alpha = \gamma = 0$

4.2.2. LCD Ternary Codes Obtained from Collinearity Graphs of GQ(n, m)

In Tables 8 and 9, we present conditions under which the constructions from collinearity graphs of GQ(n, m) produce LCD ternary codes. All congruences are given modulo 3.

Table 8. LCD ternary codes from collinearity graphs of GQ(n, m), pure construction.

r	s	t	$P^{\mathbb{F}_3}_{n,m}(r,s,t)$
$\neq 0$	0	0	Always
$\neq 0$	$\neq 0$	0	$n \equiv 0, m \equiv 2\,; rs = 2$
			$n \equiv 2, m \equiv 2\,; rs = 1$
$\neq 0$	0	$\neq 0$	$n \equiv 0, m \equiv 1$ or $n \equiv 1, m \equiv 2$ or $n \equiv 2, m \equiv 0\,; rt = 1$
			$n \equiv 0, m \equiv 2$ or $n \equiv 1, m \equiv 0\,; rt = 2$
0	$\neq 0$	0	$n \equiv 1, m \equiv 2$
0	0	$\neq 0$	$n, m \equiv 0$ or $n, m \equiv 1$
0	$\neq 0$	$\neq 0$	$n \equiv 1, m \equiv 0$
$\neq 0$	$\neq 0$	$\neq 0$	$n \equiv 1, m \equiv 2$ or $n \equiv 2, m \equiv 0, 1, 2\,; rs = 1, st = 1$
			$n, m \equiv 1\,; rs = 1, st = 2$
			$n \equiv 0, m \equiv 0, 1, 2$ or $n, m \equiv 1\,; rs = 2, st = 1$
			$n \equiv 0, m \equiv 1$ or $n \equiv 1, m \equiv 2\,; rs = 2, st = 2$

Table 9. LCD ternary codes from collinearity graphs of GQ(n, m), bordered construction.

r	s	t	$B^{\mathbb{F}_3}_{n,m}(r,s,t)$
$\neq 0$	0	0	$\beta = \gamma = 0$
$\neq 0$	$\neq 0$	0	$n, m \equiv 1\,; \alpha = 0, \gamma^2 = 1, rs = 1$
			$n, m \equiv 2\,; \beta = \gamma = 0, rs = 1$
			$n \equiv 0, m \equiv 2\,; \beta = \gamma = 0, rs = 2$
			$n \equiv 2, m \equiv 1\,; \alpha = 0, \gamma^2 = 1, rs = 2$
$\neq 0$	0	$\neq 0$	$n \equiv 0, m \equiv 1$ or $n \equiv 2, m \equiv 0\,; \beta = \gamma = 0, rt = 1$
			$n \equiv 1, m \equiv 2\,; \gamma = 0, rt = 1$
			$n \equiv 0, m \equiv 2$ or $n \equiv 1, m \equiv 0\,; \beta = \gamma = 0, rt = 2$
			$n \equiv 2, m \equiv 1\,; \alpha = 0, \gamma^2 = 1, rt = 2$
0	$\neq 0$	0	$n \equiv 0, m \equiv 1\,; \alpha = 0, \gamma^2 = 1$
			$n \equiv 1, m \equiv 2\,; \gamma = 0$

Table 9. *Cont.*

r	s	t	$B^{\mathbb{F}_3}_{n,m}(r,s,t)$
0	0	$\neq 0$	$n, m \equiv 0$; $\gamma = 0, 1 + \alpha^2 + \beta^2 \neq 0$
			$n, m \equiv 1$; $\beta = \gamma = 0$
0	$\neq 0$	$\neq 0$	$n \equiv 0, m \equiv 0,\ 1,\ 2$ or $n, m \equiv 1$; $\alpha = 0, \gamma^2 = 1, st = 1$
			$n \equiv 1, m \equiv 0$; $\beta = \gamma = 0, st = 1$
			$n \equiv 0, m \equiv 2$ or $n \equiv 2, m \equiv 1$; $\alpha = 0, \gamma^2 = 1, st = 2$
			$n \equiv 1, m \equiv 0$; $\beta = \gamma = 0, st = 2$
$\neq 0$	$\neq 0$	$\neq 0$	$n \equiv 1, m \equiv 0$; $\alpha\gamma + \beta(r+s) = 0, 1 + \alpha^2 + \beta^2 \neq 0, rs = 1, st = 1$
			$n \equiv 1, m \equiv 2$ or $n \equiv 2, m \equiv 0,\ 1,\ 2$; $\gamma = 0, rs = 1, st = 1$
			$n, m \equiv 1$; $\beta = \gamma = 0, rs = 1, st = 2$
			$n, m \equiv 2$; $\alpha\gamma + \beta(r+2t) = 0, \gamma^2 = 1, rs = 1, st = 2$
			$n \equiv 0, m \equiv 0,1,2$ or $n, m \equiv 1$; $\beta = \gamma = 0, rs = 2, st = 1$
			$n \equiv 1,\ 2, m \equiv 2$; $\alpha\gamma + \beta(r+2t) = 0, \gamma^2 = 1, rs = 2, st = 1$
			$n \equiv 2, m \equiv 0,1$; $\alpha = 0, \gamma^2 = 1, rs = 2, st = 1$
			$n \equiv 0, m \equiv 1$; $\beta = \gamma = 0, rs = 2, st = 2$
			$n \equiv 1, m \equiv 2$; $\gamma = 0, rs = 2, st = 2$
			$n \equiv 2, m \equiv 0$; $\alpha = 0, \gamma^2 = 1, rs = 2, st = 2$

