Proceedings of the 20th International Conference on Principles of Knowledge Representation and Reasoning
Main Track

A Family of Decidable Bi-intuitionistic Modal Logics

David Fernández-Duque[1] , **Brett McLean**[2] , **Lukas Zenger**[3]
[1]Department of Philosophy, University of Barcelona
[2]Department of Mathematics WE16, Ghent University
[3]Institute of Computer Science, University of Bern
fernandez-duque@ub.edu, brett.mclean@ugent.be, lukas.zenger@unibe.ch

Abstract

We investigate intuitionistic logics extended with both the co-implication connective of Hilbert–Brouwer logic and with diamond and box modalities. We use a Kripke semantics based on frames with two 'forth' confluence conditions on the modal relation with respect to the intuitionistic relation. We give sound and strongly complete axiomatisations for entailment on this class of frames, and give similar axiomatisations for the subclasses of frames satisfying any combination of reflexivity, transitivity, and seriality. We then prove that all of these logics are decidable, by proving that they have the finite frame property.

1 Introduction

Intuitionistic logic (Gödel 1932) and intermediate logics (Gabbay and Olivetti 2000) have long-established roles in the foundations of constructive reasoning and reasoning with incomplete information. More recently, two distinct ways to extend intuitionistic propositional logic have gained attention: intuitionistic *modal* logics and *bi*-intuitionistic logics.

Bi-intuitionistic logic (Rauszer 1974) adds a new connective, 'co-implication', that is order dual to standard implication, in the same way that disjunction is dual to conjunction. This logic and stronger versions of it have been proposed as a framework for modelling graded, incomplete, and inconsistent information (Bílková, Frittella, and Kozhemiachenko 2022).

The standard semantics for co-implication is based on the well-known Kripke semantics for intuitionistic logic. Intuitionistic logic is interpreted over structures $(W, \leq, V)$, where W is a set of 'worlds', $\leq$ a partial order on W, and V a valuation assigning upward-closed sets of worlds to propositions. With this, we recursively define the relation $w \models \varphi$, where φ is a formula. Conjunction and disjunction are treated in the standard way, e.g. $w \models \varphi \wedge \psi$ if $w \models \varphi$ and $w \models \psi$. The partial order is used in the evaluation of implication: $w \models \varphi \to \psi$ if for all $v \geq w$ we have $v \models \varphi$ implies $v \models \psi$. A defining characteristic of this semantics is that it is *monotone* in the sense that if $w \leq v$ and $w \models \varphi$, then $v \models \varphi$. Monotonicity is crucial for the information interpretation of intuitionistic semantics, where $v \leq w$ indicates that w has more complete information than v, so that ascending within a Kripke model can be regarding as learning.

In bi-intuitionistic logic, a connective $\leftarrow$ is added whose semantics is obtained by dualising the semantics of implication: $w \not\models \varphi \leftarrow \psi$ if for all $v \leq w$ we have $v \not\models \psi$ implies $v \not\models \varphi$, i.e. $w \models \varphi \leftarrow \psi$ if there exists $v \leq w$ such that $v \models \varphi$ but $v \not\models \psi$. This allows the language to 'look downward' from a given world, which is impossible in pure intuitionistic logic. Bi-intuitionistic logic continues to have the monotonicity property, just like its intuitionistic fragment.

Where an implication asserts the consequences of *obtaining* something, a co-implication is an assertion about *losing/relinquishing* something. For example $p \to (e \leftarrow c)$ could mean "If I obtain Polish nationality, then relinquishing my Czech nationality would not entail losing EU citizenship", or "If I get a physical key, then forgetting my code would not mean I lose access to the building".

To instead extend intuitionistic logic to a *modal logic*, one works with models $(W, \leq, R, V)$, where R is used for interpreting modalities $\Diamond$ and $\Box$. However, if one applies the standard classical definitions—e.g. $w \models \Diamond\varphi$ if there exists v such that $w\ R\ v$ and $v \models \varphi$—the resulting semantics no longer has the monotonicity property.[1] This can be remedied by either enforcing that this condition hold for *all* $w' \geq w$, or else requiring that $\leq$ and R 'commute' in some sense.

Different design choices lead to non-equivalent modal extensions of intuitionistic logic, and at least three variants are prominent in the literature. *Intuitionistic modal logics* have been studied by Plotkin and Stirling (1986), Fischer Servi (1977; 1984) and Simpson (1994). Here, the semantics mimic those of intuitionistic first-order logic via the standard translation used in classical modal logic (van Benthem 1976). *Constructive modal logics* have been studied by Fitch (1948) and Wijesekera (1990), and have the characteristic that the addition of excluded middle does not yield classical modal logic K. A third variant, which is also called *intuitionistic modal logic* but has a somewhat different flavour has been studied by Wolter and Zakharyaschev (1997; 1999).

Aside from differences in motivation, these logics vary with respect to their computational properties, particularly when modalities are interpreted via a *transitive* relation. Intuitionistic versions of the modal logic $\mathsf{S4}$ have been extensively

[1]To understand the seriousness of this, consider that since monotonic sets validate intuitionistic logic and non-monotonic sets generally do not, a semantics without monotonicity will not yield a 'logic' with even the most basic property: *closure under substitutions*.

studied. The logic IntS4 of Wolter and Zakharyaschev is decidable and enjoys the finite model property (FMP) (Wolter and Zakharyaschev 1997; Wolter and Zakharyaschev 1999). The constructive modal logic CS4 is also decidable (Alechina et al. 2001) and has the FMP (Balbiani, Diéguez, and Fernández-Duque 2021a). It has only recently been proven that the logic IS4 (from the family of Plotkin, Stirling, Fischer Servi, and Simpson) also has these properties (Girlando et al. 2023). Balbiani, Diéguez, and Fernández-Duque (2021a) also proposed a fourth variant, which they call S4I, based on frames of IS4 where the roles of the intuitionistic and modal relations are reversed.

The logic S4I (and, more generally, logics ΛI, where Λ is some set of modal axioms) are 'pseudo-classical', in the sense that the frame conditions allow modalities to be evaluated as in the classical case, without compromising monotonicity of truth. These conditions are the basis of 'expanding' intuitionistic temporal logic, which has been shown to be decidable (Balbiani et al. 2020) and enjoys applications to topological dynamics (Boudou, Diéguez, and Fernández-Duque 2022; Fernández-Duque 2018).

These developments beg the question of whether bi-intuitionistic modal logics also enjoy the FMP. As in the (mono-)intuitionistic case, this question comes in various flavors. Bi-intuitionistic logics *a la* Wolter and Zakharyaschev have been shown to have the FMP (Sano and Stell 2017; Stell, Schmidt, and Rydeheard 2016) using filtration methods, and have applications in representing spatial relations (Sindoni, Sano, and Stell 2021). However, such results are not yet known for logics in the style of Fitch or Fischer Servi.

Here we consider pseudo-classical, bi-intuitionistic logics and show that many of them are indeed decidable and enjoy the FMP. These are particularly natural given that the frame conditions are order-symmetric: $(W, \leq, R)$ is a pseudo-classical frame if and only if $(W, \geq, R)$ is—a natural property to expect in the presence of co-implication. Unfortunately, filtration does not work in our setting and we instead turn to techniques in the spirit of Balbiani et al. (2021b); however, as we will see, co-implication requires a substantial expansion of these techniques, in particular since we can no longer restrict our attention to tree-like models.

As an example, suppose $\Diamond$ and $\Box$ model temporal 'eventually' and 'henceforth' respectively. Then suppose: a power plant holds a permit to burn *gas*; it has been announced that all such plants will automatically be given a permit to burn *biomass*; plants holding gas or biomass permits are always awarded contracts in the winter *capacity* market. Then $\Diamond(\Box c \leftarrow g)$ is a true assertion that eventually the plant can choose not to renew its gas permit, but then still have winter capacity contracts henceforth.

In this paper, we prove that many standard bi-intuitionistic, pseudo-classical modal logics are decidable and have the finite frame property. Despite being a more accessible problem than the extension of IS4 with co-implication, this is by no means a straightforward result, requiring a rather delicate combinatorial analysis of the semantics. However, our techniques are robust in the sense that a uniform proof of decidability is obtained for the logics of frame classes satisfying any combination of reflexivity, transitivity, and seriality.

Structure of paper In Section 2, we syntactically define six bi-intuitionistic modal logics and give their intended semantics. In Section 3, we note that the logics are sound with respect to the semantics. In Section 4 we prove (strong) completeness, using canonical models. The remainder of the paper is devoted to proving decidability of all six logics. Section 5 gives a standard argument that models are equivalent to labelled structures whose labellings validate certain coherency conditions. Section 6 defines *dynamic simulations* and Section 7 introduces *moments*. In Section 8 we use a dynamic simulation to show that any falsifiable formula is falsifiable on a model built entirely from moments. Finally, Section 9 shows that any moment can be 'compressed' to one of uniformly bounded size, and it is the set of these bounded moments that makes up our desired finite models. We thus obtain a computable finite frame property, and hence decidability of our logics.

2 Syntax and Semantics

In this section we first introduce the propositional modal language shared by the six logics we consider. Then we define the logics syntactically, and finally we give the intended relational semantics.

2.1 Language

Fix a countably infinite set $\mathbb{P}$ of propositional variables. The **bi-intuitionistic modal language** $\mathcal{L}_{\mathsf{bIM}}$ is defined by the grammar (in Backus–Naur form):

$$\varphi \;:=\; p \mid \varphi\wedge\varphi \mid \varphi\vee\varphi \mid \varphi\to\varphi \mid \varphi\leftarrow\varphi \mid \Diamond\varphi \mid \Box\varphi$$

where $p \in \mathbb{P}$. We also use $\bot$ as a shorthand for $p \leftarrow p$ (where p is some designated element of $\mathbb{P}$), $\neg\varphi$ as shorthand for $\varphi \to \bot$, the symbol $\top$ as shorthand for $\neg\bot$, and $\varphi \leftrightarrow \psi$ as shorthand for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. As usual, the unary modalities bind tighter than the binary connectives; we also assume that $\wedge$ and $\vee$ bind tighter than $\to$.

2.2 Deductive Calculi

In this paper a **logic** means a set of formulas closed under *substitution*, *modus ponens*, and *necessitation*: $\varphi/\Box\varphi$.

