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Issues in the Expansion and Contraction of Operators

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Abstract: This paper explores fundamental issues in the correct contraction and expansion of operators, with a primary focus on the concept of symmetry within operator theory. Special attention is given to how symmetry influences the behavior of operators, particularly regarding their approximation and convergence properties. In the domains of quantum mechanics and condensed matter physics, such operators are essential for modeling phenomena like superconductivity, excitons, and surface states. The symmetric properties of operators have a profound impact on the physical interpretations and predictions these models generate. A rigorous analysis is provided regarding the existence of correct contractions and expansions for a specific class of nonlinear operators, demonstrating how symmetry affects the structural integrity of operators under natural conditions. The study presents a comprehensive description of the set of all correct contractions, expansions, and regular expansions, with an application to a third-order nonlinear differential expression. Additionally, a condition for the unique solvability of a Bitsadze–Samarskii-type problem is derived, showcasing how symmetry plays a crucial role in guiding the solution of complex physical models. Furthermore, the paper emphasizes the importance of preserving symmetry in the construction of operators, ensuring the consistency and accuracy of mathematical models. This has significant implications for both theoretical research and practical applications in various fields, including nuclear physics and quantum theory.

Keywords: operator; correct extension; correct contraction; regular extension; Bitsadze–Samarskii-type problem

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1. Introduction

Problems with extension and contraction arise mainly in the study of differential operators. Thus, in the works [1–3], questions of the regular extension of the "minimal" linear operator in the Hilbert space are considered. M.I. Vishik applies his results to the study of general boundary value problems for elliptic differential equations of the second order. In the work of A.A. Dezin, the properties of extensions for ordinary differential equations with constant coefficients and the question of the correct formulation of boundary value problems for "nonclassical" equations of mathematical physics are investigated. At the same time, A.A. Dezin simultaneously proves a statement similar to the theorem of M.I. Vishik, which allows one to describe the class of solvable ("correct") extensions of the original operator. In the mentioned works, the Hilbert property of the space and the linearity of the operators under consideration are essential. Moreover, the regular extensions under study obeyed both the "maximal" and "minimal" operators simultaneously. And in the work [4] the requirement of the Hilbert space was removed and it is sufficient that the operators under consideration are additive. In this case, correct restrictions of the "maximal" operator and the questions of extension of the "minimal" operator in a Banach space are considered separately. This approach to questions concerning the extension and restriction of operators, as subsequent publications on this topic show, turned out to be very rich in applications. For example, in the work [5] the relationship between the correct restrictions on the product of operators and the correct restrictions on its factors is established, and, using the abstract results obtained, the author manages to formulate a correct boundary value problem for a hyperbolic equation in a circle and a correct problem of Bitsadze–Samarskii type. In general, after the 1990s, numerous work applications of the abstract theories of correct restriction, correct extension, and regular extension, which were developed in the works [4,6], were published. For example, the theory of correct operators' restrictions is successfully applied in the study of singular values of differential operators. In the work [7], the correct restrictions of the Navier–Stokes system are studied in terms of the correct formulation of boundary value problems. Also worth noting is the work [8], where a general formula is considered to resolve a correct restriction of an elliptic operator of the second order. In the work [9], the author, relying on the results of work [10], proves the non-Volterra property of the Laplace operator.

However, the following questions remain open: (1) is it possible to remove the conditions of the "minimality" and "maximality" of the original operators when studying, respectively, correct extensions and correct restrictions?; (2) the existence of a correct extension and a correct restriction for the corresponding original operators; (3) is it possible to apply the theory of correct extensions and correct restrictions to nonlinear operators? In the work [11], the author, relying on the results of work [12], proves the non-Volterra property of the Laplace operator. The same problem is considered in the works [13–17].

In this paper, it is shown that these restrictions to the "minimality" and "maximality" of the corresponding initial operators are not essential, and more general approaches to the issues of operator restriction and extension are put forward. The relevance of this issue is that there are correct restrictions of the "maximal" operator that are not a correct extension of the "minimal" operator, for example, the Bitsadze–Samarskii-type problem. Also, there are correct extensions of the "minimal" operator that are not a correct restriction of the "maximal" operator, for example, the so-called "loaded equations" (see [10]). It is also proven that, if the manifold $R(\mathrm{A}_0)$ is complemented in some specially constructed Banach space, then, for the operator A_0, there is at least one regular extension, $\tilde{A}$ (or, for the operator A, there is at least one regular restriction, $\tilde{A}$).

