Pushing the Boundaries of Tractable Multiperspective Reasoning: A Deduction Calculus for Standpoint $\mathcal{EL}+$

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Abstract

Standpoint $\mathcal{EL}$ is a multi-modal extension of the popular description logic $\mathcal{EL}$ that allows for the integrated representation of domain knowledge relative to diverse standpoints or perspectives. Advantageously, its satisfiability problem has recently been shown to be in PTIME, making it a promising framework for large-scale knowledge integration.

In this paper, we show that we can further push the expressivity of this formalism, arriving at an extended logic, called Standpoint $\mathcal{EL}+$, which allows for axiom negation, role chain axioms, self-loops, and other features, while maintaining tractability. This is achieved by designing a satisfiability-checking deduction calculus, which at the same time addresses the need for practical algorithms. We demonstrate the feasibility of our calculus by presenting a prototypical Datalog implementation of its deduction rules.

1 Introduction

The Semantic Web enables the exploitation of artefacts of knowledge representation (e.g., ontologies, knowledge bases, etc.) to support increasingly sophisticated automated reasoning tasks over linked data from various sources. *Description logics* (DLs) (Baader et al. 2017; Rudolph 2011) are a prominent class of logic-based KR formalisms in this context since they provide the theoretical underpinning of the Web Ontology Language (OWL 2), the main KR standard by the W3C (Bao et al. 2009).

In particular, the lightweight description logic $\mathcal{EL}$ (Baader, Brandt, and Lutz 2005) serves as the core of a popular family of DLs which is the formal basis of OWL 2 EL (Motik et al. 2009), a widely used tractable profile of OWL 2. One major appeal of the $\mathcal{EL}$ family is that basic reasoning tasks can be performed in polynomial time with respect to the size of the ontology, enabling reasoning-supported creation and maintenance of very large ontologies. An example is SNOMED CT (Donnelly 2006), which is the largest healthcare ontology and has a broad user base including clinicians, patients, and researchers.

Beyond the scalable reasoning, the Semantic Web must provide mechanisms for the combination and integrated exploitation of the many knowledge sources available. Yet, its decentralised nature has led to the proliferation of ontologies of overlapping knowledge, which inevitably reflect different points of view and follow diverging modelling principles.

For instance, in the medical domain, some sources may use the concept Tumour to denote a process and others to denote a piece of tissue. Similarly, Allergy may denote an allergic reaction or just an allergic disposition. These issues pose well-known challenges in the area of knowledge integration.

Common ontology management approaches fully merge knowledge perspectives, which often requires logical weakening in order to maintain consistency. For instance, an initiative proposed the integration of SNOMED CT with the FMA1140 (Foundational Model of Anatomy) and the NCIt (National Cancer Institute Thesaurus) into a unified combination called LargeBio and reported ensuing challenges (Osman, Ben Yahia, and Diallo 2021). Beyond the risk of causing inconsistencies or unintended consequences, the unifying approach promotes weakly axiomatised ontologies designed to avoid conflict in any context of application at the expense of richer theories that would allow for further inferencing. Hence, while frameworks supporting the integrated representation of multiple perspectives seem preferable to recording the distinct views in a detached way, entirely merging them comes with significant downsides.

This need of handling multiple perspectives in the Semantic Web has led to several logic-based proposals. The closest regarding goals are multi-viewpoint ontologies (Hemam and Boufaïda 2011; Hemam 2018), which often model the intuition of viewpoints in a tailored extension of OWL for which no complexity bounds are given. Related problems are also addressed in work on contextuality, e.g. C-OWL, DDL, and CKR (Bouquet et al. 2003; Borgida and Serafini 2003; Serafini and Homola 2012).

Modal logics are natural frameworks for modelling contexts and perspectives (Klarman and Gutiérrez-Basulto 2013; Gómez Álvarez and Rudolph 2021), and in contrast to tailored multi-perspective frameworks, they benefit from well-understood semantics. However, the interplay between DL constructs and modalities is often not well-behaved and can easily endanger the decidability of reasoning tasks or increase their complexity (Baader and Ohlbach 1995; Mosurović 1999; Wolter and Zakharyaschev 1999). Notable examples are NEXPTIME-completeness of the multi-modal description logic $\mathbf{K}_{\mathcal{ALC}}$ (Lutz et al. 2002) and 2EXPTIME-completeness of $\mathcal{ALC}_{\mathcal{ALC}}$ (Klarman and Gutiérrez-Basulto 2013), a modal contextual logic framework in the style proposed by McCarthy and Buvac (1998).

Standpoint logic (Gómez Álvarez and Rudolph 2021) is a recently proposed formalism that is rooted in modal logic and allows for the simultaneous representation of multiple, potentially contradictory viewpoints and the establishment of alignments between them. This is achieved by extending a given base logic (propositional logic in the case of Gómez Álvarez and Rudolph, description logic $\mathcal{EL}$ herein) with labelled modal operators, where propositions $\Box_S \phi$ and $\Diamond_S \phi$ express information relative to the *standpoint* S and read, respectively: "according to S, it is *unequivocal/conceivable* that ϕ". Semantically, standpoints are represented by sets of *precisifications*,[1] such that $\Box_S \phi$ and $\Diamond_S \phi$ hold if ϕ is true in all/some of the precisifications associated with S.

The following example illustrates the use of standpoint logic for knowledge integration in the medical domain.

Example 1 (Tumour Disambiguation). A hospital H and a laboratory L have developed their own knowledge bases and aim to make them interoperable. Hospital H gathers clinical data about patients, which may be used to determine whether a person has priority at the emergency service. According to H, a `Tumour` is a process by which abnormal or damaged cells grow and multiply (Formula 1), and patients that conceivably have a `Tumour` have a `HighRisk` priority (Formula 2). The laboratory L annotates patients' radiographs, and models `Tumour` as a lump of tissue (Formula 3).

$$\Box_H[\texttt{Tumour} \sqsubseteq \texttt{Process}] \quad (1)$$
$$\Box_H[\texttt{Patient} \sqcap \Diamond_H[\exists\texttt{HasProcess.Tumour}] \sqsubseteq \texttt{HighRisk}] \quad (2)$$
$$\Box_L[\texttt{Tumour} \sqsubseteq \texttt{Tissue}] \quad (3)$$

Both institutions inherit from SN, which contains the original SNOMED CT as well as patient data (Formula 4, with the operator $\preceq$ encoding the inheritance). Among the background knowledge in SN, we have that `Tissue` and `Process` are disjoint classes (Formula 5) and that everything that has a part which has a process, has that process itself (Formula 6).

$$H \preceq SN \qquad L \preceq SN \quad (4)$$
$$\Box_{SN}[\texttt{Tissue} \sqcap \texttt{Process} \sqsubseteq \bot] \quad (5)$$
$$\Box_{SN}[\texttt{HasPart} \circ \texttt{HasProcess} \sqsubseteq \texttt{HasProcess}] \quad (6)$$

While clearly incompatible due to Formula 5, the perspectives of H and L are semantically close and and we may be aware of further complex relations between their perspectives. For instance, we might assert that whenever a clinician at L deems that a cancerous lump of tissue is large enough to conceivably be a `Tumour` (tissue), then it is unequivocally undergoing a `Tumour` (process) according to H (Formula 7). We might also want to specify negative information such as non-subsumption between the classes of unequivocal instances of `Process` according to H and to L (Formula 8).

$$\Diamond_L[\texttt{Tumour}] \sqsubseteq \Box_H[\texttt{Tissue} \sqcap \exists\texttt{HasProcess.Tumour}] \quad (7)$$
$$\neg(\Box_H[\texttt{Process}] \sqsubseteq \Box_L[\texttt{Process}]) \quad (8)$$

Finally, these sources may also have assertional knowledge:

$$\Box_{SN}[\texttt{Patient}(p1) \wedge \texttt{HasPart}(p1,a) \wedge \texttt{Colon}(a)] \quad (9)$$
$$\Box_H[\texttt{HasPart}(a,b)] \quad (10)$$
$$\Diamond_L[\texttt{Tumour}(b)] \quad (11)$$

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[1] Precisifications are analogous to the *worlds* of modal-logic frameworks with possible-worlds semantics.

That is, through SN, both H and L know of a patient $p1$ and their body parts (Formula 9) and, in view of some radiograph requested by H on part b of this patient's colon (Formula 10), L suspects there may be tumour tissue (Formula 11). ◇

In the first place, one should notice that a naive, standpoint-free integration of the knowledge bases without the standpoint infrastructure would trigger an inconsistency. Specifically, from a `Tumour`(b) instance we could infer both that `Tissue`(b) and `Process`(b) using the background knowledge of H and L, which in turn would lead to inconsistency with the SN axiom stating `Tissue` $\sqcap$ `Process` $\sqsubseteq \bot$. Instead, with Standpoint $\mathcal{EL}+$, the logical statements (3)–(11) formalising Example 1 are not inconsistent, so all axioms can be jointly represented. On the one hand, we will be able to infer that H and L are indeed incompatible, denoted by $H \cap L \preceq \mathbf{0}$ and obtained from Formulas (1), (3), (4), (5) and (11). On the other hand, beyond preserving consistency, the use of standpoint logic supports reasoning with and across individual perspectives.

Example 2 (Continued from Example 1). Assume that patient $p1$, of which laboratory L detected a tumour tissue (Formula 10), registers at emergencies in hospital H. From the knowledge expressed in Formulas (3)–(11), we can infer

via (7) and (11)	$\Box_H[(\exists\texttt{HasProcess.Tumour})(b)]$	(12)
via (6),(10) and (12)	$\Box_H[(\exists\texttt{HasProcess.Tumour})(a)]$	(13)
via (6),(9) and (13)	$\Box_H[(\exists\texttt{HasProcess.Tumour})(p1)]$	(14)
via (4) and (9)	$\Box_H[\texttt{Patient}(p1)]$	(15)
via (2),(14) and (15)	$\Box_H[\texttt{HighRisk}(p1)]$	(16)

meaning that, according to H, $p1$ has a tumour process and is classified as 'high risk'. ◇

Formally, *standpoint logics* are multi-modal logics characterised by a simplified Kripke semantics, which brings about beneficial computational properties in different settings. For instance, it is known that adding sentential standpoints (where applying modal operators to formulas with free variables is disallowed) does not increase the complexity of numerous NP-*hard* FO-fragments (Gómez Álvarez, Rudolph, and Strass 2022), including the expressive DL $\mathcal{SROIQ}b_s$, a logical basis of OWL 2 DL (Motik, Patel-Schneider, and Cuenca Grau 2009).