We define the logics we are interested in syntactically, via Hilbert-style deductive calculi. We now give the details.

Definition 1. We define the following axioms and rules.

$\mathbf{IPC}$ All intuitionistic tautologies
$\mathbf{A}_{\leftarrow}$ $p \to (q \vee (p \leftarrow q))$

$\mathbf{R1}_{\leftarrow}$ $\dfrac{\varphi \to \psi}{(\varphi \leftarrow \theta) \to (\psi \leftarrow \theta)}$ $\mathbf{R2}_{\leftarrow}$ $\dfrac{\varphi \to \psi \vee \gamma}{(\varphi \leftarrow \psi) \to \gamma}$

$\mathbf{K}_{\Box}$ $\Box(p \to q) \to (\Box p \to \Box q)$
$\mathbf{K}_{\Diamond}$ $\Box(p \to q) \to (\Diamond p \to \Diamond q)$

$\mathbf{DP}$ $\Diamond(p \vee q) \to \Diamond p \vee \Diamond q$ $\mathbf{T}_{\Box}$ $\Box p \to p$
$\mathbf{RV}$ $\Box(p \vee q) \to \Box p \vee \Diamond q$ $\mathbf{T}_{\Diamond}$ $p \to \Diamond p$
$\mathbf{N}$ $\neg\Diamond\bot$ $\mathbf{4}_{\Box}$ $\Box p \to \Box\Box p$
$\mathbf{D}$ $\Diamond\top$ $\mathbf{4}_{\Diamond}$ $\Diamond\Diamond p \to \Diamond p$

We define the logics[2]

$$\begin{aligned}
\mathsf{KbI} &:= \mathbf{IPC} + \mathbf{A}_{\leftarrow} + \mathbf{R1}_{\leftarrow} + \mathbf{R2}_{\leftarrow} + \mathbf{K}_{\Box} + \mathbf{K}_{\Diamond} \\
&\quad + \mathbf{DP} + \mathbf{RV} + \mathbf{N} \\
\mathsf{DbI} &:= \mathsf{KbI} + \mathbf{D} \\
\mathsf{TbI} &:= \mathsf{KbI} + \mathbf{T}_{\Box} + \mathbf{T}_{\Diamond} \\
\mathsf{K4bI} &:= \mathsf{KbI} + \mathbf{4}_{\Box} + \mathbf{4}_{\Diamond} \\
\mathsf{K4DbI} &:= \mathsf{K4bI} + \mathbf{D} \\
\mathsf{S4bI} &:= \mathsf{K4bI} + \mathbf{T}_{\Box} + \mathbf{T}_{\Diamond}
\end{aligned}$$

Thus KbI will serve as the 'minimal' logic in this paper, and the remaining logics are extensions.

2.3 Semantics

Before defining pseudo-classical frames, let us introduce the general context of intuitionistic modal frames.

Definition 2. An **intuitionistic frame** is a pair $(W, \leq)$, where W is a set and $\leq$ is a partial order on W.

An **intuitionistic Kripke frame** is a triple $(W, \leq, R)$, where $(W, \leq)$ is an intuitionistic frame and (W, R) is a Kripke frame (a set equipped with a binary relation).

A **valuation** on an intuitionistic Kripke frame $\mathcal{F} = (W, \leq, R)$ is a function $V : \mathbb{P} \to 2^W$ that is *monotone* in the sense that each $V(p)$ is upward closed with respect to $\leq$.

The satisfaction relation $\models$ is defined recursively (temporarily suppressing $\mathcal{F}$ and V in the notation):

- $w \models p$ if $w \in V(p)$ (for $p \in \mathbb{P}$);
- $w \not\models \bot$;
- $w \models \varphi \wedge \psi$ if $w \models \varphi$ and $w \models \psi$;
- $w \models \varphi \vee \psi$ if $w \models \varphi$ or $w \models \psi$;
- $w \models \varphi \to \psi$ if $\forall v \geq w$, $v \models \varphi$ implies $v \models \psi$;
- $w \models \varphi \leftarrow \psi$ if $\exists v \leq w$ such that $v \models \varphi$ and $v \not\models \psi$;
- $w \models \Diamond\varphi$ if $\forall w' \geq w$, $\exists v$ such that $w' \mathrel{R} v$ and $v \models \varphi$;
- $w \models \Box\varphi$ if $\forall w', v$ such that $w \leq w' \mathrel{R} v$, we have $v \models \varphi$.

It is easily proved by induction on φ that for all $w, v \in W$, if $w \leq v$, then $w \models \varphi \implies v \models \varphi$.

An **intuitionistic Kripke model** is an intuitionistic Kripke frame equipped with a valuation.

Definition 3. Let S be any class of models or class of frames. Let $\Gamma \subseteq \mathcal{L}_{\mathsf{bIM}}$ and $\varphi \in \mathcal{L}_{\mathsf{bIM}}$. We write $\Gamma \models_{\mathsf{S}} \varphi$ and say that φ is a **local semantic consequence** of Γ if, for each model $\mathcal{M} = (W, \leq, R, V)$ from S and each $w \in W$, we have

$$\forall \psi \in \Gamma, (\mathcal{M}, w) \models \psi \implies (\mathcal{M}, w) \models \varphi.$$

We say that φ is **valid** on S if $\models_{\mathsf{S}} \varphi$ (that is, $\varnothing \models_{\mathsf{S}} \varphi$), and **falsifiable** otherwise. This terminology extends to single models or frames.

Confluence conditions governing the interaction between the order and modal relations on frames will be of recurring importance.

[2]In the first definition, the meaning of the notation is the evident one. In the subsequent definitions, the defined logic is not only required to *include* the logic on the right-hand-side, but should also be closed under the *rules* of that logic.

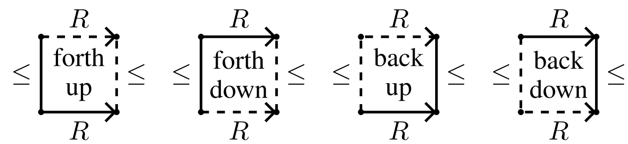

Figure 1: Confluence conditions

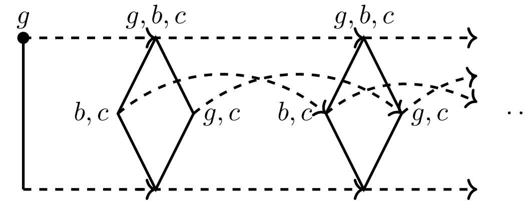

Figure 2: A pseudo-classical model, representing the power plant example from Section 1. $\Diamond(\Box c \leftarrow g)$ holds at the marked world.

Definition 4. Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be posets and $R \subseteq X \times Y$. We say R is **forth–up confluent** (for $(\leq_X, \leq_Y)$) if, whenever $w \leq_X w'$ and $w \mathrel{R} v$, there exists v' such that $w' \mathrel{R} v'$ and $v \leq_Y v'$. Three other confluence conditions are defined similarly as depicted in Figure 1.

We say $\mathcal{F} = (W, \leq, R)$ is forth–up confluent if R is forth–up confluent for $(\leq, \leq)$, and so on.

Having forth–up and forth–down confluence allows us to simplify the semantic clauses for $\Diamond$ and $\Box$, respectively.

Lemma 1 ((Balbiani, Diéguez, and Fernández-Duque 2021a))**.** *Let $\mathcal{M} = (W, \leq, R, V)$ be any intuitionistic Kripke model, $w \in W$, and $\varphi \in \mathcal{L}_{\mathsf{bIM}}$.*

1. *If $\mathcal{M}$ is forth–up confluent, then $(\mathcal{M}, w) \models \Diamond\varphi$ if and only if $\exists v$ such that $w \mathrel{R} v$ and $(\mathcal{M}, v) \models \varphi$.*
2. *If $\mathcal{M}$ is forth–down confluent, then $(\mathcal{M}, w) \models \Box\varphi$ if and only if $\forall v$, if $w \mathrel{R} v$ then $(\mathcal{M}, v) \models \varphi$.*

Definition 5.

1. The class of **pseudo-classical** (or KbI) frames is the class of forth–up and forth–down confluent, intuitionistic Kripke frames.
2. For $\Lambda \in \{\mathsf{DbI}, \mathsf{TbI}, \mathsf{K4bI}, \mathsf{K4DbI}, \mathsf{S4bI}\}$, the class of **$\Lambda$-frames** is to be the class of KbI frames such that
   - if $\mathbf{D}$ is an axiom of Λ, then R is serial;
   - if $\mathbf{T}_{\Box}$ is an axiom of Λ, then R is reflexive;
   - if $\mathbf{4}_{\Box}$ is an axiom of Λ, then R is transitive.

In view of Lemma 1, we can evaluate $\Diamond$ and $\Box$ classically on pseudo-classical frames. Figure 2 depicts an example.

3 Soundness

In this section we note the soundness of the logics KbI, DbI, TbI, K4bI, K4DbI, and S4bI with respect to the validities of the corresponding frame classes. The following is standard (see (Balbiani, Diéguez, and Fernández-Duque 2021a) and (Simpson 1994)).

Proposition 1.

(1) On any class of intuitionistic Kripke frames, substitution, modus ponens, necessitation, $\mathbf{R1}_{\leftarrow}$, and $\mathbf{R2}_{\leftarrow}$ each preserves validity, and $\mathbf{IPC}$, $\mathbf{A}_{\leftarrow}$, $\mathbf{K}_{\Box}$, and $\mathbf{K}_{\Diamond}$ are valid.
(2) Axioms $\mathbf{N}$, $\mathbf{DP}$, and $\mathbf{RV}$ are valid on the class of KbI *frames.*
(3) Axiom $\mathbf{D}$ is valid on the class of serial frames.
(4) Axioms $\mathbf{T}_{\Box}$ and $\mathbf{T}_{\Diamond}$ are valid on the class of reflexive frames.
(5) Axiom $\mathbf{4}_{\Diamond}$ is valid on the class of transitive frames.
(6) Axiom $\mathbf{4}_{\Box}$ is valid on any transitive frame that is forth–down confluent.

It follows that each of the logics we consider is sound for its respective class from Definition 5.