In the work [4], a complete description of the set of all correct restrictions of the maximal operator A, the set of all correct extensions of the minimal operator A_0, and the set of all regular extensions of the operator A_0 is also given. In this work, a similar complete description of correct restrictions to the nonlinear operator A_N, the set of all correct extensions of the nonlinear operator A_{0N}, and the set of all regular extensions of the nonlinear operator are given, A_{0N}; i.e., for the first time, the theory of extension and restriction is applied to nonlinear operators acting in a Banach space, while the methods of the work [7] are significantly used.

As an example, a nonlinear differential expression of the third order in a cube is considered, $[0,1] \times [0,1] \times [0,1]$. The condition of unique solvability of one Bitsadze–Samarskii-type problem is shown.

2. Main Results

Let A_0 and A be additive closed operators of a Banach space Binto itself, such that the following conditions are satisfied:

(i) $A_0 \subset A$;
(ii) For the operator A_0, there is a continuous left inverse, A_{0l}^{-1};
(iii) For the operator, there exists a continuous right inverse, A_r^{-1} and $\mathrm{Ker}\, A \neq 0$.

The manifold $D(A)$ with respect to the norm is as follows:

$$\|u\|_{\mathfrak{M}} = \|u\|_{\mathbb{B}} + \|Au\|_{\mathbb{B}}, u \in D(A), \tag{1}$$

In this case, we denote the resulting Banach space as $\mathfrak{M}$. It is obvious that $\overline{D(\mathrm{A})} = \mathfrak{M}$ and $R(\mathrm{A}) = \mathfrak{M}$.

Definition 1. *An operator A_1 is called a regular operator A_0 extension (or regular operator A contraction) if*

1. $A_0 \subset A_1 \subset A$;
2. *There exists a continuous inverse, A_1^{-1}, defined on $\mathfrak{M}$.*

Theorem 1. *Let A_0 and the additive operators satisfy conditions (i)–(iii), and manifolds $R(A_0)$ and $\mathrm{Ker}\, A$ are complemented in space $\mathfrak{M}$. Then, there is at least one regular operator expansion, A_0 (or at least one regular operator contraction, A).*

Proof. To prove this theorem, we first prove Lemma 1. First, we give some notation.

Let closed additive subspaces $\overline{X_1}$ and be given $\overline{X_2}$, such that the following decomposition holds:

$$\mathfrak{M} = \overline{X_1} + \overline{X_2}, \tag{2}$$

Let us introduce the notation $X_1 = D(A) \cap \overline{X_1}, X_2 = D(A) \cap \overline{X_2}$, and let $M = AX_1, N = AX_2$. Suppose that the sets M and N are complemented in space $\mathfrak{M}$; that is, there exists a projector, P_1, such that $R(P_1) = M, \mathrm{Ker}\, P_1 = N$. Then, the lemma holds. □

Lemma 1. *If the operators A_1 and A_2 are restrictions of the operator A on the manifolds X_1 and X_2, respectively, then, for the operator to be invertible, it is necessary and sufficient that the operators A_1 and are invertible A_2. In this case, the following equality holds:*

$$\mathrm{A}^{-1} = \mathrm{A}_1^{-1}\mathrm{P}_1 + \mathrm{A}_2^{-1}(I - \mathrm{P}_1) \tag{3}$$

Proof. Let there exist A^{-1}. Then, for each $y \in \mathfrak{M}$, there is a unique element, $x \in D(A)$, such that $\mathrm{A}^{-1}y = x$. According to the condition of the lemma, the following expansions are satisfied:

$$\begin{aligned} x &= x_1 + x_2, \quad x_1 \in X_1, x_2 \in X_2, \\ y &= y_1 + y_2, \quad y_1 \in M, y_2 \in N, \end{aligned}$$

and, at the same time, $y_1 = A_1x_1, y_2 = A_2x_2$. Then, $y_1 = Ax_1$. Therefore, $x_1 = \mathrm{A}^{-1}y_1 = \mathrm{A}^{-1}P_1y$. Similarly, $x_2 = \mathrm{A}^{-1}(I - P_1)y$. From these two relations, we have

$$\begin{aligned} A_1^{-1}P_1 &= A^{-1}P_1, \\ A_2^{-1}(I - P_1) &= A^{-1}(I - P_1), \end{aligned}$$