Yet, a fine-grained terminological alignment between different perspectives requires concepts preceded by modal operators, as in Axiom (7), which falls out of the sentential fragment. Recently, we introduced a *standpoint version* of the description logic $\mathcal{EL}$, called Standpoint $\mathcal{EL}$, and established that it exhibits $\mathcal{EL}$'s favourable PTIME standard reasoning (Gómez Álvarez, Rudolph, and Strass 2023a). The result was obtained by means of a variation of the quasi-model-based tableau algorithms usually employed in the literature on modal description logics (Wolter and Zakharyaschev 1999). In addition, we proved that introducing additional features like empty standpoints, rigid roles, and nominals makes standard reasoning tasks intractable.

In this paper, we show that we can push the expressivity of Standpoint $\mathcal{EL}$ further while retaining tractability. We present an extended logic, called Standpoint $\mathcal{EL}+$,

which allows, on the one hand, features of the popular, W3C-standardised OWL-2 EL that do not break tractability, i.e. self-loops and role-chain axioms (cf. Axiom 6). On the other hand, it supports additional features such as modalised axiom sets, which are motivated by modelling desiderata. For instance, a diamond in front of an ABox/TBox allows representing a viewpoint that is conceivable as a whole, in contrast to its axioms being conceivable individually but not necessarily simultaneously. The result presented here is achieved by designing a satisfiability-checking deduction calculus for a standpoint-enhanced DL, which is a novel technique in the context of the literature on modal description logics. The use of this technique, commonly employed for scalable lightweight DL reasoners (Kazakov, Krötzsch, and Simancik 2014), is fundamentally motivated by its implementability, which we demonstrate with a Datalog-based software prototype.

Our paper is structured as follows. After introducing the syntax and semantics of Standpoint $\mathcal{EL}+$ (denoted $\mathbb{S}_{\mathcal{EL}+}$) and a suitable normal form (Section 2), we establish our main result: satisfiability checking and statement entailment in $\mathbb{S}_{\mathcal{EL}+}$ is tractable. We show this by providing a particular Hilbert-style deduction calculus (Section 3) that operates on axioms of a fixed shape and bounded size, which immediately warrants that saturation can be performed in PTIME. For said calculus, we establish soundness and refutation-completeness. In Section 4, we briefly describe a proof-of-concept implementation of our approach and illustrate its key ideas. We conclude the paper in Section 5 with a discussion of future work. An extended version of the paper with proofs of all results is available as a technical report (Gómez Álvarez, Rudolph, and Strass 2023b).

2 Syntax, Semantics, and Normalisation

We now introduce syntax and semantics of Standpoint $\mathcal{EL}+$ (referred to as $\mathbb{S}_{\mathcal{EL}+}$) and propose a normal form that is useful for subsequent algorithmic considerations.

2.1 Syntax

A *Standpoint DL vocabulary* is a traditional DL vocabulary consisting of sets N_C of *concept names*, N_R of *role names*, and N_I of *individual names*, extended by an additional set N_S of *standpoint names* with $* \in N_\mathsf{S}$ the *universal standpoint*. A *standpoint operator* is of the form $\Diamond_\mathsf{s}$ ("diamond") or $\Box_\mathsf{s}$ ("box") with $\mathsf{s} \in N_\mathsf{S}$; we use $\odot_\mathsf{s}$ to refer to either.[2]

- *Concept terms* are defined via (where $A \in N_\mathsf{C}$, $R \in N_\mathsf{R}$)

  $C ::= \top \mid \bot \mid A \mid C_1 \sqcap C_2 \mid \exists R.C \mid \odot_\mathsf{s} C \mid \exists R.\mathsf{Self}$

- A *general concept inclusion (GCI)* is of the form $C \sqsubseteq D$, where C and D are concept terms.
- A *role inclusion axiom (RIA)* is of the form $R_1 \circ \ldots \circ R_n \sqsubseteq R$ where $n \geq 1$, $R_1, \ldots, R_n, R \in N_\mathsf{R}$.
- A *concept assertion* is of the form $C(a)$, where C is a concept term and $a \in N_\mathsf{I}$.
- A *role assertion* is of the form $R(a,b)$, with $a, b \in N_\mathsf{I}$ and $R \in N_\mathsf{R}$.

- An *axiom* ξ is a GCI, RIA, or concept/role assertion.
- A *literal* λ is an axiom ξ or a negated axiom $\neg\xi$.
- A *monomial* μ is a conjunction $\lambda_1 \wedge \ldots \wedge \lambda_m$ of literals.
- A *formula* φ is of the form $\odot_\mathsf{s}\mu$ for a monomial μ and $\mathsf{s} \in N_\mathsf{S}$.
- A *sharpening statement* is of the form $\mathsf{s}_1 \cap \ldots \cap \mathsf{s}_n \preceq \mathsf{s}$ where $n \geq 1$ and $\mathsf{s}_1, \ldots, \mathsf{s}_n, \mathsf{s} \in N_\mathsf{S} \cup \{\mathbf{0}\}$.[3]

Note that in particular, monomials subsume (finite) knowledge bases of the $\mathcal{EL}$ family; monomials even go beyond that in allowing for the occurence of *negated* axioms. Yet, monomials do not have the full expressive power of arbitrary Boolean combinations of axioms, which is a necessary restriction in order to maintain tractability.

A $\mathbb{S}_{\mathcal{EL}+}$ *knowledge base* (KB) is a finite set of formulae and possibly negated sharpening statements. We refer to arbitrary elements of $\mathcal{K}$ as *statements*. Note that all statements except sharpening statements are preceded by modal operators ("*modalised*" for short).

2.2 Semantics

The semantics of $\mathbb{S}_{\mathcal{EL}+}$ is defined via (description logic) standpoint structures. Given a Standpoint DL vocabulary $\langle N_\mathsf{C}, N_\mathsf{R}, N_\mathsf{I}, N_\mathsf{S} \rangle$, a *description logic standpoint structure* is a tuple $\mathfrak{D} = \langle \Delta, \Pi, \sigma, \gamma \rangle$ where:

- Δ is a non-empty set, the *domain* of $\mathfrak{D}$;
- Π is a set, called the *precisifications* of $\mathfrak{D}$;
- σ is a function mapping each standpoint name to a non-empty[4] subset of Π while we set $\sigma(\mathbf{0}) = \emptyset$ and $\sigma(*) = \Pi$;
- γ is a function mapping each precisification from Π to an "ordinary" DL interpretation $\mathcal{I} = \langle \Delta, \cdot^\mathcal{I} \rangle$ over the domain Δ, where the interpretation function $\cdot^\mathcal{I}$ maps:
  - each concept name $A \in N_\mathsf{C}$ to a set $A^\mathcal{I} \subseteq \Delta$,
  - each role name $R \in N_\mathsf{R}$ to a binary relation $R^\mathcal{I} \subseteq \Delta \times \Delta$,
  - each individual name $a \in N_\mathsf{I}$ to an element $a^\mathcal{I} \in \Delta$,

  and we require $a^{\gamma(\pi)} = a^{\gamma(\pi')}$ for all $\pi, \pi' \in \Pi$ and $a \in N_\mathsf{I}$.

Note that by this definition, individual names (also referred to as constants) are interpreted rigidly, i.e., each individual name a is assigned the same $a^{\gamma(\pi)} \in \Delta$ across all precisifications $\pi \in \Pi$. We will refer to this uniform $a^{\gamma(\pi)}$ by $a^\mathfrak{D}$.

For each $\pi \in \Pi$, we extend the interpretation mapping $\mathcal{I} = \gamma(\pi)$ to concept terms via structural induction:

$$\begin{aligned}
\top^\mathcal{I} &:= \Delta \\
\bot^\mathcal{I} &:= \emptyset \\
(\Diamond_\mathsf{s}C)^\mathcal{I} &:= \bigcup\nolimits_{\pi' \in \sigma(\mathsf{s})} C^{\gamma(\pi')} \\
(\Box_\mathsf{s}C)^\mathcal{I} &:= \bigcap\nolimits_{\pi' \in \sigma(\mathsf{s})} C^{\gamma(\pi')} \\
(C_1 \sqcap C_2)^\mathcal{I} &:= C_1^\mathcal{I} \cap C_2^\mathcal{I} \\
(\exists R.C)^\mathcal{I} &:= \left\{ \delta \in \Delta \mid \langle \delta, \varepsilon \rangle \in R^\mathcal{I} \text{ for some } \varepsilon \in C^\mathcal{I} \right\} \\
(\exists R.\mathsf{Self})^\mathcal{I} &:= \left\{ \delta \in \Delta \mid \langle \delta, \delta \rangle \in R^\mathcal{I} \right\}
\end{aligned}$$

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[2]We use brackets $[\ldots]$ to delimit the scope of the operators.

[3]$\mathbf{0}$ is used to express standpoint disjointness as in $\mathsf{s} \cap \mathsf{s}' \preceq \mathbf{0}$.

[4]As shown in our prior work (Gómez Álvarez, Rudolph, and Strass 2023a), allowing for "empty standpoints" immediately incurs intractability, even for an otherwise empty vocabulary.

A role chain expression $\rho = R_1 \circ R_2 \circ \ldots \circ R_n$ is interpreted as $\rho^{\mathcal{I}} := ((\cdots(R_1^{\mathcal{I}} \circ R_2^{\mathcal{I}}) \circ \ldots) \circ R_n^{\mathcal{I}})$, where, as usual, $R \circ U := \{\langle x, z \rangle \mid \langle x, y \rangle \in R, \langle y, z \rangle \in U\}$.