Corollary 1. *Each of* KbI, DbI, TbI, $\mathsf{K4bI}$, $\mathsf{K4DbI}$, *and* $\mathsf{S4bI}$ *is sound for its respective class of frames.*

4 Strong Completeness

In this section, we prove the strong completeness of KbI, DbI, TbI, $\mathsf{K4bI}$, $\mathsf{K4DbI}$, and $\mathsf{S4bI}$ with respect to their corresponding semantics, using canonical models.

Fix a logic Λ. We use the standard Gentzen-style notation $\Gamma \vdash \Delta$ to mean $\bigwedge \Gamma' \to \bigvee \Delta' \in \Lambda$ for some finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ (with the convention that $\bigvee \varnothing \equiv \bot$). We call $\vdash$ the **syntactic consequence** relation. The logic Λ will always be clear from context, which is why we do not reflect it in the notation. When working with $\vdash$, we follow the usual proof-theoretic convention of writing φ instead of $\{\varphi\}$.

The logic is **strongly complete** with respect to a class S of frames if for all $\Gamma \subseteq \mathcal{L}_{\mathsf{bIM}}$ and $\varphi \in \mathcal{L}_{\mathsf{bIM}}$:

$$\Gamma \models_{\mathsf{S}} \varphi \implies \Gamma \vdash \varphi.$$

4.1 Prime Theories

Definition 6. Given $\Phi, \Xi \subseteq \mathcal{L}_{\mathsf{bIM}}$, we say Φ is **Ξ-consistent** if $\Phi \nvdash \Xi$. We say Φ is **consistent** if it is $\varnothing$-consistent.

Note that if Φ is Ξ-consistent, then necessarily $\bot \notin \Phi$.

Definition 7. We say that $\Gamma \subseteq \mathcal{L}_{\mathsf{bIM}}$ is a **theory** if it is closed under syntactic consequence ($\Gamma \vdash \varphi$ implies $\varphi \in \Gamma$), and **prime** if whenever $\varphi \vee \psi \in \Gamma$, either $\varphi \in \Gamma$ or $\psi \in \Gamma$.

We say that $\Psi \subseteq \mathcal{L}_{\mathsf{bIM}}$ **extends** Φ if $\Phi \subseteq \Psi$.

Lemma 2 (Lindenbaum lemma)**.** *Any Ξ-consistent Φ can be extended to a Ξ-consistent prime theory Φ_*.*

Proof. Obtain a maximal Ξ-consistent $\Phi_* \supseteq \Phi$ by Zorn's lemma. Then prove by contradiction that Φ_* is prime, using the fact that left disjunction introduction $(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$ is derivable in intuitionistic logic. □

The proof of the following saturation lemma is standard (Alechina et al. 2001; Aguilera et al. 2022).

Lemma 3. *For each prime theory Φ:*

(1) $\varphi \wedge \psi \in \Phi$ if and only if $\varphi \in \Phi$ and $\psi \in \Phi$,
(2) $\varphi \vee \psi \in \Phi$ if and only if $\varphi \in \Phi$ or $\psi \in \Phi$,
(3) if $\varphi \to \psi \in \Phi$, then $\varphi \in \Phi \implies \psi \in \Phi$,
(4) if $\varphi \leftarrow \psi \notin \Phi$, then $\psi \notin \Phi \implies \varphi \notin \Phi$,
(5) if $\mathbf{T}_{\Diamond} \in \Lambda$ and $\varphi \in \Phi$, then $\Diamond\varphi \in \Phi$.
(6) if $\mathbf{T}_{\Box} \in \Lambda$ and $\Box\varphi \in \Phi$, then $\varphi \in \Phi$.

4.2 Canonical Models

In this subsection we show that the logics we consider are strongly complete, using standard canonical model arguments. We can uniformly define the canonical model for any logic Λ including KbI and closed under $\mathbf{R1}_{\leftarrow}$ and $\mathbf{R2}_{\leftarrow}$.

Definition 8. Let $\mathsf{KbI} \subseteq \Lambda \subseteq \mathcal{L}_{\mathsf{bIM}}$ be a logic closed under $\mathbf{R1}_{\leftarrow}$ and $\mathbf{R2}_{\leftarrow}$. We define the canonical model for Λ as $\mathcal{M}^{\Lambda}_{\mathrm{c}} = (W_{\mathrm{c}}, \leq_{\mathrm{c}}, R_{\mathrm{c}}, V_{\mathrm{c}})$, where

a) W_{c} is the set of consistent prime Λ-theories;
b) $\leq_{\mathrm{c}} \subseteq W_{\mathrm{c}} \times W_{\mathrm{c}}$ is $\subseteq$;
c) $R_{\mathrm{c}} \subseteq W_{\mathrm{c}} \times W_{\mathrm{c}}$ is defined by $\Phi \ R_{\mathrm{c}} \ \Psi$ if and only if $\{\varphi \in \mathcal{L}_{\mathsf{bIM}} \mid \Box\varphi \in \Phi\} \subseteq \Psi$ and $\{\varphi \in \mathcal{L}_{\mathsf{bIM}} \mid \Diamond\varphi \notin \Phi\} \cap \Psi = \varnothing$;
d) $V_{\mathrm{c}} \colon \mathbb{P} \to 2^{W_{\mathrm{c}}}$ is defined by $V_{\mathrm{c}}(p) \coloneqq \{\Phi \in W_{\mathrm{c}} \mid p \in \Phi\}$.

Each item of the following lemma is proven either by (Simpson 1994) or by (Aguilera et al. 2022).

Lemma 4 (witnessing lemma)**.** *Let* $\mathsf{KbI} \subseteq \Lambda \subseteq \mathcal{L}_{\mathsf{bIM}}$ *be a logic closed under $\mathbf{R1}_{\leftarrow}$ and $\mathbf{R2}_{\leftarrow}$. For any $\Phi \in W_{\mathrm{c}}$ and $\varphi, \psi \in \mathcal{L}_{\mathsf{bIM}}$:*

1. *$\varphi \to \psi \in \Phi$ if and only if, whenever $\Phi \leq_{\mathrm{c}} \Psi$ we have $\varphi \in \Psi \implies \psi \in \Psi$.*
2. *$\varphi \leftarrow \psi \in \Phi$ if and only if $\exists\Psi$ such that $\Psi \leq_{\mathrm{c}} \Phi$, $\varphi \in \Psi$, and $\psi \notin \Psi$.*
3. *$\Diamond\varphi \in \Phi$ if and only if $\exists\Psi$ such that $\Phi \ R_{\mathrm{c}} \ \Psi$ and $\varphi \in \Psi$.*
4. *$\Box\varphi \in \Phi$ if and only if, whenever $\Phi \ R_{\mathrm{c}} \ \Psi$ we have $\varphi \in \Psi$.*

Using Lemma 4 we can show that the canonical model is indeed a model for each of the logics we consider.

Lemma 5. *If* $\Lambda \in \{\mathsf{KbI}, \mathsf{DbI}, \mathsf{TbI}, \mathsf{K4bI}, \mathsf{K4DbI}, \mathsf{S4bI}\}$, *then $\mathcal{M}^{\Lambda}_{\mathrm{c}}$ is a model based on a Λ-frame.*

Proof. That $\mathcal{M}^{\Lambda}_{\mathrm{c}}$ is forth–up and forth–down confluent is proven in (Balbiani, Diéguez, and Fernández-Duque 2021a) for $\mathsf{S4I}$, and the proof works uniformly for all logics extending KI. They show further that the axioms $\mathbf{T}_{\Box}$ and $\mathbf{T}_{\Diamond}$ lead to reflexivity of R_{c}, and $\mathbf{4}_{\Diamond}$ and $\mathbf{4}_{\Box}$ lead to transitivity of R_{c}. For logics with the axiom $\mathbf{D}$, if $\Phi \in W_{\mathrm{c}}$ then $\Diamond\top \in \Phi$, so Lemma 4.3 yields Ψ such that $\Phi \ R_{\mathrm{c}} \ \Psi$ (and $\top \in \Psi$). □

The last ingredient in our proof is a standard truth lemma, which readily follows from Lemma 4 and induction on φ.

Lemma 6 (truth lemma)**.** *Let* $\Lambda \in \{\mathsf{KbI}, \mathsf{DbI}, \mathsf{TbI}, \mathsf{K4bI}, \mathsf{K4DbI}, \mathsf{S4bI}\}$. *For any $\Phi \in W_{\mathrm{c}}$ and $\varphi \in \mathcal{L}_{\mathsf{bIM}}$,*

$$\varphi \in \Phi \iff (\mathcal{M}^{\Lambda}_{\mathrm{c}}, \Phi) \models \varphi.$$

From this, we obtain strong completeness for all of the logics we consider.

Theorem 1. *Let* $\Lambda \in \{\mathsf{KbI}, \mathsf{DbI}, \mathsf{TbI}, \mathsf{K4bI}, \mathsf{K4DbI}, \mathsf{S4bI}\}$. *Let $\models_{\Lambda}$ denote semantic consequence on the class of Λ frames as given by Definition 5, and $\vdash_{\Lambda}$ denote the syntactic consequence relation for Λ. Then for any set of formulas $\Gamma \cup \{\varphi\}$,*

$$\Gamma \models_{\Lambda} \varphi \iff \Gamma \vdash_{\Lambda} \varphi.$$

5 Labelled Posets and Labelled Frames

The remainder of the paper is devoted to proving that our logics have the finite frame property, which yields decidability. Our constructions are based on labelled structures, which are essentially partially evaluated models and particularly amenable to a combinatorial analysis.

Definition 9. Let $\Sigma \subseteq \mathcal{L}_{\mathsf{bIM}}$ be closed under subformulas. A **(two-sided)** Σ**-type** is a pair $\Phi = (\Phi^+, \Phi^-)$ of disjoint subsets of Σ with the following properties:

1. $\bot \notin \Phi^+$.
2. If $\varphi \wedge \psi \in \Phi^+$, then $\varphi, \psi \in \Phi^+$.
3. If $\varphi \wedge \psi \in \Phi^-$, then $\varphi \in \Phi^-$ or $\psi \in \Phi^-$.
4. If $\varphi \vee \psi \in \Phi^+$, then $\varphi \in \Phi^+$ or $\psi \in \Phi^+$.
5. If $\varphi \vee \psi \in \Phi^-$, then $\varphi, \psi \in \Phi^-$.
6. If $\varphi \to \psi \in \Phi^+$, then $\varphi \in \Phi^-$ or $\psi \in \Phi^+$.
7. If $\varphi \leftarrow \psi \in \Phi^-$, then $\varphi \in \Phi^-$ or $\psi \in \Phi^+$.