And, via direct verification, it is easy to obtain that

$$A_1 A^{-1} P_1 = A^{-1} P_1 A_1 = I,$$
$$A_2 A^{-1}(I - P_1) = A^{-1}(I - P_1) A_2 = I.$$

Conversely, let there exist inverse operators, A_1^{-1}, A_2^{-1}. If $y \in \mathfrak{M}$, $y_1 = P_1 y \in M$, $y_2 = (I - P_1)y \in N$, then there exist unique elements, $x_1 \in X_1$ and $x_2 \in X_2$, such that $A_1^{-1} y_1 = x_1$, $A_2^{-1} y_2 = x_2$. Now, we define the operator $\tilde{A}^{-1}$ by the following formula:

$$\tilde{A}^{-1} y = A_1^{-1} P_1 y + A_2^{-1}(I - P_1) y, \quad y \in \mathfrak{M}.$$

According to this definition, $R(\tilde{A}^{-1}) = D(A)$, and $A\tilde{A}^{-1} y = y, y \in \mathfrak{M}$. On the other hand, via the expansion (2) for each $x \in D(A)$, we have

$$x = x_1 + x_2, \quad x_1 \in X_1, x_2 \in X_2.$$

That is why

$$\tilde{A}^{-1} A x = \tilde{A}^{-1} y = A_1^{-1} P_1 y + A_2^{-1}(I - P_1) y =$$
$$= A_1^{-1} y_1 + A_2^{-1} y_2 = x_1 + x_2 = x.$$

Then, the inverse operator A^{-1} is defined, and moreover, $A^{-1} = \tilde{A}^{-1}$. The lemma has been completely proven. □

Proof of Theorem 1. According to the conditions of the theorem $\mathcal{M} \subset \mathfrak{M}$, for which $\mathfrak{M} = R(A_0) + \mathcal{M}$, given $\mathcal{N}$, there is a maximal additive variety of $D(\mathrm{A})$, such that $\mathrm{A}\mathcal{N} = \mathcal{M}$ (this is possible since $R(\mathrm{A}) = \mathfrak{M}$). Obviously, $\mathrm{Ker\,A} \subset \mathcal{N}$. Set $\mathfrak{R} = \mathfrak{M} -$ Ker A, and let $\mathcal{W} = \mathcal{N} \cap \mathfrak{R}$. Hence, $A\mathcal{W} = \mathcal{M}$.

Furthermore, if A_1 is the restriction to the operator A for the manifolds $\mathcal{W}$, then we will show that the operator is invertible. In fact, let $A_1 : \mathcal{W} \to \mathcal{M}$ hold the equalities $A_1 x_1 = y$ for some $x_1, x_2 \in \mathcal{W}, y \in \mathcal{M} \quad A_1 x_2 = y$. Then, $A(x_1 - x_2) = 0$ or $x_1 - x_2 \in \mathrm{KerA}$. But, according to condition $\Re \cap \mathrm{Ker}\, A = 0$, all the more so, $\mathcal{W} \cap \mathrm{Ker}\, A = \{0\}$. Therefore, $x_1 = x_2$. Consequently, there exists A_1^{-1}. Let us define the set.

$$D(\tilde{A}) = \{x = x_0 + \omega : x_0 \in D(A_0), \omega \in \mathcal{W}\}$$

and an operator, $\tilde{A}$, with domain $D(\tilde{A})$, acting according to the law

$$\tilde{A} x = A_0 x_0 + A_1 \omega,$$

where $x \in D(\tilde{A}), x_0 \in D(A_0), \omega \in \mathcal{W}, x = x_0 + \omega$. By definition, $A_0 \subset \tilde{A} \subset A$, and $R(\tilde{A}) = \mathfrak{M}$. Therefore, based on Lemma 1, there is an inverse operator, $\tilde{A}^{-1}$, to the operator $\tilde{A}$. Since the operator $\tilde{A}^{-1}$ is closed and defined throughout the space $\mathfrak{M}$, it is continuous. The theorem is proven.