Satisfaction of a statement by a DL standpoint structure $\mathfrak{D}$ (and precisification π) is then defined as follows:

$$
\begin{aligned}
\mathfrak{D}, \pi &\models C \sqsubseteq D & :&\Longleftrightarrow C^{\gamma(\pi)} \subseteq D^{\gamma(\pi)} \\
\mathfrak{D}, \pi &\models \rho \sqsubseteq R & :&\Longleftrightarrow \rho^{\gamma(\pi)} \subseteq R^{\gamma(\pi)} \\
\mathfrak{D}, \pi &\models C(a) & :&\Longleftrightarrow a^{\mathfrak{D}} \in C^{\gamma(\pi)} \\
\mathfrak{D}, \pi &\models R(a, b) & :&\Longleftrightarrow \langle a^{\mathfrak{D}}, b^{\mathfrak{D}} \rangle \in R^{\gamma(\pi)} \\
\mathfrak{D}, \pi &\models \neg \xi & :&\Longleftrightarrow \mathfrak{D}, \pi \not\models \xi \\
\mathfrak{D}, \pi &\models \lambda_1 \wedge \ldots \wedge \lambda_n & :&\Longleftrightarrow \mathfrak{D}, \pi \models \lambda_i \text{ for all } 1 \le i \le n \\
\mathfrak{D} &\models \Box_{\mathsf{s}} \mu & :&\Longleftrightarrow \mathfrak{D}, \pi \models \mu \text{ for each } \pi \in \sigma(\mathsf{s}) \\
\mathfrak{D} &\models \Diamond_{\mathsf{s}} \mu & :&\Longleftrightarrow \mathfrak{D}, \pi \models \mu \text{ for some } \pi \in \sigma(\mathsf{s}) \\
\mathfrak{D} &\models \mathsf{s}_1 \cap \ldots \cap \mathsf{s}_n \preceq \mathsf{s} & :&\Longleftrightarrow \sigma(\mathsf{s}_1) \cap \ldots \cap \sigma(\mathsf{s}_n) \subseteq \sigma(\mathsf{s})
\end{aligned}
$$

Finally, $\mathfrak{D}$ is a *model* of a $\mathbb{S}_{\mathcal{EL}+}$ knowledge base $\mathcal{K}$ (written $\mathfrak{D} \models \mathcal{K}$) iff it satisfies every statement in $\mathcal{K}$. As usual, we call $\mathcal{K}$ *satisfiable* iff some $\mathfrak{D}$ with $\mathfrak{D} \models \mathcal{K}$ exists. A $\mathbb{S}_{\mathcal{EL}+}$ statement ψ is *entailed* by $\mathcal{K}$ (written $\mathcal{K} \models \psi$) iff $\mathfrak{D} \models \psi$ holds for every model $\mathfrak{D}$ of $\mathcal{K}$.

2.3 Normalisation

Before we can present our deduction calculus for checking satisfiability of $\mathbb{S}_{\mathcal{EL}+}$ knowledge bases, we need to introduce an appropriate normal form.

Definition 1 (Normal Form of $\mathbb{S}_{\mathcal{EL}+}$ Knowledge Bases).
A knowledge base $\mathcal{K}$ is in *normal form* iff it only contains statements of the following shapes:

- sharpening statements of the form $\mathsf{s} \preceq \mathsf{s}'$ and $\mathsf{s}_1 \cap \mathsf{s}_2 \preceq \mathsf{s}'$ for $\mathsf{s}, \mathsf{s}', \mathsf{s}_1, \mathsf{s}_2 \in N_{\mathsf{S}}$,
- modalised GCIs of the shape $\Box_{\mathsf{s}} \xi$ with $\mathsf{s} \in N_{\mathsf{S}}$ and GCI ξ being of one of the following forms:

$$
\begin{array}{ll}
C \sqsubseteq D & C_1 \sqcap C_2 \sqsubseteq D \\
\exists R.C \sqsubseteq D & C \sqsubseteq \exists R.D \\
C \sqsubseteq \Box_{\mathsf{u}} D & C \sqsubseteq \Diamond_{\mathsf{u}} D
\end{array}
$$

for $C, C_1, C_2, D \in N_{\mathsf{C}} \cup \{\top, \bot\} \cup \{\exists R.\mathsf{Self} \mid R \in N_{\mathsf{R}}\}$ with $C, C_1, C_2 \neq \bot$ and $D \neq \top$, $R \in N_{\mathsf{R}}$, and $\mathsf{u} \in N_{\mathsf{S}}$.
- modalised RIAs of the form $\Box_{\mathsf{s}}[R_1 \sqsubseteq R_2]$ and $\Box_{\mathsf{s}}[R_1 \circ R_2 \sqsubseteq R_3]$ with $R_1, R_2, R_3 \in N_{\mathsf{R}}$;
- modalised assertions of the form $\Box_{\mathsf{s}}[A(a)]$ or $\Box_{\mathsf{s}}[R(a, b)]$ for $a, b \in N_{\mathsf{I}}$, $A \in N_{\mathsf{C}}$, and $R \in N_{\mathsf{R}}$. $\Diamond$

Note that complex/nested concepts can only occur on one side of a GCI and then must not nest deeper than one level.

For a given $\mathbb{S}_{\mathcal{EL}+}$ knowledge base $\mathcal{K}$, we can compute its normal form in two phases. In the first phase, we "break down" formulas into modalised axioms, effectively compiling away negation, and in the second phase we "break down" complex concepts occurring within these axioms.

Phase 1: Modalised Axioms We obtain the (outer) normal form of axioms by exhaustively applying the transformation rules depicted in Figure 1, where "rule application" means that the statement on the left-hand side is replaced with the set of statements on the right-hand side. This eliminates statements preceded by diamonds, modalised axiom sets, and negated axioms.

Phase 2: Restricted Concept Terms To obtain the (inner) normal forms of concept terms occurring in GCIs as well as the restricted forms of sharpening statements and role inclusion axioms, we use the rules displayed in Figure 2. Transformation rules (24)–(26) and (30) are novel, the others were already proposed and formally justified in our earlier work (Gómez Álvarez, Rudolph, and Strass 2023a).

A careful analysis yields that the overall transformation (Phase 1 + Phase 2) has the desired semantic and computational properties.

Lemma 1. *Any $\mathbb{S}_{\mathcal{EL}+}$ knowledge base $\mathcal{K}$ can be transformed into a $\mathbb{S}_{\mathcal{EL}+}$ knowledge base $\mathcal{K}'$ in normal form such that:*
- *$\mathcal{K}'$ is a $\mathbb{S}_{\mathcal{EL}}$-conservative extension of $\mathcal{K}$,*
- *the size of $\mathcal{K}'$ is at most linear in the size of $\mathcal{K}$, and*
- *the transformation can be computed in PTIME.*

In particular, $\mathcal{K}'$ being a $\mathbb{S}_{\mathcal{EL}}$-conservative extension of $\mathcal{K}$ means that $\mathcal{K}$ and $\mathcal{K}'$ are equisatisfiable.

2.4 Reasoning Problems and Reductions

The deduction calculus we are going to present in the next section decides the fundamental reasoning task of satisfiability for $\mathbb{S}_{\mathcal{EL}+}$:

Problem: $\mathbb{S}_{\mathcal{EL}+}$ KNOWLEDGE BASE SATISFIABILITY
Input: $\mathbb{S}_{\mathcal{EL}+}$ knowledge base $\mathcal{K}$.
Output: YES, if $\mathcal{K}$ has a model, NO otherwise.

This reasoning task is useful in itself, e.g. for knowledge engineers to check for grave modelling errors that turn the specified knowledge base globally inconsistent. From a user's perspective, however, a more application-relevant problem is that of statement entailment, as it allows to "query" the specified knowledge for consequences:

Problem: $\mathbb{S}_{\mathcal{EL}+}$ STATEMENT ENTAILMENT
Input: $\mathbb{S}_{\mathcal{EL}+}$ knowledge base $\mathcal{K}$, $\mathbb{S}_{\mathcal{EL}+}$ statement ϕ.
Output: YES, if $\mathcal{K} \models \phi$, NO otherwise.

Typically, entailment $\mathcal{K} \models \phi$ can be (many-one-)reduced to unsatisfiability of $\mathcal{K} \cup \{\neg\phi\}$. This is not immediately possible in the case of $\mathbb{S}_{\mathcal{EL}+}$ due to its restricted syntax: note that despite the possibility to negate single axioms or sharpening statements, the negation of monomials or formulae in general is not supported by the syntax of $\mathbb{S}_{\mathcal{EL}+}$. However, it turns out that a similar technique can be applied with some additional care.

It is clear that the straightforward reduction works for arbitrary modalised literals $\odot_{\mathsf{s}}[\lambda]$, for negated formulas $\neg\odot_{\mathsf{s}}[\mu]$, and for (possibly negated) sharpening statements, so what remains to be detailed is the reduction for modalised monomials. Consider $\odot_{\mathsf{s}} \mu = \odot_{\mathsf{s}}[\lambda_1 \wedge \ldots \wedge \lambda_n]$. For $\odot_{\mathsf{s}} = \Box_{\mathsf{s}}$, we have that $\Box_{\mathsf{s}} \mu$ is logically equivalent to $\Box_{\mathsf{s}}[\lambda_1] \wedge \ldots \wedge \Box_{\mathsf{s}}[\lambda_n]$ and thus we can (Turing-)reduce checking entailment $\mathcal{K} \models \Box_{\mathsf{s}}[\mu]$ to checking whether all of $\mathcal{K} \cup \{\neg\Box_{\mathsf{s}}[\lambda_1]\}, \ldots, \mathcal{K} \cup \{\neg\Box_{\mathsf{s}}[\lambda_n]\}$ are unsatisfiable (which is still polynomial in $\mathcal{K}$ and $\Box_{\mathsf{s}}[\mu]$). Finally, for

$$\Diamond_{\mathsf{s}}[\mu] \quad \longrightarrow \quad \{\mathsf{v} \preceq \mathsf{s}, \Box_{\mathsf{v}}[\mu]\} \tag{17}$$

$$\Box_{\mathsf{s}}[\lambda_1 \wedge \ldots \wedge \lambda_n] \quad \longrightarrow \quad \{\Box_{\mathsf{s}}[\lambda_1], \ldots, \Box_{\mathsf{s}}[\lambda_n]\} \tag{18}$$

$$\Box_{\mathsf{s}}[\neg(C \sqsubseteq D)] \quad \longrightarrow \quad \{\Box_{\mathsf{s}}[A \sqsubseteq C], \Box_{\mathsf{s}}[A \sqcap D \sqsubseteq \bot], \Box_{\mathsf{s}}[\top \sqsubseteq \exists R'.A]\} \tag{19}$$

$$\Box_{\mathsf{s}}[\neg C(a)] \quad \longrightarrow \quad \{\Box_{\mathsf{s}}[A(a)], \Box_{\mathsf{s}}[A \sqcap C \sqsubseteq \bot]\} \tag{20}$$

$$\Box_{\mathsf{s}}[\neg R(a,b)] \quad \longrightarrow \quad \{\Box_{\mathsf{s}}[A_a(a)], \Box_{\mathsf{s}}[A_b(b)], \Box_{\mathsf{s}}[A_a \sqcap \exists R.A_b \sqsubseteq \bot]\} \tag{21}$$

$$\Box_{\mathsf{s}}[\neg(R_1 \circ \ldots \circ R_n \sqsubseteq R)] \quad \longrightarrow \quad \{\Box_{\mathsf{s}}[\top \sqsubseteq \exists R'.A_a], \Box_{\mathsf{s}}[A_a \sqcap \exists R.A_b \sqsubseteq \bot], \Box_{\mathsf{s}}[A_a \sqsubseteq \exists R_1. \cdots \exists R_n.A_b]\} \tag{22}$$

$$\neg(\mathsf{s}_1 \cap \ldots \cap \mathsf{s}_n \preceq \mathsf{u}) \quad \longrightarrow \quad \{\mathsf{v} \preceq \mathsf{s}_1, \ldots, \mathsf{v} \preceq \mathsf{s}_n, \mathsf{v} \cap \mathsf{u} \preceq \mathbf{0}\} \tag{23}$$

