Line-transitive point-imprimitive linear spaces with Fang-Li parameter gcd(k, r) at most 12

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Submitted: Dec 8, 2023; Accepted: Mar 12, 2024; Published: Apr 5, 2024
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Abstract

This paper investigates the finite line-transitive point-imprimitive linear spaces. Let $\mathcal{S}$ be a non-trivial finite line-transitive point-imprimitive linear space with the Fang-Li parameter $k^{(r)} = 11$ or 12. Our conclusion is that $\mathcal{S}$ is the Desarguesian projective plane $\mathrm{PG}(2, 11)$.

Mathematics Subject Classifications: 05B05, 20B25, 51E05

1 Introduction

A *finite linear space* $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ consists of a finite set $\mathcal{P}$ and a non-empty set $\mathcal{L}$ of subsets of $\mathcal{P}$, where $\mathcal{P}$ is a set of v points and $\mathcal{L}$ consists of b distinguished subsets called lines, such that any two distinct points are exactly on one line and each line contains at least two points. $\mathcal{S}$ is said to be *non-trivial* if every line has more than two points. If the lengths of all lines are equal, then we say that $\mathcal{S}$ is *regular*. An automorphism of $\mathcal{S}$ is a permutation of $\mathcal{P}$ which leaves the set $\mathcal{L}$ invariant. The full automorphism group of $\mathcal{S}$ will be denoted by $\mathrm{Aut}(\mathcal{S})$. Any subgroup of $\mathrm{Aut}(\mathcal{S})$ will be called an automorphism group of $\mathcal{S}$. An automorphism group G of $\mathcal{S}$ is called *line-transitive* (resp.*point-transitive*) if it acts transitively on $\mathcal{L}$ (resp.$\mathcal{P}$). Owing to the result of Block ([3]), $\mathcal{S}$ is point-transitive if $\mathcal{S}$ is line-transitive. Similarly, if the automorphism group G acts primitively on points, then we say that $\mathcal{S}$ is *point-primitive*, otherwise, $\mathcal{S}$ is said to be *point-imprimitive*.

In this paper, we focus on the non-trivial finite line-transitive point-imprimitive linear space. Then the size of lines is constant, say k, so $\mathcal{S}$ is regular and is a 2-$(v, k, 1)$ design. It is easily known that for a non-trivial regular linear space, every point lies on the same number of lines, and is denoted by r. For a non-trivial finite linear space with parameters v, r, k and $b \geqslant 2$, it is well known that $vr = kb$ and $r = \frac{v-1}{k-1}$.

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Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a non-trivial finite linear space admitting G as a line-transitive point-imprimitive automorphism group, and $\mathcal{C} = \{C_1, C_2, \cdots, C_d\}$ be a G-invariant partition of the point set $\mathcal{P}$ with d classes of size c. In [9], Delandtsheer and Doyen proved that there exist two positive integers x and y such that

$$c = \frac{\binom{k}{2} - x}{y}, d = \frac{\binom{k}{2} - y}{x},$$

where x represents the number of unordered pairs of points $\{\alpha, \beta\}$ of a line λ which belongs to the same class of $\mathcal{C}$. We refer to c, d, x and y as the Delandtsheer-Doyen parameters. They also demonstrated that a line-transitive point-imprimitive 2-$(v, k, 1)$ design must satisfy $v \leqslant (\frac{k(k-1)}{2} - 1)^2$. For the special case when $v = (\frac{k(k-1)}{2} - 1)^2$, Nickel, Niemeyer, O'Keefe, Penttila and Praeger proved that k must be equal to 8 and constructed lots of examples ([18]). On the other hand, the line-transitive point-imprimitive linear spaces with line size k at most 8 were classified by Camina and Mischke in [5]. Praeger and Zhou continued this work in 2008, classifying the line-transitive point-imprimitive linear spaces with line size k at most 12 ([20]).

In 1991, Fang and Li introduced some parameters of linear spaces, defining

$$k^{(v)} = \gcd(k, v),\ k^{(r)} = \gcd(k, r),\ b^{(v)} = \gcd(b, v),\ b^{(r)} = \gcd(b, r).$$

For these parameters, we have

$$b = b^{(v)} b^{(r)},\ k = k^{(v)} k^{(r)},\ v = b^{(v)} k^{(v)} \text{ and } r = k^{(r)} b^{(r)}.$$

Furthermore, they proved that there exist two positive integers γ, δ such that

$$c = \gamma b^{(r)} + 1, d = \delta b^{(r)} + 1.$$

Here $k^{(v)}, k^{(r)}, b^{(v)}, b^{(r)}, \gamma$, and δ are called the Fang-Li parameters of $\mathcal{S}$ corresponding to the partition $\mathcal{C}$([12]). Note that, v is bounded by a function of $k^{(r)}$ according to [2, Theorem 2.3]. Then Betten, Delandtsheer and Zhou et al. provided the classification of the line-transitive point-imprimitive linear spaces with the Fang-Li parameter $\gcd(k, r) \leqslant 8$ ([2]). In 2012, this classification was extended up to 10 in [14]. In this paper, we continue this work to find parameters which would yield multiple new examples of line-transitive, point-imprimitive linear spaces, like the 2-$(729, 8, 1)$ family constructed by Nickel et al. in the 1990s which gave over 400 new examples. And we obtain the following classification.

Theorem 1. *Let $\mathcal{S}$ be a non-trivial finite linear space with the Fang-Li parameter $k^{(r)} =$ 11 or 12, and the automorphism group G of $\mathcal{S}$ is line-transitive and point-imprimitive, then $\mathcal{S}$ is the Desarguesian projective plane* $\mathrm{PG}(2, 11)$.

Combining Theorem 1, [2, Theorem 1.2] and [14, Theorem 1], we can conclude the following corollary.

Corollary 2. *Let $\mathcal{S}$ be a non-trivial finite linear space with the Fang-Li parameter $k^{(r)} \leqslant 12$, and let G be a line-transitive point-imprimitive automorphism group of $\mathcal{S}$. Then $\mathcal{S}$ is the Desarguesian projective plane* $\mathrm{PG}(2,4)$, $\mathrm{PG}(2,7)$, $\mathrm{PG}(2,9)$ *or* $\mathrm{PG}(2,11)$ *or the Mills design or Colbourn-McCalla design, both with $(v,k) = (91,6)$, or one of the 467 Nickel-Niemeyer-O'Keefe-Penttila-Praeger designs with $(v,k) = (729,8)$.*

As can be seen from the corollary 2, we have not found any new examples. To find new examples, it may be necessary to calculate larger Fang-Li parameter values. Therefore, our next step is to explore new methods on the basis of the existing algorithms, find some new parameters, and thus find some new examples. The rest of this paper is structured as follows. In Section 2, we provide some preliminary results. In Section 3 and Section 4, we utilize and implement the algorithms in [2] to generate a list of potential parameters and some group theory information for $k^{(r)} = 12$ or 11. We then conduct a detailed analysis of each of these potential parameters for $k^{(r)} = 12$ or 11, respectively.

2 Preliminary results

Let G be a transitive permutation group on $\mathcal{P}$, $\mathcal{C} = \{C_1, C_2, \cdots, C_d\}$ be a G-invariant partition of $\mathcal{P}$ with d classes of size c. Then $G^{\mathcal{C}} \cong G/G_{(\mathcal{C})}$ (called the *top group*) is a transitive permutation group induced by G on $\mathcal{C}$, and $G^{C} \cong G_C/G_{(C)}$ (called the *bottom group*) is a transitive permutation group induced by G on $C \in \mathcal{C}$ ([2]). Let $\mathcal{C}$ and $\mathcal{C}'$ be the partitions of $\mathcal{P}$, if every class of $\mathcal{C}$ is contained in a class of $\mathcal{C}'$, we say that $\mathcal{C}$ *refines* $\mathcal{C}'$ and this refinement is strict if $\mathcal{C} \neq \mathcal{C}'$. We also say that $\mathcal{C}'$ is *coarser* or *strictly coarser* than $\mathcal{C}$. The partition $\mathcal{C}$ is called *minimal* if it has no non-trivial strict refinement, and $\mathcal{C}$ is called *maximal* if the only G-invariant partition that is strictly coarser than $\mathcal{C}$ has only one class. We called the group G is 2-*step imprimitive* if $\mathcal{C}$ is both maximal and minimal, in this case, both $G^{\mathcal{C}}$ and G^{C} are primitive. If a partition $\mathcal{C}$ of $\mathcal{P}$ is both maximal and minimal, then we call G is 2-*step imprimitive* relative to $\mathcal{C}$, in this case, both $G^{\mathcal{C}}$ and G^{C} are primitive.

The partition $\mathcal{C}$ is *G-normal* if the subgroup $G_{(\mathcal{C})}$ is transitive on each class of $\mathcal{C}$. By [2, Theorem 1.1], we know that if G is a line-transitive point-imprimitive group of automorphisms of a linear space, then either

1. there exists a non-trivial G-normal point-partition; or
2. G is point-quasiprimitive and almost simple.

Recall that a finite permutation group G is *quasiprimitive* if every non-trivial normal subgroup of G is transitive. Equivalently, G is quasiprimitive if every minimal normal subgroup of G is transitive (since overgroups of transitive groups are transitive).