Let $\mathcal{R}_1$ denote the set of all continuous mappings $K : \mathbb{B} \to D(\mathrm{A})$, satisfying the following condition: $KR(A_0) \subseteq D(A_0), \forall K \in \mathcal{R}_1$. □

Theorem 2. *Let A be a fixed regular extension of the operator A_0. Then, the following applies:*

(a) *If $K \in \mathcal{R}_1$, then the operator A_k^{-1}, defined by the formula*

$$\mathrm{A}_{\mathrm{k}}^{-1} f = \tilde{A}^{-1} f + K f - \tilde{A}^{-1} \mathrm{A} K f, f \in \mathbb{B}, \tag{4}$$

is the inverse of some regular extension, A_{k}, of the operator A_0 with respect to A;

(b) *And, if A_{K} is some regular extension of the operator A_0 with respect to A, then there exists $K \in \mathcal{R}_1$, such that equality (4) is satisfied.*

Proof. (a) Let $K \in \mathcal{R}_1$; then, it is clear that A_k^{-1}, defined by Formula (4), is defined on the entire space $\mathbb{B}$, and it is easy to show that there exists an inverse operator for it, A_{k}. Since $\tilde{A} \subset A$ and $A\left(Kf - \tilde{A}^{-1}\mathrm{A}Kf\right) = 0$, then $\mathrm{A}_{\kappa} \subset A$. Now, we will show that $A_0 \subset \mathrm{A}_{\mathrm{k}}$. Indeed, if $u_0 \in D(A_0), \exists f_0 \in R(A_0) : A_0^{-1}f_0 = u_0$. Therefore,

$$\mathrm{A}_{\mathrm{k}}^{-1}f_0 = \tilde{A}^{-1}f_0 + Kf_0 - \tilde{A}^{-1}\mathrm{A}Kf_0 = \tilde{A}^{-1}f_0 + Kf_0 - Kf_0 = u_0$$

i.e., $A_0 \subset \mathrm{A}_{\mathrm{k}}$ and A_{k} are a regular extension of the operator A_0.

(b) Let A_{k} be a regular extension of the operator. Then, with A_0, we can take the operator A_k^{-1} as K. It is clear that $K = \mathrm{A}_k^{-1} \in \mathcal{R}_1$. Taking into account the identities $\mathrm{A}_{\mathrm{k}}^{-1}f = \tilde{A}^{-1}f + \mathrm{A}_k^{-1}f - \tilde{A}^{-1}f$ and $\mathrm{A}\tilde{A}^{-1} = I$, we have

$$\mathrm{A}_{\mathrm{k}}^{-1}f = \tilde{A}^{-1}f + Kf - \tilde{A}^{-1}\mathrm{A}Kf, \quad f \in \mathbb{B}$$

The theorem is proven.

This theorem is also proven as the corresponding theorem from [5].

Now, under the conditions of the notation adopted above, we consider some bijective mapping, $N : \mathbb{B} \to \mathbb{B}$. Then, the $N(0) = 0$ operator is defined, $A_N = A \cdot N$, in space $\mathfrak{M}$ since there is a right inverse, A_r^{-1} and $\mathrm{Ker}\, A \neq \{0\}$; then, for the operator A_N, there is also a right inverse, A_{Nr}^{-1}, $A_{Nr}^{-1} = N^{-1} \cdot A_r^{-1}$, and $\mathrm{Ker}\, A_N = N(\mathrm{Ker}\, A) \neq 0$. If $\tilde{A}$ is a fixed regular extension of the operator A_0, then the operator $\tilde{A}_N = \tilde{A} \cdot N$ is a regular extension of the operator $A_{0N} = A_0 \cdot N$.

Let $\{\mathrm{A}_0, A\}$ be the set of all regular extensions of the operator A_0, and let $\{A_{0N}, A_N\}$ be the set of all regular extensions of the operator A_{0N}. Then, we have the case in which the operator N implements a one-to-one correspondence between the sets $\{\mathrm{A}_0, A\}$, and $\{A_{0N}, A_N\}$. Moreover, if $\mathrm{A}_1 \in \{\mathrm{A}_0, A\}$, then $\mathrm{A}_1 \cdot N \in \{A_{0N}, A_N\}$, and conversely, if $\tilde{A}_N \in \{A_{0N}, A_N\}$, then $N^{-1} \cdot \tilde{A}_N \in \{\mathrm{A}_0, A\}$. Thus, we have proven the following theorem. □

