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More on $\mathfrak{R}$

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Abstract We define the notions of *$\mathfrak{R}$-primarity* and of *$\mathfrak{R}$-connectedness*, as well as an analog of *monomial U-rank*, namely *Frattini-free*. We then derive some results concerning the Frattini-free component. In particular, the Frattini-free component of a semisimple stable group is the direct product of $\mathfrak{R}$-connected groups, and the Frattini-free component of a soluble $\mathfrak{R}$-group has nilpotent derivative. Finally, the radical of the Frattini-free component of a definable $\mathfrak{R}$-group is definable.

0 Introduction Nesin [5] proved that the derived subgroup of a soluble group of finite Morley rank is nilpotent. This was consequently generalized by Hrushovski [3] to superstable p-connected groups, where p is a regular type. In Wagner [8], many properties of superstable groups were generalized to $\mathfrak{R}$-groups (Definition 7 of the next section; examples of $\mathfrak{R}$-groups are superstable and small stable groups). However, Nesin's result resisted at the time, mainly due to the fact that the relevant connectivity condition remained unclear.

Here we shall define $\mathfrak{R}$-primarity and $\mathfrak{R}$-connectedness. As an $\mathfrak{R}$-connected component need not exist in general, we also define various approximations and show that connected components exist for these. We then define an analog of *monomial U-rank* and generalize Nesin's result, as well as a result of Baldwin and Pillay [1] on superstable semisimple groups. In the last section we prove that the radical (i.e., maximal normal soluble subgroup) of the Frattini-free component of a definable $\mathfrak{R}$-group exists and is definable; furthermore, the quotient group decomposes.

One surprising feature for us was that quite a lot of our results already hold in a merely stable context. In fact, $\mathfrak{R}$ is invoked only in the last two sections, and even there only in a very weak form (two independent generics cannot be algebraic in a single element of the group).

The underlying philosophy is that stability is a strong condition for non-abelian (and even more for nonnilpotent) groups. On the other hand, an abelian

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$\mathfrak{R}$-group can be rather unpleasant (unless of course we have additional structure, e.g. a field):

Example 1 Let T be the theory of $G = \bigoplus_{i<\omega} C_i(\omega)$, where C_i is the cyclic group of order 2^i. This is an abelian group and hence stable and one-based. The almost $\varnothing$-definable subgroups are sums and intersections of the various $G[2^n]$ and 2^nG for $n < \omega$, and finite extensions of quotients of these are all definable groups. Now $\mathfrak{R}$ in this context means that if a generic a for some such group H is algebraic over some $b \in H$, then b is also generic. In order to see this, we first note that we can assume H connected; it follows from Hrushovski and Pillay [4] that the algebraicity is given by a coset E of some definable subgroup of $H \times H$ such that $(a, b) \in E$. Now modulo a translation we can assume that E is a subgroup itself, as a is generic, and its projection to the first coordinate is surjective. If $C = \{h \in H \mid (0, h) \in E\}$, then E induces a definable endomorphism of H/C. Now, as a is algebraic over b via the formula $(a, b) \in E$, we must have $b \notin C$; furthermore the endomorphism has finite kernel. But all definable endomorphisms are given by multiplication by some natural number: they either have infinite kernel or are bijective. But then b is generic as well.

G is connected, and its generic is foreign to $G[2]$ (there is no definable homomorphism from $G[2]$ onto $G/2G$), so $x \mapsto 2x$ is a map with small kernel $G[2]$ and nongeneric image $2G$ (thus answering a question in [8] negatively). Furthermore, G is not the sum of a divisible group and a group of bounded exponent, and therefore not small (see Poizat and Wagner [7], Lemma 13). Hence G is an example of a nonsuperstable $\mathfrak{R}$-group which is not small.

All groups and homomorphisms will be infinitely definable (i.e., the intersection of definable groups or homomorphisms) except when otherwise stated. Group theoretic notation will be standard. In particular, if H is a normal subgroup and X a subset of G, then $C_G(X/H) = \{g \in G : [X, g] \subseteq H\}$ is the centralizer of X in G relative to H. We shall omit H if it is trivial, and G if it is obvious. Elements of the ascending central series of G will be denoted by $Z_i(G)$, that is $Z_0(G) = Z(G)$ and $Z_{i+1}(G) = C_G(G/Z_i(G))$. The iterated centralizers of H are defined as follows: $C_1(H) = C(H)$, and $C_{i+1}(H) = C(H/C_i(H))$. Notice that $Z_i(H) = C_i(H) \cap H$. If G and H are subgroups of some group, we shall denote by $[G, H]$ the smallest infinitely definable group containing all the commutators $[g, h] = g^{-1}h^{-1}gh$, where $g \in G$ and $h \in H$. The derived series is given by $G^{(0)} = G$, $G^{(i+1)} = [G^{(i)}, G^{(i)}]$, and the descending central series by $G^0 = G$, $G^{i+1} = [G, G^i]$. Note that although the definition is exactly the same as in group theory, we may get a different series due to the different interpretation of the commutator.