5. Strongly Regular Graphs and 3-Designs Constructed from Codes

In this section, we construct 3-designs and strongly regular graphs from the support designs of some of the self-dual or LCD codes described in this paper. The strongly regular graphs are obtained from support designs as so-called generalized block graphs.

Generalized block graphs of support designs are obtained in the following way (see [31]). Let $S = \{|x \cap y|\ x, y \in \mathcal{B}\}$, where $\mathcal{B}$ is the block set of the support design, and let $A \subset S$. Let us define a graph whose vertices are the elements of the block set $\mathcal{B}$ of the support design, and two vertices are adjacent if and only if the size of the intersection of the corresponding blocks is an element of A. This construction is a generalization of the construction of the block graph from a quasi-symmetric design, so a graph constructed in that way is called a generalized block graph.

In Table 10, we give the codes used for the construction of the 3-designs and strongly regular graphs, which are presented in Tables 11 and 12.

Table 10. Codes used for the construction of the 3-designs and SRGs given in Tables 11 and 12.

Graph	Construction	Parameters of Code	LCD or Self-Dual
$\Gamma(GQ(4,\ 2))$, $(45, 12, 3, 3)$	$P(1,\ 2,\ 1)$	C_1, $[90,\ 45,\ 5]_3$	LCD
$K(6,\ 2)$, $(15, 6, 1, 3)$	$P(1,\ 0,\ 1)$	C_2, $[30,\ 15,\ 6]_2$	self-dual
$K(6,\ 2)$, $(15, 6, 1, 3)$	$P(0,\ 0,\ 1)$	C_3, $[30,\ 15,\ 3]_2$	LCD
$K(6,\ 2)$, $(15, 6, 1, 3)$	$B(1,\ 1,\ 0)$	C_4, $[32,\ 16,\ 4]_2$	LCD
$K(6,\ 2)$, $(15, 6, 1, 3)$	$B(0,\ 1,\ 0)$	C_5, $[32,\ 16,\ 8]_2$	self-dual
$\Gamma_1(GQ(3,\ 3))$, $(40, 12, 2, 4)$	$P(0,\ 1,\ 0)$	C_6, $[80,\ 40,\ 6]_2$	LCD
$\Gamma_2(GQ(3,\ 3))$, $(40, 12, 2, 4)$	$P(0,\ 0,\ 1)$	C_7, $[80,\ 40,\ 4]_3$	LCD
$\Gamma(GQ(2,\ 4))$, $(27, 10, 1, 5)$	$P(1,\ 0,\ 1)$	C_8, $[54,\ 27,\ 6]_2$	self-dual
$K(5,\ 2)$, $(10, 3, 0, 1)$	$P(2,\ 1,\ 1)$	C_9, $[20,\ 10,\ 4]_3$	LCD
$K(5,\ 2)$, $(10, 3, 0, 1)$	$B(0,\ 1,\ 1)$	C_{10}, $[22,\ 11,\ 4]_3$	LCD
$\Gamma_1(GQ(3,\ 3))$, $(40, 12, 2, 4)$	$P(0,\ 0,\ 1)$	C_{11}, $[80,\ 40,\ 4]_3$	LCD
$K(6,\ 2)$, $(15, 6, 1, 3)$	$P(1,\ 1,\ 0)$	C_{12}, $[30,\ 15,\ 8]_3$	LCD
$K(7,\ 2)$, $(21, 10, 3, 6)$	$B(2,\ 1,\ 1)$	C_{13}, $[44,\ 22,\ 4]_3$	LCD
$K(8,\ 2)$, $(28, 15, 6, 10)$	$P(2,\ 1,\ 1)$	C_{14}, $[56,\ 28,\ 4]_3$	LCD
$K(8,\ 2)$, $(28, 15, 6, 10)$	$B(0,\ 1,\ 1)$	C_{15}, $[58,\ 29,\ 4]_3$	LCD
$K(7,\ 2)$, $(21, 10, 3, 6)$	$P(0,\ 1,\ 0)$	C_{16}, $[42,\ 21,\ 8]_3$	LCD
$K(7,\ 2)$, $(21, 10, 3, 6)$	$B(1,\ 2,\ 1)$	C_{17}, $[44,\ 22,\ 8]_3$	LCD
$\Gamma(GQ(4,\ 2))$, $(45, 12, 3, 3)$	$B(1,\ 2,\ 2)$	C_{18}, $[92,\ 46,\ 4]_3$	LCD

