ECAI 2024
U. Endriss et al. (Eds.)

doi:10.3233/FAIA240619

Belief Erasure in Propositional Logic

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Abstract. Belief change is an important topic of knowledge representation and reasoning in artificial intelligence. Within the logical framework, the AGM approach has become a standard and various belief change operations have been considered. While revision, contraction and updating have given rise to a great deal of work, erasure has so far attracted less interest. Erasure is to contraction what update is to revision.This article deals with the study of erasure within the framework of propositional logic. It extends Katsuno and Mendelzon's approach with additional postulates capturing the minimality of change and proposes two representation theorems for erasure operators, one in terms of total preorders on interpretations, the other in terms of partial preorders on interpretations. Finally, it completes the work of Caridroit, Konieczny and Marquis for contraction by proposing a new representation theorem for contraction operators in terms of partial preorders on interpretations.

1 Introduction

Belief change is an important topic in the field of knowledge representation and reasoning in artificial intelligence. The logical approaches have given rise to numerous works since the AGM approach, which has been considered as a standard [2, 11]. Different belief change operations have been studied, revision [2, 11, 7, 21], update [10, 22, 15, 13] and contraction [1, 11, 4] [2].

In order to characterize different semantic approaches of revision in the same framework, Katsuno and Mendelzon [14] restricted the AGM approach to propositional logic. They reformulated the AGM postulates, and proposed two representation theorems, one in terms of total preorders on interpretations, the other in terms of partial preorders on interpretations. This semantic framework has made it possible to clearly distinguish between revision and update [15]. Indeed, belief revision consists in incorporating into an agent's beliefs new information in a static environment, while belief update occurs in a changing environment where new information reflects a change in the agent's environment. In other words, when an agent's beliefs are represented by a logical formula, revision makes the models of this formula evolve as a whole towards the closest models of new information. In contrast, update makes each model of this formula locally evolve towards the closest models of new information.

More recently Caridroit et al. [4] provided a complete reformulation of the AGM postulates for contraction and proposed a representation theorem that characterizes contraction operations in terms of total preorders over interpretations. Belief contraction consists in retracting an agent's existing beliefs in a static environment, while belief erasure occurs in a changing environment where the erased beliefs are not anymore true after the environment has changed. Therefore, erasure is to contraction as what update is to revision.

Consider the example inspired by the one in [15] where beliefs describe two objects A and B in a room. There is a table in the room and the objects may or may not be on the table. Formula a means that *"object A is on the table"* and formula b means that *"object B is on the table"*. Let us assume that an agent's beliefs are represented by the formula $\psi = (a \wedge \neg b) \vee (\neg a \wedge b)$, which expresses that either object A or object B is on the table, but not both. The contraction of ψ by b, i.e., removing the information that B is on the table, does not change the agent's beliefs and $\psi - b \equiv \psi$. On the other hand, suppose the agent's environment has changed, a robot has been sent out with instructions to ensure that B is not on the table. This change in environment results in b being erased from ψ and in this case $\psi \lhd b \equiv (\neg a \wedge b) \vee \neg b$. More formally, $\mathrm{Mod}(\psi) = \{a, b\}$, $\mathrm{Mod}(b) = \{b, ab\}$ and $\mathrm{Mod}(\neg b) = \{a, \emptyset\}$, the models of the agent's beliefs after contraction by b are $\mathrm{Mod}(\psi - b) = \{a, b\}$ while the models of the agent's beliefs after the erasure of b are $\mathrm{Mod}(\psi \lhd b) = \{a, b, \emptyset\}$.

The intuitive difference between contraction and erasure can be explained as follows. Contraction by b means that nothing has changed in the room, object A or object B is on the table but not both, contraction has no effect on the agent's beliefs. On the other hand, erasure by b means that the state of the room has changed. If object B was on the table before the change, it has been moved, but nothing can be deduced about object A's position from the fact that object B is not on the table anymore.

Katsuno and Mendelzon in [15] proposed a semantic definition of erasure as well as basic postulates that this operation should satisfy.

Erasure has been studied like many belief change operations in the context of propositional fragments [6]. However, studies of erasure have remained incomplete because only a set of basic postulates has been proposed for this operation, which has not been sufficient to lead to a representation theorem. The purpose of this paper is to complete the study of erasure by proposing:

- additional postulates capturing the minimality of change of erasure operation,
- a representation theorem for erasure operators in terms of total preorders on interpretations,

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[2] Further details about these belief change operations can be found for example in [19] or [20].

- a representation theorem for erasure operators in terms of partial preorders on interpretations.

Moreover, this paper completes the work of Caridroit, Konieczny and Marquis for contraction by proposing a new representation theorem for contraction operators in terms of partial preorders on interpretations that has been missing so far. Thus, this work puts a final touch to the panorama of the semantic study of belief change operators for contraction and erasure. Furthermore, it provides an unified presentation of the logical properties of four well-known belief change operations, namely revision, contraction update and erasure, highlighting their relationships and differences, which are set out in a summary table. The remainder of this paper is organized as follows: After preliminaries in Section 2 which recall some useful concepts, Section 3 displays a brief overview of the state of the art on belief revision, belief contraction and belief update. Section 4 presents the contribution of the paper on erasure (additional postulates, representation theorems) as well as a representation theorem for contraction operators in terms of partial preorders on interpretations. In Section 5 we give a summary of the semantic study of the four well-known belief change operations considered in this paper, namely revision, update, contraction and erasure. Finally, in Section 6, we draw some relevant conclusions and provide new perspectives for future work.

2 Preliminaries

We consider a finite propositional language $\mathcal{L}$ built up from a finite set of variables (atoms) $\mathcal{U}$ and equipped with standard connectives. An interpretation is a mapping $w : \mathcal{U} \to \{0, 1\}$ that associates each atom with a truth value, 0 for False or 1 for True.

An *interpretation* is represented either by a set $w \subseteq \mathcal{U}$ of atoms (corresponding to the variables set to True) or by its corresponding characteristic bit-vector of length $|\mathcal{U}|$.

As usual, if an interpretation w satisfies a formula $\varphi \in \mathcal{L}$ we call w a *model* of φ. By $\mathrm{Mod}(\varphi)$ we denote the set of all models of the formula φ. Moreover, $\varphi \models \psi$ if $\mathrm{Mod}(\varphi) \subseteq \mathrm{Mod}(\psi)$ and $\psi \equiv \varphi$ if $\mathrm{Mod}(\psi) = \mathrm{Mod}(\varphi)$.

A formula $\psi \in \mathcal{L}$ is *complete* if for any formula $\mu \in \mathcal{L}$, we have $\psi \models \mu$ or $\psi \models \neg\mu$. In an equivalent way, a satisfiable formula ψ is complete if it has exactly one model. A formula $\mu \in \mathcal{L}$ is *a tautology* if it is satisfied by all interpretations.

Let $\mathcal{I}$ be a set of interpretations. We will use binary relations $\leq$ on the set $\mathcal{I}$ to encode preference or plausability relations over interpretations. A *preorder* $\leq : \mathcal{I} \times \mathcal{I}$ is a reflexive and transitive binary relation. The strict preorder associated to $\leq$ is defined as $w < w'$ if $w \leq w'$ and $w' \not\leq w$. An equivalence relation induced by $\leq$, denoted by $\approx$, is defined as $w \approx w'$ if $w \leq w'$ and $w' \leq w$. A preorder $\leq$ on $\mathcal{I}$ is *total* if, for any interpretations w and w', $w \leq w'$ or $w' \leq w$. The set of minimal elements of $\mathcal{I}$ with respect to $\leq$ is defined as $\min_{\leq}(\mathcal{I}) = \{w \in \mathcal{I} \mid \text{ there is no } w' \in \mathcal{I} \text{ such that } w' < w\}$.

