General Linear Recurrence Sequences and Their Convolution Formulas

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Abstract: We extend a technique recently introduced by Chen Zhuoyu and Qi Lan in order to find convolution formulas for second order linear recurrence polynomials generated by $\left(\frac{1}{1+at+bt^2}\right)^x$. The case of generating functions containing parameters, even in the numerator is considered. Convolution formulas and general recurrence relations are derived. Many illustrative examples and a straightforward extension to the case of matrix polynomials are shown.

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1. Introduction

Generating functions [1] constitute a bridge between continuous analysis and discrete mathematics. Linear recurrence relations are satisfied by many special polynomials of classical analysis. A wide scenario including special sequences of polynomials and numbers, combinatorial analysis, and application of mathematics is related to the above mentioned topics.

It would be impossible to list in the Reference section all of even the most important articles dedicated to these subjects. As a first example, we recall the Chebyshev polynomials of the first and second kind, which are powerful tools used in both theoretical and applied mathematics. Their links with the Lucas and Fibonacci polynomials have been studied and many properties have been derived. Connections with Bernoulli polynomials have been highlighted in [2].

In particular, the important calculation of sums of several types of polynomials have been recently studied (see e.g., [3–5] and the references therein). This kind of subject has attracted many scholars. For example, W. Zhang [6] proved an identity involving Chebyshev polynomials and their derivatives.

Fibonacci and Lucas polynomias and their extensions have been studied for a long time, in particular within the Fibonacci Association, which has contributed to the study of this and similar subjects. As an applications of a results proved by Y. Zhang and Z. Chen [3], Y. Ma and W. Zhang [4] obtained some identities involving Fibonacci numbers and Lucas numbers.

Convolution techniques are connected with combinatorial identities, and many results have been obtained in this direction [2,7,8]. Convolution sums using second kind Chebyshev polynomials are contained in [7].

Recently, Taekyun Kim et al. [8] studied properties of Fibonacci numbers by introducing the so called convolved Fibonacci numbers. By using the genereting function:

$$\left(\frac{1}{1 - t - bt^2} \right)^x = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!},$$

for $x \in \mathbf{R}$ and $r \in \mathbf{N}$, they proved the interesting relation

$$p_n(x) = \sum_{\ell=0}^{n} p_\ell(r) p_{n-\ell}(x - r) = \sum_{\ell=0}^{n} p_{n-\ell}(r) p_\ell(x - r).$$

Furthermore, they derived a link between $p_n(x)$ and a particular combination of sums of Fibonacci numbers, so that complex sums of Fibonacci numbers have been converted to the easier calculation of $p_n(x)$.

In a recent article Chen Zhuoyu and Qi Lan [9] introduced convolution formulas for second order linear recurrence sequences related to the generating function [1] of the type

$$f(t) = \frac{1}{1 + at + bt^2},$$

deriving coefficient expressions for the series expansion of the function $f^x(t)$, $(x \in \mathbf{R})$. In this article, motivated by this research, we continue the study of possible applications of the considered method, by analyzing the general situation of a generating function of the type

$$G(t, x) = \left(\frac{1}{1 + a_1 t + a_2 t^2 + \cdots + a_r t^r} \right)^x,$$

and we deduce the recurrence relation for the generated polynomials.

Several illustrative examples are shown in Section 6. In the last section the results are extended, in a straightforward way, to the case of matrix polynomials.

2. Generating Functions

We start from the generating function considered by Chen Zhuoyu and Qi Lan:

$$G(t, x) = \left(\frac{1}{1 + at + bt^2} \right)^x = \left[\frac{1}{(1 - \alpha t)(1 - \beta t)} \right]^x =$$

$$= \exp\left\{ -x \log\left[(1 - \alpha t)(1 - \beta t) \right] \right\}, \tag{1}$$

with

$$a = -(\alpha + \beta), \qquad b = \alpha \beta \tag{2}$$

$$G(t, x) = \sum_{k=0}^{\infty} g_k(x; \alpha, \beta) \frac{t^k}{k!} =$$

$$= \exp\left\{ -x \log(1 - \alpha t) \right\} \cdot \exp\left\{ -x \log(1 - \beta t) \right\} = G_\alpha(t, x) \cdot G_\beta(t, x), \tag{3}$$

where

$$G_\alpha(t, x) = \exp\left[-x \log\left(1 - \alpha t\right)\right] = \sum_{k=0}^\infty p_k(x, \alpha) \frac{t^k}{k!}, \tag{4a}$$

$$G_\beta(t, x) = \exp\left[-x \log\left(1 - \beta t\right)\right] = \sum_{k=0}^\infty q_k(x, \beta) \frac{t^k}{k!}. \tag{4b}$$