Figure 1: Normalisation rules for Phase 1. Therein, $\mathsf{s}_1, \ldots, \mathsf{s}_n, \mathsf{u} \in N_{\mathsf{S}} \cup \{\mathbf{0}\}$, the A, A_a, A_b denote fresh concept names, R' a fresh role name, and v a fresh standpoint name.

$$\mathsf{s}_1 \cap \ldots \cap \mathsf{s}_n \preceq \mathsf{s} \quad \longrightarrow \quad \{\mathsf{s}_1 \cap \mathsf{s}_2 \preceq \mathsf{s}', \mathsf{s}' \cap \mathsf{s}_3 \cap \ldots \cap \mathsf{s}_n \preceq \mathsf{s}\} \tag{24}$$

$$\mathsf{s}_1 \cap \ldots \cap \mathsf{s}_n \preceq \mathbf{0} \quad \longrightarrow \quad \{\Box_{\mathsf{s}_1}[\top \sqsubseteq A_1], \ldots, \Box_{\mathsf{s}_n}[\top \sqsubseteq A_n], \Box_{*}[A_1 \sqcap \ldots \sqcap A_n \sqsubseteq \bot]\} \tag{25}$$

$$\Box_{\mathsf{s}}[R_1 \circ \ldots \circ R_n \sqsubseteq R] \quad \longrightarrow \quad \{\Box_{\mathsf{s}}[R_1 \circ R_2 \sqsubseteq R'], \Box_{\mathsf{s}}[R' \circ R_3 \circ \ldots \circ R_n \sqsubseteq R]\} \tag{26}$$

$$\Box_{\mathsf{s}}[\bar{C}(a)] \quad \longrightarrow \quad \{\Box_{\mathsf{s}}[A(a)], \Box_{\mathsf{s}}[A \sqsubseteq \bar{C}]\} \tag{27}$$

$$\Box_{\mathsf{s}}[C \sqsubseteq \top] \quad \longrightarrow \quad \emptyset \tag{28}$$

$$\Box_{\mathsf{s}}[\bot \sqsubseteq D] \quad \longrightarrow \quad \emptyset \tag{29}$$

$$\Box_{\mathsf{s}}[\bar{C} \sqsubseteq \bar{D}] \quad \longrightarrow \quad \{\bar{C} \sqsubseteq A, A \sqsubseteq \bar{D}\} \tag{30}$$

$$\Box_{\mathsf{s}}[B \sqsubseteq \exists R.\bar{C}] \quad \longrightarrow \quad \{\Box_{\mathsf{s}}[B \sqsubseteq \exists R.A], \Box_{\mathsf{s}}[A \sqsubseteq \bar{C}]\} \tag{31}$$

$$\Box_{\mathsf{s}}[B \sqsubseteq C \sqcap D] \quad \longrightarrow \quad \{\Box_{\mathsf{s}}[B \sqsubseteq A], \Box_{\mathsf{s}}[A \sqsubseteq C], \Box_{\mathsf{s}}[A \sqsubseteq D]\} \tag{32}$$

$$\Box_{\mathsf{s}}[C \sqsubseteq \odot_{\mathsf{u}}\bar{D}] \quad \longrightarrow \quad \{\Box_{\mathsf{s}}[C \sqsubseteq \odot_{\mathsf{u}}A], \Box_{\mathsf{s}}[A \sqsubseteq \bar{D}]\} \tag{33}$$

$$\Box_{\mathsf{s}}[\exists R.\bar{C} \sqsubseteq D] \quad \longrightarrow \quad \{\Box_{\mathsf{s}}[\bar{C} \sqsubseteq A], \Box_{\mathsf{s}}[\exists R.A \sqsubseteq D]\} \tag{34}$$

$$\Box_{\mathsf{s}}[\bar{C} \sqcap D \sqsubseteq E] \quad \longrightarrow \quad \{\Box_{\mathsf{s}}[\bar{C} \sqsubseteq A], \Box_{\mathsf{s}}[A \sqcap D \sqsubseteq E]\} \tag{35}$$

$$\Box_{\mathsf{s}}[\Diamond_{\mathsf{u}}C \sqsubseteq D] \quad \longrightarrow \quad \{\Box_{\mathsf{u}}[C \sqsubseteq \Box_{*}A], \Box_{\mathsf{s}}[A \sqsubseteq D]\} \tag{36}$$

$$\Box_{\mathsf{s}}[\Box_{\mathsf{u}}C \sqsubseteq D] \quad \longrightarrow \quad \{\mathsf{v}_0 \preceq \mathsf{u}, \mathsf{v}_1 \preceq \mathsf{u}, \Box_{\mathsf{u}}[C \sqsubseteq A], \Box_{\mathsf{s}}[\Diamond_{\mathsf{v}_0}A \sqcap \Diamond_{\mathsf{v}_1}A \sqsubseteq D]\} \tag{37}$$

Figure 2: Normalisation rules for Phase 2. Therein, $\mathsf{u} \in N_{\mathsf{S}}$, $\bar{C}$ and $\bar{D}$ stand for complex concept terms not contained in $N_{\mathsf{C}} \cup \{\top, \bot\} \cup \{\exists R.\mathsf{Self} \mid R \in N_{\mathsf{R}}\}$, whereas each occurrence of A (possibly with subscript) on a right-hand side denotes the introduction of a fresh concept name; each occurrence of R' on a right-hand side denotes the introduction of a fresh role name; likewise, v, v_0, and v_1 denote the introduction of a fresh standpoint name. In rule (24), we implicitly assume that $n \geq 3$. Rule (35) is applied modulo commutativity of $\sqcap$.

$\odot_{\mathsf{s}} = \Diamond_{\mathsf{s}}$, we employ the idea underlying normalisation rule (17) to obtain the following:

Lemma 2. *Let $\mathcal{K}$ be a $\mathbb{S}_{\mathcal{EL}+}$ knowledge base with normal form $\mathcal{K}'$ and μ be a monomial. It holds that $\mathcal{K} \models \Diamond_{\mathsf{s}}[\mu]$ if and only if $\mathcal{K}' \models \Box_{\mathsf{u}}[\mu]$ for some $\mathsf{u} \in N_{\mathsf{S}}$ with $\mathcal{K}' \models \mathsf{u} \preceq \mathsf{s}$.*

Proof sketch. The main idea is that for $\mathcal{K} \models \Diamond_{\mathsf{s}}[\mu]$ to hold there is a formula $\Diamond_{\mathsf{s}'}[\mu'] \in \mathcal{K}$ with $\mathcal{K} \models \mathsf{s}' \preceq \mathsf{s}$ and formulas $\Box_{\mathsf{s}_1}[\mu_1], \ldots, \Box_{\mathsf{s}_m}[\mu_m] \in \mathcal{K}$ with $\mathcal{K} \models \mathsf{s}' \preceq \mathsf{s}_i$ for all $1 \leq i \leq n$ (where neither type of formula is strictly required, and $\mathsf{s}' = \mathsf{s}$ in case no $\Diamond_{\mathsf{s}'}$ formula is involved), such that $\{\mu', \mu_1, \ldots, \mu_m\} \models \mu$. In case all relevant formulas are of the form $\Box_{\mathsf{s}_i}[\mu_i]$, then $\mathcal{K} \models \Box_{\mathsf{s}}[\mu]$ and the claim trivially holds (with $\mathsf{u} = \mathsf{s}$). In case some $\Diamond_{\mathsf{s}'}[\mu'] \in \mathcal{K}$ is involved, normalisation rule (17) will introduce the new standpoint name u that can serve as witness in the normalised KB $\mathcal{K}'$. ☐

So to decide $\mathcal{K} \models \Diamond_{\mathsf{s}}[\mu]$, we normalise $\mathcal{K}$ into $\mathcal{K}'$ and then successively enumerate $\mathsf{s}' \in N_{\mathsf{S}}$ occurring in $\mathcal{K}'$ for which $\mathcal{K}' \models \mathsf{s}' \preceq \mathsf{s}$ and test $\mathcal{K} \models \Box_{\mathsf{s}'}[\mu]$ for each. In view of these considerations, we arrive at the following reducibility result.

Theorem 3. *There exists a* PTIME *Turing reduction from* $\mathbb{S}_{\mathcal{EL}+}$ STATEMENT ENTAILMENT *to* $\mathbb{S}_{\mathcal{EL}+}$ KNOWLEDGE BASE SATISFIABILITY.

Thus, every tractable decision procedure for satisfiability can be leveraged to construct a tractable entailment checker. Therefore, we will concentrate on a method for the former.

3 Refutation-Complete Deduction Calculus for Normalised KBs

In this section, we present the Hilbert-style deduction calculus for $\mathbb{S}_{\mathcal{EL}+}$.[5] Premises and consequents of the calculus' deduction rules will be axioms in normal form with one notable exception: We allow for extended versions of modalised GCIs of the general shape

$$\Box_{\mathsf{t}}[A \sqsubseteq \Box_{\mathsf{s}}[B \Rightarrow C]], \tag{38}$$

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[5]We recall that the proofs of all results are available in the extended version (Gómez Álvarez, Rudolph, and Strass 2023b).

the meaning of which should be intuitively clear, despite the fact that $\Rightarrow$ is not a connective available in $\mathbb{S}_{\mathcal{EL}+}$. In terms of more expressive Standpoint DLs, such an axiom could be written $\Box_t[A \sqsubseteq \Box_s[\neg B \sqcup C]]$, but this would obfuscate the "Horn nature" of the statement. Note that the axiom can be expressed in $\mathbb{S}_{\mathcal{EL}+}$ by the two axioms $\Box_t[A \sqsubseteq \Box_s D]$ and $\Box_s[D \sqcap B \sqsubseteq C]$ using an auxiliary fresh concept D. Yet, for better treatment in the calculus, we need all the information "bundled" within one axiom. With this new axiom type in place, we dispense with axioms of the shapes $\Box_s[A \sqsubseteq B]$ and $\Box_s[A \sqsubseteq \Box_{s'} B]$, replacing them by $\Box_*[\top \sqsubseteq \Box_s[A \Rightarrow B]]$ and $\Box_s[A \sqsubseteq \Box_{s'}[\top \Rightarrow B]]$, respectively. Similarly, we will replace concept assertions of the form $\Box_s A(a)$ by $\Box_*[\{a\} \sqsubseteq \Box_s[\top \Rightarrow A]]$, where $\{a\}$ is understood as a "nominal concept", to be interpreted by the singleton set $\{a^{\mathfrak{D}}\}$ in the usual way.[6] Then, it should be clear that these are equivalent axiom replacements.