Let ψ and μ be two propositional formulas and w and w' be two interpretations seen as sets of atoms. Let $w \Delta w'$ be the set of atoms defined as the symmetric difference between w and w'. We denote by $|\Delta|_w^{min}(\mu)$ the minimal number of atoms on which a model of μ and w differ. More formally, $|\Delta|_w^{min}(\mu) = min\{|w\Delta w'| : w' \in \mathrm{Mod}(\mu)\}$. Besides, we denote by $\Delta_w^{min}(\mu)$, the minimal subsets of atoms w.r.t. set inclusion on which the models of a formula μ and w differ. More formally, $\Delta_w^{min}(\mu) = min_{\subseteq}(\{w\Delta w' : w' \in \mathrm{Mod}(\mu)\})$.

3 State of Art

3.1 Belief revision

Belief revision consists in incorporating a new belief, changing the initial beliefs as little as possible, while preserving consistency. More formally, a revision operator denoted by $\circ$, is a function from $\mathcal{L} \times \mathcal{L}$ to $\mathcal{L}$ that maps two formulas ψ (the initial agent's beliefs) and μ (new information) to a new formula $\psi \circ \mu$ (the revised agent's beliefs). Alchourròn, Gärdenfors and Makinson [2] studied belief revision when an agent's beliefs are represented by a theory (or belief set) and proposed a set of postulates, called *AGM postulates*, any "rational" revision operator should satisfy. Within the propositional framework, Katsuno and Mendelzon reformulated the AGM postulates when a theory is represented by the models of a propositional formula. Within this framework, revising ψ by μ means looking for the models of μ that are closest to those of ψ. We recall these postulates known as *KM postulates* [14].

Let $\psi, \psi_1, \psi_2, \mu, \mu_1, \mu_2 \in \mathcal{L}$.

(R1) $\psi \circ \mu \models \mu$.
(R2) If $(\psi \wedge \mu)$ is satisfiable then $\psi \circ \mu \equiv \psi \wedge \mu$.
(R3) If μ is satisfiable then $\psi \circ \mu$ is satisfiable.
(R4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$ then $\psi_1 \circ \mu_1 \equiv \psi_2 \circ \mu_2$.
(R5) $(\psi \circ \mu_1) \wedge \mu_2 \models \psi \circ (\mu_1 \wedge \mu_2)$.
(R6) If $(\psi \circ \mu_1) \wedge \mu_2$ is satisfiable then $\psi \circ (\mu_1 \wedge \mu_2) \models (\psi \circ \mu_1) \wedge \mu_2$.
(R7) If $(\psi \circ \mu_1) \models \mu_2$ and $(\psi \circ \mu_2) \models \mu_1$, then $\psi \circ \mu_1 \equiv \psi \circ \mu_2$.
(R8) $(\psi \circ \mu_1) \wedge (\psi \circ \mu_2) \models \psi \circ (\mu_1 \vee \mu_2)$.

A detailed description of these postulates can be found in [14]. Katsuno and Mendelzon showed that a revision operator satisfying their version of the AGM postulates may result in a total preorder or a partial preorder over interpretations, which reflects a plausibility ordering on interpretations. More formally, a *faithful assignment* is a function that maps any propositional formula ψ to a preorder over interpretations, denoted by $\leq_\psi$, such that:

1. If $w \models \psi$ and $w' \models \psi$, then $w \approx_\psi w'$.
2. If $w \models \psi$ and $w' \not\models \psi$, then $w <_\psi w'$.
3. If $\psi_1 \equiv \psi_2$ then $\leq_{\psi_1} = \leq_{\psi_2}$.

They provided the following representation theorem.

Theorem 1. [14]

1. *A revision operator $\circ$ satisfies the postulates* (R1)–(R6) *if and only if there exists a faithful assignment that maps each formula ψ to a total preorder $\leq_\psi$ such that* $\mathrm{Mod}(\psi \circ \mu) = min_{\leq_\psi}(\mathrm{Mod}(\mu))$.
2. *A revision operator $\circ$ satisfies the postulates* (R1)–(R5), (R7) *and* (R8) *if and only if there exists a faithful assignment that maps each formula ψ to a partial preorder $\leq_\psi$ such that* $\mathrm{Mod}(\psi \circ \mu) = min_{\leq_\psi}(\mathrm{Mod}(\mu))$.

There exist many revision operators in the literature, we limit ourselves to recalling two well-known model-based revision operators, namely Dalal's [7] and Satoh's operators [21]. For these revision operators the closeness between models relies on the symmetric difference between models, that is the set of propositional variables on which they differ. Dalal's revision operator, denoted by $\circ_D$, focuses on cardinality and is defined as $\mathrm{Mod}(\psi \circ_D \mu) = \{m \in \mathrm{Mod}(\mu) : \exists m' \in \mathrm{Mod}(\psi) \text{ such that } |m\Delta m'| = |\Delta|_m^{min}(\mu)\}$. While Satoh's revision operator, denoted by $\circ_S$, is based on set inclusion and is defined as $\mathrm{Mod}(\psi \circ_S \mu) = \{m \in \mathrm{Mod}(\mu) : \exists m' \in \mathrm{Mod}(\psi)$ such that $m\Delta m' \in \Delta_m^{min}(\mu)\}$. Let us mention that Dalal's revision operator satisfies (R1)–(R6) [9, 14], while the Satoh's revision operator satisfies (R1)–(R5), (R7) and (R8) [14].

3.2 Belief contraction

Belief contraction consists in reducing or retracting beliefs without adding any new information and changing as little as possible of the initial beliefs. More formally, a contraction operator, denoted by $-$, is a function from $\mathcal{L} \times \mathcal{L}$ to $\mathcal{L}$ that maps two formulas ψ (the initial agent's beliefs) and μ (the belief to be removed) to a new formula $\psi - \mu$ (the contracted agent's beliefs). Likewise belief revision, postulates were proposed by Alchourron, Gädenfors and Makinson [2] for belief contraction when beliefs are represented by a theory (or belief set), Katsuno and Mendelzon reformulated some of them, when a theory is represented by the models of a propositional formula [15]. More recently, Caridroit, Konieczny and Marquis [4] revisited theses postulates, by proposing, in particular, additional postulates capturing the minimality of change.

Let $\psi, \psi_1, \psi_2, \mu, \mu_1, \mu_2 \in \mathcal{L}$.

(C1) $\psi \models \psi - \mu$.
(C2) If $\psi \not\models \mu$, then $\psi - \mu \models \psi$.
(C3) If $\psi - \mu \models \mu$, then μ is a tautology.
(C4) If $\psi \models \mu$, then$(\psi - \mu) \wedge \mu \models \psi$.
(C5) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, then $\psi_1 - \mu_1 \equiv \psi_2 - \mu_2$.
(C6) $\psi - (\mu_1 \wedge \mu_2) \models (\psi - \mu_1) \vee (\psi - \mu_2)$.
(C7) If $\psi - (\mu_1 \wedge \mu_2) \not\models \mu_1$, then $\psi - \mu_1 \models \psi - (\mu_1 \wedge \mu_2)$.

A detailed description of these postulates can be found in [15] and [4]. The following theorem, formulated by Caridroit, Konieczny and Marquis [4], is a representation theorem for model-based contraction operators in the same spirit as Katsuno et Mendelzon's representation theorem for revision in terms of total preorders.