In Table 11, we list details on the 3-designs obtained as support designs of the code C_5. As shown in Table 10, the code C_5 is obtained by using the bordered construction from the Kneser graph $K(6,2)$, which is the strongly regular graph with parameters $(15,6,1,3)$.

Table 11. The 3-designs from the code C_5.

3-(v,k,λ)	Aut. Group of a Design
3-(32, 8, 7), $b = 620$	$E_{32}.PSL(5,\ 2)$
3-(32, 12, 616), $b = 13,888$	$E_{32}.PSL(5,\ 2)$
3-(32, 16, 4123), $b = 36,518$	$E_{32}.PSL(5,\ 2)$

In Table 12, we list strongly regular graphs constructed as generalized block graphs from support designs of the codes given in Table 10. The first column provides information on the code that we used, and in the second column, we present information about support designs. The next column gives all intersections of blocks of support designs and we point out the intersections used for defining generalized block graphs. For example, the support design of the code C_1 for the weight 5 has block intersections 0 and 1, and intersection 1 is used to construct the graph Γ_1. Further, we give the parameters of the strongly regular graphs and a structure of their automorphism groups.

Table 12. SRGs constructed as generalized block graphs of support designs.

Code	Support Design	Intersections	SRG	Aut. Group of SRG
C_1	$k=5$	$\{0,\ 1\},\ 1$	$\Gamma_1,\ (27,10,1,5)$	$O(5,\ 3){:}Z_2$
C_2	$1-(30,\ 6,\ 7)$	$\{0,\ 2\},\ 0$	$\Gamma_2,\ (35,16,6,8)$	S_8
C_7	$k=4$	$\{0,\ 1\},\ 1$	$\Gamma_3,\ (40,12,2,4)_1$	$O(5,\ 3){:}Z_2$
C_6	$k=76$	$\{72,\ 73\},\ 72$	$\Gamma_4,\ (40,12,2,4)_2$	$O(5,\ 3){:}Z_2$
C_8	$1-(54,\ 6,\ 5)$	$\{0,\ 2\},\ 2$	$\Gamma_5,\ (45,12,3,3)$	$O(5,\ 3){:}Z_2$
C_9	$1-(20,\ 4,\ 9)$	$\{0,\ 2\},\ 2$	$\Gamma_6,\ (45,16,8,4)$	S_{10}
C_{10}	$1-(22,\ 4,\ 10)$	$\{0,\ 2\},\ 2$	$\Gamma_7,\ (55,18,9,4)$	S_{11}
C_9	$1-(20,\ 18,\ 90)$	$\{16,\ 17\},\ 17$	$\Gamma_8,\ (100,18,8,2)$	$S_{10}\wr Z_2$
C_3	$k=21$	$\{15,\ 16,\ 19\},\ 16$	$\Gamma_9,\ (105,26,13,4)$	S_{15}
C_9	$1-(20,\ 6,\ 36)$	$\{0,\ 2,\ 4\},\ 0,\ 4$	$\Gamma_{10},\ (120,56,28,24)$	S_{10}
C_4	$k=22$	$\{16,\ 17,\ 20\},\ 17$	$\Gamma_{11},\ (120,28,14,4)$	S_{16}
C_{10}	$1-(22,\ 20,\ 110)$	$\{18,\ 19\},\ 19$	$\Gamma_{12},\ (121,20,9,2)$	$S_{11}\wr Z_2$
C_{11}	$k=4$	$\{0,\ 1\},\ 1$	$\Gamma_{13},\ (130,48,20,16)$	$(PSL(4,\ 3){:}Z_2){:}Z_2$
C_{12}	$1-(30,\ 28,\ 210)$	$\{26,\ 27\},\ 27$	$\Gamma_{14},\ (225,28,13,2)$	$S_{15}\wr Z_2$
C_{13}	$1-(44,\ 4,\ 21)$	$\{0,\ 2\},\ 2$	$\Gamma_{15},\ (231,40,20,4)$	S_{22}
C_{14}	$1-(56,\ 4,\ 27)$	$\{0,\ 2\},\ 2$	$\Gamma_{16},\ (378,52,26,4)$	S_{28}
C_{15}	$1-(58,\ 4,\ 28)$	$\{0,\ 2\},\ 2$	$\Gamma_{17},\ (406,54,27,4)$	S_{29}
C_{16}	$1-(42,\ 40,\ 420)$	$\{38,\ 39\},\ 39$	$\Gamma_{18},\ (441,40,19,2)$	$S_{21}\wr Z_2$
C_{17}	$1-(44,\ 42,\ 462)$	$\{40,\ 41\},\ 41$	$\Gamma_{19},\ (484,42,20,2)$	$S_{22}\wr Z_2$
C_{18}	$1-(92,\ 4,\ 45)$	$\{0,\ 2\},\ 2$	$\Gamma_{20},\ (1035,88,44,4)$	S_{46}