We emphasise that it is not necessary that $\Phi^+ \cup \Phi^- = \Sigma$; in this sense our types are *partial*. The set of all Σ-types is denoted by $\mathbb{T}_\Sigma$.

We define two partial orders on $\mathbb{T}_\Sigma$:

1. $\Phi \leq_{\mathbb{T}} \Psi$ if and only if $\Phi^+ \subseteq \Psi^+$ and $\Psi^- \subseteq \Phi^-$,
2. $\Phi \subseteq_{\mathbb{T}} \Psi$ if and only if $\Phi^+ \subseteq \Psi^+$ and $\Phi^- \subseteq \Psi^-$,

and we define $\Phi\upharpoonright_\Delta = (\Phi^+ \cap \Delta, \Phi^- \cap \Delta)$.

Definition 10. Let $\Sigma \subseteq \mathcal{L}_{\mathsf{bIM}}$ be closed under subformulas. A Σ**-labelled poset** is a tuple $\mathcal{X} = (X, \leq_\mathcal{X}, \ell_\mathcal{X})$ where:

- $(X, \leq_\mathcal{X})$ is a poset.
- $\ell_\mathcal{X} \colon X \to \mathbb{T}_\Sigma$ such that:
  - For all $x, y \in X$ if $x \leq_\mathcal{X} y$, then $\ell_\mathcal{X}(x) \leq_{\mathbb{T}} \ell_\mathcal{X}(y)$.
  - If $\varphi \to \psi \in \ell_\mathcal{X}(x)^-$, then there exists $y \geq_\mathcal{X} x$ with $\varphi \in \ell_\mathcal{X}(y)^+$ and $\psi \in \ell_\mathcal{X}(y)^-$.
  - If $\varphi \leftarrow \psi \in \ell_\mathcal{X}(x)^+$, then there exists $y \leq_\mathcal{X} x$ with $\varphi \in \ell_\mathcal{X}(y)^+$ and $\psi \in \ell_\mathcal{X}(y)^-$.

If the structure $\mathcal{X}$ is clear, we may drop $\mathcal{X}$ as subscript.

Next we define conditions that will allow us to interpret the modalities on labelled posets. For example, if $x\ R\ y$ and $\Box\varphi \in \ell(x)$, we will want $\varphi \in \ell(y)$. However, for transitive logics, it is *also* convenient to have $\Box\varphi \in \ell(y)$. In order to accommodate the possible variations that may be needed, we consider 'sensibility conditions' that the pair $(\ell(x), \ell(y))$ must satisfy in order to relate them via R.

Definition 11. A binary relation $S \subseteq \mathbb{T}_\Sigma \times \mathbb{T}_\Sigma$ is a **sensibility condition** if whenever $\Phi\ S\ \Psi$ and Δ is any set of formulas closed under subformulas then $\Phi\upharpoonright_\Delta\ S\ \Psi\upharpoonright_\Delta$ and, moreover, if $\Phi\ S\ \Psi$ and $\Psi \subseteq_{\mathbb{T}} \Psi'$, then $\Phi\ S\ \Psi'$.

We define the **standard condition** by setting $\Phi\ S_{\rm st}\ \Psi$ if whenever $\Diamond\varphi \in \Phi^-$, it follows that $\varphi \in \Psi^-$, and whenever $\Box\varphi \in \Phi^+$, it follows that $\varphi \in \Psi^+$. The **transitive condition** is defined by setting $\Phi\ S_{\rm tr}\ \Psi$ if whenever $\Diamond\varphi \in \Phi^-$, it follows that $\varphi, \Diamond\varphi \in \Psi^-$, and whenever $\Box\varphi \in \Phi^+$, it follows that $\varphi, \Box\varphi \in \Psi^+$.

Definition 12. Fix a sensibility condition S. Let $\Sigma \subseteq \mathcal{L}_{\mathsf{bIM}}$ be subformula-closed, and let $\mathcal{X} = (X, \leq_\mathcal{X}, \ell_\mathcal{X})$ and $\mathcal{Y} = (Y, \leq_\mathcal{Y}, \ell_\mathcal{Y})$ be Σ-labelled posets. A relation $R \subseteq X \times Y$ is **sensible** if it is both forth–up and forth–down confluent and validates $w\ R\ v \implies \ell_\mathcal{X}(w)\ S\ \ell_\mathcal{Y}(v)$.

Definition 13. Fix $\Sigma \subseteq \mathcal{L}_{\mathsf{bIM}}$ and a sensibility condition S. A Σ**-labelled frame with respect to** S is a Σ-labelled poset $\mathcal{X} = (X, \leq_\mathcal{X}, \ell_\mathcal{X})$ equipped with a sensible relation $R_\mathcal{X} \subseteq X \times X$.

When the sensibility condition is not relevant to the discussion we may omit mention of S and write simply Σ**-labelled frame**. Observe that models can be regarded as Σ-labelled frames by labelling worlds with the sets of formulas they satisfy/falsify. The converse is not true in general;[3] it requires an additional condition on our labelled frames.

Definition 14. Let $\mathcal{X} = (X, \leq_\mathcal{X}, \ell_\mathcal{X}, R_\mathcal{X})$ be a Σ-labelled frame. We say that $R_\mathcal{X}$ is **witnessed** if

- Whenever $\Diamond\varphi \in \ell_\mathcal{X}(w)^+$, there is v such that $w\ R_\mathcal{X}\ v$ and $\varphi \in \ell_\mathcal{X}(v)^+$.
- Whenever $\Box\varphi \in \ell_\mathcal{X}(w)^-$, there is v such that $w\ R_\mathcal{X}\ v$ and $\varphi \in \ell_\mathcal{X}(v)^-$.

If $R_\mathcal{X}$ is witnessed, we say that $\mathcal{X}$ is a Σ**-labelled model**.

A Σ-labelled model $\mathcal{X}$ *can* reasonably be regarded as an intuitionistic Kripke model by setting $V(p) = \{x \in X \mid p \in \ell_\mathcal{X}(x)^+\}$.[4] Then by structural induction on formulas:

$$\varphi \in \ell_\mathcal{X}(x)^+ \implies x \models \varphi \text{ and } \varphi \in \ell_\mathcal{X}(x)^- \implies x \not\models \varphi.$$

The following lemma is now immediate by regarding models as Σ-labelled models or vice-versa. Below, a formula φ is **falsified** on a Σ-labelled model $\mathcal{X}$ if there is v with $\varphi \in \ell_\mathcal{X}(v)^-$. The formula φ is **valid** over a class M of Σ-labelled models if it is not falsified on any element of M.

Lemma 7.

1. *If* $\Lambda \in \{\mathsf{Kbl}, \mathsf{Dbl}, \mathsf{Tbl}\}$*, then a formula* φ *is valid over the class of* Λ*-frames if and only if it is valid over the class* Σ*-labelled models with respect to the standard condition* $S_{\rm st}$ *based on a* Λ*-frame.*
2. *If* $\Lambda \in \{\mathsf{K4bl}, \mathsf{K4Dbl}, \mathsf{S4bl}\}$*, then a formula* φ *is valid over the class of* Λ*-frames if and only if it is valid over the class* Σ*-labelled models with respect to the transitive condition* $S_{\rm tr}$ *based on a* Λ*-frame.*

Thus our strategy for proving decidability will be to construct (for finite Σ) a finite Σ-labelled model from an arbitrary Σ-labelled model.

6 Simulations

It is crucial for our proof to identify the correct notion of 'embedding' in the setting of labelled models. This is given by *dynamic simulations*. We first define the component notion of *simulation*.

[3]Specifically, if a Σ-labelled frame is regarded as a model, the truth lemma may fail: for example, $\Diamond p \in \ell(w)^+$ does not imply $w \models \Diamond p$, since no witness may be available.

[4]Other valuations compatible with the labelling are possible, the maximal such valuation being $V(p) = \{x \in X \mid p \notin \ell_\mathcal{X}(x)^-\}$.

Definition 15. Let $\Sigma \subseteq \mathcal{L}_{\mathsf{bIM}}$ be subformula-closed and let $\mathcal{X} = (X, \leq_{\mathcal{X}}, \ell_{\mathcal{X}})$ and $\mathcal{Y} = (Y, \leq_{\mathcal{Y}}, \ell_{\mathcal{Y}})$ be Σ-labelled posets. A binary relation $E \subseteq X \times Y$ is a **simulation** if:

1. Whenever $x \mathrel{E} y$, we have $\ell_{\mathcal{X}}(x) \subseteq_{\mathbb{T}} \ell_{\mathcal{Y}}(y)$.
2. E is forth–up and forth–down confluent for $\leq_{\mathcal{X}}$ and $\leq_{\mathcal{Y}}$.

Lemma 8. *Unions and compositions of simulations are simulations.*

Suppose $\mathcal{M}, \mathcal{N}$ are labelled frames, $w \in M$ and $v \in N$, and there is a simulation $E \subseteq M \times N$ with $w \mathrel{E} v$. Then in general $\mathcal{M}$ can be much smaller than $\mathcal{N}$, and thus simulations help us to 'compress' models. However, it may be that $\mathcal{N}$ is a labelled model, but $\mathcal{M}$ is only a labelled frame. In order to avoid this situation, we work with dynamic simulations.

Definition 16. Let $\Sigma \subseteq \mathcal{L}_{\mathsf{bIM}}$ be subformula-closed, and let $\mathcal{X} = (X, \leq_{\mathcal{X}}, R_{\mathcal{X}}, \ell_{\mathcal{X}})$ and $\mathcal{Y} = (Y, \leq_{\mathcal{Y}}, R_{\mathcal{Y}}, \ell_{\mathcal{Y}})$ be Σ-labelled frames. A simulation $E \subseteq X \times Y$ is a **dynamic simulation** if whenever $x \mathrel{E} y \mathrel{R_{\mathcal{Y}}} y'$, there is x' such that $x \mathrel{R_{\mathcal{X}}} x' \mathrel{E} y'$.