Theorem 2. [3] *A contraction operator $-$ satisfies the postulates* (C1)–(C7) *if and only if there exists a faithful assignment that maps each formula ψ to a total preorder $\leq_\psi$ such that* $\mathrm{Mod}(\psi - \mu) = \mathrm{Mod}(\psi) \cup min_{\leq_\psi}(\mathrm{Mod}(\neg\mu))$.

Revision and contraction operators are closely related. Indeed, as a revision operator $\circ$ can be defined from a contraction operator via Levi's identity [18] ($\psi \circ \mu \equiv (\psi - \neg\mu) \wedge \mu$), a contraction operator can be likewise defined from a revision operator via Harper's identity [12] ($\psi - \mu \equiv \psi \vee (\psi \circ \neg\mu)$). These indenties allowed Caridroit, Konieczny and Marquis to show the correspondence between the postulates of revision and those of contraction [4].

Theorem 3. [4]

- *If a revision operator $\circ$ satisfies* (R1)–(R4) *then its corresponding contraction operator $-$ defined by Harper's identity satisfies* (C1)–(C5). *In addition, if $\circ$ satisfies* (R5)*, then $-$ satisfies* (C6) *and if $\circ$ satisfies* (R6)*, then $-$ satisfies* (C7).
- *If a contraction operator $-$ satisfies* (C1)–(C5) *then its corresponding revision operator $\circ$ defined by Levi's identity satisfies* (R1) – (R4). *In addition, if $-$ satisfies* (C6)*, then $\circ$ satisfies* (R5) *and if $-$ satisfies* (C7)*, then $\circ$ satisfies* (R6).

Harper's identity naturally leads to the definition of contraction operators from revision ones. Dalal's contraction operator, denoted by $-_D$, can thus be defined from Dalal's revision operator $\circ_D$ and Satoh's contraction operator, denoted by $-_S$, from Satoh's revision operator $\circ_S$):

$$\mathrm{Mod}(\psi -_D \mu) = \mathrm{Mod}(\psi) \cup \mathrm{Mod}(\psi \circ_D \neg\mu),$$

$$\mathrm{Mod}(\psi -_S \mu) = \mathrm{Mod}(\psi) \cup \mathrm{Mod}(\psi \circ_S \neg\mu).$$

Furthermore, the contraction operator $-_D$ satisfies (C1)–(C7) while the contraction operator $-_S$ satisfies (C1)–(C6), but does not satisfy (C7) [4] .

3.3 Belief update

Belief update consists in incorporating into an agent's beliefs new information reflecting a change in her environment. More formally, an update operator, denoted by $\diamond$, is a function from $\mathcal{L} \times \mathcal{L}$ to $\mathcal{L}$ that maps two formulas ψ (the initial agent's beliefs) and μ (new information) to a new formula $\psi \diamond \mu$ (the updated agent's beliefs). Keller and Winslett [16] then Katsuno and Mendelzon [15] contributed to a better understanding regarding the distinction between belief revision and belief update. Revision makes the models of the formula representing the initial beliefs evolve as a whole towards the closest models of new information, while update makes each model of the formula representing the initial beliefs locally evolve towards the closest models of new information. We recall the KM postulates for belief update [14, 15]. Let $\psi, \psi_1, \psi_2, \mu, \mu_1, \mu_2 \in \mathcal{L}$.

(U1) $\psi \diamond \mu \models \mu$.
(U2) If $\psi \models \mu$, then $\psi \diamond \mu \equiv \psi$.
(U3) If ψ and μ are satisfiable then so is $\psi \diamond \mu$.
(U4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, then $\psi_1 \diamond \mu_1 \equiv \psi_2 \diamond \mu_2$.
(U5) $(\psi \diamond \mu) \wedge \phi \models \psi \diamond (\mu \wedge \phi)$.
(U6) If $(\psi \diamond \mu_1) \models \mu_2$ and $(\psi \diamond \mu_2) \models \mu_1$, then $\psi \diamond \mu_1 \equiv \psi \diamond \mu_2$.
(U7) If ψ is complete, then $(\psi \diamond \mu_1) \wedge (\psi \diamond \mu_2) \models \psi \diamond (\mu_1 \vee \mu_2)$.
(U8) $(\psi_1 \vee \psi_2) \diamond \mu \equiv (\psi_1 \diamond \mu) \vee (\psi_2 \diamond \mu)$.
(U9) If ψ is complete and $(\psi \diamond \mu) \wedge \phi$ is satisfiable, then $\psi \diamond (\mu \wedge \phi) \models (\psi \diamond \mu) \wedge \phi$.

A detailed description of these postulates can be found in [15]. These postulates have been discussed in several papers (see for example [13, 5]).

The four postulates (U1), (U4), (U5) and (U6) directly correspond to the revision postulates (R1), (R4), (R5) and (R7), respectively. The postulate (U8) is specific to update and expresses that an update operator should give each of the models of the initial beliefs equal consideration. The postulates (U7) and (U9) correspond to (R8) and (R9), respectively, but are restricted to complete formulas (which makes sense in presence of (U8)). The postulates (U2) and (U3) differ from (R2) and (R3), they are weaker versions of the revision postulates. A consequence for update is that once an inconsistency is introduced into the initial beliefs, there is not way to eliminate it. Note that this is not the case for revision.

Katsuno and Mendelzon showed that an update operator, depending on the set of KM postulates it satisfies, may result in a total preorder or a partial preorder. More formally, a *pointwise faithful assignment* is a function that maps any interpretation w to a preorder over interpretations $\leq_w$, such that for any interpretation w', if $w \neq w'$ then $w <_w w'$. They provided the following representation theorem.

Theorem 4. [15]

1. *An update operator $\diamond$ satisfies the postulates* (U1)–(U9) *if and only if there exists a pointwise faithful assignment that maps each interpretation w to a total preorder $\leq_w$ such that* $\mathrm{Mod}(\psi \diamond \mu) = \bigcup_{w \in \mathrm{Mod}(\psi)} min(\mathrm{Mod}(\mu), \leq_w)$.
2. *An update operator $\diamond$ satisfies the postulates* (U1)–(U8) *if and only if there exists a pointwise faithful assignment that maps each*

interpretation w *to a partial preorder* $\leq_w$ *such that* $\mathrm{Mod}(\psi \diamond \mu) = \bigcup_{w \in \mathrm{Mod}(\psi)} min(\mathrm{Mod}(\mu), \leq_w)$.

The representation theorems, Theorem 1 and Theorem 4, pinpoint the differences between revision and update. Update stems from a pointwise minimization, model by model of ψ, while revision stems from a global minimization on all the models of ψ. Update operators, for each model w of ψ, select the set of models of μ that are the closest to w, while revision operators select the set of models of μ that are the closest to the set of models of ψ. Note that when there exists only one model of ψ (which is the case when ψ is complete) revision and update coincide.

Several update operators have been proposed. We will focus on Forbus' [10] and Winslett's operators [22], which are the update counterpart of Dalal's and Satoh's revision operators, respectively. The Forbus update operator, denoted by $\diamond_F$, is then defined by:

$$\mathrm{Mod}(\psi \diamond_F \mu) = \bigcup_{w \in \mathrm{Mod}(\psi)} \{w' \in \mathrm{Mod}(\mu) : |w \Delta w'| = |\Delta|_w^{min}(\mu)\}.$$

Similarly the Winslett operator, also called *PMA (Possible Models Approach)*, denoted by $\diamond_W$, is then defined by:

$$\mathrm{Mod}(\psi \diamond_W \mu) = \bigcup_{w \in \mathrm{Mod}(\psi)} \{w' \in \mathrm{Mod}(\mu) : w \Delta w' \in \Delta_w^{min}(\mu)\}.$$

The Forbus operator $\diamond_F$ satisfies (U1)–(U8) [14] and (U9) [13] while the Winslett operator satisfies (U1)–(U8) [14] but does not satisfy (U9) [17].