Remark 1. *Strongly regular graphs given in Table 12 have been previously known. The graphs $\Gamma_i, i \in \{6,7,9,11,15,16,17,20\}$ are unique strongly regular graphs with the given parameters, known as triangular graphs. Further, for $i \in \{8,12,14,18,19\}$, the graphs Γ_i are constructed as the Cartesian product of two complete graphs on n vertices, and are known as n^2 graphs. The graphs n^2 are also unique SRGs with the given parameters. The graph Γ_1 is the collinearity graph of a GQ(2,4), Γ_5 is the collinearity graph of a GQ(4,2), and Γ_3 and Γ_4 are two non-isomorphic collinearity graphs of a $GQ(3,3)$. Strongly regular graphs having the parameters of Γ_2 are all classified. More information about these graphs one can find at [27]. For information about strongly regular graphs Γ_{10} and Γ_{13}, see [27].*

6. Conclusions

In this paper, we determine the conditions under which the codes constructed from Kneser graphs K(n, 2) and collinearity graphs of generalized quadrangles using the so-called pure and bordered construction are self-dual or LCD. Further, for the codes over $\mathbb{Z}_{2k}$, we give the conditions which ensure that they are Type II. The binary and ternary codes are studied in more detail. We also investigate the support designs of the constructed codes, and from the support designs for certain weights of some of the codes, we obtain strongly regular graphs and 3-designs. Some of the 3-designs and strongly regular graphs have interesting automorphism groups, namely, the linear groups $\mathrm{PSL}(5,\ 2)$ and $\mathrm{PSL}(4,\ 3)$, and the orthogonal group $\mathrm{O}(5,\ 3)$.