As one final preprocessing step, we introduce, for every concept $\Diamond_s B$ that occurs in the normalised KB and every ABox individual a, a fresh standpoint name denoted $s[a, B]$ and a fresh concept name $P_{s,a,B}$ and add the following background axioms:

$$s[a, B] \preceq s \tag{39}$$

$$\Box_*[\{a\} \sqsubseteq \Box_s[B \Rightarrow P_{s,a,B}]] \tag{40}$$

$$\Box_*[P_{s,a,B} \sqsubseteq \Box_{s[a,B]}[\top \Rightarrow B]] \tag{41}$$

Intuitively, the purpose of this conservative extension is that, whenever a is required to satisfy $\Diamond_s B$, it will satisfy B in all $s[a, B]$-precisifications, this way arranging for a concrete, "addressable" witness for $\Diamond_s B(a)$.

Given a $\mathbb{S}_{\mathcal{EL}+}$ knowledge base $\mathcal{K}$, let $\mathcal{K}^{\text{prep}}$ denote its preprocessed variant, obtained through normalisation and the steps described above. Again note that $\mathcal{K}^{\text{prep}}$ can be computed in deterministic polynomial time. Now let $\mathcal{K}^{\vdash}$ denote the set of axioms obtained from $\mathcal{K}^{\text{prep}}$ by saturating it under the deduction rules of Figure 3. We note that each axiom type has a bounded number of parameters, each of which can be instantiated by a polynomial number of elements (concepts, roles, individuals, standpoints) occurring in $\mathcal{K}^{\text{prep}}$. Consequently, the overall number of distinct inferrable axioms is polynomial and therefore the saturation process to obtain $\mathcal{K}^{\vdash}$ runs in deterministic polynomial time. We will see in the next section that these observations give rise to a worst-case optimal Datalog implementation of the saturation procedure.

Theorem 4. *Computing the closure of* $\mathbb{S}_{\mathcal{EL}+}$ *knowledge bases under the deduction calculus displayed in Figure 3 terminates and can be done in* PTIME.

We next argue that the presented calculus has the desired properties. As usual, soundness of the calculus is easy to show and can be argued for each deduction rule separately by referring to the definition of the semantics.

Theorem 5. *The deduction calculus displayed in Figure 3 is sound for* $\mathbb{S}_{\mathcal{EL}+}$ *knowledge bases.*

What remains to be shown is a particular type of completeness: Among the inferrable axioms, the particular intrinsically contradictory statement $\Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \bot]]$ will play the pivotal role of indicating unsatisfiability of $\mathcal{K}$ (also referred to as refutation). We will show that our calculus is *refutation-complete*, meaning that for any unsatisfiable $\mathbb{S}_{\mathcal{EL}+}$ knowledge base $\mathcal{K}$, we have that $\Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \bot]] \in \mathcal{K}^{\vdash}$. More concretely, we prove the contrapositive by establishing the existence of a model whenever $\Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \bot]] \notin \mathcal{K}^{\vdash}$. This model is canonical in a sense but, as opposed to canonical models of the $\mathcal{EL}$ family, it will typically be infinite.

We now provide the construction of the canonical model. Given a $\mathbb{S}_{\mathcal{EL}+}$ knowledge base $\mathcal{K}$ with $\Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \bot]] \notin \mathcal{K}^{\vdash}$, we construct a model $\mathfrak{D}$ of $\mathcal{K}$ in an infinite process: we start from an initialised model $\mathfrak{D}_0$ and extend it (both by adding domain elements and precisifications) in a stepwise fashion, resulting in a "monotonic" sequence of models. The result of the process is the "limit" of this sequence, which can be expressed via an infinite union.

For the initialisation, we choose the standpoint structure $\mathfrak{D}_0 = \langle \Delta_0, \Pi_0, \sigma_0, \gamma_0 \rangle$ where:

- Δ_0 consists of one element δ_a for every individual name a mentioned in $\mathcal{K}$;
- Π_0 consists of one precisification π_s for every standpoint s mentioned in $\mathcal{K}$ (including $*$);
- σ_0 maps each standpoint s to $\{\pi_s\} \cup \{\pi_{s'} \mid s' \preceq s \in \mathcal{K}^{\vdash}\}$;
- γ_0 maps each π_s to the description logic interpretation $\mathcal{I}$ over Δ_0, where
  - $a^{\mathcal{I}} = \delta_a$ for each individual name a,
  - $A^{\mathcal{I}} = \{\delta_a \mid \Box_u[\{a\} \sqsubseteq \Box_s[\top \Rightarrow A]] \in \mathcal{K}^{\vdash}, s \in \sigma_0^{-1}(\pi_s)\}$ for every concept name A, and
  - $R^{\mathcal{I}} = \{(\delta_a, \delta_b) \mid \Box_s[R(a,b)] \in \mathcal{K}^{\vdash}, s \in \sigma_0^{-1}(\pi_s)\}$ for each role name R.

It can be shown that the obtained structure $\mathfrak{D}_0$ satisfies all axioms of $\mathcal{K}$ except for those of the shape $\Box_s[E \sqsubseteq \exists R.F]$. This will also be the case for all structures $\mathfrak{D}_1, \mathfrak{D}_2, \ldots$ subsequently produced. The sequence arises by iteratively adding a fresh domain element in order to satisfy a previously unsatisfied axiom of the shape $\Box_s[E \sqsubseteq \exists R.F]$, while preserving the satisfaction of all axioms of other shapes. Thereby, the concept and role memberships of pre-existing elements with respect to pre-existing standpoints will remain unchanged. This justifies the definition of a labelling function Λ_π immutably assigning to every domain element δ a set of concepts, satisfied in π. For $\mathfrak{D}_0$, we let Λ_π map elements δ_a according to

$$\Lambda_\pi(\delta_a) = \{C \mid \Box_u[\{a\} \sqsubseteq \Box_s[\top \Rightarrow C]] \in \mathcal{K}^{\vdash}, s \in \sigma_0^{-1}(\pi)\}.$$

As discussed above, after having arrived at a structure $\mathfrak{D}_i = \langle \Delta_i, \Pi_i, \sigma_i, \gamma_i \rangle$, we inspect if $\mathfrak{D}_i$ satisfies all axioms of the form $\Box_t[E \sqsubseteq \exists R.F]$. If so, $\mathfrak{D}_i$ is a model of $\mathcal{K}$ and we are done. Otherwise, for some axiom of the

Tautologies

$(T.1)$ $\dfrac{}{\mathsf{s} \preceq *}$ $(T.2)$ $\dfrac{}{\mathsf{s} \preceq \mathsf{s}}$ $(T.3)$ $\dfrac{}{\Box_*[\top \sqsubseteq \Box_*[C \Rightarrow C]]}$ $(T.4)$ $\dfrac{}{\Box_*[\top \sqsubseteq \Box_*[C \Rightarrow \top]]}$ $(T.5)$ $\dfrac{}{\Box_*[R \sqsubseteq R]}$

Standpoint hierarchy rules (for all $\mathsf{s} \in N_S$, ξ being any extended GCI, RIA, or role assertion)

$(S.1)$ $\dfrac{\mathsf{s} \preceq \mathsf{s}' \quad \mathsf{s}' \preceq \mathsf{s}''}{\mathsf{s} \preceq \mathsf{s}''}$ $(S.2)$ $\dfrac{\mathsf{s} \preceq \mathsf{s}_1 \quad \mathsf{s} \preceq \mathsf{s}_2 \quad \mathsf{s}_1 \cap \mathsf{s}_2 \preceq \mathsf{s}'}{\mathsf{s} \preceq \mathsf{s}'}$ $(S.3)$ $\dfrac{\Box_{\mathsf{s}'}\xi \quad \mathsf{s} \preceq \mathsf{s}'}{\Box_{\mathsf{s}}\xi}$ $(S.4)$ $\dfrac{\Box_{\mathsf{t}}[C \sqsubseteq \Box_{\mathsf{s}'}[D \Rightarrow E]] \quad \mathsf{s} \preceq \mathsf{s}'}{\Box_{\mathsf{t}}[C \sqsubseteq \Box_{\mathsf{s}}[D \Rightarrow E]]}$

Internal inferences for extended GCIs | **Role subsumptions**

$(I.1)$ $\dfrac{\Box_{\mathsf{s}}[C \sqsubseteq \Box_{\mathsf{s}}[\top \Rightarrow D]]}{\Box_*[\top \sqsubseteq \Box_{\mathsf{s}}[C \Rightarrow D]]}$ $(I.2)$ $\dfrac{\Box_{\mathsf{u}}[\top \sqsubseteq \Box_{\mathsf{s}}[C \Rightarrow D]]}{\Box_*[\top \sqsubseteq \Box_{\mathsf{s}}[C \Rightarrow D]]}$ $(R.1)$ $\dfrac{\Box_{\mathsf{s}}[R \sqsubseteq R''] \quad \Box_{\mathsf{s}}[R'' \sqsubseteq R']}{\Box_{\mathsf{s}}[R \sqsubseteq R']}$

Forward chaining

$(C.1)$ $\dfrac{\Box_{\mathsf{t}}[B \sqsubseteq \Box_{\mathsf{s}}[C \Rightarrow D]] \quad \Box_{\mathsf{t}}[B \sqsubseteq \Box_{\mathsf{s}}[D \Rightarrow E]]}{\Box_{\mathsf{t}}[B \sqsubseteq \Box_{\mathsf{s}}[C \Rightarrow E]]}$ $(C.2)$ $\dfrac{\Box_{\mathsf{u}}[\top \sqsubseteq \Box_{\mathsf{t}}[B \Rightarrow C]] \quad \Box_{\mathsf{t}}[C \sqsubseteq \Box_{\mathsf{s}}[D \Rightarrow E]]}{\Box_{\mathsf{t}}[B \sqsubseteq \Box_{\mathsf{s}}[D \Rightarrow E]]}$

$(C.3)$ $\dfrac{\Box_{\mathsf{u}}[\top \sqsubseteq \Box_{\mathsf{t}}[C \Rightarrow D]] \quad \Box_{\mathsf{t}}[D \sqsubseteq \Diamond_{\mathsf{s}}E]}{\Box_{\mathsf{t}}[C \sqsubseteq \Diamond_{\mathsf{s}}E]}$ $(C.4)$ $\dfrac{\Box_{\mathsf{t}}[C \sqsubseteq \Diamond_{\mathsf{s}}D] \quad \Box_{\mathsf{t}}[C \sqsubseteq \Box_{\mathsf{s}}[D \Rightarrow E]]}{\Box_{\mathsf{t}}[C \sqsubseteq \Diamond_{\mathsf{s}}E]}$