4 Belief erasure

Belief erasure, introduced by Katsuno and Mendelzon [15], is to contraction what update is to revision. Intuitively, erasing a belief means the world may have changed in such a way that this belief is not true anymore.

From a logical point of view, when the agent's beliefs are represented by a logical formula ψ, erasing the belief μ from ψ means selecting the models of ψ and adding the models of $\neg\mu$ that are "closest" to each model of ψ. More formally, an erasure operator, denoted by $\triangleleft$ is a function from $\mathcal{L} \times \mathcal{L}$ to $\mathcal{L}$ that maps two formulas ψ (the initial agent's beliefs) and μ (the belief to erase) to a new $\psi \triangleleft \mu$ (the erased agent's beliefs).

4.1 Basic postulates for belief erasure

Postulates characterizing the rational behavior of erasure operators have also been proposed by Katsuno and Mendelzon [15] in the same spirit as the ones they proposed for belief contraction and belief update. Let $\psi, \psi_1, \psi_2, \mu, \mu_1, \mu_2 \in \mathcal{L}$.

(E1) $\psi \models \psi \triangleleft \mu$.
(E2) If $\psi \models \neg\mu$, then $\psi \triangleleft \mu \equiv \psi$.
(E3) If ψ is satisfiable and μ is not a tautology, then $\psi \triangleleft \mu \not\models \mu$.
(E4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, then $\psi_1 \triangleleft \mu_1 \equiv \psi_2 \triangleleft \mu_2$.
(E5) $(\psi \triangleleft \mu) \wedge \mu \models \psi$.
(E8) $(\psi_1 \vee \psi_2) \triangleleft \mu \equiv (\psi_1 \triangleleft \mu) \vee (\psi_2 \triangleleft \mu)$.

A detailed description of these postulates can be found in [15, 6]. Similarly to revision and update, the postulate (E8) is specific to erasure, while the other postulates for contraction and erasure differ only on the second and third postulates. The postulates (E2) and (E3) are weaker than the postulates (C2) and (C3). This is illustrated by the example given in the introduction. As for update a consequence for erasure is that once an inconsistency is introduced into the initial beliefs, there is no way to eliminate it.

Similarly to Harper's and Levi's identities, Katsuno and Mendelzon [15] proposed two other identities (Id_1 and Id_2) allowing the crossing from update to erasure and reciprocally. (Id_1) is defined as $\psi \triangleleft \mu \equiv \psi \vee (\psi \diamond \neg\mu)$ and (Id_2) is defined as $\psi \diamond \mu \equiv (\psi \triangleleft \neg\mu) \wedge \mu$. Moreover, the following theorem gives the correspondence between update and erasure similarly to the correspondence between revision and contraction.

Theorem 5. [3] [15]

- *If an update operator* $\diamond$ *satisfies* (U1)–(U4) *and* (U8)*, then the corresponding erasure operator* $\triangleleft$ *defined by the identity (*Id_1*) satisfies* (E1)–(E5) *and* (E8).
- *If an erasure operator* $\triangleleft$ *satisfies* (E1)–(E4) *and* (E8)*, then the update operator* $\diamond$ *defined by the identity (*Id_2*) satisfies* (U1)–(U4) *and* (U8).

The (Id_1) identity allows one to define two model-based erasure operators from well-known update operators, namely the Forbus's and Winslett's erasure operators, denoted respectively by $\triangleleft_F$ and $\triangleleft_W$ and defined by

$$\mathrm{Mod}(\psi \triangleleft_F \mu) = \mathrm{Mod}(\psi) \cup \mathrm{Mod}(\psi \diamond_F \neg\mu),$$

$$\mathrm{Mod}(\psi \triangleleft_W \mu) = \mathrm{Mod}(\psi) \cup \mathrm{Mod}(\psi \diamond_W \neg\mu).$$

According to Theorem 5 the erasure operators $\triangleleft_F$ and $\triangleleft_W$ both satisfy (E1)–(E5) and (E8). The erasure operators $\triangleleft_F$ and $\triangleleft_W$ are illustrated in the following example.

Example 1. *Let* ψ, μ *two propositional formulas such that* $Mod(\psi) = \{abcd, a\}$ *and* $Mod(\mu) = \{a, c, d, ab, ac, ad, bc, cd, abc, abd, bcd, abcd, \emptyset\}$. *We have* $Mod(\neg\mu) = \{acd, bd, b\}$ *and according to Table 1,* $Mod(\psi \diamond_F \neg\mu) = \{acd, b\}$ *and* $Mod(\psi \diamond_W \neg\mu) = \{acd, bd, b\}$. *Erasure results with* $\triangleleft_F$ *and* $\triangleleft_W$ *are respectively* $Mod(\psi \triangleleft_F \mu) = \{abcd, a, acd, b\}$ *and* $Mod(\psi \triangleleft_W \mu) = \{abcd, a, acd, bd, b\}$.

Table 1. Symmetric difference; per column, minimal sets according to cardinality are noted in bold and minimal sets according to set inclusion are noted with an asterisk.

		$Mod(\psi)$	
	Δ	$abcd$	a
$Mod(\neg\mu)$	acd	$\boldsymbol{b}$*	$\boldsymbol{cd}$*
	bd	ac*	abd
	b	acd	$\boldsymbol{ab}$*

4.2 New postulates capturing the minimality of change

We can add two additional postulates for capturing the minimality of change. They are equivalent to (C6) and (C7), with the only difference that due to the disjunctive rule (E8), postulate (E7) can be restricted to complete formulas.

[3] Note that the second item of this theorem seems different from its analogue in Theorem 3, where five postulates (C1)-(C5) are required. Actually, as here (C1)-(C3) plus (C5) are sufficient to preserve the basic postulates, (C4) is only needed when it comes to the preservation of the postulates dealing with the minimality of change.

(E6) $\psi \lhd (\mu_1 \wedge \mu_2) \models (\psi \lhd \mu_1) \vee (\psi \lhd \mu_2)$.
(E7) If ψ is complete and $\psi \lhd (\mu_1 \wedge \mu_2) \not\models \mu_1$, then $\psi \lhd \mu_1 \models \psi \lhd (\mu_1 \wedge \mu_2)$.

Observe that the erasure operator $\lhd_F$ satisfies (E6) and (E7), while $\lhd_W$ satisfies (E6) but does not satisfy (E7).

Not surprisingly we can state a representation theorem for model-based erasure operators, which is the counterpart of the representation theorem of Caridroit *et al.*. The proof requires the two following lemma. The first one makes clear the result of erasure by a formula which has only one countermodel and uses only the basic postulates.

Lemma 6. *Let $\lhd$ be an erasure operator satisfying the postulates* (E1)–(E5) *and* (E8), *ψ be a satisfiable formula, w be an interpretation and α_w be a formula having w as unique model, then $\psi \lhd \neg\alpha_w \equiv \psi \vee \alpha_w$.*

Proof. Since the operator satisfies (E8) it is sufficient to prove it when ψ is a complete formula, having say w_0 as unique model.

If $w_0 \neq w$, From (E5) $(\psi \lhd \neg\alpha_w) \wedge \neg\alpha_w \models \psi$, thus $(\psi \lhd \neg\alpha_w) \models \psi \vee \alpha_w$. From (E1), $\psi \models (\psi \lhd \neg\alpha_w)$. Furthermore, according to (E3), $(\psi \lhd \neg\alpha_w) \not\models \neg\alpha_w$. Thus we obtain $\alpha_w \models (\psi \lhd \neg\alpha_w)$. Therefore $\psi \lhd \neg\alpha_w \equiv \psi \vee \alpha_w$. □

The second one uses the two additional postulates that deal with the minimality of change.