Below, for $Z \subseteq X$, the notation $\mathcal{X}{\upharpoonright_Z}$ denotes the substructure obtained by restricting $\mathcal{X}$ to Z, i.e. $\mathcal{X}{\upharpoonright_Z} = (Z, \leq_{\mathcal{X}} \cap Z^2, R_{\mathcal{X}} \cap Z^2, \ell_{\mathcal{X}} \cap (Z \times \mathbb{T}_{\Sigma}))$.

Theorem 2. *If $\Sigma \subseteq \mathcal{L}_{\mathsf{bIM}}$ is subformula-closed, $\mathcal{X} = (X, \leq_{\mathcal{X}}, R_{\mathcal{X}}, \ell_{\mathcal{X}})$ is a Σ-labelled frame, $\mathcal{Y} = (Y, \leq_{\mathcal{Y}}, R_{\mathcal{Y}}, \ell_{\mathcal{Y}})$ is a Σ-labelled model, and $E \subseteq X \times Y$ is a dynamic simulation, then $\mathcal{X}{\upharpoonright_{E^{-1}(Y)}}$ is a Σ-labelled model.*

Proof. Proven in (Fernández-Duque 2018). The key is to observe that E being dynamic implies that $R_{\mathcal{X}}{\upharpoonright_{E^{-1}(Y)}}$ is witnessed. ☐

7 Moments

Let $(P, \leq)$ be a poset. Let $x, y \in P$. Recall that y **covers** x means $x < y$ and there is no z with $x < z < y$. We say that x and y are **neighbours** if either y covers x or x covers y. The poset $(P, \leq)$ is **discrete** if whenever $x \leq y$, there are finitely many $x = x_0 < x_1 < \ldots < x_n = y$ where x_{i+1} covers x_i, for each i.

Definition 17. Let $(P, \leq)$ be a poset. A **path** through $(P, \leq)$ is a finite sequence $(x_i)_{0 \leq i \leq n}$ of elements of P such that for all $0 \leq i < n$ either x_{i+1} covers x_i or vice versa. The **length** of a path $(x_i)_{0 \leq i \leq n}$ is n.

Given two paths ρ_1 and ρ_2, denote by $\rho_1 \sqsubseteq \rho_2$ that ρ_1 is an initial segment of ρ_2. When $\rho_1 \sqsubseteq \rho_2$, we denote by $\rho_2 - \rho_1$ the final segment of ρ_2 disjoint from ρ_1. Furthermore, given a path $\rho = (x_i)_{i \leq n}$, we write $\uparrow(\rho)$ if $x_i < x_{i+1}$ for all i. The notation $\downarrow(\rho)$ is defined similarly.

We define the **zigzag width** hierarchy on paths recursively: a length 0 path is both Π_0 and Σ_0. A path is Π_1 if it is decreasing and it is Σ_1 if it is increasing. A path ρ is Π_{m+1} if there exists a decreasing path τ such that $\tau \sqsubseteq \rho$ and $\rho - \tau$ is Σ_m. A path ρ is Σ_{m+1} if there exists an increasing path τ such that $\tau \sqsubseteq \rho$ and $\rho - \tau$ is Π_m.

We call a path $(x_i)_{i \leq n}$ **acyclic** if all its elements are distinct. A discrete poset $(P, \leq)$ is called **acyclic** if the undirected graph induced by the *neighbours* relation is acyclic. A discrete poset $(P, \leq)$ has **zigzag width** m if all acyclic paths through P are both Π_m and Σ_m.

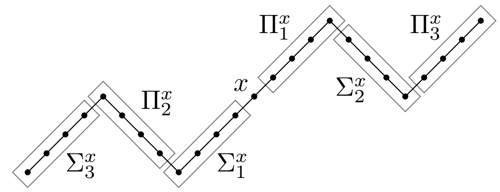

Figure 3: Zigzag hierarchy relative to x

Given $x, y \in P$ we say y is Π_m^x if the acyclic path from y to x is Π_m. Similarly, y is Σ_m^x if the acyclic path from y to x is Σ_m. When we use this terminology, it should usually implicitly be understood that no lower classification in the hierarchy is possible (see Figure 3). We write $\mathrm{zz}_x(y) = m$ if the acyclic path from y to x is both Π_m and Σ_m (and nothing lower in the hierarchy).

Definition 18. The **depth** $d(\varphi)$ of a formula $\varphi \in \mathcal{L}_{\mathsf{bIM}}$ is inductively defined as follows:

$$
\begin{array}{lclcl}
d(\bot) & = & d(p) & = & 0 \\
d(\varphi \wedge \psi) & = & d(\varphi \vee \psi) & = & \max\{d(\varphi), d(\psi)\} \\
d(\varphi \to \psi) & = & d(\varphi \leftarrow \psi) & = & \max\{d(\varphi), d(\psi)\} + 1 \\
d(\Diamond\varphi) & = & d(\Box\varphi) & = & d(\varphi)
\end{array}
$$

Given a *finite* set of formulas Σ, the **depth** $d(\Sigma)$ of Σ is defined by

$$d(\Sigma) = \max\{d(\varphi) \mid \varphi \in \Sigma\}.$$

Below, we fix a finite subformula-closed $\Sigma \subseteq \mathcal{L}_{\mathsf{bIM}}$.

Definition 19. A **Σ-moment** is a tuple $\mathcal{M} = (M, \leq_{\mathcal{M}}, \ell_{\mathcal{M}}, m)$ where

1. $(M, \leq_{\mathcal{M}}, \ell_{\mathcal{M}})$ is a Σ-labelled discrete poset;
2. $(M, \leq_{\mathcal{M}})$ is acyclic and has zigzag width bounded by $4d(\Sigma) + 2$;
3. $m \in M$ is called the **initial world**.

Let $\mathbb{M}_{\Sigma}$ denote the class of all Σ-moments.[5] We define a partial order $\leq_{\Sigma}$ on $\mathbb{M}_{\Sigma}$ as follows:

$$\mathcal{M} \leq_{\Sigma} \mathcal{N} \iff \mathcal{M} = (M, \leq_{\mathcal{M}}, \ell_{\mathcal{M}}, m), \\ \mathcal{N} = (M, \leq_{\mathcal{M}}, \ell_{\mathcal{M}}, n) \text{ and } m \leq_{\mathcal{M}} n.$$

Define a labelling function $\ell_{\Sigma} \colon \mathbb{M}_{\Sigma} \to \mathbb{T}_{\Sigma}$ as follows:

$$\ell_{\Sigma}(M, \leq_{\mathcal{M}}, \ell_{\mathcal{M}}, m) := \ell_{\mathcal{M}}(m).$$

Fix a sensibility condition S. Given two moments $\mathcal{M} = (M, \leq_{\mathcal{M}}, \ell_{\mathcal{M}}, m)$ and $\mathcal{N} = (N, \leq_{\mathcal{N}}, \ell_{\mathcal{N}}, n)$, we say that $\mathcal{N}$ is a **modal successor** of $\mathcal{M}$ if there exists a sensible relation $R \subseteq M \times N$ such that $(m, n) \in R$.

Define the relation $R_{\Sigma} \subseteq \mathbb{M}_{\Sigma} \times \mathbb{M}_{\Sigma}$ as follows:

$$(\mathcal{M}, \mathcal{N}) \in R_{\Sigma} \iff \mathcal{N} \text{ is a modal successor of } \mathcal{M}.$$

Definition 20. Define $\mathcal{M}_{\Sigma} := (\mathbb{M}_{\Sigma}, \leq_{\Sigma}, \ell_{\Sigma}, R_{\Sigma})$.

The following are easy to check, by reasoning about initial worlds.

[5]We treat $\mathbb{M}_{\Sigma}$ as a set; it will not matter that it is a proper class.

Lemma 9. $(\mathbb{M}_\Sigma, \leq_\Sigma, \ell_\Sigma)$ *is a* Σ*-labelled poset.*

Lemma 10. *For any sensibility condition* S*, the relation* $R_\Sigma \subseteq \mathbb{M}_\Sigma \times \mathbb{M}_\Sigma$ *is sensible with respect to* S*.*

Corollary 2. $\mathcal{M}_\Sigma$ *is a* Σ*-labelled frame.*

Lemma 11. *If the sensibility condition for* $\mathcal{M}_\Sigma$ *is* S_{tr}*, then* $\mathcal{M}_\Sigma$ *is transitive.*

Proof. Check that if $R \subseteq \mathcal{M}_1 \times \mathcal{M}_2$, and $R' \subseteq \mathcal{M}_2 \times \mathcal{M}_3$ are sensible, then the composition $R' \circ R$ is sensible. □

8 Constructing Surjective Dynamic Simulations

In this section we first show that without loss of generality we can assume labelled frames are acyclic, by describing the *path unravelling* $\mathcal{Q}^*$ of a labelled frame $\mathcal{Q}$. Then we show that given an arbitrary acyclic labelled frame $\mathcal{Q}^*$, there exists a surjective dynamic simulation from $\mathcal{M}_\Sigma$ to $\mathcal{Q}^*$.

Definition 21. Let $\mathcal{Q} = (Q, \leq_\mathcal{Q}, \ell_\mathcal{Q}, R_\mathcal{Q})$ be a labelled frame with respect to S. Its **path unravelling** is defined as $\mathcal{Q}^* = (Q^*, \leq_\mathcal{Q}^*, \ell_\mathcal{Q}^*, R_\mathcal{Q}^*)$ where

1. Q^* is the set of all paths through $\mathcal{Q}$;
2. $\leq_\mathcal{Q}^* \subseteq Q^* \times Q^*$ is defined by
$$\rho_1 \leq_\mathcal{Q}^* \rho_2 \iff (\rho_1 \sqsubseteq \rho_2 \text{ and } \uparrow(\rho_2 - \rho_1)) \text{ or } (\rho_2 \sqsubseteq \rho_1 \text{ and } \downarrow(\rho_1 - \rho_2));$$
3. For $\rho = (x_i)_{i \leq n} \in Q^*$, $\ell_\mathcal{Q}^*(\rho) := \ell_\mathcal{Q}(x_n)$;
4. For $\rho = (x_i)_{i \leq n}, \rho' = (y_i)_{i \leq n'} \in Q^*$,
$$\rho \, R_\mathcal{Q}^* \, \rho' \iff n = n' \text{ and } \forall i \leq n, x_i \, R_\mathcal{Q} \, y_i.$$

The proof of the following is then routine.