In [10], the authors gave a systematic analysis of finite line-transitive linear spaces with normal point-partition. Among their work, they defined the *intersection type*.

Definition 3. ([10]) Let $\mathcal{S}$ be a linear space, $\lambda \in \mathcal{L}$. Define

$$d_i = |\{C \in \mathcal{C} : |C \cap \lambda| = i\}|,$$

and

$$(0^{d_0}, 1^{d_1}, \cdots, k^{d_k})$$

to be the intersection type of $\mathcal{S}$.

We let the set of non-zero intersection sizes

$$\mathrm{Spec}(\mathcal{S}) := \{i > 0 : d_i \neq 0\}$$

be the spectrum of $\mathcal{S}$. Since the transitivity of G on $\mathcal{L}$, the intersection type and spectrum are independent of the choice of λ. We sometimes write $Spec_{\mathcal{C}}(\mathcal{S})$ if we need to specify the partition $\mathcal{C}$. For $d_0 = d - \sum_{i=1}^{k} d_i$, hence it doesn't matter to denote the intersection type of $\mathcal{S}$ by $(1^{d_1}, \cdots, k^{d_k})$.

In our discussion, t_{max} plays an important role.

Definition 4. ([2]) For a given intersection type $(1^{d_1}, \cdots, k^{d_k})$, and non-empty subset $S \subseteq$ Spec $(\mathcal{S})$, set $d(S) = \sum_{i \in S} d_i$. Define t_{max} to be the largest positive integer t such that, for all $S \subseteq$ Spec $(\mathcal{S})$, and all positive integers $h \leqslant min\{t, d(S)\}$,

$$\prod_{j=0}^{h-1} (d-j) \mid b \prod_{j=0}^{h-1} (d(\mathcal{S}) - j).$$

If no such positive integer exists then define t_{max} to be 0.

According to the [2, Lemma 4.9], the condition holds for $t = 1$, so t_{max} is positive, and then t_{max} is well-defined. For the sake of convenience, we state some results about linear spaces and assume the following hypothesis throughout the paper.

Hypothesis: Let $\mathcal{S}$ be a non-trivial finite linear space with v points and b lines, each of line size k, and with r lines through each point. Assumes that $G \leqslant \mathrm{Aut}(\mathcal{S})$ be line-transitive and point-imprimitive and leave invariant a non-trivial partition $\mathcal{C} = \{C_1, C_2, \cdots, C_d\}$ of $\mathcal{P}$ with d classes of size c where $c > 1$ and $d > 1$. Also, $\mathcal{S}$ has the Delandtsheer-Doyen parameters c, d, x, y, the Fang-Li parameter $k^{(v)}, k^{(r)}, b^{(v)}, b^{(r)}, \gamma, \delta$, intersection type $(1^{d_1}, \cdots, k^{d_k})$, t_{max}. Moreover, let $K = G_{(\mathcal{C})}$, $S = \mathrm{Soc}(K)$, $X = C_G(K)$, and $Y = C_G(S)$ if $\mathcal{C}$ is G-normal.

Next, we will present some lemmas regarding subgroups of the automorphism group G.

Lemma 5. [5, Lemma 5] *Assume that the* Hypothesis *holds and let p be a prime divisor of $|G|$.*

(i) *If $p \mid |G_{(\lambda)}|$ and $k^2 - k + 1 > max(r + k - p + 1, r)$, then $p \mid |G^{\lambda}|$ for any line λ.*

(ii) *If $p > k$ and $k^2 - k + 1 > r$, then $p \mid v$ or $p \mid (v - 1)$. Furthermore, if T is a Sylow p-subgroup of G, then $|T|$ divides v or $v - 1$, respectively.*

The following lemma is well known.

Lemma 6. [8, Lemma 2] *Assume that* $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ *is a non-trivial linear space admitting a line-transitive automorphism group* G. *Let* $\lambda \in \mathcal{L}$ *and* $H \leqslant G_\lambda$ *such that, for* $F := \mathrm{Fix}_{\mathcal{P}}(H)$,

(i) $2 \leqslant |F \cap \lambda| < |F|$, *and*

(ii) *If* $K \leqslant G_\lambda, |\mathrm{Fix}_{\mathcal{P}}(K) \cap \lambda| \geqslant 2$, *and* H *and* K *are conjugate in* G, *then* H *and* K *are conjugate in* G_λ.

Then the deduced linear space $\mathcal{S}|_F = (F, \mathcal{L}|_F)$ *has constant line size and* $N_G(H)$ *acts line-transitively on* $\mathcal{S}|_F$, *where* $\mathcal{L}|_F = \{\lambda \cap F : \lambda \in \mathcal{L}, |\lambda \cap F| \geqslant 2\}$.

Since two Sylow p-subgroups of G_λ must be conjugate in G_λ, we get:

Corollary 7. [2, Corollary 4.6] *Assume that* HYPOTHESIS *holds. Let* $\lambda \in \mathcal{L}$, *and let* p *be a prime dividing* $|G_\lambda|$. *Let* P *be a Sylow* p*-subgroup of* G_λ, *and let* $F = \mathrm{Fix}_{\mathcal{P}}(P)$. *Suppose that* $2 \leqslant |F \cap \lambda| < |F|$. *Then*

(i) $N_G(P)$ *is line-transitive on* $\mathcal{S}|_F$;

(ii) $\mathcal{C}|_F = \{C \cap F : C \in \mathcal{C}, C \cap F \neq \emptyset\}$ *is an* $N_G(P)$*-invariant partition of* F;

(iii) $|F| = f \cdot |C \cap F|$, *where* $f = |\mathcal{C}|_F|$, $C \cap F \in \mathcal{C}|_F$, *and* $|C \cap F| \geqslant 3$.

Subsequently, some results about the top group $G^{\mathcal{C}}$ and the bottom group G^{C} were given.

Lemma 8. [2, Lemma 4.9] *Assume that the* HYPOTHESIS *holds. Then the corresponding parameter* t_{max} *is positive. Moreover, if* $G^{\mathcal{C}}$ *is* t*-transitive, then* $t \leqslant t_{max}$.

Lemma 9. [10, Proposition 2.6] *Assume that the* HYPOTHESIS *holds.*

(i) *The number* $b^{(r)}$ *divides each non-trivial subdegree of* $G^{\mathcal{C}}$, *and in particular, rank* $(G^{\mathcal{C}}) \leqslant 1 + \delta$.

(ii) *The number* $b^{(r)}$ *divides each non-trivial subdegree of* G^{C}, *and in particular, rank* $(G^{C}) \leqslant 1 + \gamma$.

(iii) *Moreover, for* $\alpha \in \mathcal{P}$, $b^{(r)}$ *divides each non-trivial subdegree of* $G^{\mathcal{P}}$ *and each orbit length of* G_α *in* $\lambda \in \mathcal{L}$ *and* $\alpha \in \lambda$.

Lemma 10. [16] *Assume that the* HYPOTHESIS *holds.*

(i) *If* $\gamma = 2$, $b^{(r)}$ *is odd, and* c *is not a prime power, then* G^{C} *is* 2*-transitive on* C.

(ii) *If* $\delta = 2$, $b^{(r)}$ *is odd, and* d *is not a prime power, then* $G^{\mathcal{C}}$ *is* 2*-transitive on* $\mathcal{C}$.

Finally, we collect some lemmas about line-transitive point-primitive linear spaces if the partition $\mathcal{C}$ is G-normal.

Lemma 11. [2, Theorem 6.1] *Assume that the* HYPOTHESIS *holds and that* $\mathcal{C}$ *is* G*-normal.*

(i) *If* $k > 2x + \frac{3}{2} + \sqrt{4x - \frac{7}{4}}$*, then* $G_{(\mathcal{C})}$ *is semiregular on points and lines,* $|G_{(\mathcal{C})}| = c$ *is odd, and* $d_1 > 0$*.*

(ii) *If* $x \leqslant 8$*, then* $G_{(\mathcal{C})}$ *has an abelian subgroup* S *of index at most* 2 *such that* S *is normal in* G*, semiregular on points, and* $|S| = c$ *is odd.*

(iii) *If either of the conditions of* (i) *or* (ii) *holds, and if* $\mathcal{C}$ *is minimal, then* c *is an odd prime power and* $G^{\mathcal{C}}$ *is affine.*

Lemma 12. [20, Lemma 2.2] *Assume that the* HYPOTHESIS *holds,* $\mathcal{C}$ *is* G*-normal and minimal.*

(a) *Then* $\mathcal{C}$ *is the set of* S*-orbits in* $\mathcal{P}$ *and*

 (i) *Either* $Y \cap K = 1$*, or* S *is elementary abelian and* $Y \cap K = S$*.*

 (ii) *Either* $X \cap K = 1$*, or* S *is elementary abelian and* $X \cap K = K = S$*.*

(b) *Suppose in addition that* $\mathcal{C}$ *is maximal, that* $Y \cap K = S \neq Y$*, and that one of the following conditions holds:*

Condition	$\mathrm{Soc}(G^{\mathcal{C}})$	*Extra Property*
1	*abelian*	$\gcd(c, d) = 1$
2	*non-abelian*	*Schur multiplier of a minimal normal subgroup of* $G^{\mathcal{C}}$ *has no section isomorphic to* S