Lemma 7. *Let $\lhd$ be an erasure operator satisfying the postulates* (E1)–(E8), *ψ be a complete formula, and α and β be formulas that are not tautologies, then $\psi \lhd (\alpha \wedge \beta) \equiv \psi \lhd \alpha$ or $\psi \lhd \beta$ or $(\psi \lhd \alpha) \vee (\psi \lhd \beta)$.*

Proof. Under these assumptions made on the formulas the proof is similar to the one given in [4, proof of Proposition 9]. □

In the following for any interpretation w, ψ_w (or α_w) denotes a complete formula having w as unique model. Also given an interpretation w_i, we denote by α_i a formula that has w_i as unique model.

4.3 Erasure in terms of total preorders

We are now in a position to prove the following representation theorem, which shows that the postulates capture all erasure operators defined by a total preorder.

Theorem 8. *An erasure operator $\lhd$ satisfies the postulates* (E1)–(E8) *if and only if there exists a pointwise faithful assignment that maps each interpretation w to a total preorder $\leq_w$ such that* $\mathrm{Mod}(\psi \lhd \mu) = \mathrm{Mod}(\psi) \cup \bigcup_{w \in \mathrm{Mod}(\psi)} min_{\leq_w}(\mathrm{Mod}(\neg\mu))$.

Proof. $\Leftarrow$) Suppose we have a pointwise faithful assignment that maps each interpretation w to a total preorder $\leq_w$. Consider the erasure operator $\lhd$ defined by

$$\mathrm{Mod}(\psi \lhd \mu) = \mathrm{Mod}(\psi) \cup \bigcup_{w \in \mathrm{Mod}(\psi)} min_{\leq_w}(\mathrm{Mod}(\neg\mu)).$$

We prove that $\lhd$ satisfies the postulates (E1) to (E8). It is obvious that the erasure operator $\lhd$ satisfies (E1), (E3), (E4), (E5) and (E8). If ψ is inconsistent then (E2), (E6) and (E7) trivially hold. We assume in the following that ψ is consistent.

We show (E2). It follows from the definition of a pointwise faithful assignment that if w is a model of $\neg\mu$, then $\psi_w \lhd \mu$ is equivalent to ψ_w. Hence we obtain (E2) in using (E8).

We show (E6). We have $\mathrm{Mod}(\psi \lhd (\mu_1 \wedge \mu_2)) = \mathrm{Mod}(\psi) \cup \bigcup_{w \in \mathrm{Mod}(\psi)} min_{\leq_w}(\mathrm{Mod}(\neg\mu_1 \vee \neg\mu_2))$. Since $min_{\leq_w}(\mathrm{Mod}(\neg\mu_1 \vee \neg\mu_2)) \subseteq min_{\leq_w}(\mathrm{Mod}(\neg\mu_1)) \cup min_{\leq_w}(\mathrm{Mod}(\neg\mu_2))$, (E6) holds.

We show (E7). Let ψ be a complete formula such that $\mathrm{Mod}(\psi) = \{w_0\}$. Let us suppose that $\psi \lhd (\mu_1 \wedge \mu_2) \not\models \mu_1$. This means that there is $w \in \mathrm{Mod}(\psi) \cup min_{\leq_{w_0}}(\mathrm{Mod}(\neg(\mu_1 \wedge \mu_2)))$ such that $w \in \mathrm{Mod}(\neg\mu_1)$. If $w \in \mathrm{Mod}(\psi \wedge \neg\mu_1)$, then $\mathrm{Mod}(\psi) = \{w\}$ and in this case since the assignment is pointwise faithful $\mathrm{Mod}(\psi \lhd \mu_1) = \mathrm{Mod}(\psi)$ and the conclusion follows from (E1). If $w \notin \mathrm{Mod}(\psi)$, to prove that (E7) holds it is sufficient to show that $min_{\leq_{w_0}}(\mathrm{Mod}(\neg\mu_1)) \subseteq min_{\leq_{w_0}}(\mathrm{Mod}(\neg(\mu_1 \wedge \mu_2)))$. Let $w' \in min_{\leq_{w_0}}(\mathrm{Mod}(\neg\mu_1))$. Since $\leq_{w_0}$ is a total preorder and $w \in \mathrm{Mod}(\neg\mu_1)$, we have $w' \leq_{w_0} w$. Suppose that there exists $w'' \in \mathrm{Mod}(\neg(\mu_1 \wedge \mu_2))$ such that $w'' <_{w_0} w'$, then also $w'' <_{w_0} w$, thus contradicting the fact that $w \in min_{\leq_{w_0}}(\mathrm{Mod}(\neg(\mu_1 \wedge \mu_2)))$. Therefore (E7) is satisfied.

$\Rightarrow$) Let $\lhd$ be an erasure operator that satisfies the postulates (E1) to (E8). We define the binary relation $\leq_w$ on interpretations by :

$$w_1 \leq_w w_2 \text{ if either } w_1 = w \text{ or } w_1 \in \mathrm{Mod}(\psi_w \lhd \neg(\alpha_1 \vee \alpha_2)).$$

We first show that $\leq_w$ is a total preorder. From (E3) it follows that either w_1 or w_2 is in $\mathrm{Mod}(\psi_w \lhd \neg(\alpha_1 \vee \alpha_2))$, thus proving that the binary relation is total. The fact that $\leq_w$ is reflexive follows from Lemma 6.

The proof that $\leq_w$ is transitive is completely similar to the one given by Caridroit *et al.* in the case of contraction (see [4, Proof of Theorem 14]. The later builds on a lemma similar to Lemme 7 for contraction and then only uses postulates (C1), (C6) and (C7), which are the same as (E1), (E6) and (E7) when restricted to a complete formula (for the last one).

It follows from (E2) that the mapping $w \mapsto \leq_w$ is a pointwise faithful assignment.

It remains to prove that $\mathrm{Mod}(\psi \lhd \mu) = \mathrm{Mod}(\psi) \cup \bigcup_{w \in \mathrm{Mod}(\psi)} min_{\leq_w}(\mathrm{Mod}(\neg\mu))$. If ψ is inconsistent, then both sides of the equation are empty and the equality holds. If ψ is consistent, then according to (E8) given some interpretation w it is sufficient to prove that $\mathrm{Mod}(\psi_w \lhd \mu) = \{w\} \cup min_{\leq_w}(\mathrm{Mod}(\neg\mu))$. If $w \in \mathrm{Mod}(\neg\mu)$ then it follows from (E2) that $\psi_w \lhd \mu \equiv \psi$ and the equality holds since we use a faithful assignment. So we assume in the following that $w \notin \mathrm{Mod}(\neg\mu)$. If μ is a tautology, according to (E1) and (E5) $\psi_w \lhd \mu \equiv \psi$ and the equality holds. Suppose now that there exists $w_1 \in \mathrm{Mod}(\psi_w \lhd \mu)$ which is in $\mathrm{Mod}(\neg\mu)$ but not minimal in $\mathrm{Mod}(\neg\mu)$ w.r.t. $\leq_w$. Then there is $w_2 \in \mathrm{Mod}(\neg\mu)$ such that $w_2 <_w w_1$. We have then $w_1 \notin \mathrm{Mod}(\psi_w \lhd \neg(\alpha_1 \vee \alpha_2))$.