Note that, by Equation (2) we could write, in equivalent form:

$$g_k(x; \alpha, \beta) = g_k(x; a, b), \qquad p_k(x, \alpha) = p_k(x, a), \qquad q_k(x, \beta) = q_k(x, b), \tag{5}$$

but, in what follows, we put for shortness:

$$g_k(x; \alpha, \beta) = g_k(x), \qquad p_k(x, \alpha) = p_k(x), \qquad q_k(x, \beta) = q_k(x). \tag{6}$$

By Equations (3), (4a) and (4b) we find the convolution formula:

$$g_k(x) = \sum_{h=0}^k \binom{k}{h} p_{k-h}(x) \, q_h(x). \tag{7}$$

3. Recurrence Relation

Note that

$$\frac{\partial G(t, x)}{\partial t} = \frac{\partial G_\alpha(t, x)}{\partial t} \cdot G_\beta(t, x) + G_\alpha(t, x) \cdot \frac{\partial G_\beta(t, x)}{\partial t} =$$

$$= \left(\frac{\alpha x}{1 - \alpha t} + \frac{\beta x}{1 - \beta t}\right) G(t, x) = -x \left(\frac{a + 2bt}{1 + at + bt^2}\right) G(t, x), \tag{8}$$

as can be derived directly from Equation (1).

Then we have

$$(1 + at + bt^2)\frac{\partial G(t, x)}{\partial t} = -x \, a \, G(t, x) - 2 \, b \, x \, t \, G(t, x), \tag{9}$$

$$\sum_{k=0}^\infty g_{k+1}(x)\frac{t^k}{k!} + a \sum_{k=0}^\infty g_{k+1}(x)\frac{t^{k+1}}{k!} + b \sum_{k=0}^\infty g_{k+1}(x)\frac{t^{k+2}}{k!} =$$

$$= -x \, a \sum_{k=0}^\infty g_k(x)\frac{t^k}{k!} - 2 \, b \, x \sum_{k=0}^\infty g_k(x)\frac{t^{k+1}}{k!},$$

that is

$$\sum_{k=0}^\infty g_{k+1}(x)\frac{t^k}{k!} + a \sum_{k=1}^\infty k g_k(x)\frac{t^k}{k!} + b \sum_{k=2}^\infty k(k-1) g_{k-1}(x)\frac{t^k}{k!} =$$

$$= -a \, x \sum_{k=0}^\infty g_k(x)\frac{t^k}{k!} - 2 \, b \, x \sum_{k=1}^\infty k g_{k-1}(x)\frac{t^k}{k!},$$

and therefore, we can conclude with the theorem:

Theorem 1. *The sequence $\{g_k(x)\}_{k \in \mathbf{N}}$ satisfies the linear recurrence relation*

$$g_k(x) + a(x + k - 1)g_{k-1}(x) + b(k-1)(k + 2x - 2)g_{k-2}(x) = 0.\tag{10}$$

3.1. Properties of the Basic Generating Function

We consider now a few properties of the basic generating functions $G_\alpha(t, x)$. According to the definition (4a), the polynomials $p_k(x)$ are recognized as associated Sheffer polynomials [10] and quasi-monomials, according to the Dattoli [11,12] definition.

3.1.1. Differential Equation

We have:

$$G_\alpha(t, x) = \exp\left[-xH(t)\right] = \sum_{k=0}^{\infty} p_k(x, \alpha)\frac{t^k}{k!},\tag{11}$$

where

$$H(t) = -\log(1 - \alpha t), \qquad H'(t) = \frac{\alpha}{1 - \alpha t},\tag{12}$$

and its functional inverse is given by

$$H^{-1}(t) = \frac{1}{\alpha}\left(1 - e^{-t}\right),\tag{13}$$

so that, recalling the results by Y. Ben Cheikh [13], we find the derivative and multiplication operators of the quasi-monomials $p_k(x)$, in the form:

$$\hat{P} = \frac{1}{\alpha}\left(1 - e^{-D_x}\right) \qquad \hat{M} = xH'\left(H^{-1}(D_x)\right) = \alpha\, x\, e^{D_x},\tag{14}$$

and we can conclude that

Theorem 2. *The polynomials $p_k(x)$ satisfy the differential equation:*

$$\hat{M}\hat{P}\, p_n(x) = x\,\left(e^{D_x} - 1\right)\, p_n(x) = n\, p_n(x),\tag{15}$$

that is, $\forall n \geq 1$:

$$x\left(\frac{1}{n!}p_n^{(n)} + \frac{1}{(n-1)!}p_n^{(n-1)} + \cdots + p_n'(x)\right) = n\, p_n(x).\tag{16}$$

3.1.2. Differential Identity

Differentiating Equation (11) with respect to x, we find

$$\frac{\partial G_\alpha(t, x)}{\partial x} = -G_\alpha(t, x)\log(1 - \alpha t) = \sum_{k=1}^{\infty} p_k'(x, \alpha)\frac{t^k}{k!}\tag{17}$$

that is

$$\sum_{k=1}^{\infty} p_k'(x, \alpha)\frac{t^k}{k!} = -\log(1 - \alpha t)\sum_{k=0}^{\infty} p_k(x, \alpha)\frac{t^k}{k!},$$

$$\sum_{k=1}^{\infty} p'_k(x,\alpha)\, \frac{t^k}{k!} = \sum_{k=1}^{\infty} \frac{(\alpha t)^k}{k} \sum_{k=0}^{\infty} p_k(x,\alpha)\, \frac{t^k}{k!} = \sum_{k=1}^{\infty} \frac{(\alpha t)^k}{k}\left[1 + \sum_{k=1}^{\infty} p_k(x,\alpha)\, \frac{t^k}{k!}\right],$$

$$\sum_{k=1}^{\infty} p'_k(x,\alpha)\, \frac{t^k}{k!} = \sum_{k=1}^{\infty} (k-1)!\, \alpha^k \frac{t^k}{k} + \sum_{k=1}^{\infty} (k-1)!\, \alpha^k \frac{t^k}{k} \sum_{k=1}^{\infty} p_k(x,\alpha)\, \frac{t^k}{k!},$$

$$\sum_{k=1}^{\infty} p'_k(x,\alpha)\, \frac{t^k}{k!} = \sum_{k=1}^{\infty} (k-1)!\, \alpha^k \frac{t^k}{k!} + \sum_{k=1}^{\infty} (k-1)!\, \alpha^k \frac{t^k}{k!} \sum_{k=1}^{\infty} p_k(x,\alpha)\, \frac{t^k}{k!} =$$

$$= \sum_{k=1}^{\infty} (k-1)!\, \alpha^k \frac{t^k}{k!} + \sum_{k=1}^{\infty} \sum_{h=1}^{k} (k-h-1)!\, \alpha^{k-h}\, p_h(x,\alpha)\, \frac{t^k}{k!},$$

so that we can conclude with the theorem:

Theorem 3. *The polynomials $p_k(x)$ satisfy the differential identity:*

$$p'_k(x,\alpha) = (k-1)!\, \alpha^k + \sum_{h=1}^{k} (k-h-1)!\, \alpha^{k-h}\, p_h(x,\alpha). \tag{18}$$

3.2. Extension by Convolution

We now consider the case of a generating function of the type:

$$G(t,x) = \left(\frac{1+ct}{1+at+bt^2} \right)^x = \sum_{k=0}^{\infty} q_k(x;c;a,b) = \frac{\displaystyle\sum_{k=0}^{\infty} p_k(x;c)}{\displaystyle\sum_{k=0}^{\infty} g_k(x;a,b)}. \tag{19}$$

A straightforward consequence is the convolution formula for the resulting polynomials:

$$p_k(x;c) = \sum_{h=0}^{k} \binom{k}{h} g_{k-h}(x;a,b)\, q_h(x;c;a,b), \tag{20}$$

so that the $q_h(x;c;a,b)$ can be found recursively by solving the infinite system

$$\begin{cases} q_0(x;c;a,b) = 1, \\ q_k(x;c;a,b) = p_k(x;a,b) - \displaystyle\sum_{h=0}^{k-1} \binom{k}{h} g_{k-h}(x;a,b)\, q_h(x;c;a,b). \end{cases} \tag{21}$$