Flattening of modalities

$(F.1)$ $\dfrac{\Box_{\mathsf{t}}[C \sqsubseteq \Box_{\mathsf{s}'}[\top \Rightarrow D]] \quad \Box_{\mathsf{s}'}[D \sqsubseteq \Box_{\mathsf{s}}[E \Rightarrow F]]}{\Box_{\mathsf{t}}[C \sqsubseteq \Box_{\mathsf{s}}[E \Rightarrow F]]}$ $(F.2)$ $\dfrac{\Box_{\mathsf{t}}[C \sqsubseteq \Box_{\mathsf{s}'}[\top \Rightarrow D]] \quad \Box_{\mathsf{s}'}[D \sqsubseteq \Diamond_{\mathsf{s}}E]}{\Box_{\mathsf{t}}[C \sqsubseteq \Diamond_{\mathsf{s}}E]}$

$(F.3)$ $\dfrac{\Box_{\mathsf{t}}[C \sqsubseteq \Diamond_{\mathsf{s}'}D] \quad \Box_{\mathsf{s}'}[D \sqsubseteq \Box_{\mathsf{s}}[E \Rightarrow F]]}{\Box_{\mathsf{t}}[C \sqsubseteq \Box_{\mathsf{s}}[E \Rightarrow F]]}$ $(F.4)$ $\dfrac{\Box_{\mathsf{t}}[C \sqsubseteq \Diamond_{\mathsf{s}'}D] \quad \Box_{\mathsf{s}'}[D \sqsubseteq \Diamond_{\mathsf{s}}E]}{\Box_{\mathsf{t}}[C \sqsubseteq \Diamond_{\mathsf{s}}E]}$

Inferences involving existential quantifiers and conjunction

$(E.1)$ $\dfrac{\Box_{\mathsf{s}}[C \sqsubseteq \exists R.D] \quad \Box_{\mathsf{u}}[\top \sqsubseteq \Box_{\mathsf{s}}[D \Rightarrow E]] \quad \Box_{\mathsf{s}}[R \sqsubseteq R']}{\Box_{\mathsf{s}}[C \sqsubseteq \exists R'.E]}$ $(E.2)$ $\dfrac{\Box_{\mathsf{s}}[C \sqsubseteq \exists R_1.D] \quad \Box_{\mathsf{s}}[D \sqsubseteq \exists R_2.E] \quad \Box_{\mathsf{s}}[R_1 \circ R_2 \sqsubseteq R']}{\Box_{\mathsf{s}}[C \sqsubseteq \exists R'.E]}$

$(E.3)$ $\dfrac{\Box_{\mathsf{s}}[C \sqsubseteq \exists R.D] \quad \Box_{\mathsf{s}}[\exists R.D \sqsubseteq F]}{\Box_*[\top \sqsubseteq \Box_{\mathsf{s}}[C \Rightarrow F]]}$ $(E.4)$ $\dfrac{\Box_{\mathsf{t}}[B \sqsubseteq \Box_{\mathsf{s}}[C \Rightarrow C_1]] \quad \Box_{\mathsf{t}}[B \sqsubseteq \Box_{\mathsf{s}}[C \Rightarrow C_2]] \quad \Box_{\mathsf{s}}[C_1 \sqcap C_2 \sqsubseteq D]}{\Box_{\mathsf{t}}[B \sqsubseteq \Box_{\mathsf{s}}[C \Rightarrow D]]}$

Individual-based inferences

$(A.1)$ $\dfrac{\Box_{\mathsf{u}}[\top \sqsubseteq \Box_{\mathsf{s}}[B \Rightarrow C]]}{\Box_*[\{a\} \sqsubseteq \Box_{\mathsf{s}}[B \Rightarrow C]]}$ $(A.2)$ $\dfrac{\Box_{\mathsf{u}}[\{a\} \sqsubseteq \Box_{\mathsf{s}}[\top \Rightarrow C]]}{\Box_*[\top \sqsubseteq \Box_{\mathsf{s}}[\{a\} \Rightarrow C]]}$ $(A.3)$ $\dfrac{\Box_{\mathsf{u}}[\{a\} \sqsubseteq \Box_{\mathsf{s}}[B \Rightarrow C]]}{\Box_*[\{a\} \sqsubseteq \Box_{\mathsf{s}}[B \Rightarrow C]]}$

$(A.4)$ $\dfrac{\Box_{\mathsf{s}}[R(a,b)] \quad \Box_{\mathsf{s}}[R \sqsubseteq R']}{\Box_{\mathsf{s}}[R'(a,b)]}$ $(A.5)$ $\dfrac{\Box_{\mathsf{s}}[R_1(a,b)] \quad \Box_{\mathsf{s}}[R_2(b,c)] \quad \Box_{\mathsf{s}}[R_1 \circ R_2 \sqsubseteq R']}{\Box_{\mathsf{s}}[R'(a,c)]}$

$(A.6)$ $\dfrac{\Box_{\mathsf{s}}[R(a,b)] \quad \Box_{\mathsf{u}}[\{b\} \sqsubseteq \Box_{\mathsf{s}}[\top \Rightarrow B]]}{\Box_{\mathsf{s}}[\{a\} \sqsubseteq \exists R.B]}$ $(A.7)$ $\dfrac{\Box_{\mathsf{s}}[R_1(a,b)] \quad \Box_{\mathsf{s}}[\{b\} \sqsubseteq \exists R_2.C] \quad \Box_{\mathsf{s}}[R_1 \circ R_2 \sqsubseteq R']}{\Box_{\mathsf{s}}[\{a\} \sqsubseteq \exists R'.C]}$

$(A.8)$ $\dfrac{\Box_{\mathsf{s}}[R_1(a,b)] \quad \Box_{\mathsf{u}}[\{b\} \sqsubseteq \Box_{\mathsf{s}}[\top \Rightarrow B]] \quad \Box_{\mathsf{s}}[B \sqsubseteq \exists R_2.C] \quad \Box_{\mathsf{s}}[R_1 \circ R_2 \sqsubseteq R']}{\Box_{\mathsf{s}}[\{a\} \sqsubseteq \exists R'.C]}$

Interaction of self-loops with other statements

$(L.1)$ $\dfrac{\Box_{\mathsf{u}}[\{a\} \sqsubseteq \Box_{\mathsf{s}}[\top \Rightarrow \exists R.\mathsf{Self}]]}{\Box_{\mathsf{s}}[R(a,a)]}$ $(L.2)$ $\dfrac{\Box_{\mathsf{u}}[\top \sqsubseteq \Box_{\mathsf{s}}[C \Rightarrow \exists R.\mathsf{Self}]]}{\Box_{\mathsf{s}}[C \sqsubseteq \exists R.C]}$ $(L.3)$ $\dfrac{\Box_{\mathsf{s}}[\exists R.D \sqsubseteq C]}{\Box_{\mathsf{s}}[\exists R.\mathsf{Self} \sqcap D \sqsubseteq C]}$

$(L.4)$ $\dfrac{\Box_{\mathsf{s}}[R(a,a)]}{\Box_*[\{a\} \sqsubseteq \Box_{\mathsf{s}}[\top \Rightarrow \exists R.\mathsf{Self}]]}$ $(L.5)$ $\dfrac{\Box_{\mathsf{s}}[R \sqsubseteq R']}{\Box_*[\top \sqsubseteq \Box_{\mathsf{s}}[\exists R.\mathsf{Self} \Rightarrow \exists R'.\mathsf{Self}]]}$ $(L.6)$ $\dfrac{\Box_{\mathsf{s}}[R_1 \circ R_2 \sqsubseteq R']}{\Box_{\mathsf{s}}[\exists R_1.\mathsf{Self} \sqcap \exists R_2.\mathsf{Self} \sqsubseteq \exists R'.\mathsf{Self}]}$

Backpropagation of $\perp$-inferences

$(B.1)$ $\dfrac{\Box_{\mathsf{s}}[C \sqsubseteq \exists R.\perp]}{\Box_*[\top \sqsubseteq \Box_{\mathsf{s}}[C \Rightarrow \perp]]}$ $(B.2)$ $\dfrac{\Box_{\mathsf{t}}[C \sqsubseteq \Box_{\mathsf{s}}[\top \Rightarrow \perp]]}{\Box_*[\top \sqsubseteq \Box_{\mathsf{t}}[C \Rightarrow \perp]]}$ $(B.3)$ $\dfrac{\Box_{\mathsf{t}}[C \sqsubseteq \Diamond_{\mathsf{s}}\perp]}{\Box_*[\top \sqsubseteq \Box_{\mathsf{t}}[C \Rightarrow \perp]]}$ $(B.4)$ $\dfrac{\Box_{\mathsf{u}}[\{a\} \sqsubseteq \Box_{\mathsf{s}}[\top \Rightarrow \perp]]}{\Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \perp]]}$

Figure 3: Deduction calculus for $\mathbb{S}_{\mathcal{EL}+}$

form $\Box_t[E \sqsubseteq \exists R.F]$ in $\mathcal{K}$ that is unsatisfied in $\mathfrak{D}_i$, we pick some $\delta^* \in \Delta_i$ and some $\pi^* \in \Pi_i$ with $\pi^* \in \sigma(t)$ and $\delta^* \in E^{\gamma(\pi^*)} \setminus (\exists R.F)^{\gamma(\pi^*)}$. Among the eligible pairs δ^*, π^*, we pick one for which the value $\min\{j \leq i \mid \delta^* \in \Delta_j\} + \min\{j \leq i \mid \pi^* \in \Pi_j\}$ is minimal; with this, we ensure fairness in the sense that any axiom violation will ultimately be addressed.