Now let us consider the formula $\beta = \neg\mu \wedge \neg\alpha_1 \wedge \neg\alpha_2$. Clearly we have $\neg\mu \equiv \beta \vee (\alpha_1 \vee \alpha_2)$. Since (E4) is satisfied, $\psi_w \lhd \mu \equiv \psi_w \lhd (\neg\beta \wedge \neg\alpha_1 \wedge \neg\alpha_1)$. Hence by (E6), $\psi_w \lhd \mu \models (\psi_w \lhd \neg\beta) \vee (\psi_w \lhd \neg(\alpha_1 \vee \alpha_2))$. We have supposed $w_1 \in \mathrm{Mod}(\psi_w \lhd \mu)$ and $w_1 \notin \mathrm{Mod}(\psi_w \lhd \neg(\alpha_1 \vee \alpha_2))$, thus $w_1 \in \mathrm{Mod}(\psi_w \lhd \neg\beta)$. By (E5) we have $(\psi_w \lhd \neg\beta) \wedge \neg\beta \models \psi_w$. Since we have also $w_1 \in \mathrm{Mod}(\neg\beta)$, we get $w_1 \in \mathrm{Mod}(\psi_w)$, i.e., $w_1 = w$ a contradiction.

This shows that $\mathrm{Mod}(\psi_w \lhd \mu) \subseteq \{w\} \cup min_{\leq_w}(\mathrm{Mod}(\neg\mu))$.

Let us now show the converse inclusion. According to (E1), $w \in \mathrm{Mod}(\psi_w \lhd \mu)$. Consider now w_1 minimal in $\mathrm{Mod}(\neg\mu)$ w.r.t. $\leq_w$ and in seek of contradiction suppose $w_1 \notin \mathrm{Mod}(\psi_w \lhd \mu)$. In this case μ is not a tautology and by (E3) $\psi_w \lhd \mu \not\models \mu$, so there is w_2 a model of $\neg\mu$ that is in $\mathrm{Mod}(\psi_w \lhd \mu)$. If $w_2 = w$ then $w_2 <_w w_1$, thus contradicting the minimality of w_1.

If $w_2 \neq w$. Since $w_2 \in \mathrm{Mod}(\psi_w \lhd \mu)$ we have $\psi_w \lhd \mu \not\models \neg(\alpha_1 \vee \alpha_2)$. Since both w_1 and w_2 are models of $\neg\mu$ observe that $\mu \wedge \neg(\alpha_1 \vee \alpha_2) \equiv \mu$, therefore by (E7), $\psi_w \lhd \neg(\alpha_1 \vee \alpha_2) \models \psi_w \lhd \mu$. Since by assumption $w_1 \notin \mathrm{Mod}(\psi_w \lhd \mu)$, we have $w_1 \notin \mathrm{Mod}(\psi_w \lhd \neg(\alpha_1 \vee \alpha_2)$. Hence, according to Lemma 7 and Lemma 6, $\mathrm{Mod}(\psi_w \lhd \neg(\alpha_1 \vee \alpha_2) = \{w, w_2\}$, thus contradicting the minimality of w_1. □

4.4 Erasure in terms of partial preorders

Recall that Winslett's erasure operator does not satisfy (E7), so this operator is not captured by the previous theorem. This operator is induced by a partial preorder. We can modify erasure postulates so that they accommodate partial preorders.

In the proof of Theorem 8, given an erasure operator associated with a total preorder only postulate (E7) requires that the preorder is total. Therefore, as it was done by Katsuno and Mendelzon for revision, in order to get a representation theorem through partial preorders we remove the postulate (E7) and replace it by two weaker postulates.

(E9) If $\psi \models \mu_1 \wedge \mu_2$, $(\psi \lhd \mu_1) \models \psi \vee \neg\mu_2$
and $(\psi \lhd \mu_2) \models \psi \vee \neg\mu_1$, then $(\psi \lhd \mu_1) \equiv (\psi \lhd \mu_2)$.

(E10) If ψ is complete
then $(\psi \lhd \mu_1) \wedge (\psi \lhd \mu_2) \models \psi \lhd (\mu_1 \vee \mu_2)$.

The erasure operators $\lhd_F$ and $\lhd_W$ satisfy (E9) and (E10).

Having defined these postulates allows to design a class of erasure operators based on partial preorders. The following theorem shows that erasure operators based on partial preorders are completely characterized by postulates (E1) to (E6) and (E8) to (E10).

Theorem 9. *An erasure operator* $\lhd$ *satisfies the postulates* (E1)–(E6) *and* (E8)–(E10) *if and only if there exists a pointwise faithful assignment that maps each interpretation* w *to a partial preorder* $\leq_w$ *such that* $\mathrm{Mod}(\psi \lhd \mu) = \mathrm{Mod}(\psi) \cup \bigcup_{w\in \mathrm{Mod}(\psi)} min_{\leq_w}(\mathrm{Mod}(\neg\mu))$.

Proof. $\Leftarrow$) Suppose we have a pointwise faithful assignment that maps each interpretation w to a partial preorder $\leq_w$. Consider the erasure operator $\lhd$ defined by $\mathrm{Mod}(\psi \lhd \mu) = \mathrm{Mod}(\psi) \cup \bigcup_{w\in \mathrm{Mod}(\psi)} min_{\leq_\psi}(\mathrm{Mod}(\neg\mu))$.

We prove that $\lhd$ satisfies the postulates (E1)–(E6) and (E8)–(E10). The proof that $\lhd$ satisfies the postulates (E1)–(E6)and (E8) is similar to the proof of Theorem 8 above, the fact that the preorder is partial (and not necessarily total) has no impact on the proof for these seven postulates.

We show (E9). Let us suppose that $\psi \models \mu_1 \wedge \mu_2$, $(\psi \lhd \mu_1) \models \psi \vee \neg\mu_2$ and $(\psi \lhd \mu_2) \models \psi \vee \neg\mu_1$. In order to get a contradiction suppose that there exists an interpretation w such that $w \in \mathrm{Mod}(\psi \lhd \mu_1)$ and $w \notin \mathrm{Mod}(\psi \lhd \mu_2)$. Observe that $w \notin \mathrm{Mod}(\psi)$. Since $(\psi \lhd \mu_1) \models \psi \vee \neg\mu_2$, we have $w \in \mathrm{Mod}(\neg\mu_2)$. By definition of the operator since $w \notin \mathrm{Mod}(\psi \lhd \mu_2)$, for each model w_i of ψ there exists $w'_i \in min_{\leq_{w_i}}(\mathrm{Mod}(\neg\mu_2))$ such that $w'_i <_{w_i} w$ and w'_i is minimal in $\mathrm{Mod}(\neg\mu_2)$. Then each w'_i is a model of $(\psi \lhd \mu_2)$, but $(\psi \lhd \mu_2) \models \psi \vee \neg\mu_1$. Therefore either $w'_i \in \mathrm{Mod}(\psi)$ or $w'_i \in \mathrm{Mod}(\neg\mu_1)$. If $w'_i \in \mathrm{Mod}(\psi)$, then $w'_i \in \mathrm{Mod}(\psi) \cap \mathrm{Mod}(\neg\mu_2)$, which contradicts the fact that $\psi \models \mu_2$. If $w'_i \in \mathrm{Mod}(\neg\mu_1)$, then the fact that $w'_i <_{w_i} w$ contradicts the minimality of w in $\mathrm{Mod}(\neg\mu_1)$ w.r.t $\leq_{w_i}$. So in both cases we reach a contradiction, thus $\mathrm{Mod}(\psi \lhd \mu_1) \subseteq \mathrm{Mod}(\psi \lhd \mu_2)$. The converse inclusion is proved in a similar way, thus proving that (E9) is satisfied.