Then G *has a normal subgroup* $M = T \times S$*, where* T *is a minimal normal subgroup of* G *and* $T^{\mathcal{C}}$ *is minimal normal in* $G^{\mathcal{C}}$*. Moreover, either*

(i) T *is non-abelian and transitive on* $\mathcal{P}$*, or*

(ii) *the set* $\mathcal{C}'$ *of* T*-orbits in* $\mathcal{P}$ *is a* G*-normal partition of* $\mathcal{P}$ *with* $|\mathcal{C}'| = c$ *such that for* $C \in \mathcal{C}$ *and* $C' \in \mathcal{C}'$*,* $|C'| = d$*,* $|C \cap C'| = 1$*. Moreover, either* M *is regular on* $\mathcal{P}$ *or* T *is not semiregular on* $\mathcal{P}$*.*

3 Case $k^{(r)} = 12$

In this section, we will deal with the non-trivial finite linear spaces with $k^{(r)} = 12$. By applying Algorithms 1 and 2 in [1] which are implemented in GAP([13]), we obtain 16 cases listed in Table 1, which are potential parameter sequences

$$(d, c, x, y, \gamma, \delta, k^{(v)}, k^{(r)}, b^{(v)}, b^{(r)}, t_{max}).$$

For each case, there is no other partition of $\mathcal{P}$ which is neither a strict refinement nor strict coarser than $\mathcal{C}$, so all of their line-transitive and point-imprimitive automorphism groups are 2-step imprimitive relative to $\mathcal{C}$. According to the classification of the primitive groups (cf.[7],[11],[21]), we get the potential structures of the bottom groups and the top groups for each case and list them in Table 2.

Table 1: Potential parameter sequences for $k^{(r)} = 12$

Case	$d{\cdot}c$	(x,y)	(γ,δ)	$k^{(v)}{\cdot}k^{(r)}$	$b^{(v)}{\cdot}b^{(r)}$	t_{max}	Reference
1	65·65	(1,1)	(2,2)	1·12	4225·32	2	[20]
2	497·497	(7,7)	(2,2)	7·12	35287·248	2	$(T, 491)$
3	3521·3521	(49,49)	(2,2)	49·12	253009·1760	2	$(T, 3517)$
4	61·13	(1,5)	(2,10)	1·12	793·6	2	[20]
5	781·157	(11,55)	(2,10)	11·12	11147·78	2	KSXY
6	53·5	(1,13)	(2,26)	1·12	265·2	1	[20]
7	1633·97	(23,391)	(2,34)	23·12	6887·48	1	$(T, 1627)$
8	2201·89	(31,775)	(2,50)	31·12	6319·44	1	$(T, 2179)$
9	385·49	(22,176)	(4,32)	11·12	1715·12	1	$(T, 383)$
10	19·7	(3,9)	(6,18)	1·12	133·1	1 or 2	[20]
11	13·61	(5,1)	(10,2)	1·12	793·6	2	[20]
12	157·781	(55,11)	(10,2)	11·12	11147·78	2	(T, t_{max})
13	7·19	(9,3)	(18,6)	1·12	133·1	1 or 2	[20]
14	5·53	(13,1)	(26,2)	1·12	265·2	5	[20]
15	49·385	(176,22)	(32,4)	11·12	1715·12	1	$(B, 383)$
16	97·1633	(391,23)	(34,2)	23·12	6887·48	2	$(B, 1627)$

Table 2: Candidate groups with $k^{(r)} = 12$

Case	$\mathrm{Soc}(G^{\mathcal{C}})$	$\mathrm{Soc}(G^{C})$
1	A_{65}, $\mathrm{PSL}(2,2^6)$, $\mathrm{PSL}(2,5^2)$, $\mathrm{PSL}(2,5^2)$, $\mathrm{PSU}(3,2^2)$, $Sz(2^3)$	A_{65}, $\mathrm{PSL}(2,2^6)$, $\mathrm{PSL}(2,5^2)$, $\mathrm{PSU}(3,2^2)$, $Sz(2^3)$
2	A_{497}	A_{497}
3	A_{3521}	A_{3521}
4	A_{61}, affine	A_{13}, $\mathrm{PSL}(3,3)$, affine
5	A_{781}, $\mathrm{PSL}(5,5)$	A_{157}, affine
6	A_{53}, affine	A_5, affine
7	A_{1633}	A_{97}, affine
8	A_{2201}	A_{89}, affine
9	A_{385}	A_{49}, $A_7 \times A_7$, $\mathrm{PSL}(2,7) \times \mathrm{PSL}(2,7)$, affine
10	A_{19}, affine	A_7, $\mathrm{PSL}(2,7) = \mathrm{PGL}(3,2)$, affine
11	A_{13}, $\mathrm{PSL}(3,3)$, affine	A_{61}, affine
12	A_{157}, affine	A_{781}, $\mathrm{PSL}(5,5)$
13	A_7, $\mathrm{PSL}(2,7) = \mathrm{PSL}(3,2)$	A_{19}, affine
14	A_5, affine	A_{53}, affine
15	A_{49}, $A_7 \times A_7$, $\mathrm{PSL}(2,7) \times \mathrm{PSL}(2,7)$, affine	A_{385}
16	A_{97}, affine	A_{1633}

Proposition 13. *Assume that the* HYPOTHESIS *holds and* $k^{(r)} = 12$. *Then* $\mathcal{S}$ *is the Desarguesian projective plane* $\mathrm{PG}(2,11)$.

Proof. Since $k^{(r)} = 12$, the parameter sequence of $\mathcal{S}$

$$(d, c, x, y, \gamma, \delta, k^{(v)}, k^{(r)}, b^{(v)}, b^{(r)}, t_{\max})$$

is one of lines in Table 1. By [20, Theorem 1.1], Cases 1, 4, 6, 11, 14 can be ruled out, and $\mathcal{S}$ is the Desarguesian projective plane $\mathrm{PG}(2,11)$ for Cases 10, 13. (The parameter sequence of Case 1 was omitted in [20, Table 3], but it also can be excluded by the method of Case 12.)

Assume that Case 7 holds. Here $d = 23{\cdot}71$ is not a prime power, then $\mathrm{Soc}(G^{\mathcal{C}}) = A_{1633}$ according to Table 2. It follows that $1627 \mid |\mathrm{Soc}(G^{\mathcal{C}})|$, then $1627 \mid |G|$. Let $p = 1627$, then $p > k = 276$ and $k^2 - k + 1 = 75901 > r = 576$. However, $p \nmid v(v-1)$ for $v(v-1) = 2^6{\cdot}3^2{\cdot}5^2{\cdot}11{\cdot}23{\cdot}71{\cdot}97$, a contradiction by Lemma 5. Similarly, $\mathrm{Soc}(G^{\mathcal{C}}) = A_{1633}$ for Case 16. Then $p = 1627$ satisfies the conditions of Lemma 5, but $p \nmid v(v-1)$. Thus, Case 16 can be ruled out. Cases 2, 3, 8, 9 and 15 can be excluded in the same way as Cases 3 and 16. In Table 1, the symbol "(T, p)" (or "(B, p)") denotes the contradiction by the top group (or bottom group) and the prime p.

For Case 12, we have

$$k = 132 > 2x + \frac{3}{2} + \sqrt{4x - \frac{7}{4}} \approx 126.3.$$

If the partition $\mathcal{C}$ is G-normal, then c must be an odd prime power by Lemma 11, a contradiction since here $c = 11 \cdot 71$. Therefore, $G^{\mathcal{C}} \cong G$ is quasiprimitive and almost simple. Then $\mathrm{lcm}(b, v) \mid |G|$, thus, $G \geqslant A_{157}$ according to Table 2. However, $t_{max} = 2$ for this case, which is impossible by Lemma 8. The symbol "(T, t_{max})" refers to this situation.

Suppose that Case 5 holds, then $\mathrm{Soc}(G^{\mathcal{C}}) = \mathrm{PSL}(5,5)$ for $t_{max} = 2$ by Table 2 and Lemma 8. If $G^{\mathcal{C}} \cong G$ is an almost simple primitive group of degree $d = 781$, then $\mathrm{lcm}(v, b) \mid |G^{\mathcal{C}}|$, a contradiction. Thus, the partition $\mathcal{C}$ is G-normal relative to K. Note that

$$k = 132 > 2x + \frac{3}{2} + \sqrt{4x - \frac{7}{4}} = 30,$$

then $K = Z_{157}$ is semiregular by Lemma 11, and

$$Y \cap K = K = S,\ G/Y \leqslant \mathrm{Aut}(S) = Z_{156}.$$

Since $781 \mid |G|$, but $781 \nmid 156$, thus we have $781 \mid |Y|$, hence $Y^{\mathcal{C}} \neq 1$. Therefore, there exists a normal subgroup M of G by Lemma 12, such that

$$S < M \leqslant Y,\ M/S \cong M^{\mathcal{C}},\ M = T \times S.$$

Also, we have $\gcd(c, d) = 1$, thus, either T is transitive on $\mathcal{P}$ or T induces a new partition of $\mathcal{P}$ with 157 parts of size 781. The latter cannot occur because the corresponding case with parameters $(d, c) = (157, 781)$ in Table 1 has been ruled out. Thus, T is transitive on $\mathcal{P}$, and then $c \mid |T|$, we obtain the desired contradiction since $|T| = |M^{\mathcal{C}}|$ and $|M^{\mathcal{C}}| \mid |\mathrm{PSL}(5,5)|$. In Table 1, the symbol" KSXY " means that the corresponding case is ruled out by Lemma 12. This completes the proof of Proposition 13. □

4 Case $k^{(r)} = 11$

This section deals with the non-trivial finite linear spaces with $k^{(r)} = 11$. By applying Algorithms 1 and 2 of [1], a list of feasible parameters for $k^{(r)} = 11$ is obtained. Table 9 (The table is placed in the Appendix) lists 180 cases, each providing values of the following parameters:

$$(d, c, x, y, \gamma, \delta, k^{(v)}, k^{(r)}, b^{(v)}, b^{(r)}, t_{max}).$$

The intersection types for each case are not displayed in Table 9 due to limited space. For each case, there is no other partition of $\mathcal{P}$ which is neither strictly refined nor strictly coarser than $\mathcal{C}$, so all of their automorphism groups admitting line-transitive and point-imprimitive are 2-step imprimitive relative to $\mathcal{C}$, this provides a prerequisite for our proof in this section.