Noting that $p_0(x;a,b) = g_0(x;c;a,b) = 1$, the very first polynomials are given by

$$\begin{aligned}
q_0(x;c;a,b) &= 1, \\
q_1(x;c;a,b) &= p_1(x;a,b) - g_1(x;c;a,b), \\
q_2(x;c;a,b) &= p_2(x;a,b) - 2g_1(x;c;a,b)\, p_1(x;a,b) + 2g_1^2(x;c;a,b) - g_2(x;c;a,b), \\
q_3(x;c;a,b) &= p_3(x;a,b) - 3g_1(x;c;a,b)\, p_2(x;a,b) + 6g_1^2(x;c;a,b)\, p_1(x;a,b) \\
&\quad - 6g_1^3(x;c;a,b) + 6g_1(x;c;a,b)\, g_2(x;c;a,b) - 3g_2(x;c;a,b)\, p_1(x;a,b) \\
&\quad - g_3(x;c;a,b).
\end{aligned} \tag{22}$$

Further values can be obtained by using symbolic computation.

4. The General Case

Note that the above results can be extended to the general case, considering the generating function:

$$G(t,x) = \left(\frac{1}{1 + a_1 t + a_2 t^2 + \cdots + a_r t^r} \right)^x = \left[\frac{1}{(1 - \alpha_1 t)(1 - \alpha_2 t) \cdots (1 - \alpha_r t)} \right]^x =$$

$$= \exp\left\{ -x \log\left[(1 - \alpha_1 t)(1 - \alpha_2 t) \cdots (1 - \alpha_r t) \right] \right\} = \sum_{k=0}^{\infty} g_k(x; \alpha_1, \alpha_2, \ldots, \alpha_r) \frac{t^k}{k!}, \tag{23}$$

where

$$a_1 = \sigma_1 = -(\alpha_1 + \alpha_2 + \cdots + \alpha_r),$$

$$\cdots,$$

$$a_s = \sigma_r = (-1)^s \sum_{j_1, j_2, \ldots, j_s} \alpha_{j_1} \alpha_{j_2} \cdots \alpha_{j_s}, \tag{24}$$

$$\cdots,$$

$$a_r = \sigma_r = \alpha_1 \alpha_2 \cdots \alpha_r,$$

are the elementary symmetric functions of the zeros.

Putting as before:

$$G_{\alpha_h}(t,x) = \exp\left[-x \log(1 - \alpha_h t) \right] = \sum_{k=0}^{\infty} p_{1,k}(x, \alpha_h) \frac{t^k}{k!}, \qquad (h = 1, 2, \ldots, r), \tag{25}$$

since

$$G(t,x) = G_{\alpha_1}(t,x) \cdot G_{\alpha_2}(t,x) \cdots G_{\alpha_r}(t,x),$$

we find the result:

Theorem 4. *The sequence* $\{g_k(x)\}_{k \in \mathbf{N}}$ *satisfies the convolution formula:*

$$g_k(x) = \sum_{\substack{k_1 + k_2 + \cdots + k_r = k \\ 0 \leq k_i \leq k}} \binom{k}{k_1, k_2, \ldots, k_r} p_{1,k_1}(x) p_{2,k_2}(x) \cdots p_{r,k_r}(x), \tag{26}$$

where, according to our position,

$$g_k(x) = g_k(x; \alpha_1, \alpha_2, \ldots, \alpha_r), \quad p_{1,k_1}(x) = p_{1,k_1}(x, \alpha_1), \ldots, p_{r,k_r}(x) = p_{r,k_r}(x, \alpha_r).$$