We show (E10). Note that $\mathrm{Mod}((\psi_w \lhd \mu_1) \wedge (\psi \lhd \mu_2)) = \{w\} \cup min_{\leq_w}(\mathrm{Mod}(\neg\mu_1)) \cap min_{\leq_w}(\mathrm{Mod}(\neg\mu_2)) \subseteq \{w\} \cup min_{\leq_w}(\mathrm{Mod}(\neg\mu_1 \wedge \neg\mu_2))$. Hence $(\psi \lhd \mu_1) \wedge (\psi \lhd \mu_2) \models \psi - \lhd (\mu_1 \vee \mu_2)$, thus proving that (E10) is satisfied.

$\Rightarrow$) Let $\lhd$ be an erasure operator that satisfies the postulates (E1) to (E6) and (E8) to (E10). Remember that in the following given an interpretation w_i, we denote by α_i a formula that has w_i as a unique model. For each interpretation w we define the binary relation $\leq_w$ on interpretations by :

$$w_1 \leq_w w_2 \text{ if } \mathrm{Mod}(\psi_w \lhd \neg(\alpha_1 \vee \alpha_2)) = \{w\} \cup \{w_1\}.$$

We first show that $\leq_w$ is a preorder. According to Lemma 6, $\mathrm{Mod}(\psi_w \lhd \neg\alpha_1) = \mathrm{Mod}(\psi_w) \cup \{w_1\}$. Hence $\leq_w$ is reflexive.

Let us now prove that it is transitive. Let us consider three pairwise distinct interpretations w_1, w_2 and w_3 such that $w_1 \leq_w w_2$ and $w_2 \leq_w w_3$. So we know that $\mathrm{Mod}(\psi_w \lhd \neg(\alpha_1 \vee \alpha_2)) = \mathrm{Mod}(\psi_w) \cup \{w_1\}$, i.e., $\psi \lhd \neg(\alpha_1 \vee \alpha_2) \equiv \psi_w \vee \alpha_1$ and $\mathrm{Mod}(\psi_w \lhd \neg(\alpha_2 \vee \alpha_3)) = \mathrm{Mod}(\psi_w) \cup \{w_2\}$, i.e., $\psi_w \lhd \neg(\alpha_2 \vee \alpha_3) \equiv \psi_w \vee \alpha_2$. Suppose first that one of the three interpretations is equal to w. If $w_1 = w$, then by (E2), $\mathrm{Mod}(\psi_w \lhd \neg(\alpha_1 \vee \alpha_3)) = \mathrm{Mod}(\psi_w)$ and $w_1 \leq_w w_3$. If $w_2 = w$, then by the assumptions and by (E2), $w_1 = w$ as well and $w_1 = w_2$. If $w_3 = w$ then by the assumptions and by (E2) both $w_2 = w_1 = w_3 = w$. Suppose now that none of w_1, w_2 and w_3 is equal to w, that is $\psi_w \models (\neg\alpha_1 \wedge \neg\alpha_2 \wedge \neg\alpha_3)$. On the one hand $\psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_2) \equiv \psi_w \vee \alpha_1$, hence $\psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_2) \models \psi_w \vee (\alpha_1 \vee \alpha_2 \vee \alpha_3)$. On the other hand by (E6), $\psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_2 \wedge \neg\alpha_3) \models (\psi_w \lhd \neg\alpha_1) \vee (\psi \lhd (\neg\alpha_2 \wedge \neg\alpha_3))$. Hence according to our assumptions $\psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_2 \wedge \alpha_3) \models (\psi_w \lhd \neg\alpha_1) \vee \psi_w \vee \alpha_2$. According to Lemma 6, $\psi_w \lhd \neg\alpha_1 \equiv \psi \vee \alpha_1$. Thus $\psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_2 \wedge \neg\alpha_3) \models \psi_w \vee \alpha_1 \vee \alpha_2$. Therefore by (E9) $\psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_2) \equiv \psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_2 \wedge \neg\alpha_3)$. Hence on the one hand, $\psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_2 \wedge \neg\alpha_3) = \{w, w_1\}$, thus $\psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_2 \wedge \neg\alpha_3) \models \psi_w \vee \alpha_1 \vee \alpha_3$. On the other hand by (E5), $(\psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_3)) \wedge (\neg\alpha_1 \wedge \neg\alpha_3)) \models \psi_w$. Hence, $(\psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_3)) \models \psi_w \vee (\alpha_1 \vee \alpha_3) \models \psi_w \vee (\alpha_1 \vee \alpha_2 \vee \alpha_3)$. By (E9) we obtain that $\psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_3) \equiv \psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_2 \wedge \neg\alpha_3)$. Therefore $\psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_3) \equiv \psi_w \lhd (\neg\alpha_1 \wedge \neg\alpha_2)$. Hence $\mathrm{Mod}(\psi_w \lhd \neg(\alpha_1 \vee \alpha_3)) = \mathrm{Mod}(\psi_w) \cup \{w_1\}$, that is $w_1 \leq_w w_3$, thus proving the transitivity.

It follows from (E2) that the mapping $w \mapsto \leq_w$ is a pointwise faithful assignment.

It remains to prove that $\mathrm{Mod}(\psi \lhd \mu) = \mathrm{Mod}(\psi) \cup \bigcup_{w\in \mathrm{Mod}(\psi)} min_{\leq_w}(\mathrm{Mod}(\neg\mu))$. If ψ is inconsistent, then both sides of the equation are empty and the equality holds. If ψ is consistent, then according to (E8) given some interpretation w it is sufficient to prove that $\mathrm{Mod}(\psi_w \lhd \mu) = \{w\} \cup min_{\leq_w}(\mathrm{Mod}(\neg\mu))$. If $w \in \mathrm{Mod}(\neg\mu)$ then it follows from (E2) that $\psi_w \lhd \mu \equiv \psi$ and the equality holds since we use a faithful assignment. If μ is a tautology, according to (E5) $\psi_w \lhd \mu \equiv \psi$ and the equality holds. So we assume in the following that $w \notin \mathrm{Mod}(\neg\mu)$ and that μ is not a tautology.

Let us first prove $\mathrm{Mod}(\psi_w) \cup min_{\leq_w}(\mathrm{Mod}(\neg\mu)) \subseteq \mathrm{Mod}(\psi_w \lhd \mu)$. By (E1) $w \in \mathrm{Mod}(\psi_w \lhd \mu)$. So let us now consider $w_0 \neq w$ and $w_0 \in min_{\leq_w}(\mathrm{Mod}(\neg\mu))$.

Suppose that $\mathrm{Mod}(\neg\mu) = \{w_0, w_1, \ldots, w_n\}$. For any $w_i \in \mathrm{Mod}(\neg\mu)$, since neither $\neg\alpha_0$ nor $\neg\alpha_i$ is a tautology by Lemma 7 $\psi_w \lhd (\neg\alpha_0 \wedge \neg\alpha_i) \equiv \psi_w \vee \alpha_0$ or $\psi_w \vee \alpha_i$ or $\psi_w \vee \alpha_0 \vee \alpha_i$. Since $w_0 \in min_{\leq_w}(\mathrm{Mod}(\neg\mu))$ there exists no $w_i \in \mathrm{Mod}(\neg\mu)$ such that $\psi_w \lhd (\neg\alpha_0 \wedge \neg\alpha_i) \equiv \psi_w \vee \alpha_i$, therefore $w_0 \in \psi_w \lhd (\neg\alpha_0 \wedge \neg\alpha_i)$.