Theorem 3. *A bijective mapping, $N : \mathbb{B} \to \mathbb{B}$, establishes a one-to-one correspondence between sets $\{\mathrm{A}_0, A\}$ and $\{A_{0N}, A_N\}$. Moreover, if $\mathrm{A}_1 \in \{\mathrm{A}_0, A\}$, then $\mathrm{A}_1 \cdot N \in \{A_{0N}, A_N\}$, and conversely, if $\tilde{A}_N \in \{A_{0N}, A_N\}$, then $N^{-1} \cdot \tilde{A}_N \in \{\mathrm{A}_0, A\}$.*

If Theorem 2 describes a set, $\{\mathrm{A}_0, A\}$, in the language of some continuous mappings, $K : \mathbb{B} \to D(\mathrm{A}), (KR(A_0) \subseteq D(A_0), \forall K \in \mathcal{R}_1)$, then a similar theorem is valid to describe the set $\{A_{0N}, A_N\}$.

Theorem 4. *Let A be a fixed regular extension of the operator A_0. Then, the following applies:*

(a) *If $K \in \mathcal{R}_1$, then the operator A_{NK}^{-1} defined by the formula*

$$\mathrm{A}_{NK}^{-1}f = N^{-1}\left(\tilde{A}^{-1}f + Kf - \tilde{A}^{-1}\mathrm{A}Kf\right), \quad f \in \mathfrak{M}, \tag{5}$$

is the inverse of some regular operator expansion, A_{NK} A_{0N}.

(b) *And, if A_{NK} is some regular extension of the operator A_{0N}, then there exists $K \in \mathcal{R}_1$, such that equality (5) is satisfied.*

Similar statements are true for the correct contraction and correct expansion of the corresponding operators. For completeness, we recall some definitions and concepts. An operator is given, $A_0 : D(A_0) \to \mathbb{B}$, in a Banach space, $\mathbb{B}$, for which there exists a continuous left inverse, A_{0l}^{-1}; i.e., on $R(\mathrm{A}_0) \subset \mathbb{B}$, there is a continuous left inverse, A_{0l}^{-1} if the operator $A_1 : D(A_1) \to \mathbb{B}s$ satisfies the following conditions:

1. $A_0 \subset A_1$;
2. In the space $\mathbb{B}$, there exists a continuous inverse, A_1^{-1}; then, the operator A_1 is called a correct extension of the operator A_0.

Now, let a Ban operator be defined, $A : D(A) \to \mathbb{B}$, in a Banach space, $D(A) \subset \mathbb{B}$, for which there exists a continuous right inverse, A_r^{-1} and $\operatorname{Ker} A \neq \{0\}.D(A)$. We close the manifold with respect to the norm:

$$\|u\|_{\mathfrak{M}} = \|u\|_{\mathbb{B}} + \|Au\|_{\mathbb{B}}, u \in D(A), \tag{6}$$

In this case, we denote the resulting Banach space as $\mathfrak{M}$. It is obvious that $\overline{D(\mathrm{A})} = \mathfrak{M}$, and $R(\mathrm{A}) = \mathfrak{M}$ if the operator $A_1 : D(A_1) \to \mathfrak{M}$ satisfies the following conditions:

1. $A_1 \subset A$;
2. In the space $\mathfrak{M}$, there exists a continuous inverse, A_1^{-1}; then, the operator A_1 is called a correct restriction of the operator A.

If the relation$A_0 \subset A_1 \subset A$ holds, then the operator A_1 is a regular extension of the operator A_0 (or a regular contraction of the operator A). That is, the operator A_1 is simultaneously both a regular extension of the operator A_0 and a regular contraction of the operator A.

Let it be as before with $N : \mathbb{B} \to \mathbb{B}$ being a bijective mapping, such that $N(0) = 0$. Then, the operator $\mathfrak{M}$ is defined in the space $A_N = A \cdot N$ since there is a right inverse, A_r^{-1} and $\operatorname{Ker} A \neq \{0\}$; then, for A_N, there is also a right inverse for the operator A_{Nr}^{-1}, and $A_{Nr}^{-1} = N^{-1} \cdot A_r^{-1}$ and $\operatorname{Ker} A_N = N(\operatorname{Ker} A) \neq \{0\}$. If $\tilde{A}$, a fixed correct restriction of the operator A, then $\tilde{A}_N = \tilde{A} \cdot N$ is a correct restriction of the operator A_N. As $\{A\}$ and $\{A_N\}$, we denote the set of all correct restrictions of the operators A and A_N, respectively. Then a statement similar to Theorem 4 is true. Namely, if $P, \mathrm{Q} \subset \mathfrak{M}$, then $\mathcal{H}(P, \mathrm{Q})$; we denote the set of all continuous mappings, $\mathrm{K} : \mathrm{P} \to \mathrm{Q}$, and Then the theorem is valid:

Theorem 5. *Let $\tilde{A}$ be a fixed correct restriction of the operator A. Then, the following applies:*

(a) *If $K \in \mathcal{H}(\mathfrak{M}, D(\mathrm{A}))$, then the operator A_{NK}^{-1}, defined by the formula*

$$\mathrm{A}_{NK}^{-1} f = N^{-1}\left(\tilde{A}^{-1} f + Kf - \tilde{A}^{-1}\mathrm{A}Kf\right), \quad f \in \mathfrak{M} \tag{7}$$

is the inverse of some correct restriction, A_{NK}, of the operator A_N.

(b) *And, if $A_{NK} \in \{A_N\}$, then there exists $K \in \mathcal{H}(\mathfrak{M}, D(\mathrm{A}))$, such that equality (7) is satisfied.*

3. Example

Here, as an application of the above results, we consider the cubed differential expression $Q = \{(x, y, z) \in R^3 : 0 \leq x, y, z \leq 1\}$.

$$Au \equiv \frac{\partial^3 u}{\partial x \partial y \partial z} \tag{8}$$

Let A be the closure of the operator $\frac{\partial^3}{\partial x \partial y \partial z}$, initially defined in $C^\infty(Q)$, followed by the closure in the norm of the space $C(Q)$.

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = f(x,y,z), \quad f \in C(Q) \tag{9}$$

$$\frac{\partial^2 u}{\partial y \partial z} = 0, \quad 0 \leq y,z \leq 1 \tag{10}$$

$$\frac{\partial u}{\partial z} = 0, \quad 0 \leq x,z \leq 1 \tag{11}$$

$$u_{/y=0} = 0, \quad 0 \leq x,y \leq 1 \tag{12}$$

is uniquely solvable for any right-hand-side $f \in C(Q)$, and the only solution to the specified problem has the form

$$u(x,y,z) = \int_0^z \int_0^y \int_0^x f(t,\tau,\vartheta)dtd\tau d\vartheta$$

For any solution to the equation

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = 0$$

in space $C(Q)$ we can represent as

$$\text{Ker } A = u(x,y,z) = \int_0^z \int_0^y \varphi_3(t,\tau)dtd\tau + \int_0^z \varphi_2(x,\tau)d\tau + \varphi_1(x,y),$$

where $\varphi_i(t,\tau) \in C([0,1] \times [0,1]), i = 1,2,3.$

Next, integrating the boundary condition (10) twice in the intervals $(0,x)$ and $(0,y)$, we obtain the following relation.

$$u(0,y,z) = u(0,y,0) + u(0,0,z) - u(0,0,0), \quad 0 \leq y,z \leq 1 \tag{13}$$

Integrating the boundary condition (11), we obtain

$$u(x,0,z) = u(x,0,0), \quad 0 \leq x,z \leq 1 \tag{14}$$

and from condition (12), we come to the conclusion,

$$u(0,y,0) = u(x,0,0) = u(x,0,0) = u(0,0,0) = 0$$

From this and from (13) and (14), it follows that problems (9)–(12) are equivalent to the following problem:

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = f(x,y,z), \quad f \in C(Q),$$

$$u_{/x=0} = 0, \quad 0 \leq y,z \leq \ 1$$

$$u_{/y=0} = 0, \quad 0 \leq x,z \leq \ 1$$

$$u_{/z=0} = 0,$$

Now consider the bijective mapping $N : C(Q) \to C(Q)$, acting according to the formula $Nu = u^3, u \in C(Q)$. The operator generated via the last boundary value problem is denoted as $\tilde{A}$, which is a correct restriction to the original operator A. Therefore, the operator $\tilde{A}_N = \tilde{A} \cdot N$ is generated via the following boundary value problem:

$$u^2 \frac{\partial^3 u}{\partial x \partial y \partial z} + 2Pu = f(x,y,z), \quad f \in C(Q) \tag{15}$$

$$u_{/x=0} = 0, \quad 0 \leq y,z \leq 1 \tag{16}$$

$$u_{/y=0} = 0, \quad 0 \leq x,z \leq 1 \tag{17}$$

$$u_{/z=0} = 0, \quad 0 \leq x,y \leq 1 \tag{18}$$

where P denotes the following non-linear operator:

$$Pu = u\left[\frac{\partial^2 u}{\partial x \partial y} \cdot \frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial x \partial z} \cdot \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial z \partial y} \cdot \frac{\partial u}{\partial x}\right] + \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial z} \tag{19}$$