5. The General Recurrence Relation

From Equation (17) we find:

$$\frac{\partial G(t,x)}{\partial t} = -x \left(\frac{a_1 + 2a_2 t + \cdots + r a_r t^{r-1}}{1 + a_1 t + a_2 t^2 + \cdots + a_r t^r} \right) G(t,x), \tag{27}$$

$$\left(1 + a_1 t + a_2 t^2 + \cdots + a_r t^r \right) \frac{\partial G(t,x)}{\partial t} = -x \left(a_1 + 2a_2 t + \cdots + r a_r t^{r-1} \right) G(t,x),$$

$$\sum_{k=0}^{\infty} g_{k+1}(x)\frac{t^k}{k!} + a_1 \sum_{k=0}^{\infty} g_{k+1}(x)\frac{t^{k+1}}{k!} + a_2 \sum_{k=0}^{\infty} g_{k+1}(x)\frac{t^{k+2}}{k!} + \cdots + a_r \sum_{k=0}^{\infty} g_{k+1}(x)\frac{t^{k+r}}{k!} =$$

$$= -a_1 x \sum_{k=0}^{\infty} g_k(x)\frac{t^k}{k!} - 2a_2 x \sum_{k=0}^{\infty} g_k(x)\frac{t^{k+1}}{k!} - \cdots - r\,a_r\,x \sum_{k=0}^{\infty} g_k(x)\frac{t^{k+r-1}}{k!},$$

that is

$$\sum_{k=0}^{\infty} g_{k+1}(x)\frac{t^k}{k!} + a_1 \sum_{k=1}^{\infty} kg_k(x)\frac{t^k}{k!} + a_2 \sum_{k=2}^{\infty} k(k-1)g_{k-1}(x)\frac{t^k}{k!} + \cdots$$

$$+ a_r \sum_{k=r}^{\infty} k(k-1)\cdots(k-r+1)g_{k-r+1}(x)\frac{t^k}{k!} =$$

$$= -a_1 x \sum_{k=0}^{\infty} g_k(x)\frac{t^k}{k!} - 2a_2 x \sum_{k=0}^{\infty} kg_{k-1}(x)\frac{t^k}{k!} - \cdots$$

$$- r\,a_r\,x \sum_{k=0}^{\infty} k(k-1)\cdots(k-r+2)g_{k-r+1}(x)\frac{t^k}{k!}.$$

Therefore, we can conclude that

Theorem 5. *The sequence $\{g_k(x)\}_{k \in \mathbf{N}}$ satisfies the linear recurrence relation*

$$g_k(x) + a_1(x + k - 1)\,g_{k-1}(x) + a_2(k-1)(2x + k - 2)\,g_{k-2}(x) + \dots$$

$$+ a_r(k-1)(k-2)\cdots(k-r+1)(rx + k - r)\,g_{k-r}(x) = 0.$$

(28)

Extension to the General Case

We now generalize the convolution formula in Section 3.2, putting for shortness
$$[c]_{r-1} = c_1, c_2, \dots, c_{r-1}, \quad [a]_r = a_1, a_2, \dots, a_r,$$

and considering the generating function:

$$G(t,x) = \left(\frac{1 + c_1 t + c_2 t^2 + \cdots + c_{r-1}t^{r-1}}{1 + a_1 t + a_2 t^2 + \cdots + a_r t^r}\right)^x = \sum_{k=0}^{\infty} q_k(x; [c]_{r-1}; [a]_r)\,\frac{t^k}{k!} =$$

$$= \frac{\displaystyle\sum_{k=0}^{\infty} p_k(x; [c]_{r-1})\,\frac{t^k}{k!}}{\displaystyle\sum_{k=0}^{\infty} g_k(x; [a]_r)\,\frac{t^k}{k!}},$$

(29)

so that we find the convolution formula:

$$p_k(x; [c]_{r-1}) = \sum_{h=0}^{k} \binom{k}{h} g_{k-h}(x; [a]_r)\,q_h(x; [c]_{r-1}; [a]_r),$$

(30)

and the $q_h(x; [c]_{r-1}; [a]_r)$ can be found recursively by solving the infinite system

$$\begin{cases} q_0(x; [c]_{r-1}; [a]_r) = 1, \\ q_k(x; [c]_{r-1}; [a]_r) = p_k(x; [a]_r) - \displaystyle\sum_{h=0}^{k-1} \binom{k}{h} g_{k-h}(x; [a]_r)\,q_h(x; [c]_{r-1}; [a]_r). \end{cases}$$

(31)

6. Illustrative Examples—Second Order Recurrences

- **Gegenbauer polynomials** [14], defined by

$$(1 - 2yt + t^2)^{-\lambda} = \sum_{k=0}^{\infty} C_k^{(\lambda)}(y)\, t^k\,,$$

  $x = \lambda, a = -2y, b = 1, g_k(\lambda; -2y, 1) = k!\, C_k^{(\lambda)}(y)$.