Proposition 14. *There is no non-trivial finite line-transitive point-imprimitive linear space $\mathcal{S}$ with the Fang-Li parameter $k^{(r)} = 11$.*

Proof. Suppose for the contrary that there exists a non-trivial finite line-transitive point-imprimitive linear space $\mathcal{S}$ with the Fang-Li parameter $k^{(r)} = 11$. Then the parameter sequence of $\mathcal{S}$

$$(d, c, x, y, \gamma, \delta, k^{(v)}, k^{(r)}, b^{(v)}, b^{(r)}, t_{max})$$

is one of the lines in Table 9. Cases 67, 73, 80, 96, 117, 129 and 137 can be ruled out by [20, Theorem 1.1]. According to [7, Table 2-8], [11, Table B.4] and [21, Table 2-12], there are 22 cases with $\mathrm{Soc}(G^{\mathcal{C}}) = A_d$. However, $t_{max} = 1$ or 2 for these cases, which is impossible by Lemma 8. We denote this contradiction in Table 9 with the symbol "(T, t_{max})".

For Case 5, $d = 1440 = 2^5 \cdot 3^2 \cdot 5$ is not a prime power, then $\mathrm{Soc}(G^{\mathcal{C}}) = A_{1440}$ or $\mathrm{PSL}(2, 1439)$ according to [21, Table 2-12]. For both cases, the prime $719 \mid |\mathrm{Soc}(G^{\mathcal{C}})|$. Let $p = 719$, then $p > k = 132$ and $k^2 - k + 1 = 17293 > r = 15829$. However, $p \nmid v(v-1) = 2^{10} \cdot 3^4 \cdot 5^2 \cdot 11 \cdot 131 \cdot 1439$, a contradiction by Lemma 5. Similarly, there are 80 cases which can be ruled out. This contradiction is represented in Table 9 by the symbols "(T, p)" or "(B, p)".

Next, we consider the remaining cases in Table 9.

Step 1. Cases 1, 9-12, 16, 25, 28, 33, 44, 57, 66, 72, 76, 82, 92, 95, 101, 105, 106, 108, 115, 132, 145 and 155 cannot occur.

For Cases 9-12, 66, 105, 132, 145 and 155, we have $\delta = 1$. Then $G^{\mathcal{C}}$ is 2-transitive by Lemma 9. For Cases 16 and 106, $\delta = 2$, $b^{(r)}$ is odd, and d is not a prime power, then $G^{\mathcal{C}}$ is 2-transitive by Lemma 10. Thus, we can get the potential top groups for these cases according to the classification of 2-transitive permutation groups ([4, Theorem 5.3]). For the rest of cases, we get the potential top groups by [11, Table B.4] and [21, Table 2-12]. The information for their top groups is listed in Table 3.

Firstly, Cases 9, 10, 12, 16, 106, 132, 135 and 155 are ruled out by Lemma 8. Secondly, for Cases 1, 11, 25, 28, 33, 44, 57, 66, 72, 76, 82, 92, 95, 101, 105, 108 and 115, we have

$$k > 2x + \frac{3}{2} + \sqrt{4x - \frac{7}{4}}.$$

If the partition $\mathcal{C}$ is G-normal, then c is a prime power, a contradiction by Lemma 11. Therefore, $G^{\mathcal{C}} \cong G$ is quasiprimitive and almost simple. According to [6, MAIN THEOREM], $\mathrm{Soc}(G^{\mathcal{C}}) \neq A_n$ for an integer $n > 8$. However, $\mathrm{lcm}(b, v) \nmid |\mathrm{Soc}(G^{\mathcal{C}})||\,\mathrm{Out}(\mathrm{Soc}(G^{\mathcal{C}}))|$ for each other potential top group, which is impossible.

Table 3: Relevant information for "Step 1"

Case	$\mathrm{Soc}(G^{\mathcal{C}})$	k	$2x + \frac{3}{2} + \sqrt{4x - \frac{7}{4}}$	c	t_{max}
1	A_{230}, PSL(2,229) ([11])	22	$= 5$	$2 \cdot 5 \cdot 23$	3
9	A_{4344} ([4])				2
10	A_{5796} ([4])				2
11	A_{8700}, PSL(2,8699) ([4])	792	≈ 85.4	$2^2 \cdot 3 \cdot 5^2 \cdot 29$	2
12	A_{17412} ([4])				2
16	A_{18863} ([4])				2
25	A_{1435}, PSL(2,41) ([21])	132	≈ 18.2	$2^4 \cdot 3 \cdot 5$	2
28	A_{1433}, affine ([21])	132	≈ 18.2	$2^2 \cdot 3^2 \cdot 5$	1
33	A_{1429}, affine ([21])	132	≈ 18.2	$2^3 \cdot 3 \cdot 5$	2
44	A_{208}, $U_3(2^2)$ ([11])	22	$= 5$	$2 \cdot 5$	1
57	A_{1381}, affine ([21])	132	≈ 18.2	$2^3 \cdot 3$	2
66	A_{9432}, PSL(2,9431) ([4])	1716	≈ 338.4	$13 \cdot 1451$	2
72	A_{171}, A_{19}, PSL(2,19) ([11])	33	≈ 10.7	$5 \cdot 7$	2
76	A_{769}, affine ([11])	143	≈ 34.6	$5 \cdot 13$	1
82	A_{761}, affine ([11])	143	≈ 34.6	$3 \cdot 13$	1
92	A_{553}, PSL(3,23) ([11])	154	≈ 52.6	$2 \cdot 5 \cdot 7$	1
95	A_{1812}, PSL(2,1811) ([21])	660	≈ 263.4	$3^2 \cdot 5 \cdot 7 \cdot 23$	2
101	A_{449}, affine ([11])	165	≈ 72.4	$3^2 \cdot 5^2$	1
105	A_{4644}, PSL(2,4643) ([4])	2112	≈ 1005.3	$2^4 \cdot 1451$	2
106	A_{5031} ([4])				2
108	A_{190}, A_{20}, PSL(2,19) ([11])	88	≈ 50.3	$2^3 \cdot 17$	1
115	A_{240}, PSL(2,239) ([11])	132	≈ 85.4	$5 \cdot 7 \cdot 41$	2
132	A_{144} ([4])				2
145	A_{816} ([4])				2
155	A_{276} ([4])				2

Step 2. Cases 31, 38, 41, 43, 52, 54, 55, 59, 69, 75, 79, 81, 93, 109, 111, 113, 114 and 135 cannot occur.

For Cases 31, 38 and 43, we have $\gamma = 1$, then G^C is 2-transitive by Lemma 9. For Case 69, $\gamma = 2$, $b^{(r)}$ is odd, and c is not a prime power, then G^C is also 2-transitive by Lemma 10. Therefore, $G^C \geqslant A_c$ for Cases 31, 38, 43 and 69 by [4, Theorem 5.3].

For Case 109, G^C is a primitive group of degree $c = 2^4 \cdot 311$. Let $p_1 = 311$, then G^C has an element g of order p_1. Suppose that g is a product of q p_1-cycles and $|\mathrm{Fix}_{\mathcal{C}}(g)| = m$. Then $q \leqslant 16$. According to [22, Theorem 13.10], we have $|\mathrm{Fix}_{\mathcal{C}}(g)| \leqslant 4q - 4$, it follows that $m = 0$ since $d \equiv m \pmod{p_1}$. Thus $G^C \geqslant A_c$ according to [17, Theorem1.1, Table 3]). Similarly, for Case 135, there is also $G^C \geqslant A_c$. For the rest of cases, we also can get $\mathrm{Soc}(G^C) = A_c$ from [7, Table 2-8], [11, TABLE B.4] and [21, Table 2-12].