Lemma 12. *If* $\mathcal{Q}$ *is a* Σ*-labelled frame, then* $\mathcal{Q}^*$ *is a* Σ*-labelled frame.*

Lemma 13. *The poset* $(Q^*, \leq_\mathcal{Q}^*)$ *is discrete and acyclic.*

Proof. By construction. □

Clearly a formula φ is falsified on $\mathcal{Q}$ if and only if it is falsified on $\mathcal{Q}^*$. Thus, to check validity, it suffices to check validity on acyclic labelled frames.

Now let $\mathcal{Q}$ be an acyclic Σ-labelled frame. We show that there exists a dynamic simulation $E \subseteq \mathbb{M}_\Sigma \times Q$ that is surjective, i.e. for each $\rho \in Q$ there exists a moment $\mathcal{M} \in \mathbb{M}_\Sigma$ such that $\mathcal{M} \; E \; \rho$. To that end we are going to show that given $\rho \in Q$, we can inductively define a substructure of Q that corresponds to a moment.

Definition 22. Let $\mathcal{Q} = (Q, \leq_\mathcal{Q}, \ell_\mathcal{Q})$ be an acyclic Σ-labelled poset, and let $\tau \in Q$. The **connected component** of τ is the substructure $(C(\tau), \leq_\mathcal{Q}\restriction_{C(\tau)}, \ell_\mathcal{Q}\restriction_{C(\tau)})$ where $\rho \in C(\tau)$ if and only if there exists a path from τ to ρ.

Observe that $C(\tau)$ does not in general have finite zigzag width. We define a Σ-labelled substructure $\mathcal{M}^{C(\tau)} = (M, \leq_\mathcal{M}, \ell_\mathcal{M})$ of $C(\tau)$ with bounded zigzag width as follows:

- $M := \{\rho \in Q \mid \mathrm{zz}_\tau(\rho) \leq 2d(\Sigma) + 1\}$
- $\leq_\mathcal{M} := \leq_\mathcal{Q}\restriction_M$.
- if $\rho \in M$ is Π_m^τ, then $\ell_\mathcal{M}(\rho)^+ := \ell_\mathcal{Q}(\rho)^+\restriction_{\Sigma_{d(\Sigma)-\lfloor m/2 \rfloor}}$ and $\ell_\mathcal{M}(\rho)^- := \ell_\mathcal{Q}(\rho)^-\restriction_{\Sigma_{d(\Sigma)-\lfloor (m+1)/2 \rfloor}}$ ($\lfloor\ \rfloor$ is the floor function)
  if $\rho \in M$ is Σ_m^τ, then $\ell_\mathcal{M}(\rho)^+ := \ell_\mathcal{Q}(\rho)^+\restriction_{\Sigma_{d(\Sigma)-\lfloor (m+1)/2 \rfloor}}$ and $\ell_\mathcal{M}(\rho)^- := \ell_\mathcal{Q}(\rho)^-\restriction_{\Sigma_{d(\Sigma)-\lfloor m/2 \rfloor}}$

Lemma 14. *The zigzag width of* $\mathcal{M}^{C(\tau)}$ *is bounded by* $4d(\Sigma) + 2$*.*

Proof. By construction it is only possible for a state $\rho \in C(\tau)$ to occur in M if there exists an acyclic path from τ to ρ that has zigzag width of at most $2d(\Sigma) + 1$. Therefore, the maximal zigzag width of $\mathcal{M}^{C(\tau)}$ is at most $2(2d(\Sigma) + 1) = 4d(\Sigma) + 2$. □

Lemma 15. $\ell_\mathcal{M} \colon M \to \mathbb{T}_\Sigma$ *is well defined.*

Proof. Check that $\ell_\mathcal{M}$ assigns to each $\rho \in M$ a Σ-type. □

Lemma 16. $\mathcal{M}^{C(\tau)} = (M, \leq_\mathcal{M}, \ell_\mathcal{M}, \tau)$ *is a* Σ*-moment.*

Proof. Check all the defining conditions of a Σ-moment.

1. For monotonicity of $\ell_\mathcal{M} \colon (M, \leq_\mathcal{M}) \to (\mathbb{T}_\Sigma, \leq_\mathbb{T})$, suppose $\rho \leq_\mathcal{M} \rho'$. If ρ and ρ' are both Π_i^τ or both Σ_i^τ, for some i, then as $\ell_\mathcal{Q}(\rho) \leq_\mathbb{T} \ell_\mathcal{Q}(\rho')$, we have $\ell_\mathcal{M}(\rho) \leq_\mathbb{T} \ell_\mathcal{M}(\rho')$. Otherwise, either ρ is Σ_i^τ and ρ' is Π_{i+1}^τ, or ρ is Σ_{i+1}^τ and ρ is Π_i^τ. In either case, one can check that $\ell_\mathcal{M}(\rho) \leq_\mathbb{T} \ell_\mathcal{M}(\rho')$. It may by instructive to refer again to Figure 3.
2. Suppose $\varphi \to \psi \in \ell_\mathcal{M}(\rho)^-$. Then $\varphi \to \psi \in \ell_\mathcal{Q}(\rho)^-$. As $\mathcal{Q}$ is a Σ-labelled poset, $\exists \rho' \geq_\mathcal{Q} \rho$ with $\varphi \in \ell_\mathcal{Q}(\rho')^+$ and $\psi \in \ell_\mathcal{Q}(\rho')^-$. Since $0 \leq d(\varphi), d(\psi) < d(\varphi \to \psi)$ and $\mathrm{zz}_\tau(\rho') \leq \mathrm{zz}_\tau(\rho) + 1$, we know $\rho' \in M$ and both $\varphi \in \ell_\mathcal{M}(\rho')^+$ and $\psi \in \ell_\mathcal{M}(\rho')^-$. $\leftarrow$-formulas are similar. □

Proposition 2. *There exists a surjective simulation* $E \subseteq \mathbb{M}_\Sigma \times Q$*.*

Proof. Let $\rho \in Q$. By Lemma 16, $\mathcal{M}^{C(\rho)} = (M, \leq_\mathcal{M}, \ell_\mathcal{M}, \rho) \in \mathbb{M}_\Sigma$. We first define a simulation $E_\rho \subseteq \mathbb{M}_\Sigma \times Q$ that includes the pair $(\mathcal{M}^{C(\rho)}, \rho)$ as follows:
$$E_\rho := \{((M, \leq_\mathcal{M}, \ell_\mathcal{M}, \rho'), \rho') \mid \rho' \in M\}.$$

Clearly, $\mathcal{M}^{C(\rho)} \; E_\rho \; \rho$ and $E_\rho \subseteq \mathbb{M}_\Sigma \times Q$. In order to show that E_ρ is a simulation we check the defining conditions.

1. Suppose $\mathcal{M}' \; E_\rho \; \rho'$. By construction $\mathcal{M}' = (M, \leq_\mathcal{M}, \ell_\mathcal{M}, \rho')$. Then $\ell_\Sigma(\mathcal{M}') = \ell_\mathcal{M}(\rho') \subseteq_\mathbb{T} \ell_\mathcal{Q}(\rho')$.
2. For forth–up confluence, suppose $\mathcal{M}' \; E_\rho \; \rho'$ and $\mathcal{M}' \leq_\Sigma \mathcal{M}''$. This implies that $\mathcal{M}' = (M, \leq_\mathcal{M}, \ell_\mathcal{M}, \rho')$ and $\mathcal{M}'' = (M, \leq_\mathcal{M}, \ell_\mathcal{M}, \rho'')$ where $\rho' \leq_\mathcal{M} \rho''$. Thus $\rho'' \in Q$ and $\rho' \leq_\mathcal{Q} \rho''$, and by definition $\mathcal{M}'' \; E_\rho \; \rho''$.
3. The proof for forth–down confluence is similar.

Thus for each $\rho \in Q$ we have a simulation E_ρ such that $\mathcal{M}^{C(\rho)} \; E_\rho \; \rho$. Now define
$$E := \bigcup_{\rho \in Q} E_\rho.$$

By Lemma 8, E is a simulation, and by construction E is surjective. □

The next step is to show that E is dynamic.

Let $\rho, \tau \in Q$ be such that $\rho\ R_{\mathcal{Q}}\ \tau$, and write $\mathcal{M}^{C(\rho)} = (M, \leq_{\mathcal{M}}, \ell_{\mathcal{M}}, \rho)$ and $\mathcal{M}^{C(\tau)} = (N, \leq_{\mathcal{N}}, \ell_{\mathcal{N}}, \tau)$.

Lemma 17. *There exists a sensible relation $R \subseteq M \times N$ such that $(\rho, \tau) \in R$.*

Proof. Using the confluence conditions, inductively build up a total function $R\colon M \to N$, starting by setting $R(\rho) = \tau$, and then proceeding by induction on the distance of points $\rho' \in M$ from ρ. We can ensure that $\mathrm{zz}_\tau(R(\rho')) \leq \mathrm{zz}_\rho(\rho')$ always holds.

It remains to show that R is sensible. By construction, R is forth–up and forth–down confluent. Let S be the sensibility condition, and suppose $(\rho', \tau') \in R$. Then $(\rho', \tau') \in R_{\mathcal{Q}}$, so $\ell_{\mathcal{Q}}(\rho')\ S\ \ell_{\mathcal{Q}}(\tau')$. Since $\ell_{\mathcal{M}}(\rho')$ and $\ell_{\mathcal{N}}(\tau')$ are given by restrictions of $\ell_{\mathcal{Q}}(\rho')$ and $\ell_{\mathcal{Q}}(\tau')$ respectively, and $\ell_{\mathcal{Q}}(\rho')$ is restricted more than $\ell_{\mathcal{Q}}(\tau')$ (since $\mathrm{zz}_\tau(\tau') \leq \mathrm{zz}_\rho(\rho')$), by the properties of sensibility conditions $\ell_{\mathcal{M}}(\rho')\ S\ \ell_{\mathcal{N}}(\tau')$. Therefore R is sensible. □

Lemma 18. $R^{-1}(N) = M$.