- **Sinha polynomials** [15], defined by

$$[1 - 2yt + (2y - 1)t^2]^{-\nu} = \sum_{k=0}^{\infty} S_k^{(\nu)}(y)\, t^k\,,$$

  $x = \nu, a = -2y, b = (2y - 1), g_k(\nu; -2y, 2y - 1) = k!\, S_k^{(\nu)}(y)$.

- **Fibonacci polynomials** [16], defined by

$$\frac{t}{1 - yt - t^2} = \sum_{k=0}^{\infty} F_k(y)\, t^k\,, \qquad F_k(1) = F_k \quad (\text{Fibonacci numbers})\,.$$

  We have:

$$\frac{t}{1 - yt - t^2} = t \sum_{k=0}^{\infty} g_k(1; -y, -1)\, \frac{t^k}{k!} = \sum_{k=0}^{\infty} k!\, F_k(y)\, \frac{t^k}{k!}\,,$$

  so that

$$\sum_{k=1}^{\infty} k g_{k-1}(1; -y, -1)\, \frac{t^k}{k!} = \sum_{k=0}^{\infty} k!\, F_k(y)\, \frac{t^k}{k!}\,.$$

  Since $F_0(y) = 0$, we find

$$F_k(y) = \frac{1}{(k-1)!}\, g_{k-1}(1; -y, -1)\,.$$

- **Lucas polynomials** [16], defined by

$$\frac{2 - yt}{1 - yt - t^2} = \sum_{k=0}^{\infty} L_k(y)\, t^k\,, \qquad L_k(1) = L_k \quad (\text{Lucas numbers})\,.$$

  We have:

$$\frac{2 - yt}{1 - yt - t^2} = 2 \sum_{k=0}^{\infty} g_k(1; -y, -1)\, \frac{t^k}{k!} - y \sum_{k=1}^{\infty} k\, g_{k-1}(1; -y, -1)\, \frac{t^k}{k!} = \sum_{k=0}^{\infty} k!\, L_k(y)\, \frac{t^k}{k!}\,.$$

  Since $L_0(y) = 0$, we find

$$L_k(y) = \left(\frac{2}{k!} - \frac{y}{(k-1)!}\right) g_{k-1}(1, -y, -1)\,.$$

Illustrative Examples—Higher Order Recurrences

- **Humbert polynomials** [14], defined by

$$(1 - 3yt + t^3)^{-\lambda} = \sum_{k=1}^{\infty} u_k(y)\, t^k\,,$$

$x = \lambda,\, a_1 = -3y,\, a_2 = 0,\, a_3 = 1,\, g_k(\lambda; -3y, 0, 1) = k!\, u_k(y).$

- **First kind Chebyshev polynomials in several variables** [17–20], defined by

$$\frac{r - (r-1)u_1 t + (r-2)u_2 t^2 + \cdots + (-1)^{r-1} u_{r-1} t^{r-1}}{1 - u_1 t + u_2 t^2 - \cdots + (-1)^{r-1} u_{r-1} t^{r-1} + (-1)^r t^r} = \sum_{k=0}^{\infty} T_k(u_1, \ldots, u_{r-1})\, t^k\,,$$

$x = 1,\, c_1 = -\frac{r-1}{r} u_1,\, \ldots,\, c_{r-1} = \frac{(-1)^{r-1}}{r} u_{r-1},\ \ a_1 = -u_1,\, \ldots,\, a_{r-1} = (-1)^{r-1} u_{r-1},\, a_r = (-1)^r,$

$$q_k(1; [c]_{r-1}; [a]_r) = \tfrac{1}{r}\, k!\, T_k(u_1, \ldots, u_{r-1}).$$

- **Second kind Chebyshev polynomials in several variables** [17–20], defined by

$$\frac{1}{1 - u_1 t + u_2 t^2 - \cdots + (-1)^{r-1} u_{r-1} t^{r-1} + (-1)^r t^r} = \sum_{k=0}^{\infty} U_k(u_1, \ldots, u_{r-1})\, t^k\,,$$

$x = 1,\, a_1 = -u_1,\, \ldots,\, a_{r-1} = (-1)^{r-1} u_{r-1},\, a_r = (-1)^r,\ \ g_k(1; [a]_r) = k!\, U_k(u_1, \ldots, u_{r-1}).$