We now obtain $\mathfrak{D}_{i+1} = \langle \Delta_{i+1}, \Pi_{i+1}, \sigma_{i+1}, \gamma_{i+1} \rangle$ from $\mathfrak{D}_i = \langle \Delta_i, \Pi_i, \sigma_i, \gamma_i \rangle$ given δ^* and π^* in the following way:

- $\Delta_{i+1} = \Delta_i \cup \{\delta'\}$, where δ' is a fresh domain element.
- Let $\mathrm{Con}(F, \pi^*)$ denote the concepts subsumed by F under π^*, i.e., $\{A \mid \Box_u[\top \sqsubseteq \Box_s[F \Rightarrow A]] \in \mathcal{K}^\vdash, s \in \sigma^{-1}(\pi^*)\}$.
- Π_{i+1} is obtained from Π_i by adding a fresh precisification $\pi_{\delta', \Diamond_s D}$ whenever there is some $C \in \mathrm{Con}(F, \pi^*)$ with $\Box_t[C \sqsubseteq \Diamond_s D] \in \mathcal{K}^\vdash$ for some $t \in \sigma^{-1}(\pi^*)$.
- Let σ_{i+1} be such that $\sigma_{i+1}(s'') = \sigma_i(s'') \cup \{\pi_{\delta', \Diamond_s D}\}$ if $s'' \in \{s\} \cup \{s' \mid s \preceq s' \in \mathcal{K}^\vdash\}$ and $\sigma_{i+1}(s'') = \sigma_i(s'')$ otherwise.
- For $\pi \in \Pi_{i+1}$ we let $\Lambda_\pi(\delta')$ be

$$\begin{cases} \mathrm{Con}(F, \pi^*) & \text{if } \pi = \pi^*, \\[2ex] \bigcup_{\substack{G \in \mathrm{Con}(F, \pi^*),\, s \in \sigma_{i+1}^{-1}(\pi),\, t \in \sigma_{i+1}^{-1}(\pi^*)}} \{A \mid \Box_t[G \sqsubseteq \Box_s[\top \Rightarrow A]] \in \mathcal{K}^\vdash\} & \text{if } \pi \in \Pi_i \setminus \{\pi^*\}, \\[2ex] \bigcup_{\substack{G \in \mathrm{Con}(F, \pi^*),\, s \in \sigma_{i+1}^{-1}(\pi),\, t \in \sigma_{i+1}^{-1}(\pi^*)}} \{A \mid \Box_t[G \sqsubseteq \Box_s[D \Rightarrow A]] \in \mathcal{K}^\vdash\} & \text{if } \pi = \pi_{\delta', \Diamond_s D}. \end{cases}$$

- Let γ_i' be the extension of γ_i to the domain of Π_{i+1} such that for all $\pi_{\delta, \Diamond_s D} \notin \Pi_i$, we have $\gamma_i'(\pi_{\delta, \Diamond_s D}) = \gamma_i(\pi_s)$.
- Let γ_{i+1} be the interpretation function defined as follows:
  - $a^{\gamma_{i+1}(\pi)} = \delta_a$ for each individual name a and for each $\pi \in \Pi_{i+1}$.
  - for concept names A, we let $A^{\gamma_{i+1}(\pi)} = A^{\gamma_i'(\pi)} \cup \{\delta'\}$ if $A \in \Lambda_\pi(\delta')$, and $A^{\gamma_{i+1}'(\pi)} = A^{\gamma_i(\pi)}$ otherwise.
  - for all role names T, we obtain $T^{\gamma_{i+1}(\pi)}$ essentially by performing a concurrent saturation process under all applicable RIAs, that is,

$$T^{\gamma_{i+1}(\pi)} = \bigcup_{k \in \mathbb{N}} [T^{\gamma_{i+1}(\pi)}]_k,$$

where we let $[T^{\gamma_{i+1}(\pi)}]_0 = Self \cup Other$, with

$$Self = \begin{cases} \{(\delta', \delta')\} & \text{if } \exists T.\mathsf{Self} \in \Lambda_\pi(\delta'), \\ \emptyset & \text{otherwise.} \end{cases}$$

$$Other = \begin{cases} \emptyset & \text{whenever } \pi \in \Pi_{i+1} \setminus \Pi_i \\ T^{\gamma_i'(\pi)} & \text{for } \pi \in \Pi_i, \text{ if } T \neq R \text{ or } \pi \neq \pi^* \\ R^{\gamma_i'(\pi^*)} \cup (\delta^*, \delta') & \text{if } T = R \text{ and } \pi = \pi^*, \end{cases}$$

and obtain

$$[T^{\gamma_{i+1}(\pi)}]_{k+1} = [T^{\gamma_{i+1}(\pi)}]_k \cup \bigcup_{\substack{s \in \sigma_{i+1}^{-1}(\pi),\\ \Box_s[R_0 \sqsubseteq T] \in \mathcal{K}^\vdash}} [R_0^{\gamma_{i+1}(\pi)}]_k \;\cup\; \bigcup_{\substack{s \in \sigma_{i+1}^{-1}(\pi),\\ \Box_s[R_1 \circ R_2 \sqsubseteq T] \in \mathcal{K}^\vdash}} [R_1^{\gamma_{i+1}(\pi)}]_k \circ [R_2^{\gamma_{i+1}(\pi)}]_k.$$

The interpretation function in γ_{i+1} is defined using γ_i', which extends γ_i to include the fresh precisifications in $\mathfrak{D}_{i+1}$, and the functions Λ_π, which collect the concept memberships of the new element γ' for all $\pi \in \Pi_{i+1}$. Finally, new role memberships involving γ' are computed in a second saturation process that is triggered by the introduction of (δ^*, δ') to $R^{\gamma_{i+1}(\pi^*)}$, as well as any self-loops on γ'.

After producing the (potentially infinite) sequence $\mathfrak{D}_0, \mathfrak{D}_1, \ldots$ we obtain the wanted model $\mathfrak{D}$ via

$$\mathfrak{D} = \langle \Delta, \Pi, \sigma, \gamma \rangle = \langle \textstyle\bigcup_i \Delta_i, \bigcup_i \Pi_i, \bigcup_i \sigma_i, \bigcup_i \gamma_i \rangle.$$

We then establish that the $\mathfrak{D}$ resulting from this construction indeed is a well-defined structure that satisfies all axioms of $\mathcal{K}$. To this end, an important observation is that for all domain elements $\delta \in \Delta$ and precisifications $\pi \in \Pi$ of $\mathfrak{D}$, it holds that $C \in \Lambda_\pi(\delta)$ implies $\delta \in C^{\gamma(\pi)}$. Furthermore, we show that if $\bot \in \Lambda_\pi(\delta)$ were to hold for any $\delta \in \Delta$ and $\pi \in \Pi$ (which is the only way the model construction could possibly fail, by declaring an existing domain element to be contradictory), then this would necessarily imply $\mathcal{K}^\vdash \models \Box_*[\top \sqsubseteq \Box_*[\top \Rightarrow \bot]]$. By virtue of these considerations, we arrive at the aspired result.

Theorem 6. *The deduction calculus displayed in Figure 3 is refutation-complete for $\mathbb{S}_{\mathcal{EL}+}$ knowledge bases.*

Then, together with Theorem 3, we can use Theorems 4, 5 and 6 to establish tractability of the fundamental standard reasoning tasks in $\mathbb{S}_{\mathcal{EL}+}$.

Corollary 7. $\mathbb{S}_{\mathcal{EL}+}$ KNOWLEDGE BASE SATISFIABILITY *and* $\mathbb{S}_{\mathcal{EL}+}$ STATEMENT ENTAILMENT *are* PTIME-*complete.*

Therein, PTIME-hardness follows immediately from the PTIME-hardness of reasoning in plain $\mathcal{EL}$.

4 Datalog-Based Implementation

We have prototypically implemented our approach in the Datalog-based language SOUFFLÉ (Jordan, Scholz, and Subotić 2016). The prototype's source code is available from our group's github site at `https://github.com/cl-tud/standpoint-el-souffle-reasoner`. The implementation currently does not scale well, so optimising both calculus and implementation is an important issue for future work.

4.1 Calculus

The calculus of Section 3 is implemented in the pure Datalog fragment of SOUFFLÉ's input language. Following the common approach, as e.g. detailed by Krötzsch (2010), we introduce a predicate symbol for each possible (normal-form) formula shape as follows:

$$\Box_s[C \sqsubseteq \Box_{s'}[D \Rightarrow E]] \rightsquigarrow \texttt{gci\_nested(s, C, s', D, E)}$$
$$\Box_s[C \sqcap D \sqsubseteq E] \rightsquigarrow \texttt{gci\_conj\_left(s, C, D, E)}$$
$$\Box_s[\exists R.C \sqsubseteq D] \rightsquigarrow \texttt{gci\_ex\_left(s, R, C, D)}$$
$$\Box_s[C \sqsubseteq \exists R.D] \rightsquigarrow \texttt{gci\_ex\_right(s, C, R, D)}$$
$$\Box_s[C \sqsubseteq \Diamond_{s'} D] \rightsquigarrow \texttt{gci\_diamond\_right(s, C, s', D)}$$
$$s_1 \cap s_2 \preceq s_3 \rightsquigarrow \texttt{sharper\_intersect(s}_1\texttt{, s}_2\texttt{, s}_3\texttt{)}$$
$$s_1 \preceq s_2 \rightsquigarrow \texttt{sharper(s}_1\texttt{, s}_2\texttt{)}$$
$$R_1 \circ R_2 \sqsubseteq R_3 \rightsquigarrow \texttt{ria3(R}_1\texttt{, R}_2\texttt{, R}_3\texttt{)}$$
$$R_1 \sqsubseteq R_2 \rightsquigarrow \texttt{ria2(R}_1\texttt{, R}_2\texttt{)}$$

Therein, $s_1, s_2, s_3, s, s' \in N_S$, and $C, D, E \in N_C$, as well as $R_1, R_2, R_3, R \in N_R$. Implementing the deduction calculus then boils down to writing Datalog rules for all deduction rules. This is straightforward; for example, rule (C.1) is implemented as

```
gci_nested(t, b, s, c, e) :-
    gci_nested(t, b, s, c, d),
    gci_nested(t, b, s, d, e).
```

while rule (E.1) becomes

```
gci_ex_right(s, c, r2, e) :-
    gci_ex_right(s, c, r1, d),
    gci_nested(_, ⊤, s, d, e),
    ria2(s, r1, r2).
```

For the axiom schemas (Tautologies, T.1–T.5) we make use of helper predicates (with prefix "is_") that keep track of the vocabulary:

$$C \in N_C \rightsquigarrow \texttt{is\_cn(C)} \qquad R \in N_R \rightsquigarrow \texttt{is\_rn(R)}$$
$$s \in N_S \rightsquigarrow \texttt{is\_sn(s)} \qquad a \in N_I \rightsquigarrow \texttt{is\_in(a)}$$

Then, for instance, axiom schema (T.4) is expressed via

```
gci_nested(*, ⊤, *, c, ⊤) :- is_cn(c).
```

where the symbols $*$ and $\top$ are used for readability here. For nominals and self-loops we use binary predicates to translate back and forth between individual names/nominal concepts, and role names/self-loop concepts, respectively, hence treating nominals and self-loops as "ordinary" concept names.

4.2 Normalisation

For obtaining the normal form of a given $\mathbb{S}_{\mathcal{EL}+}$ knowledge base in its full expressiveness, we employ several SOUFFLÉ features that are not strictly Datalog. For one, we use algebraic data types[7] to define term-based encodings of all structured constructs involved in representing knowledge bases, such as concept terms, axioms, formulas, etc., where the base types "standpoint name", "role name", "concept name", and "individual name" are subtypes of the built-in type `symbol` (i.e., string). More importantly, during normalisation we employ SOUFFLÉ's built-in functor `cat`[8] for concatenating strings to create unique identifiers for newly introduced standpoint, concept, role, and individual names.