Proof. The construction of R adds in step 0 the world ρ to $R^{-1}(N)$ and in step 1 every world different from ρ which has zigzag width 0. Moreover, it adds in each step $n > 1$ every world which has zigzag width $n - 1$. Therefore after $2d(\Sigma) + 2$ steps, $R^{-1}(N) = M$. □

Note that for every $\rho_0 \in M$, there exists $\tau_0 \in N$ with $(\rho_0, \tau_0) \in R$. Thus, using the notation $\mathcal{M}^{C(\rho)}_{\rho_0}$ to mean the moment formed by changing the initial world of $\mathcal{M}^{C(\rho)}$ to ρ_0, we obtain the following.

Corollary 3. *$\mathcal{N}^{C(\tau)}_{\tau_0}$ is a modal successor of $\mathcal{M}^{C(\rho)}_{\rho_0}$.*

Proof. Witnessed by the sensible relation $R \subseteq M \times N$ with $(\rho_0, \tau_0) \in R$. □

Lemma 19. *Suppose $\rho_0 \in \mathcal{M}^{C(\rho)}$ and $\rho_0\ R_{\mathcal{Q}}\ \tau_0$. Then there exists τ such that $\rho\ R_{\mathcal{Q}}\ \tau$ and $\tau_0 \in \mathcal{N}^{C(\tau)}$.*

Proof. Since $\rho_0 \in \mathcal{M}^{C(\rho)}$, there exists an acyclic path $\alpha = (x_i)_{i \leq n}$ such that $x_0 = \rho$ and $x_n = \rho_0$. Let $\overline{\alpha} = (x_{n-i})_{i \leq n}$. Since $\rho_0\ S_{\mathcal{Q}}\ \tau_0$ we find, by using forward confluence of $S_{\mathcal{Q}}$, a path $\beta = (y_i)_{i \leq n}$ such that for all $0 \leq i \leq n$ it holds that $x_{n-i}\ S_{\mathcal{Q}}\ y_i$. In particular $\rho\ S_{\mathcal{Q}}\ y_n = \tau$. Observe that the zigzag width of β is at most the zigzag width of α. Therefore, since $\rho_0 \in \mathcal{M}^{C(\rho)}$ and α connects ρ to ρ_0, $\tau_0 \in \mathcal{N}^{C(\tau)}$. □

Putting everything together, we obtain the desired result.

Proposition 3. *The simulation $E \subseteq \mathbb{M}_\Sigma \times Q$ is dynamic.*

Proof. Suppose $\mathcal{M}\ E\ \rho_0$ and $\rho_0\ R_{\mathcal{Q}}\ \tau_0$. Then $\mathcal{M} = \mathcal{M}^{C(\rho)}_{\rho_0}$. Let $\tau \in Q$ such that $\rho\ R_{\mathcal{Q}}\ \tau$ and $\tau_0 \in \mathcal{N}^{C(\tau)}$. As shown above, the moment $\mathcal{N}^{C(\tau)}_{\tau_0}$ is a modal successor of $\mathcal{M}^{C(\rho)}_{\rho_0}$. Moreover, $\mathcal{N}^{C(\tau)}_{\tau_0}\ E\ \tau_0$. □

9 Succinct Moments

In order to obtain *finite* Σ-labelled frames, we will restrict $\mathbb{M}_\Sigma$ to moments that are, in a sense, no bigger than necessary. Specifically, they should not be 'bimersive' to a moment of strictly smaller cardinality. Below, we make this precise.

In order to prove the main results in this section, we will need to consider labels that are not necessarily types. Let $(C, \leq)$ be a finite poset, which we identify with its domain C. A **C-moment** is a triple $\mathcal{M} = (M, \leq_{\mathcal{M}}, \ell_{\mathcal{M}})$, where $(M, \leq_{\mathcal{M}})$ is an acyclic discrete poset and $\ell_{\mathcal{M}}\colon M \to C$ is order preserving: $w \leq_{\mathcal{M}} v \implies \ell_{\mathcal{M}}(w) \leq \ell_{\mathcal{M}}(v)$. The class of C-moments of zigzag width n is denoted M^n_C. We will refer to the structure $(C, \leq)$ as the **set of colors**.

Definition 23. Let C be a set of colors and $\mathcal{M}$, $\mathcal{N}$ be C-moments. A relation $\sigma \subseteq M \times N$ is a **simulation** from $\mathcal{M}$ to $\mathcal{N}$ if $\mathrm{dom}(\sigma) = M$ and whenever $w\ \sigma\ v$:

1. $\ell_{\mathcal{M}}(w) = \ell_{\mathcal{N}}(v)$;
2. σ is forth–up and forth–down confluent (for $\leq_{\mathcal{M}}$ and $\leq_{\mathcal{N}}$).

A simulation is called an **immersion** if it is a function. If an immersion $\sigma\colon \mathcal{M} \to \mathcal{N}$ exists, we write $\mathcal{M} \trianglelefteq \mathcal{N}$. If, in addition, there is an immersion $\tau\colon \mathcal{M} \to \mathcal{N}$, we say that $\mathcal{M}$ and $\mathcal{N}$ are **bimersive**, write $\mathcal{M} \triangleq \mathcal{N}$, and call the pair (σ, τ) a **bimersion**.

If $\mathcal{M}$ is such that $\mathcal{M} \triangleq \mathcal{N}$ implies that $|M| \leq |N|$, we say that $\mathcal{M}$ is **succinct**.

Note that every moment is bimersive to a succinct moment, simply because there must be a minimum cardinality among all moments bimersive to it.

We wish to show that the number of bimersion classes of M^n_C is computably bounded, and there is a computable bound on the cardinality of the succinct moments. We will prove this via an inductive argument, in which worlds of a moment of maximal height are labelled by moments of smaller height, in order to apply the induction hypothesis and reduce these simpler moments. Thus we need to state the following lemma for an arbitrary set of colors. Below, say that a moment $\mathcal{M}$ is **tree-like with root** r if either $\forall w \in M$, $w \leq r$ or else $\forall w \in M, r \leq w$.

Lemma 20 (Boudou, Diéguez, and Fernández-Duque (2017, Theorem 23))**.** *Let C be a finite set of colors with $|C| = c$. Then there are computable functions F and G such that*

1. *Given a treelike C-moment $\mathcal{M}$, there is a treelike C-moment $\mathcal{M}_*$ of cardinality bounded by $F(c)$ such that $\mathcal{M}_* \triangleq \mathcal{M}$.*
2. *Given a sequence of treelike C-moments $\mathcal{M}_1, \ldots, \mathcal{M}_n$ with $n > G(c)$, there are indexes $i < j \leq n$ such that $\mathcal{M}_i \triangleq \mathcal{M}_j$.*

Remark 1. Boudou et al. use a slightly different presentation. They do not assume that C is partially ordered, but instead consider labelling functions of a fixed *level* k; this is the maximal length of a chain $w_1, \ldots, w_k$ of worlds such that $\ell(w_i) \neq \ell(w_{i+1})$. Our C-moments automatically have level at most c, since our labelling functions are monotone.

Proposition 4. *Let $\Sigma \subseteq \mathcal{L}_{\mathsf{bIM}}$ be finite and closed under subformulas. Then there are natural numbers κ_Σ and λ_Σ such that:*

1. *Given $\mathcal{M} \in \mathbb{M}_\Sigma$, there is a Σ-moment $\mathcal{M}_*$ of cardinality bounded by κ_Σ such that $\mathcal{M}_* \triangleq \mathcal{M}$.*
2. *Given a sequence $\mathcal{M}_1, \ldots, \mathcal{M}_n \in \mathbb{M}_\Sigma$ with $n > \lambda_\Sigma$, there are indexes $i < j \leq n$ such that $\mathcal{M}_i \triangleq \mathcal{M}_j$.*

Proof. Let $\mathcal{M} = (M, \leq, \ell, r)$ be a Σ-moment. For $w \in M$ let M^w be the subset of M starting at w and away from r; formally, if we let $(w_0, \ldots, w_s)$ be the unique zigzag path from w to r (with $w_0 = w$), then M^w is the connected component of w in $M \setminus \{w_1, \ldots, w_s\}$. Then we define $\mathsf{zzh}(w)$, the zigzag height of w, to be the greatest m so that every $v \in M^w$ is both Π^w_m and Σ^w_m.

First we consider a case where whenever $w < r$, it follows that $\mathsf{zzh}(w) < \mathsf{zzh}(r)$. Let C be the set of pairs (c, B), where $c \in C$ and B is a set of bimersion classes of $(n-1)$-depth Σ-moments (where $B = \varnothing$ if $n = 0$), with $(c, B) \leq^+ (c', B')$ if and only if both $c \leq c'$ and $B' \subseteq B$. For each $w \geq r$, we define a label $L(w) = (\ell(w), B(w))$, where $B(w)$ is defined as follows. Let $\mathcal{M}^- = \mathcal{M}\restriction_{M \setminus w^\uparrow}$. For each $v \in \mathcal{M}^-$, let $\mathcal{M}_v$ be the restriction of $\mathcal{M}^-$ to the connected component of v. Then let $B(w)$ be the set of bimersion classes of frames of the form $\mathcal{M}_v$ with $v \in \mathcal{M}^- \cap w^\downarrow$. It is readily checked that $\mathcal{M}\restriction_{r^\uparrow}$ is a treelike C-moment, since transitivity of $\leq$ ensures that $B(w)$ is inversely monotone. Hence by Lemma 20, $\mathcal{M}$ is bimersive to a C-moment of size at most $F(|C|)$, and there are at most $G(|C|)$ bimersion classes for such $\mathcal{M}$.

In the general case, we merely view M as $\hat{M} \cup \check{M}$, where $\hat{M}$ is the set of worlds accessible from r by a zigzag path that first goes up and $\check{M}$ is the set of worlds accessible by a zigzag path that first goes down. By applying the previous case to each of the two sides, we obtain $\mathcal{M}' \triangleq \mathcal{M}$ with $|M'| \leq 2F(|C|)$. Hence there is $\mathcal{M}_* \triangleq \mathcal{M}'$ with at most this number of worlds. Since the bimersion class of $\mathcal{M}$ is determined by the two bimersion classes, there can be at most $G(|C|)^2$ bimersion classes. ☐

Let I_Σ be the substructure formed by restricting $\mathbb{M}_\Sigma$ to succinct moments (and with isomorphic moments identified). It follows from Proposition 4 that I_Σ is finite and only contains finite moments. Next, it would be convenient if, whenever E is a simulation and $\mathcal{M}$ is any Σ-moment such that $\mathcal{M}\ E\ x$, we could replace $\mathcal{M}$ by some succinct $\mathcal{M}' \trianglelefteq \mathcal{M}$ and still have $\mathcal{M}'\ E\ x$. The following operations on simulations will help us achieve this.