- **Tribonacci polynomials** [21], defined by

$$\frac{t}{1 - y^2 t - y t^2 - t^3} = \sum_{k=0}^{\infty} \tau_k(y)\, t^k\,.$$

We have:

$$\frac{t}{1 - y^2 t - y t^2 - t^3} = t \sum_{k=0}^{\infty} g_k(1, -y^2, -y, -1)\, \frac{t^k}{k!} = \sum_{k=0}^{\infty} k!\, \tau_k(y)\, \frac{t^k}{k!}\,,$$

so that

$$\sum_{k=1}^{\infty} k\, g_{k-1}(1, -y^2, -y, -1)\, \frac{t^k}{k!} = \sum_{k=0}^{\infty} k!\, \tau_k(y)\, \frac{t^k}{k!}\,.$$

Since $\tau_0(y) = 0$, we find

$$\tau_k(y) = \frac{1}{(k-1)!}\, g_{k-1}(1, -y^2, -y, -1)\,.$$

7. Extension to Matrix Polynomials

Extensions to Matrix polynomials have become a fashionable subject recently (see e.g., [22] and the references therein).

The above results can be easily extended to Matrix polynomials assuming, in Equations (1), (7), (10), (17), (20), and (22), instead of x, a complex $N \times N$ matrix A, satisfying the condition:

A is stable, that is, denoting by $\sigma(A)$ the spectrum of A, this results in: $\forall \lambda \in \sigma(A),\, \Re\lambda > 0.$

Since all powers of a matrix A commute, even every matrix polynomial commute. More generally, if $\sigma(A) \subset \Omega$, where Ω is an open set of the complex plane, for any holomorphic functions f and g, this results in:

$$f(A)g(A) = g(A)f(A),$$

that is, the involved matrix functions commute.

Under these conditions, considering the generating function:

$$G(t, A) = \left(\frac{1}{1 + a_1 t + a_2 t^2 + \cdots + a_r t^r} \right)^A =$$
$$= \exp\left\{ -A \log\left[1 + a_1 t + a_2 t^2 + \cdots + a_r t^r \right] \right\} = \sum_{k=0}^{\infty} g_k(A; a_1, a_2, \ldots, a_r) \frac{t^k}{k!}, \tag{32}$$

recalling positions (18), and putting as before:

$$G_{\alpha_h}(t, A) = \exp\left[-A \log\left(1 - \alpha_h t \right) \right] = \sum_{k=0}^{\infty} p_{1,k}(A, \alpha_h) \frac{t^k}{k!}, \qquad (h = 1, 2, \ldots, r), \tag{33}$$

we find the result:

Theorem 6. *The sequence $\{g_k(A)\}_{k \in \mathbf{N}}$ satisfies the convolution formula:*

$$g_k(A; \alpha_1, \alpha_2, \ldots, \alpha_r) =$$
$$= \sum_{\substack{k_1 + k_2 + \cdots + k_r = k \\ 0 \leq k_i \leq k}} \binom{k}{k_1, k_2, \ldots, k_r} p_{1,k_1}(A, \alpha_1) p_{2,k_2}(A, \alpha_2) \cdots p_{r,k_r}(A, \alpha_r). \tag{34}$$

Furthermore, denoting by I the identity matrix, we can proclaim the theorem:

Theorem 7. *The sequence $\{g_k(A) := g_k(A; a_1, a_2, \ldots, a_r)\}_{k \in \mathbf{N}}$ satisfies the linear recurrence relation*

$$g_k(A) + a_1[A + (k-1)I] g_{k-1}(A) + a_2(k-1)[2A + (k-2)I] g_{k-2}(A) + \ldots$$
$$+ a_r(k-1)(k-2) \cdots (k-r+1)[rA + (k-r)I] g_{k-r}(A) = 0. \tag{35}$$

8. Conclusions

Starting from the results by Chen Zhuoyu and Qi Lan [9], we have shown convolution formulas and linear recurrence relations satisfied by a generating function containing several parameters. This can be used for number sequences (assuming $x = 1$) or polynomial sequences, depending on several parameters. Illustrative examples are shown both in case of second order or high order recurrence relations.

An extension to the case of matrix polynomials is also included.

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