Finally, a word about the methods used. A considerable emphasis is placed on formulas (as opposed to types), and often an inductive argument on the rank of types is replaced by a compactness argument in order to get some finiteness condition. In this light, the importance of superstability in the study of groups may lie less in the existence of rank or finite bases than in the fact that generic types can be replaced by formulas (defining a group in which they are generic). Thus a stable group with descending chain condition on connected components will already give most of the results we have for superstable groups.

1 Preliminaries A good introduction to stable groups is Poizat [6]; the material on $\mathfrak{R}$-groups can be found in [8]. Here we shall just give the definitions and cite the results required.

Definition 2 Let p be a complete type over A and Σ a set of partial types. We say that p is *foreign to* Σ if for every $B \supseteq A$ no realization a of a nonforking extension q of p over B can fork with a realization b of any extension of any type in Σ.

Definition 3 Let π be a partial type and Σ a set of partial types over some set A. We say that π is *Σ-internal* if there is a set $B \supseteq A$ such that every realization a of π is definable over B and finitely many realizations b_i of types $\sigma_i \in \Sigma$.

More generally, π is *Σ-analyzable* if there is some set $B \supseteq A$ such that for any realization a of π there is a sequence a_i, $i < \alpha$ in $\mathrm{acl}(Ba)$, such that $a \in \mathrm{acl}(B \cup \{a_i : i < \alpha\})$ and for all $i < \alpha$, $\mathrm{tp}(a_i/B \cup \{a_j : j < i\})$ is Σ-internal.

If $\Sigma = \{\sigma\}$ is a single type, we shall leave off the braces.

Fact 4 *If Σ is a set of formulas and a type p is Σ-analyzable, then there is a finite subset Σ_0 of Σ such that p is Σ_0-analyzable in finitely many steps.*

We shall also need a result of Hrushovski [2]:

Fact 5 *Let G be a stable group with generic type p, and let q be any type. Then p is not foreign to q if and only if there is a normal subgroup N of G of infinite index such that G/N is q-internal. In fact, we can choose N to be invariant under finitely many automorphisms of G fixing* $\mathrm{dom}(q)$.

Finally, we shall require a lemma from [8]:

Fact 6 *If a type q is G-internal via some formula of the form $y = \varphi(x_1, \dots, x_n)$ but foreign to G/N for some subgroup $N < G$, then q is N-internal via $y = \varphi(g_1x_1, \dots, g_nx_n)$ for some $g_i \in G$, $i = 1, \dots, n$. Furthermore, $\varphi(g_1, \dots, g_n) \models q$.*

Now we shall collect the results on $\mathfrak{R}$-groups:

Definition 7 An *$\mathfrak{R}$-theory* is a stable theory such that for all definable transitive group actions of some definable group G on some definable set X, if $x \in X$ is a generic for that action and x is algebraic over some element $y \in X$, then y is generic as well.

An *$\mathfrak{R}$-group* is a group whose theory is an $\mathfrak{R}$-theory.

Definition 8 Let G acting on M be groups. $A \leq M$ is *G-minimal* if it is infinitely definable, infinite, G-invariant and minimal with these properties.

We shall call a group *centralizer-connected* if there is no proper centralizer (of some element) of finite index.

Fact 9 (see [8]) *Let G be an infinite definable centralizer-connected $\mathfrak{R}$-group acting faithfully on the abelian G-minimal group A. Suppose that G has a normal abelian subgroup M and that the generic type of A is not foreign to M. Then there is an infinite definable field K and a definable finite dimensional K-vector space structure on A such that G acts K-linearly, and M is central in G.*

Occasionally we shall talk about quotients (groups or merely sets of cosets) G/H, where H is infinitely definable. If $H = \bigcap_{i<\omega} H_i$, an element $aH \in G/H$ can be seen as a sequence of elements aH_i, $i < \omega$ in T^{eq}, and for $i < j$, obviously $aH_i \in \mathrm{acl}(aH_j)$. By the type of aH we shall mean the collection of the types of the aH_i, and the generic type of G/H is the type of aH, where $a \in G$ is a generic of G. (If the language is not countable, we may have to replace sequences by filters.)