For Cases 31, 69, 109 and 135, we have $k^2 - k + 1 > r$, and a prime $p > k$ which $p \mid |G^C|$, but $p \nmid v(v-1)$, contradicting Lemma 5. If the parameter sequence of $\mathcal{S}$ is one of Cases 38, 41, 43, 52, 54, 55, 59, 75, 79, 81, 93, 111, 113 and 114 then there is a prime

$p \mid |G|$, but $p \nmid b$. Let P be a Sylow p-subgroup of G, then there is a line $\lambda \in \mathcal{L}$ such that $P \leqslant G_\lambda$. Let $F := \mathrm{Fix}_{\mathcal{P}}(P)$. For the line size k, let k_1 be the minimum integer such that $0 \leqslant k_1 < k$ and $k \equiv k_1 (\mathrm{mod}\ p)$. For each case, we have $k_1 > 2$ and $(v-k) \nmid p$, thus there exist points not on λ which are fixed by P, and so $F \nsubseteq \lambda$. Then by Lemma 6, F induces a linear space $\mathcal{S}|_F = (F, \mathcal{L}|_F)$. For this linear space, the number of the points

$$v_0 = |F| \leqslant k + r - p - 1,$$

the line-size

$$k_0 = |\lambda \cap F| \geqslant k_1.$$

Since

$$v_0 - 1 \geqslant k_0(k_0 - 1) \geqslant k_1(k_1 - 1),$$

we have

$$k_1(k_1 - 1) \leqslant k + r - p - 1,$$

a contradiction. The corresponding information about these cases is listed in Table 4.

Table 4: Relevant information for "Step 2"

Case	Group	p	k_1	$v-k$	$k_1(k_1-1)$	$k+r-p-1$
38	$G^C \geqslant A_{816}$ ([4])	811	377	$2^2 \cdot 3^3 \cdot 83 \cdot 1187$	141752	9341
41	$G^C \geqslant A_{2292}$ ([21])	2287	1673	$2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 37 \cdot 107$	2797256	26873
43	$G^C \geqslant A_{276}$ ([4])	271	257	$2^4 \cdot 3^3 \cdot 7 \cdot 17 \cdot 31$	65792	3281
52	$G^C \geqslant A_{372}$ ([11])	367	219	$2^4 \cdot 3 \cdot 5 \cdot 17 \cdot 1319$	47742	5033
54	$G^C \geqslant A_{222}$ ([11])	211	146	$2 \cdot 3^5 \cdot 5 \cdot 23 \cdot 43$	21170	3209
55	$G^C \geqslant A_{283}$ ([11])	283	276	$2^7 \cdot 3 \cdot 5^2 \cdot 7 \cdot 67$	75900	4325
59	$G^C \geqslant A_{1680}$ ([21])	1669	678	$2^2 \cdot 3^5 \cdot 19 \cdot 10691$	459006	27491
75	$G^C \geqslant A_{2483}$ ([21])	2477	1813	$2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 13 \cdot 4289$	3285156	15463
79	$G^C \geqslant A_{2067}$ ([21])	2063	389	$2 \cdot 13 \cdot 19 \cdot 23 \cdot 6577$	150932	15877
81	$G^C \geqslant A_{403}$([11])	401	227	$2 \cdot 5 \cdot 13 \cdot 17 \cdot 1429$	51302	3239
93	$G^C \geqslant A_{448}$([11])	443	57	$2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 277$	3192	2581
111	$G^C \geqslant A_{3456}$ ([7])	3449	951	$2^4 \cdot 5^2 \cdot 19 \cdot 53 \cdot 83$	903450	8551
113	$G^C \geqslant A_{3056}$ ([7])	3049	2231	$2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 5279$	4975130	8951
114	$G^C \geqslant A_{96}$ ([11])	89	87	$2^4 \cdot 5^2 \cdot 7 \cdot 13$	7482	295

Step 3. Cases 24, 36, 46, 48, 49, 51, 58, 61, 62, 123 and 124 cannot occur.

For Case 24, since $t_{max} = 2$, we have $G^{\mathcal{C}} \ngeqslant A_{23216}$ by Lemma 8. Now $G^{\mathcal{C}}$ is a primitive group of degree $d = 4^2 \cdot 1451$. Let $p = 1451$, then $G^{\mathcal{C}}$ has an element g of order p. Suppose that g is a product of q p-cycles and $|\mathrm{Fix}_{\mathcal{C}}(g)| = m$. Then $q \leqslant 16$. According to [22, Theorem 13.10], we have $|\mathrm{Fix}_{\mathcal{C}}(g)| \leqslant 4q - 4$, it follows that $m = 0$ since $d \equiv m \pmod p$. However, there is no such primitive group of degree d according to [17, Theorem1.1, Table 3], a contradiction. Similarly, Cases 36, 46, 48, 49, 51, 58, 61, 62, 123, and 124 can also be ruled out, and we list the corresponding information for these cases in Table 5.

Table 5: Relevant information for "Step 3"

Case	$G^{\mathcal{C}}$	t_{max}	d	p
36	$\ngeqslant A_{36275}$	1	$5^2 \cdot 1451$	1451
46	$\ngeqslant A_{8676}$	1	$2^2 \cdot 3^2 \cdot 241$	241
48	$\ngeqslant A_{13023}$	1	$3^2 \cdot 1447$	1447
49	$\ngeqslant A_{66746}$	1	$2 \cdot 23 \cdot 1451$	1451
51	$\ngeqslant A_{71099}$	1	$7^2 \cdot 1451$	1451
58	$\ngeqslant A_{5728}$	1	$2^5 \cdot 179$	179
61	$\ngeqslant A_{10059}$	1	$3 \cdot 7 \cdot 479$	479
62	$\ngeqslant A_{153806}$	1	$2 \cdot 53 \cdot 1451$	1451
123	$\ngeqslant A_{4976}$	1 or 2	$2^4 \cdot 311$	311
124	$\ngeqslant A_{6531}$	1 or 2	$3 \cdot 7 \cdot 311$	311

Step 4. Cases 20 and 60 cannot occur.

For Cases 20 and 60, $c-1$ is prime. Let $p = c-1$, then $\mathrm{Soc}(G^C) = A_c$ or $\mathrm{PSL}(2,p)$ by [21, Table 2-12] and [11, Table B.4]. If $\mathrm{Soc}(G^C) = A_c$, then there is a prime p_1 ($p_1 = 1801$ for Case 20, and $p_1 = 541$ for Case 60) such that $p_1 \mid |G^C|$. Similar to Step 2, $G^C \ngeqslant A_c$. Thus, $G^C \geqslant \mathrm{PSL}(2, c-1)$. Now let $P \in Syl_p(G)$, if $P^{\mathcal{C}} = 1$, then $P \leqslant G_C$ for each $C \in \mathcal{C}$, and $|\mathrm{Fix}_C(P)| \geqslant 1$ since $p \nmid d$. It follows that $|\mathrm{Fix}_{\mathcal{P}}(P)| \geqslant d$, thus there must exist a line $\lambda \in \mathcal{L}$ such that $P \leqslant G_\lambda$. On the other hand, $P^C \leqslant G^C \leqslant \mathrm{PGL}(2,p)$ for each $C \in \mathcal{C}$. Since $\mathrm{PGL}(2,p)$ is sharply 3-transitive, the non-trivial subgroup P^C can fix at most 2 points of C. But according to Lemma 7, $|\mathrm{Fix}_C(P)| \geqslant 3$, a contradiction. Therefore, $P^{\mathcal{C}} \neq 1$, and then $p \mid |G^{\mathcal{C}}|$. Since $G^{\mathcal{C}}$ is a primitive group of degree $d = q \cdot p + 1$ ($q = 4$ for Case 20, $q = 83$ for Case 60) and contains an element of order p which is a product of m ($1 \leqslant m \leqslant q$) p-cycles and has at leat one fixed part of $\mathcal{C}$. Then we get $G^{\mathcal{C}} \geqslant A_d$ according to [17, Theorem1.1, Table 2]. However, both Case 20 and 60 have $t_{max} = 1$, a contradiction by Lemma 8.

Step 5. Cases 125, 128, 136, and 159 cannot occur.

For Case 125, $\mathrm{Soc}(G^{\mathcal{C}}) \cong A_{180}$ or $\mathrm{PSL}(2,179)$ by [11, Table B.4]. If $G^{\mathcal{C}} \cong G$ is an almost simple primitive group of degree $d = 180$, then $\mathrm{lcm}(v,b) \mid |G^{\mathcal{C}}|$, which leads a contradiction because $|G^{\mathcal{C}}|$ divides $|\mathrm{Soc}(G^{\mathcal{C}})| \cdot |\mathrm{Out}(\mathrm{Soc}(G^{\mathcal{C}}))|$. Thus, the partition $\mathcal{C}$ is G-normal relative to K. Note that

$$k = 132 > 2x + \frac{3}{2} + \sqrt{4x - \frac{7}{4}} \approx 111.29,$$

then $K = Z_{1433}$ is semiregular by Lemma 11, and

$$Y \cap K = K = S, \ G/Y \leqslant \mathrm{Aut}(S) = Z_{1432}.$$

Since $180 \mid |G|$, and $180 \nmid 1432$, we have $180 \mid |Y|$. Hence $Y^{\mathcal{C}} \neq 1$. Therefore, there exists a normal subgroup M of G by Lemma 12, such that

$$S < M \leqslant Y, M/S \cong M^{\mathcal{C}}, M = T \times S.$$

Also, we have $\gcd(c,d) = 1$, thus, either T is transitive on $\mathcal{P}$ or T induces a new partition of $\mathcal{P}$ with 1433 parts of size 180. The latter case cannot occur because the corresponding case with parameters $(d,c) = (1433, 180)$ in Table 9 has been ruled out. Thus, T is transitive on $\mathcal{P}$, and $c \mid |T|$, we obtain the desired contradiction since $|T| = |M^{\mathcal{C}}|$ and $|M^{\mathcal{C}}| \mid |\mathrm{Soc}(G^{\mathcal{C}})|$. Therefore, this case is ruled out. Case 128 can be ruled out by the same method as Case 125 (See Table 6).