5 Conclusion and Future Work

In this paper, we presented the knowledge representation formalism Standpoint $\mathcal{EL}+$, which extends the formerly proposed Standpoint $\mathcal{EL}$ language (Gómez Álvarez, Rudolph, and Strass 2023a) by a row of new modelling features: role chain axioms and self-loops, extended sharpening statements including standpoint disjointness, negated axioms, and modalised axiom sets. We designed a deduction calculus that is sound and refutation-complete when applied to appropriately pre-processed $\mathbb{S}_{\mathcal{EL}+}$ knowledge bases. As both the preprocessing of $\mathcal{K}$ and the exhaustive application of the

deduction rules are shown to run in PTIME, we thereby established the tractability of satisfiability checking of $\mathbb{S}_{\mathcal{EL}+}$ knowledge bases and – by virtue of a PTIME Turing reduction – also the tractability of checking the entailment of $\mathbb{S}_{\mathcal{EL}+}$ statements from $\mathbb{S}_{\mathcal{EL}+}$ knowledge bases, notably also allowing negated statements.

We note that, if tractability is to be preserved, the options of further extending the expressivity of $\mathbb{S}_{\mathcal{EL}+}$ are limited. Clearly, any modelling feature that would turn the description logic $\mathcal{EL}$ intractable – atomic negation, disjunction, cardinality restrictions, universal quantification as well as inverse or functional roles (Baader, Brandt, and Lutz 2005) – would also destroy tractability of $\mathbb{S}_{\mathcal{EL}+}$. But also the free use of nominal concepts, which is known to still warrant PTIME reasoning when added to $\mathcal{EL}$ with role chain axioms and self-loops, has been shown to be computationally detrimental as soon as standpoints are involved. The same holds if one allows for the declaration of roles to be rigid or for a more liberal semantics that would admit empty standpoints (Gómez Álvarez, Rudolph, and Strass 2023a). On the other hand, *range restrictions* are a modelling feature that is part of the OWL 2 EL profile (Motik et al. 2009) and needs special treatment in lightweight DLs (Konev et al. 2012);[9] we expect to be able to accommodate them in $\mathbb{S}_{\mathcal{EL}+}$ at no cost.

Beyond the theoretical advancement, we also believe that the developed deduction calculus can pave the way to practical reasoner implementations, either by extending existing reasoners or by means of Datalog materialisation, a method already proven to be competitive for reasoning in lightweight description logics. In order to demonstrate the principled feasibility of the latter approach, we implemented a prototype in SOUFFLÉ and made it publicly available.

There are numerous avenues for future work. While the calculus is adequate to show our theoretical results and demonstrate feasibility, we are confident that there is much room for improvement when it comes to optimisation. We expect that refactoring the set of deduction rules can significantly improve the performance of its implementations. In Datalog terms, it would be beneficial to reduce the number and arity of the predicates involved, the number of variables per rule, and the number of alternative derivations of the same fact. These goals may be in conflict and it is typically not straightforward to find the optimal sweet spot. In this regard, realistic benchmarks can provide guidance, and, while no off-the-shelf standpoint ontologies exist yet, we expect that sensible test cases can be effectively generated from linked open data, ontology alignment settings, or ontology repositories with versioning (Konev et al. 2012).

Likewise, the calculus can be analysed and improved in terms of more comprehensive completeness guarantees; in fact, we conjecture that it already yields all entailed "boxed" assertions and concept inclusions over concept names. More generally, we will investigate standpoint extensions of other light- or heavyweight ontology languages regarding computational properties and efficient reasoning.

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[7]https://souffle-lang.github.io/types#algebraic-data-types-adt

[8]https://souffle-lang.github.io/arguments#intrinsic-functor

[9]In DLs with value restriction, a range restriction $\mathsf{ran}(R) \sqsubseteq C$ "comes for free" via $\top \sqsubseteq \forall R.C$.

Acknowledgements

References

Baader, F., and Ohlbach, H. J. 1995. A multi-dimensional terminological knowledge representation language. *Journal of Applied Non-Classical Logics* 5(2):153–197.

Baader, F.; Horrocks, I.; Lutz, C.; and Sattler, U. 2017. *An Introduction to Description Logic*. Cambridge University Press.

Baader, F.; Brandt, S.; and Lutz, C. 2005. Pushing the EL envelope. In Kaelbling, L. P., and Saffiotti, A., eds., *Proceedings of the 19th International Joint Conference on Artificial Intelligence*, 364–369. Professional Book Center.

Bao, J.; Calvanese, D.; Grau, B. C.; Dzbor, M.; Fokoue, A.; Golbreich, C.; Hawke, S.; Herman, I.; Hoekstra, R.; Horrocks, I.; Kendall, E.; Krötzsch, M.; Lutz, C.; McGuinness, D. L.; Motik, B.; Pan, J.; Parsia, B.; Patel-Schneider, P. F.; Rudolph, S.; Ruttenberg, A.; Sattler, U.; Schneider, M.; Smith, M.; Wallace, E.; Wu, Z.; and Zimmermann, A. 2009. OWL 2 Web Ontology Language: Document Overview. W3C Recommendation. http://www.w3.org/TR/owl2-overview/. Accessed: 2023-01-01.

Borgida, A., and Serafini, L. 2003. Distributed Description Logics: Assimilating information from peer sources. *Journal on Data Semantics* 2800:153–184.

Bouquet, P.; Giunchiglia, F.; van Harmelen, F.; Serafini, L.; and Stuckenschmidt, H. 2003. C-OWL: contextualizing ontologies. In Fensel, D.; Sycara, K. P.; and Mylopoulos, J., eds., *Proceedings of the 2nd International Semantic Web Conference*, volume 2870 of *LNCS*, 164–179. Springer.

Donnelly, K. 2006. SNOMED-CT: The advanced terminology and coding system for eHealth. *Studies in health technology and informatics* 121:279.

Gómez Álvarez, L., and Rudolph, S. 2021. Standpoint logic: Multi-perspective knowledge representation. In Neuhaus, F., and Brodaric, B., eds., *Proceedings of the 12th International Conference on Formal Ontology in Information Systems*, volume 344 of *FAIA*, 3–17. IOS Press.

Gómez Álvarez, L.; Rudolph, S.; and Strass, H. 2022. How to Agree to Disagree: Managing Ontological Perspectives using Standpoint Logic. In Sattler, U.; Hogan, A.; Keet, C. M.; Presutti, V.; Almeida, J. P. A.; Takeda, H.; Monnin, P.; Pirrò, G.; and d'Amato, C., eds., *Proceedings of the 21st International Semantic Web Conference*, 125–141. Springer.

Gómez Álvarez, L.; Rudolph, S.; and Strass, H. 2023a. Tractable Diversity: Scalable Multiperspective Ontology Management via Standpoint EL. In Elkind, E.; Agmon, N.;

An, B.; and Das, S., eds., *Proceedings of the 32nd Joint Conference on Artificial Intelligence*. IJCAI Inc. In press.

Gómez Álvarez, L.; Rudolph, S.; and Strass, H. 2023b. Pushing the Boundaries of Tractable Multiperspective Reasoning: A Deduction Calculus for Standpoint $\mathcal{EL}+$. *arXiv.org*. https://arxiv.org/abs/2304.14323.

Hemam, M., and Boufaïda, Z. 2011. MVP-OWL: A multi-viewpoints ontology language for the Semantic Web. *International Journal of Reasoning-based Intelligent Systems* 3(3-4):147–155.

Hemam, M. 2018. An extension of the ontology web language with multi-viewpoints and probabilistic reasoning. *International Journal of Advanced Intelligence Paradigms* 10(3):247–265.

Jordan, H.; Scholz, B.; and Subotić, P. 2016. Soufflé: On Synthesis of Program Analyzers. In Chaudhuri, S., and Farzan, A., eds., *Proceedings of the 28th International Conference on Computer Aided Verification*, volume 9780 of *LNCS*, 422–430. Springer.

Kazakov, Y.; Krötzsch, M.; and Simancik, F. 2014. The incredible ELK - from polynomial procedures to efficient reasoning with $\mathcal{EL}$ ontologies. *J. Autom. Reason.* 53(1):1–61.

Klarman, S., and Gutiérrez-Basulto, V. 2013. Description logics of context. *Journal of Logic and Computation* 26(3):817–854.

Konev, B.; Ludwig, M.; Walther, D.; and Wolter, F. 2012. The logical difference for the lightweight description logic EL. *J. Artif. Intell. Res.* 44:633–708.

Krötzsch, M. 2010. Efficient inferencing for OWL EL. In Janhunen, T., and Niemelä, I., eds., *Proceedings of the 12th European Conference on Logics in Artificial Intelligence*, volume 6341 of *LNCS*, 234–246. Springer.

Lutz, C.; Sturm, H.; Wolter, F.; and Zakharyaschev, M. 2002. A tableau decision algorithm for modalized ALC with constant domains. *Studia Logica* 72(2):199–232.

McCarthy, J., and Buvac, S. 1998. Formalizing context (expanded notes). *CSLI Lecture Notes* 81:13–50.

Mosurović, M. 1999. *On the complexity of description logics with modal operators*. PhD thesis, University of Belgrade. In Serbian.

Motik, B.; Cuenca Grau, B.; Horrocks, I.; Wu, Z.; Fokoue, A.; and Lutz, C. 2009. OWL 2 Web Ontology Language: Profiles. W3C Recommendation. http://www.w3.org/TR/owl2-profiles/. Accessed: 2023-01-01.

Motik, B.; Patel-Schneider, P. F.; and Cuenca Grau, B. 2009. OWL 2 Web Ontology Language: Direct Semantics. W3C Recommendation. http://www.w3.org/TR/owl2-direct-semantics/. Accessed: 2023-01-01.

Osman, I.; Ben Yahia, S.; and Diallo, G. 2021. Ontology Integration: Approaches and Challenging Issues. *Information Fusion* 71:38–63.

Rudolph, S. 2011. Foundations of description logics. In Polleres, A.; d'Amato, C.; Arenas, M.; Handschuh, S.; Kroner, P.; Ossowski, S.; and Patel-Schneider, P. F., eds.,

Lecture Notes of the 7th International Reasoning Web Summer School, volume 6848 of *LNCS*, 76–136. Springer.

Serafini, L., and Homola, M. 2012. Contextualized knowledge repositories for the semantic web. *Journal of Web Semantics* 12-13:64–87.

Wolter, F., and Zakharyaschev, M. 1999. Multi-dimensional description logics. In Dean, T., ed., *Proceedings of the 16th International Joint Conference on Artificial Intelligence*, volume 1, 104–109. Morgan Kaufmann Publishers Inc.