Author Contributions: Investigation, D.C. and A.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Dougherty, S.T.; Kim, J.-L.; Solé, P. Double circulant codes from two class association schemes. *Adv. Math. Commun.* **2007**, *1*, 45–64. [CrossRef]
2. Gaborit, P. Quadratic double circulant codes over fields. *J. Combin. Theory Ser. A* **2002**, *97*, 85–107. [CrossRef]
3. Crnković, D.; Grbac, A.; Švob, A. Formally self-dual LCD codes from two-class association schemes. *Appl. Algebra Eng. Comm. Comput.* **2023**, *34*, 183–200. [CrossRef]
4. Miezaki, T. Design-theoretic analogies between codes, lattices, and vertex operator algebras. *Des. Codes Cryptogr.* **2021**, *89*, 763–780. [CrossRef]
5. Pless, V. On a new family of symmetry codes and related new five-designs. *Bull. Am. Math. Soc.* **1969**, *75*, 1339–1342. [CrossRef]
6. Nebe, G.; Rains, E.M.; Sloane, N.J.A. Self-dual codes and invariant theory. In *Algorithms and Computation in Mathematics*; Springer: Berlin/Heidelberg, Germany, 2006; Volume 17.
7. Alahmadi, A.; Alshuhail, A.; Betty, R.A.; Galvez, L.; Solé, P. Mass Formula for Self-Orthogonal and Self-Dual Codes over Non-Unital Rings of Order Four. *Mathematics* **2023**, *11*, 4736. [CrossRef]
8. Bouyuklieva, S.; de la Cruz, J.; Villar, D. Extremal Binary and Ternary Codes of Length 60 with an Automorphism of Order 29 and a Generalization. *Mathematics* **2022**, *10*, 748. [CrossRef]
9. Harada, M.; Ishizuka, K.; Kharaghani, H. Ternary extremal four-negacirculant self-dual codes. *Graphs Combin.* **2024**, *40*, 59. [CrossRef]
10. Pan, X.; Chen, H.; Liu, H. Some self-dual codes and isodual codes constructed by matrix product codes. *Des. Codes Cryptogr.* **2024**, *92*, 3533–3559. [CrossRef]
11. Prakash, O.; Yadav, S.; Islam, H.; Solé, P. Self-dual and LCD double circulant codes over a class of non-local rings. *Comput. Appl. Math.* **2022**, *41*, 245. [CrossRef]
12. Roberts, A.M. Self-dual codes from a block matrix construction characterised by group rings. *Des. Codes Cryptogr.* **2024**, *92*, 1599–1617. [CrossRef]
13. Shi, M.; Lu, H.; Kim, J.-L.; Solé, P. Triorthogonal codes and self-dual codes. *Quantum Inf. Process.* **2024**, *23*, 280. [CrossRef]
14. Massey, J.L. Linear codes with complementary duals. *Discrete Math.* **1992**, *106–107*, 337–342. [CrossRef]
15. Sendrier, N. Linear codes with complementary duals meet the Gilbert-Varshamov bound. *Discrete Math.* **2004**, *304*, 345–347. [CrossRef]
16. Carlet, C.; Guilley, S. Complementary dual codes for counter-measures to side-channel attacks. *Adv. Math. Commun.* **2016**, *10*, 131–150. [CrossRef]
17. Alahmadi, A.N.; Alhazmi, H.S.; Shoaib, H.; Glynn, D.G.; Rehman, S.U.; Solé, P. Connections between Linear Complementary Dual Codes, Permanents and Geometry. *Mathematics* **2023**, *11*, 277. [CrossRef]
18. Araya, M.; Harada, M. On the classification of linear complementary dual codes. *Discrete Math.* **2019**, *342*, 270–278. [CrossRef]
19. Crnković, D.; Egan, R.; Rodrigues, B.G.; Švob, A. LCD codes from weighing matrices. *Appl. Algebra Eng. Comm. Comput.* **2021**, *32*, 175–189. [CrossRef]
20. Dougherty, S.T.; Kim, J.-L.; Ozkaya, B.; Sok, L.; Solé, P. The combinatorics of LCD codes: Linear Programming bound and orthogonal matrices. *Int. J. Inf. Coding Theory* **2017**, *4*, 116–128. [CrossRef]
21. Harada, M.; Saito, K. Binary linear complementary dual codes. *Cryptogr. Commun.* **2019**, *11*, 677–696.
22. Shi, M.; Li, S.; Kim, J.-L.; Solé, P. LCD and ACD codes over a noncommutative non-unital ring with four elements. *Cryptogr. Commun.* **2022**, *14*, 627–640. [CrossRef]
23. Sok, L.; Shi, M.; Solé, P. Constructions of optimal LCD codes over large finite fields. *Finite Fields Appl.* **2018**, *50*, 138–153. [CrossRef]
24. Bosma, W.; Cannon, J. *Handbook of Magma Functions*; Department of Mathematics, University of Sydney: Sydney, Australia, 1994. Available online: http://magma.maths.usyd.edu.au/magma (accessed on 15 September 2024).

25. Bose, R.C. Strongly regular graphs, partial geometries and partially balanced designs. *Pac. J. Math.* **1963**, *13*, 389–419. [CrossRef]
26. Brouwer, A.E.; Haemers, W.H. *Spectra of Graphs, Universitext*; Springer: New York, NY, USA, 2012.
27. Brouwer, A.E.; Maldeghem, H.V. Strongly regular graphs. In *Encyclopedia of Mathematics and Its Applications*; Cambridge University Press: Cambridge, UK, 2022; Volume 182.
28. Beth, T.; Jungnickel, D.; Lenz, H. *Design Theory*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1999.
29. Khosrovshahi, G.B.; Laue, R. t-Designs with $t \geq 3$. In *Handbook of Combinatorial Designs*, 2nd ed.; Colbourn, C.J., Dinitz, J.H., Eds.; Chapman & Hall; CRC: Boca Raton, FL, USA, 2007; pp. 79–101.
30. Payne, S.E.; Thas, J.A. *Finite Generalized Quadrangles*; EMS Press, European Mathematical Society: Berlin, Germany, 2009.
31. Crnković, D.; Mikulić, V. Self-orthogonal doubly-even codes from Hadamard matrices of order 48. *Adv. Appl. Discrete Math.* **2008**, *1*, 159–170.