2 $\mathfrak{R}$-primarity et cetera We shall define $\mathfrak{R}$-primarity, $\mathfrak{R}$-connectedness, and the Frattini-free component G^Φ and derive some basic properties about these notions. Note that we do not suppose our groups to be $\mathfrak{R}$. The underlying idea is to take notions from stability theory and replace *forking* by *algebraicity, internality*, or *analyzability*. Furthermore, *finite index* is replaced by some other notion of smallness (for the quotient), namely *foreign to a certain group generic.*

Definition 10 Let P be some property. A group G is P *$\mathfrak{R}$-connected* if it has no proper nontrivial infinitely definable subgroup H satisfying P and such that the generic of H is foreign to G/H.

Possible choices for P include *infinite* (∞), *nonsoluble* (s), *non-nilpotent* (np), *noncentral in* $G^0(c)$ (not contained in the center $Z(G^0)$), and *nonhypercentral in* G^0 (hc) (not contained in the hypercenter $\bigcup Z_i(G^0)$). We shall also write $\mathfrak{R}$-connected for ∞ $\mathfrak{R}$-connected.

Lemma 11 *Let H be a subgroup of G such that the generic of H is foreign to G/H. Then H^0 is invariant under all definable automorphisms of G (and in particular, it is normal).*

Proof: Let σ be a definable automorphism of G. As the generic of H is foreign to G/H and $G/\sigma H = \sigma G/\sigma H$ is definably isomorphic to G/H, the generic of H is also foreign to $G/\sigma H$, and hence to $G/H \cap \sigma H$ (as $g(H \cap \sigma H) \in \mathrm{dcl}\{gH, g\sigma H\}$). But this can only be if $H \cap \sigma H$ has finite index in H, i.e., $\sigma H \geq H^0$.

Definition 12 A type p is *strongly $\mathfrak{R}$-primary* if for any partial type π, if p is not foreign to π, then it is π-internal.

More generally, p is $\mathfrak{R}$-primary, if for any partial type π, if p is not foreign to π, then p is π-analyzable.

By Fact 5, the generic type of a simple group or a field is strongly $\mathfrak{R}$-primary. In a superstable theory an $\mathfrak{R}$-primary type r is primary (i.e., domination-equivalent to a power of some regular type): suppose r is nonorthogonal to the two regular types p_1 and p_2. We may assume that both p_1 and p_2 have monomial U-rank. We may assume $U(p_1) \geq U(p_2)$, so p_1 is foreign to p_2. But r is p_2-analyzable, hence p_1 is orthogonal to r as well, a contradiction. The converse of course is false: a primary type need not be $\mathfrak{R}$-primary.

Definition 13 Let $G = G_0$ be a stable group and Σ a set of partial types. Define, by ordinal induction, $G_{\alpha+1} = \bigcap\{H \leq G_\alpha : G_\alpha/H \text{ is } \Sigma\text{-internal}\}$ and $G_\lambda = \bigcap_{\alpha<\lambda} G_\alpha$ for λ a limit ordinal. As this is a descending sequence of infinitely definable groups, it has to become stationary at some α_0. This G_{α_0} is the Σ-*analyzable divisor* G^Σ *of* G.

Obviously, G/G^Σ is Σ-analyzable (and G^Σ is minimal with this property). Furthermore, if $H \lhd G$, then $(G/H)^\Sigma = G^\Sigma/(H \cap G^\Sigma)$.

Lemma 14 *G^Σ is a connected normal subgroup of G whose generic is foreign to Σ and indeed to G/G^Σ.*

Proof: Connectedness and normality are obvious, and foreignness to Σ follows from stationarity and Fact 5. So suppose the generic of G^Σ is not foreign to G/G^Σ. By Fact 5 there is a proper normal subgroup K of G^Σ such that G^Σ/K is G/G^Σ-internal via some formula $y = \varphi(x_1, \dots, x_n)$. But G^Σ/K is foreign to Σ and hence to $G_\alpha/G_{\alpha+1}$ for all $\alpha < \alpha_0$. Hence by Fact 6 and compactness there are $g_1, \dots, g_n \in G$ such that G^Σ/K is G_{α_0}/G^Σ-internal via $y = \varphi(g_1x_1, \dots, g_nx_n)$, clearly a contradiction.

Lemma 15 *A connected group G is $\mathfrak{R}$-connected if and only if its generic is $\mathfrak{R}$-primary.*

Proof: Suppose the generic of G is $\mathfrak{R}$-primary but G is not $\mathfrak{R}$-connected, as witnessed by some subgroup H. As the generic of G is not foreign to G/H, it (and hence G) is G/H-analyzable by $\mathfrak{R}$-primarity, and so is H. Hence the generic of H cannot be foreign to G/R, a contradiction.