Observe that $\mu \equiv \bigvee_{i=1}^{n}(\alpha_0 \vee \alpha_i)$. Hence by (E4) and repeated applications of (E10) we obtain that $w_0 \in \mathrm{Mod}(\psi_w \lhd \mu)$, thus proving that $\mathrm{Mod}(\psi_w) \cup min_{\leq_\psi}(\mathrm{Mod}(\neg\mu)) \subseteq \mathrm{Mod}(\psi_w \lhd \mu)$.

Let us now prove that $\mathrm{Mod}(\psi_w \lhd \mu) \subseteq \mathrm{Mod}(\psi_w) \cup min_{\leq_w}(\mathrm{Mod}(\neg\mu))$. Let us consider $w_0 \in \mathrm{Mod}(\psi_w \lhd \mu)$ such that $w_0 \neq w$. By (E5) $w_0 \in \mathrm{Mod}(\neg\mu)$. In order to reach a contradiction let us suppose that $w_0 \notin min_{\leq_w}(\mathrm{Mod}(\neg\mu))$. This means that there exists $w_1 \in \mathrm{Mod}(\neg\mu)$ such that $w_1 <_w w_0$, i.e., $\psi_w \lhd (\neg\alpha_0 \wedge \neg\alpha_1)) \equiv \psi_w \vee \alpha_1$. Now let us consider the formula $\beta = \neg\mu \wedge \neg\alpha_0 \wedge \neg\alpha_1$. Clearly we have $\neg\mu \equiv \beta \vee \alpha_0 \vee \alpha_1$. According to (E4), $\psi_w \lhd \mu \equiv \psi \lhd (\neg\beta \wedge \neg\alpha_0 \wedge \neg\alpha_1)$. By (E6), $\psi_w \lhd (\neg\beta \wedge \neg\alpha_0 \wedge \neg\alpha_1) \models (\psi_w \lhd \neg\beta) \vee (\psi_w \lhd (\neg\alpha_0 \wedge \neg\alpha_1))$. Since $w_1 <_w w_0$, $w_0 \notin \mathrm{Mod}(\psi_w \lhd (\neg\alpha_0 \wedge \neg\alpha_1)$, thus $w_0 \in \mathrm{Mod}(\psi_w \lhd \neg\beta)$. Moreover, $w_0 \in \mathrm{Mod}(\neg\mu)$ and a fortiori $w_0 \in \mathrm{Mod}(\neg\beta)$. Then by (E5), $(\psi_w \lhd \neg\beta) \wedge (\neg\beta) \models \psi_w$, so we obtain $w_0 = w$, which provides a contradiction. □

4.5 Contraction in terms of partial pre-orders

As observed in Section 3.2 a representation theorem for contraction operators through partial preorders is still missing. We aim at filling this gap. Similarly as it was done for erasure we can give a version of contraction postulates that accommodates partial preorders and we can design a class of contraction operators based on partial preorders.

We remove the postulate (C7) and replace it by two weaker postulates. They are similar to postulates (E9) and (E10), except for the last one which is not restricted to complete formulas anymore.

(C8) If $\psi \models \mu_1 \wedge \mu_2$, $(\psi - \mu_1) \models \psi \vee \neg\mu_2$ and $(\psi - \mu_2) \models \psi \vee \neg\mu_1$, then $(\psi - \mu_1) \equiv (\psi - \mu_2)$.

(C9) $(\psi - \mu_1) \wedge (\psi - \mu_2) \models \psi - (\mu_1 \vee \mu_2)$.

Thus we obtain the following representation theorem.

Theorem 10. *A contraction operator* $-$ *satisfies the postulates* (C1)–(C6) *and* (C8)–(C9) *if and only if there exists a faithful assignment that maps each formula* ψ *to a partial preorder* $\leq_\psi$ *such that* $\mathrm{Mod}(\psi - \mu) = \mathrm{Mod}(\psi) \cup min_{\leq_\psi}(\mathrm{Mod}(\neg\mu))$.

The proof of this theorem follows exactly the same lines as the proof of Theorem 9.

5 Panorama of belief change operations

We proposed new postulates capturing the minimal change principle for erasure, namely (E6) and (E7). This allowed us to establish a first representation theorem showing that an erasure operator satisfying the set of postulates (E1)–(E8) corresponds to a total preorder on interpretations. Moreover, replacing the postulate (E7) by two weaker postulates (E9) and (E10) allowed us to establish a second representation theorem showing that an erasure operator satisfying the set of postulates (E1)–(E6) and (E8)–(E10) corresponds to a partial preorder on interpretations.

Furthermore, for the contraction operation, we showed that by replacing the postulate (C7) by two weaker postulates (C8) and (C9), we can establish a representation theorem, which has been missing until now, showing that a contraction operator satisfying the set of (C1)–(C6) and (C8)–(C9) corresponds to a partial preorder on interpretations.

Our contribution thus allows us to draw an interesting panorama of four well-known belief change operations, summarised in the following table.

	Basic post.	Min. post.	Weak min. post.
Revision	(R1)-(R4)	(R5), **(R6)**	(R7), (R8)
Contraction	(C1)-(C5)	(C6), **(C7)**	<u>(C8)</u>, <u>(C9)</u>
Update	(U1)-(U4), (U8)	(U5), **(U9)**	(U6), (U7)
Erasure	(E1)-(E5), (E8)	<u>(E6)</u>, <u>**(E7)**</u>	<u>(E9)</u>, <u>(E10)</u>

Table 2. Panorama of belief change operations

Each row corresponds to an operation. The first column represents the basic postulates for each operation. Note that update and erasure have a specific postulate, namely (U8) and (E8). These postulates indicate that both update and erasure give each of the models of the initial belief equal consideration. The postulates in the second column capture the minimality of change principle. For each belief change operation, the postulates appearing in these first two columns are those required to state a representation theorem in terms of total preorders on interpretations. For each operation, the postulate in bold has to be removed and replaced by two weaker postulates, given in the third column, to account for partial preorders. All the representation theorems given in this paper can be easily read from this table. The underlined postulates are those we have introduced in this paper.

6 Conclusion

In this article, devoted to belief erasure in propositional logic, we continued and completed the work initiated by Katsuno and Mendelzon [15]. They formally defined belief erasure in a semantic framework and proposed a set of basic postulates. For revision and update, they proposed additionnal postulates that capture the minimality of change principle. They then showed that a revision (or update) operator satisfies these postulates if and only if it is induced by a total or partial preorder on interpretations. In 2017, Caridroit, Konieczny and Marquis [4] continued this study by looking at contraction. They obtained a representation theorem for contraction in terms of total preorders. In this paper we considered erasure and first adapted their work in defining postulates similar to theirs for capturing the minimality of change. In a second step, we weakened one of these postulates to account for partial preorders, and thus obtained a representation theorem for erasure operators in terms of partial preorders. As a by-product, we obtained a similar result for contraction. Finally, we drew a complete picture of four fundamental belief change operations, namely revision, update, contraction, and erasure, in the semantic setting, as shown in Table 2.

A natural continuation of this work would be the study of the operation called *Forget*, proposed by Winslett [23], which she compares to contraction. If ψ and μ are two propositional formulas and $\diamond$ is an update operator, the Forget operation is equivalent to $(\psi \diamond \mu) \vee (\psi \diamond \neg\mu)$. Another perspective would be the study of iterated contraction and erasure. While much work has been developed on iterated revision following the work of Darwiche and Pearl [8], iterated contraction has so far attracted little interest. Finally, a more ambitious study would be to investigate the complexity of decision problems such as model checking for contraction and erasure operators.

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