For Case 136, $\delta = 1$, then $G^{\mathcal{C}}$ is 2-transitive by Lemma 9. Thus, $\mathrm{Soc}(G^{\mathcal{C}}) \cong A_{120}$ or $\mathrm{PSp}(8,2)$ by [4, Theorem 5.3]. If $G^{\mathcal{C}} \cong G$ is an almost simple primitive group of degree $d = 120$, then $\mathrm{lcm}(v,b) \mid |G^{\mathcal{C}}|$, a contradiction. Thus, the partition $\mathcal{C}$ is G-normal relative to K. If K is not semiregular, we have

$$1 + \frac{r}{k} d_1(d_1 - 1) \leqslant |\mathrm{Fix}_{\mathcal{P}}(K_\alpha)| \leqslant d$$

by [19, Corollary 4.2] and [19, Lemma 5.1]. But in this case, the intersection type is $(0^{24}, 1^{84}, 4^{12})$, $(0^{36}, 1^{60}, 3^{24})$ or $(0^{48}, 1^{24}, 2^{36}, 3^{12})$, a contradiction. Hence K is semiregular. Since $K \trianglelefteq G^{C}$, K is the unique soluble minimal regular normal subgroup of primitive group G^{C}. Thus,

$$K = Z_{1429},\ Y \cap K = K = S,\ G/Y \leqslant \mathrm{Aut}(S) = Z_{1428}.$$

Since

$$5 \mid |G^{\mathcal{C}}|, \quad 5 \nmid |G/Y|,$$

we have $Y^{\mathcal{C}} \neq 1$. There exists a normal subgroup M of G by Lemma 12, such that

$$S < M \leqslant Y, M/S \cong M^{\mathcal{C}}, M = T \times S.$$

Then the conditions of Lemma 12 hold by [15, Theorem 5.1.4]. Thus, either T is transitive on $\mathcal{P}$ or T induces a G-normal partition of $\mathcal{P}$ with 1429 parts of size 120. The latter case cannot occur because the corresponding case with parameters $(d,c) = (1429, 120)$ has been ruled out. Thus, T is transitive on $\mathcal{P}$, and $c \mid |T|$, we obtain the desired contradiction. Therefore, this case is ruled out. Cases 159 can be ruled out by the same method as Case 136 (See Table 7).

Table 6: Relevant information for Case 128

Case	$\mathrm{Soc}(G^{\mathcal{C}})$	$\mathrm{lcm}(b,v)$	k	$2x + \frac{3}{2} + \sqrt{4x - \frac{7}{4}}$	v
128	A_{225}, $A_{15} \times A_{15}$, $A_8 \times A_8$, $A_7 \times A_7$, $A_6 \times A_6$	$2^3 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 449$	165	≈ 136.9	$3^2 \cdot 5^2 \cdot 449$

Table 7: Relevant information for Case 159

Case	$\mathrm{Soc}(G^{\mathcal{C}})$	$\mathrm{lcm}(b,v)$	v	intersection type
159	A_{65},$\mathrm{PSL}(2,2^6)$,$\mathrm{PSL}(2,5^2)$, $\mathrm{PSU}(3,2^2)$,$Sz(2^3)$	$2^5 \cdot 5 \cdot 13 \cdot 769$	$5 \cdot 13 \cdot 769$	$(1^{26}, 2^{26}, 5^{13})$, $(1^{39}, 4^{26})$, $(0^{13}, 1^{13}, 3^{26}, 4^{13})$

Step 6. Cases 122, 141, 157, 161, 164, 167, 170, 178 and 180 cannot occur.

For Case 122, we have $\mathrm{Soc}(G^{\mathcal{C}}) = A_{17}$, $\mathrm{PSL}(2, 2^4)$, $\mathrm{PSL}(2, 17)$, $S_4(2^2)$, $S_8(2)$ or $O_8^-(2)$ for $t_{max} = 2$ according to [11, Table B.4] and Lemma 8. If $G \cong G^{\mathcal{C}}$ is quasiprimitive and almost simple, we have $19 \mid |G^{\mathcal{C}}|$ for $19 \mid \mathrm{lcm}(b, v)$, a contradiction. Thus the partition is G-normal. Suppose that K is semiregular, then K is regular since K is normal in the primitive group G^C, hence K is a minimal normal subgroup of G^C by [22, Proposition 11.4]. Then

$$K = M_1 \times M_2 \times \cdots \times M_s,$$

where M_i is isomorphic to M_j for all $i \leqslant i, j \leqslant s$, and M_i is simple. Therefore, $|K| = c = |M_1|^s$. However,

$$c = 190 = 2 \cdot 5 \cdot 19,$$

thus, $s = 1$, then K is simple. But there is no simple group of order c, this contradiction leads to that K is not semiregular.

According to [19, Proposition 4.3], we have

$$d_1 \leqslant \frac{1}{2} + \sqrt{2k - \frac{7}{4}} \approx 13.7,$$

however, the intersection type of this case is (0^{64}, 1^{64}, 2^4, 4^4), (0^{68}, 1^{56}, 2^4, 3^8), or (0^{76}, 1^{32}, 2^{28}), a contradiction, so Case 122 is ruled out.

Similarly, Cases 141, 157, 161, 164, 167, 170, 178 and 180 can be ruled out. Please refer to Table 8 for detailed information.

Table 8: Relevant information for "Step 6"

Case	$\mathrm{Soc}(G^C)$	$\mathrm{lcm}(b, v)$	c	$\frac{1}{2}+\sqrt{2k-\frac{7}{4}}$	intersection type
141	A_{2592}, $\mathrm{PSL}(2, 2591)$	$2^5 \cdot 3^4 \cdot 5^2 \cdot 1451 \cdot 2591$	$5^2 \cdot 1451$	≈ 81.7	(0^{1032},2^{1500},5^{60}), (0^{1092},2^{1200},3^{300})
157	A_{10}, A_6, A_5	$2^5 \cdot 3^2 \cdot 5 \cdot 13$	$2^4 \cdot 13$	$= 7$	(0^1,2^8,6^1), (0^2,2^5,4^3)
161	A_{348}, $\mathrm{PSL}(2, 347)$	$2^4 \cdot 3^3 \cdot 29 \cdot 241 \cdot 347$	$2^2 \cdot 3^2 \cdot 241$	≈ 40.3	(0^6,2^{324},6^{12},12^6)
164	A_{384}, $\mathrm{PSL}(2, 383)$	$2^7 \cdot 3^3 \cdot 383 \cdot 1447$	$3^2 \cdot 1447$	≈ 49.2	(2^{216},3^{36},4^{108},9^{24}), (0^{12},2^{216},4^{108},6^{36},9^{12}), (0^{12},2^{216},3^{36},5^{108},9^{12}), (0^{24},2^{216},5^{108},6^{36},9^{12}), (0^{36},3^{324},9^{24}), (0^{48},3^{288},6^{36},9^{12}), (0^{48},2^{108},4^{216},9^{12}), (0^{60},3^{252},6^{72},9^{12}), (0^{60},2^{108},4^{108},5^{108},9^{12})
167	A_{1908}, $\mathrm{PSL}(2, 1907)$	$2^3 \cdot 3^2 \cdot 23 \cdot 53 \cdot 1451 \cdot 1907$	$2 \cdot 23 \cdot 1451$	≈ 110.7	(0^{252},2^{552},4^{828},6^{276})
170	A_{1872}, $\mathrm{PSL}(2, 1871)$	$2^4 \cdot 3^2 \cdot 7^2 \cdot 13 \cdot 1451 \cdot 1871$	$7^2 \cdot 1451$	≈ 114.2	(0^{108},2^{588},3^{588},6^{588})
178	A_{84}, $\mathrm{PSL}(2, 83)$	$2^7 \cdot 3 \cdot 7 \cdot 83 \cdot 179$	$2^5 \cdot 179$	≈ 32.97	(4^{24},6^{48},12^{12}), (4^{36},6^{24},10^{24}), (0^3,4^{12},6^{48},8^{18},16^3), (0^6,4^{24},8^{54})
180	A_{108}, $\mathrm{PSL}(2, 107)$	$2^2 \cdot 3^4 \cdot 7 \cdot 107 \cdot 479$	$3 \cdot 7 \cdot 479$	≈ 43.5	(7^{84},14^{24})

This completes the proof of Proposition 14. □

Proof of Theorem 1 and Concluding Remarks

The result follows from Propositions 13 and 14. □

In our future work, we will explore new methods, and then continue to expand the classification up to the case of $k^{(r)} = 16$ to find new examples. Moreover, we will try to explore whether there are other line-transitive point-imprimitive linear spaces which are not Desarguesian projective planes when $k^{(r)} \geqslant 13$ and $k = q+1$, here q is a prime power.

Acknowledgements

The authors thank the referees for pointing out errors in the original version of this paper and for providing their valuable suggestions and comments, which led to the improvement of this article.

This work is supported by the National Natural Science Foundation of China (Grant No.12271173) and the Natural Science Foundation of Henan Province of China (No. 2423 0042 1688).