Definition 24. Let $\Sigma \subseteq \mathcal{L}_{\mathsf{bIM}}$ be finite and closed under subformulas, $\mathfrak{X}$ be a Σ-labelled poset, and $E \subseteq \mathbb{M}_\Sigma \times X$.

1. Define the **bimersion closure** of E by $\check{E} := E \circ \triangleq$. If $\check{E} = E$, we say E is **bimersion invariant**.
2. Define the **succinct part** of E by $E_0 := E\restriction_{\mathrm{I}_\Sigma}$.

In other words, $\mathcal{M}\ \check{E}\ x$ means that there is $\mathcal{N}$ such that $\mathcal{M} \triangleq \mathcal{N}\ E\ x$. Let us see that these operations indeed produce new simulations. The following is proven by (Fernández-Duque 2018).

Lemma 21. *Suppose that $\Sigma \subseteq \mathcal{L}_{\mathsf{bIM}}$ is finite and closed under subformulas, $\mathfrak{X}$ is a Σ-labelled frame, and $E \subseteq \mathbb{M}_\Sigma \times X$ is a simulation. Then*

1. *$\check{E}$ and E_0 are also simulations.*

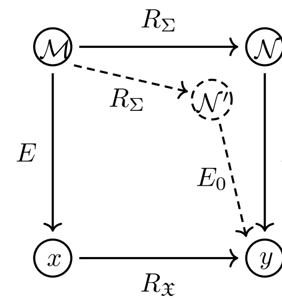

Figure 4: A diagram illustrating Lemma 21.2b.

2. *If E is bimersion invariant, then*
   (a) *$E_0(\mathrm{I}_\Sigma) = E(\mathbb{M}_\Sigma)$;*
   (b) *if $\mathcal{M}\ E\ x\ R_\mathfrak{X}\ y$ and $\mathcal{M}\ R_\Sigma\ \mathcal{N}\ E\ y$, then there exists $\mathcal{N}'$ such that $\mathcal{M}\ R_\Sigma\ \mathcal{N}'$ and $\mathcal{N}'\ E_0\ y$ (see Figure 4).*

Theorem 3. *Each $\Lambda \in \{\mathsf{KbI}, \mathsf{DbI}, \mathsf{TbI}, \mathsf{K4bI}, \mathsf{K4DbI}, \mathsf{S4bI}\}$ has the finite frame property and hence is decidable.*

Proof. If φ is derivable, it is valid on finite Λ-frames by soundness. Otherwise, it is not valid on the canonical model, so not valid on $\mathbb{M}_\Sigma\restriction_{E^{-1}(W_c)}$, and hence by Lemma 21, not valid on $\mathrm{I}_\Sigma\restriction_{E^{-1}(W_c)}$, which by Proposition 4 is finite. By Theorem 2, φ is not valid on the class of finite Λ-frames. ☐

10 Concluding Remarks

We have shown that many bi-intuitionistic modal logics interpreted on intuitionistic Kripke models satisfying forth-up and forth-down confluence are decidable. To the best of our knowledge this is the first such result in this context. The logics we have considered are inspired by intuitionistic temporal logic, also based on forward confluence, so one may ask whether similar decidability results hold for logics in the spirit of *intuitionistic modal logics* in the sense of Fischer Servi or *constructive modal logics* in the sense of Fitch.

In either case, we would argue that the intended duality between implication and co-implication should give preference to symmetric frame conditions for $\leq$, i.e. semantics for constructive bi-intuitionistic logics should require back–down confluence as well as back–up confluence, and for intuitionistic modal logics, all four confluence conditions.

For constructive logics, this should yield a conservative extensions of logics without co-implication, and we expect our techniques can be adapted to this setting: roughly speaking, simulations can preserve forth conditions or back conditions, but not both at the same time.

In contrast, for the just-mentioned reason we do not expect that the finite frame property for transitive intuitionistic modal logics (with or without co-implication) can be obtained from our techniques. Moreover, unlike in the constructive case, the forth–down confluence property does lead to new valid formulas in the original mono-intuitionistic language. However, much as has been the case for logics without co-implication, we expect that a combination of our techniques with currently existing proofs (as in e.g. (Simpson 1994)) should yield decidability results for IK with co-implication.

Acknowledgments

This work has been partially supported by FWO–FWF grant G030620N/I4513N (Fernández-Duque), COST action CA-17124 (Fernández-Duque), and SNSF–FWO Lead Agency Grant 200021L_196176/G0E2121N (Fernández-Duque, McLean, and Zenger).

References

Aguilera, J. P.; Diéguez, M.; Fernández-Duque, D.; and McLean, B. 2022. A Gödel calculus for linear temporal logic. In *Proceedings of the 19th International Conference on Principles of Knowledge Representation and Reasoning - 19th International Conference (KR 2022), Haifa, Israel*. IJCAI.

Alechina, N.; Mendler, M.; de Paiva, V.; and Ritter, E. 2001. Categorical and Kripke semantics for constructive S4 modal logic. In *Proceedings of 15th International Workshop on Computer Science Logic (CSL 2001), 10th Annual Conference of the EACSL, Paris, France*, volume 2142 of *Lecture Notes in Computer Science*, 292–307. Springer.

Balbiani, P.; Boudou, J.; Diéguez, M.; and Fernández-Duque, D. 2020. Intuitionistic linear temporal logics. *ACM Transactions on Computational Logic* 21(2):14:1–14:32.

Balbiani, P.; Diéguez, M.; and Fernández-Duque, D. 2021a. Some constructive variants of S4 with the finite model property. In *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021*, 1–13. IEEE.

Balbiani, P.; Diéguez, M.; and Fernández-Duque, D. 2021b. Some constructive variants of S4 with the finite model property. In *Proceedings of the 36th Annual ACM/IEEE Symposium on Logic in Computer Science, (LICS 2021)*, 1–13. IEEE.

Bílková, M.; Frittella, S.; and Kozhemiachenko, D. 2022. Paraconsistent Gödel modal logic. In Blanchette, J.; Kovács, L.; and Pattinson, D., eds., *Automated Reasoning - 11th International Joint Conference, IJCAR 2022, Haifa, Israel, August 8-10, 2022, Proceedings*, volume 13385 of *Lecture Notes in Computer Science*, 429–448. Springer.

Boudou, J.; Diéguez, M.; and Fernández-Duque, D. 2017. A decidable intuitionistic temporal logic. In *26th EACSL Annual Conference on Computer Science Logic, CSL 2017, August 20-24, 2017, Stockholm, Sweden*, 14:1–14:17.

Boudou, J.; Diéguez, M.; and Fernández-Duque, D. 2022. Complete intuitionistic temporal logics for topological dynamics. *Journal of Symbolic Logic* 87(3):995–1022.

Fernández-Duque, D. 2018. The intuitionistic temporal logic of dynamical systems. *Logical Methods in Computer Science* 14(3).

Fischer Servi, G. 1977. On modal logic with an intuitionistic base. *Studia Logica* 36(3):141–149.

Fischer Servi, G. 1984. Axiomatizations for some intuitionistic modal logics. *Rendiconti del Seminario Matematico Università e Politecnico di Torino* 42(3):179–194.

Fitch, F. B. 1948. Intuitionistic modal logic with quantifiers. *Portugaliae mathematica* 7(2):113–118.

Gabbay, D. M., and Olivetti, N. 2000. *Goal-Directed Proof Theory*. Dordrecht: Springer Netherlands. chapter Intermediate Logics, 93–115.

Girlando, M.; Kuznets, R.; Marin, S.; Morales, M.; and Straßburger, L. 2023. Intuitionistic S4 is decidable. In *38th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2023, Boston, USA, 26–29 June 2023*. IEEE.

Gödel, K. 1932. Zum intuitionistischen aussagenkalkül. *Anzeiger der Akademie der Wissenschaften in Wien* 69:65–66.

Plotkin, G. D., and Stirling, C. 1986. A framework for intuitionistic modal logics. In *Proceedings of the 1st Conference on Theoretical Aspects of Reasoning about Knowledge (TARK 1986), Monterey, CA, USA*, 399–406. Morgan Kaufmann.

Rauszer, C. 1974. Semi-boolean algebras and their applications to intuitionistic logic with dual operations. *Fundamenta Mathematicae* 83(3):219–249.

Sano, K., and Stell, J. 2017. Strong completeness and the finite model property for bi-intuitionistic stable tense logics. In *Ninth Workshop on Methods for Modalities (M4M9 2017), 08–09 Jan 2017, Indian Institute of Technology, Kanpur, India*, 105–121. Electronic Proceedings in Theoretical Computer Science.

Simpson, A. 1994. *The proof theory and semantics of intuitionistic modal logic*. Ph.D. Dissertation, University of Edinburgh, Edinburgh, UK.

Sindoni, G.; Sano, K.; and Stell, J. 2021. Expressing discrete spatial relations under granularity. *Journal of Logical and Algebraic Methods in Programming* 122:100682.

Stell, J.; Schmidt, R.; and Rydeheard, D. 2016. A bi-intuitionistic modal logic: Foundations and automation. *Journal of Logical and Algebraic Methods in Programming* 85(4):500–519.

van Benthem, J. 1976. *Modal Correspondence Theory*. Ph.D. Dissertation, University of Amsterdam.

Wijesekera, D. 1990. Constructive modal logics I. *Annals of Pure and Applied Logic* 50(3):271–301.

Wolter, F., and Zakharyaschev, M. 1997. The relation between intuitionistic and classical modal logics. *Algebra and Logic* 36(2):73–92.

Wolter, F., and Zakharyaschev, M. 1999. Intuitionistic modal logic. In *Logic and Foundations of Mathematics: Selected Contributed Papers of the 10th International Congress of Logic, Methodology and Philosophy of Science, Florence, Italy*. Dordrecht, Netherlands: Springer. 227–238.