The only solution to the boundary value problem (15)–(18) is written as follows:

$$u(x,y,z) = \left(3 \cdot \int_0^z \int_0^y \int_0^x f(t,\tau,\vartheta) dt d\tau d\vartheta\right)^{\frac{1}{3}} \tag{20}$$

Let equation $z = \varphi(x,y)$ be a certain smooth surface located in a cube, Q, such that each line parallel to the axis O opposite the cube intersects this surface at only one point. It is also easy to show the following theorem.

Theorem 6. *The operator A_1 is generated via the following Bitsadze–Samarskii-type problem:*

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = f(x,y,z), \quad f \in C(Q),$$
$$u(0,y,z) = 0, \quad 0 \leq y,z \leq 1,$$
$$u(x,0,z) = 0, \quad 0 \leq x,z \leq 1,$$
$$a_1 u(x,y,0) + a_2 u(x,y,\varphi) + a_3 u(x,y,1) = 0, \quad 0 \leq x,y \leq 1,$$

where $a_i = a_i(x,y) \in C([0,1] \times [0,1]), i = 1,2,3$ is a correct contraction of operator A that is necessary and sufficient for $a^*(x,y) \neq 0$. Here,

$$a^*(x,y) = a_1(x,y) + a_2(x,y) + a_3(x,y).$$

Moreover, the inverse operator A_1^{-1} has the following form:

$$\begin{aligned}
&\left(A_1^{-1} f\right)(x,y,z) = \\
&= \int_0^x \int_0^y \left[\frac{a_2 + a_3}{a^*} \int_0^z f(t,\tau,\vartheta) dt d\tau + \frac{a_2}{a^*} \int_0^{\varphi(t,\tau)} f(t,\tau,\vartheta) dt d\tau \right. \\
&\left. + \frac{a_3}{a^*} \int_0^1 f(t,\tau,\vartheta) dt d\tau \right] d\vartheta
\end{aligned} \tag{21}$$

Based on Theorems 5 and 6, we have the following theorem.

Theorem 7. *In order for the Bitsadze–Samarskii-type problem*

$$u^2 \frac{\partial^3 u}{\partial x \partial y \partial z} + 2Pu = f(x,y,z), \quad f \in C(Q),$$
$$u(0,y,z) = 0, \quad 0 \le y,z \le 1,$$
$$u(x,0,z) = 0, \quad 0 \le x,z \le 1,$$
$$a_1 u(x,y,0) + a_2 u(x,y,\varphi) + a_3 u(x,y,1) = 0, \quad 0 \le x,y \le 1,$$

to occur and be uniquely solvable, it is necessary and sufficient that $a^*(x,y) \neq 0$, the unique solution to this problem, has the following form:

$$u(x,y,z) = \left(3 \cdot \left(A_1^{-1} f\right)(x,y,z)\right)^{\frac{1}{3}}$$

Here, the non-linear operator P is defined by Formula (19), and the value $\left(A_1^{-1} f\right)(x,y,z)$ is found via equality (21).

4. Conclusions

There is no doubt that the general approach to the theory of correct contraction and correct expansion considered here will significantly expand the scope of application of this theory. It can be successfully applied in the construction of "explicitly solvable" models of physics and technology.

The role of "explicitly solvable" models in physics is well known, and attention to them has been growing in recent years. It is worth paying attention to the possibility of modeling a section of operator theory, the theory of correct contraction and correct expansion. For example, the diffraction problem has been studied, and here, a model of scattering on a resonator with a small hole has been constructed using the correct expansion.

Among the mathematical problems that arise in connection with the scheme described here, we note that it can be successfully applied in the correct formulation of boundary value problems, Bitsadze–Samarskii-type problems, various so-called "multipoint" problems for nonlinear equations of mathematical physics, and many loaded equations.

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