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Appendix

Table 9: Potential parameter sequences for $k^{(r)} = 11$

Case	$d{\cdot}c$	(x,y)	(γ,δ)	$k^{(v)}{\cdot}k^{(r)}$	$b^{(v)}{\cdot}b^{(r)}$	t_{max}	Reference
1	230·230	(1,1)	(1,1)	2·11	26450·229	3	(Step 1)
2	472·472	(2,2)	(1,1)	4·11	55696·471	2	(T, t_{max})
3	714·714	(3,3)	(1,1)	6·11	84966·713	2	(T, t_{max})
4	956·956	(4,4)	(1,1)	8·11	114242·955	2	(T, t_{max})
5	1440·1440	(6,6)	(1,1)	12·11	172800·1439	2	$(T, 719)$
6	1924·1924	(8,8)	(1,1)	16·11	231361·1923	2	$(T, 1913)$
7	2166·2166	(9,9)	(1,1)	18·11	260642·2165	2	$(T, 2161)$
8	2892·2892	(12,12)	(1,1)	24·11	348486·2891	2	$(T, 2887)$

Case	$d{\cdot}c$	(x,y)	(γ,δ)	$k^{(v)}{\cdot}k^{(r)}$	$b^{(v)}{\cdot}b^{(r)}$	t_{max}	Reference
9	4344·4344	(18,18)	(1,1)	36·11	524176·4343	2	(Step 1)
10	5796·5796	(24,24)	(1,1)	48·11	699867·5795	2	(Step 1)
11	8700·8700	(36,36)	(1,1)	72·11	1051250·8699	2	(Step 1)
12	17412 · 17412	(72,72)	(1,1)	144·11	2105401·17411	2	(Step 1)
13	471 · 236	(2,4)	(1,2)	4·11	27789·235	2	(T, t_{max})
14	1439·720	(6,12)	(1,2)	12·11	86340·719	2	$(B, 359)$
15	6279·3140	(26,52)	(1,2)	52·11	379155·3139	2	$(B, 3137)$
16	18863·9432	(78,156)	(1,2)	156·11	1140486·9431	2	(Step 1)
17	1438·480	(6,18)	(1,3)	12·11	57520·479	2	$(T, 1433)$
18	10150·3384	(42,126)	(1,3)	84·11	408900·3383	2	$(B, 3373)$
19	1437·360	(6,24)	(1,4)	12·11	43110·359	2	$(T, 1433)$
20	7245·1812	(30,120)	(1,4)	60·11	218799·1811	1	(Step 4)
21	226·46	(1,5)	(1,5)	2·11	5198·45	2	(T, t_{max})
22	1436·288	(6,30)	(1,5)	12·11	34464·287	2	$(B, 283)$
23	3856·772	(16,80)	(1,5)	32·11	93026·771	2	$(B, 769)$
24	23216·4644	(96,480)	(1,5)	192·11	561537·4643	2	(Step 3)
25	1435·240	(6,36)	(1,6)	12·11	28700·239	2	(Step 1)
26	708·102	(3,21)	(1,7)	6·11	12036·101	1	$(T, 701)$
27	26118·3732	(108,756)	(1,7)	216·11	451261·3731	1	$(B, 3727)$
28	1433·180	(6,48)	(1,8)	12·11	21495·179	1	(Step 1)
29	9177·1148	(38,304)	(1,8)	76·11	138621·1147	1	$(B, 1129)$
30	3610·402	(15,135)	(1,9)	30·11	48374·401	1	$(T, 3607)$
31	1431·144	(6,60)	(1,10)	12·11	17172·143	2	(Step 2)
32	30471·3048	(126,1260)	(1,10)	252·11	368554·3047	2	$(B, 3041)$
33	1429·120	(6,72)	(1,12)	12·11	14290·119	2	(Step 1)
34	1912·148	(8,104)	(1,13)	16·11	17686·147	1	$(T, 1907)$
35	2395·172	(10,140)	(1,14)	20·11	20597·171	1	$(T, 2393)$
36	36275·2592	(150,2100)	(1,14)	300·11	313416·2591	1	(Step 3)
37	1426·96	(6,90)	(1,15)	12·11	11408·95	2	$(T, 1423)$
38	13041·816	(54,864)	(1,16)	108·11	98532·815	1	(Step 2)
39	698·42	(3,51)	(1,17)	6·11	4886·41	1	$(T, 691)$
40	2148·114	(9,171)	(1,19)	18·11	13604·113	2	$(T, 2143)$
41	43530·2292	(180,3420)	(1,19)	360·11	277141·2291	1	(Step 2)
42	1421·72	(6,120)	(1,20)	12·11	8526·71	2	$(T, 1409)$
43	5776·276	(24,504)	(1,21)	48·11	33212·275	1	(Step 2)
44	208·10	(1,23)	(1,23)	2·11	1040·9	1	(Step 1)
45	1417·60	(6,144)	(1,24)	12·11	7085·59	1	$(T, 1409)$
46	8676·348	(36,900)	(1,25)	72·11	41934·347	1	(Step 3)
47	1411·48	(6,180)	(1,30)	12·11	5644·47	2	$(T, 1409)$
48	13023·384	(54,1836)	(1,34)	108·11	46304·383	1	(Step 3)
49	66746·1908	(276,9660)	(1,35)	552·11	230709·1907	1	(Step 3)
50	1888·52	(8,296)	(1,37)	16·11	6136·51	1	$(T, 1879)$
51	71099·1872	(294,11172)	(1,38)	588·11	226356·1871	1	(Step 3)
52	14470·372	(60,2340)	(1,39)	120·11	44857·371	1	(Step 2)
53	1401·36	(6,240)	(1,40)	12·11	4203·35	1	$(T, 1399)$
54	10830·222	(45,2205)	(1,49)	90·11	26714·221	1	(Step 2)
55	15424·292	(64,3392)	(1,53)	128·11	35186·291	1	(Step 2)
56	414·8	(2,118)	(1,59)	4·11	828·7	1	$(T, 409)$
57	1381·24	(6,360)	(1,60)	12·11	2762·23	2	(Step 1)
58	5728·84	(24,1656)	(1,69)	48·11	10024·83	1	(Step 3)
59	117531·1680	(486,34020)	(1,70)	972·11	203140·1679	2	(Step 2)

Case	$d{\cdot}c$	(x,y)	(γ,δ)	$k^{(v)}{\cdot}k^{(r)}$	$b^{(v)}{\cdot}b^{(r)}$	t_{max}	Reference
60	45402·548	(188,15604)	(1,83)	376·11	66171·547	1	(Step 4)
61	10059·108	(42,3948)	(1,94)	84·11	12933·107	1	(Step 3)
62	153806·1620	(636,60420)	(1,95)	1272·11	195885·1619	1	(Step 3)
63	236·471	(4,2)	(2,1)	4·11	27789·235	2	(T, t_{max})
64	720·1439	(12,6)	(2,1)	12·11	86340·719	2	$(T, 359)$
65	3140·6279	(52,26)	(2,1)	52·11	379155·3139	2	$(T, 3137)$
66	9432·18863	(156,78)	(2,1)	156·11	1140486·9431	2	(Step 1)
67	53·27	(1,2)	(2,4)	1·11	1431·13	1	([20])
68	3925·1963	(65,130)	(2,4)	65·11	118535·981	1	$(T, 3923)$
69	12576·5031	(208,520)	(2,5)	208·11	304182·2515	1	(Step 2)
70	14148·4043	(234,819)	(2,7)	234·11	244446·2021	1	$(B, 4027)$
71	777·195	(13,52)	(2,8)	13·11	11655·97	1	$(T, 773)$
72	171·35	(3,15)	(2,10)	3·11	1995·17	2	(Step 1)
73	$49{\cdot}3^2$	(1,6)	(2,12)	1·11	441·4	1	([20])
74	590·63	(10,95)	(2,19)	10·11	3717·31	1	$(T, 587)$
75	23580·2483	(390,3705)	(2,19)	390·11	150126·1241	1	(Step 2)
76	769·65	(13,156)	(2,24)	13·11	3845·32	1	(Step 1)
77	342·23	(6,93)	(2,31)	6·11	1311·11	1	$(T, 337)$
78	885·53	(15,255)	(2,34)	15·11	3127·26	1	$(T, 883)$
79	36156·2067	(598,10465)	(2,35)	598·11	124974·1033	1	(Step 2)
80	37·3	(1,18)	(2,36)	1·11	111·1	1	([20])
81	7840·403	(130,2535)	(2,39)	130·11	24304·201	2	(Step 2)
82	761·39	(13,260)	(2,40)	13·11	2283·19	1	(Step 1)
83	480·1438	(18,6)	(3,1)	12·11	57520·479	2	$(B, 1433)$
84	3384·10150	(126,42)	(3,1)	84·11	408900·3383	2	$(T, 3373)$
85	76·76	(3,3)	(3,3)	2·11	2888·25	2	$(T, 73)$
86	237·178	(9,12)	(3,4)	6·11	7031·59	2	$(T, 233)$
87	156·94	(6,10)	(3,5)	4·11	3666·31	1	$(B, 89)$
88	559·280	(21,42)	(3,6)	14·11	11180·93	2	$(T, 557)$
89	316·106	(12,36)	(3,9)	8·11	4187·35	1	$(T, 313)$
90	556·112	(21,105)	(3,15)	14·11	4448·37	1	$(T, 547)$
91	2248·322	(84,588)	(3,21)	56·11	12926·107	2	$(T, 2243)$
92	553·70	(21,168)	(3,24)	14·11	2765·23	1	(Step 1)
93	5067·448	(189,2142)	(3,34)	126·11	18016·149	1	(Step 2)
94	360·1437	(24,6)	(4,1)	12·11	43110·359	2	$(B, 1433)$
95	1812·7245	(120,30)	(4,1)	60·11	218799·1811	2	(Step 1)
96	27·53	(2,1)	(4,2)	1·11	1431·13	2	([20])
97	1963·3925	(130,65)	(4,2)	65·11	118535·981	2	$(B, 3923)$
98	178·237	(12,9)	(4,3)	6·11	7031·59	1 or 2	$(B, 233)$
99	2265·2265	(150,150)	(4,4)	75·11	68403·566	1	$(T, 2251)$
100	540·309	(36,63)	(4,7)	18·11	9270·77	1 or 2	$(B, 307)$
101	449·225	(30,60)	(4,8)	15·11	6735·56	1	(Step 1)
102	46·226	(5,1)	(5,1)	2·11	5198·45	2	(T, t_{max})
103	288·1436	(30,6)	(5,1)	12·11	34464·287	2	$(T, 283)$
104	772·3856	(80,16)	(5,1)	32·11	93026·771	2	$(T, 769)$
105	4644·23216	(480,96)	(5,1)	192·11	561537·4643	2	(Step 1)
106	5031·12576	(520,208)	(5,2)	208·11	304182·2515	2	(Step 1)
107	94·156	(10,6)	(5,3)	4·11	3666·31	2	$(T, 89)$
108	190·136	(20,28)	(5,7)	8·11	3230·27	1	(Step 1)
109	6966·4976	(720,1008)	(5,7)	288·11	120357·995	1	(Step 2)
110	92·36	(10,26)	(5,13)	4·11	828·7	1	$(T, 89)$

Case	$d{\cdot}c$	(x,y)	(γ,δ)	$k^{(v)}{\cdot}k^{(r)}$	$b^{(v)}{\cdot}b^{(r)}$	t_{max}	Reference
111	9675·3456	(1000,2800)	(5,14)	400·11	83592·691	1	(Step 2)
112	188·56	(20,68)	(5,17)	8·11	1316·11	1	$(T, 181)$
113	11610·3056	(1200,4560)	(5,19)	480·11	73917·611	1	(Step 2)
114	381·96	(40,160)	(5,20)	16·11	2286·19	1 or 2	(Step 2)
115	240·1435	(36,6)	(6,1)	12·11	28700·239	2	(Step 1)
116	280·559	(42,21)	(6,3)	14·11	11180·93	1 or 2	$(B, 557)$
117	17·13	(3,4)	(6,8)	1·11	221·2	1 or 2	([20])
118	102·708	(21,3)	(7,1)	6·11	12036·101	2	$(B, 701)$
119	3732·26118	(756,108)	(7,1)	216·11	451261·3731	2	$(T, 3727)$
120	4043·14148	(819,234)	(7,2)	234·11	244446·2021	2	$(T, 4027)$
121	309·540	(63,36)	(7,4)	18·11	9270·77	2	$(T, 307)$
122	136·190	(28,20)	(7,5)	8·11	3230·27	2	(Step 6)
123	4976·6966	(1008,720)	(7,5)	288·11	120357·995	1 or 2	(Step 3)
124	6531·4572	(1323,1890)	(7,10)	378·11	78994·653	1 or 2	(Step 3)
125	180·1433	(48,6)	(8,1)	12·11	21495·179	2	(Step 5)
126	1148·9177	(304,38)	(8,1)	76·11	138621·1147	2	$(T, 1129)$
127	195·777	(52,13)	(8,2)	13·11	11655·97	2	$(B, 773)$
128	225·449	(60,30)	(8,4)	15·11	6735·56	1 or 2	(Step 5)
129	13·17	(4,3)	(8,6)	1·11	221·2	1 or 2	([20])
130	402·3610	(135,15)	(9,1)	30·11	48374·401	2	$(B, 3607)$
131	106·316	(36,12)	(9,3)	8·11	4187·35	1 or 2	$(B, 313)$
132	144·1431	(60,6)	(10,1)	12·11	17172·143	2	(Step 1)
133	3048·30471	(1260,126)	(10,1)	252·11	368554·3047	2	$(T, 3041)$
134	35·171	(15,3)	(10,2)	3·11	1995·17	2	(T, t_{max})
135	4572·6531	(1890,1323)	(10,7)	378·11	78994·653	1 or 2	(Step 2)
136	120·1429	(72,6)	(12,1)	12·11	14290·119	2	(Step 5)
137	$3^2{\cdot}7^2$	(6,1)	(12,2)	1·11	441·4	9	([20])
138	148·1912	(104,8)	(13,1)	16·11	17686·147	2	$(B, 1907)$
139	36·92	(26,10)	(13,5)	4·11	828·7	1 or 2	$(B, 89)$
140	172·2395	(140,10)	(14,1)	20·11	20597·171	2	$(B, 2393)$
141	2592·36275	(2100,150)	(14,1)	300·11	313416·2591	2	(Step 6)
142	3456·9675	(2800,1000)	(14,5)	400·11	83592·691	2	(T, t_{max})
143	96·1426	(90,6)	(15,1)	12·11	11408·95	2	$(B, 1423)$
144	112·556	(105,21)	(15,3)	14·11	4448·37	1 or 2	$(B, 547)$
145	816·13041	(864,54)	(16,1)	108·11	98532·815	2	(Step 1)
146	42·698	(51,3)	(17,1)	6·11	4886·41	2	$(B, 691)$
147	56·188	(68,20)	(17,5)	8·11	1316·11	1 or 2	$(B, 181)$
148	114·2148	(171,9)	(19,1)	18·11	13604·113	2	$(B, 2143)$
149	2292·43530	(3420,180)	(19,1)	360·11	277141·2291	2	(T, t_{max})
150	63·590	(95,10)	(19,2)	10·11	3717·31	2	$(B, 587)$
151	2483·23580	(3705,390)	(19,2)	390·11	150126·1241	2	(T, t_{max})
152	3056·11610	(4560,1200)	(19,5)	480·11	73917·611	1 or 2	(T, t_{max})
153	72·1421	(120,6)	(20,1)	12·11	8526·71	2	$(B, 1409)$
154	96·381	(160,40)	(20,5)	16·11	2286·19	1	(T, t_{max})
155	276·5776	(504,24)	(21,1)	48·11	33212·275	2	(Step 1)
156	322·2248	(588,84)	(21,3)	56·11	12926·107	1 or 2	$(B, 2243)$
157	10·208	(23,1)	(23,1)	2·11	1040·9	3 or 10	(Step 6)
158	60·1417	(144,6)	(24,1)	12·11	7085·59	2	$(B, 1409)$
159	65·769	(156,13)	(24,2)	13·11	3845·32	2	(Step 5)
160	70·553	(168,21)	(24,3)	14·11	2765·23	1 or 2	(T, t_{max})
161	348·8676	(900,36)	(25,1)	72·11	41934·347	2	(Step 6)

Case	$d{\cdot}c$	(x,y)	(γ,δ)	$k^{(v)}{\cdot}k^{(r)}$	$b^{(v)}{\cdot}b^{(r)}$	t_{max}	Reference
162	48·1411	(180,6)	(30,1)	12·11	5644·47	2	$(B, 1409)$
163	23·342	(93,6)	(31,2)	6·11	1311·11	2	$(B, 337)$
164	384·13023	(1836,54)	(34,1)	108·11	46304·383	2	(Step 6)
165	53·885	(255,15)	(34,2)	15·11	3127·26	2	$(B, 883)$
166	448·5067	(2142,189)	(34,3)	126·11	18016·149	1 or 2	(T, t_{max})
167	1908·66746	(9660,276)	(35,1)	552·11	230709·1907	2	(Step 6)
168	2067·36156	(10465,598)	(35,2)	598·11	124974·1033	2	(T, t_{max})
169	52·1888	(296,8)	(37,1)	16·11	6136·51	2	$(B, 1879)$
170	1872·71099	(11172,294)	(38,1)	588·11	226356·1871	2	(Step 6)
171	372·14470	(2340,60)	(39,1)	120·11	44857·371	2	(T, t_{max})
172	403·7840	(2535,130)	(39,2)	130·11	24304·201	2	(T, t_{max})
173	36·1401	(240,6)	(40,1)	12·11	4203·35	2	$(B, 1399)$
174	39·761	(260,13)	(40,2)	13·11	2283·19	2	(T, t_{max})
175	222·10830	(2205,45)	(49,1)	90·11	26714·221	2	(T, t_{max})
176	292·15424	(3392,64)	(53,1)	128·11	35186·291	2	(T, t_{max})
177	8·414	(118,2)	(59,1)	4·11	828·7	3	$(B, 409)$
178	84·5728	(1656,24)	(69,1)	48·11	10024·83	2	(Step 6)
179	1680·117531	(34020,486)	(70,1)	972·11	203140·1679	2	(T, t_{max})
180	108·10059	(3948,42)	(94,1)	84·11	12933·107	2	(Step 6)