On the other hand, suppose that G is $\mathfrak{R}$-connected, and its generic is not foreign to some partial type π. Let H be the π-analyzable divisor of G. By $\mathfrak{R}$-connectedness, H is trivial. That is, G is π-analyzable.

Note that the same lemma holds with $\mathfrak{R}$ replaced by some regular type p.

Lemma 16 *Let H_α, $\alpha < \kappa$, be a descending chain of infinitely definable normal subgroups of the definable group G such that for all $\alpha < \kappa$, the generic of H_α is foreign to G/H_α. Then the generic of $H = \bigcap_{\alpha<\kappa} H_\alpha$ is foreign to G/H.*

Proof: Suppose not. Then there is a normal subgroup K of infinite index in H such that H/K is G/H-internal. We clearly may assume that H is connected. There is a definable function $\varphi(\bar{x})$ from $\prod_n G$ onto a superset of H/K which depends only on the coset modulo H of $\bar{x}$. By compactness, this means that there are definable supergroups H_i and H_j of H such that φ induces a function from $\prod_n (G/H_i)$ to a superset of H_j/K. But then there is $\alpha_0 < \kappa$ such that $H_i, H_j \geq H_{\alpha_0}$, and φ induces a function from $\prod_n (G/H_{\alpha_0})$ to a superset of H_{α_0}/K, a contradiction.

Definition 17 Let P be any property true for G^0, such that P does not hold for $\{1\}$ and is preserved under intersections of descending chains (in a saturated model!). A minimal infinitely definable connected subgroup H with P such that the generic of H is foreign to G/H will be called a $P\,\mathfrak{R}$-component G^P.

By Lemma 16, in a definable group such a component exists for any P. However, it need not be unique. It is connected; by Lemma 11 it is invariant under definable automorphisms of G. Obviously, $G^s \geq G^{np} \geq G^{hc} \geq G^c \geq G^\infty$ (if they are comparable), and G is $\mathfrak{R}$-connected if and only if $G^\infty = G$. Notice that a noncentral and a nonhypercentral $\mathfrak{R}$-component depend on the ambient G. That is, if $G \geq H$, then $G^P \geq H^P$ for $P = s$ or np, but it might happen that $G^P < H^P$ for $P = c$ or hc.

Lemma 18 *If G and H are normal subgroups of some stable group and G is connected, then $[G,H]$ is connected.*

Proof: Suppose K is a definable GH-invariant subgroup such that $|[G,H]: [G,H] \cap K| < \omega$. For any $h \in H$, $\{g \in G : [g,h] \in K\}$ is a subgroup of finite index in G and hence equal to G. Therefore $K \geq [G,H]$. But if $[G,H]$ has no GH-invariant subgroups of finite index, then by Baldwin-Saxl it is connected.

Lemma 19 *Let G and H be normal subgroups of some stable group. Suppose that G is connected and its generic is foreign to some partial type π. Then the generic of $[G,H]$ is foreign to π.*

Proof: Suppose not, and let $K < [G,H]$ be GH-invariant such that $[G,H]/K$ is π-internal. Take arbitrary $h \in H$ and two independent generics $g, g' \in G$. Then (modulo K)

$$[g,h] = [g',h] = [gg',h] = [g,h]^{g'}[g',h] = 1,$$

as this is an element of $[G,H]/K$ and hence independent from g and g'. Thus $[G,H] \leq K$, a contradiction.

Theorem 20 *If G_1 is a $P\mathfrak{R}$-component of G and G_2 is a connected proper subgroup such that the generic of G_2 is foreign to G_1/G_2, then $[G_1, G_2]$ is contained in an infinitely definable $\neg P$-group.*

Proof: Let $G = G_0$, G_1, and G_2 be as in the theorem, and let us define inductively a descending sequence of normal subgroups G_α of G such that:

(1) $G_\lambda = \bigcap_{\alpha<\lambda} G_\alpha$ for λ a limit ordinal, and
(2) $G_{\alpha+1}$ is the G/G_1-analyzable (G_1/G_2-analyzable) divisor of G_α for α even (odd).

Here a limit ordinal is considered even (odd) if it is not foreign to G/G_1 (G_1/G_2). Notice that this definition gives the old G_1 and G_2, and that the generic of $G_{\alpha+1}$ is foreign to G/G_1 (G_1/G_2) for α even (odd). Furthermore, the G_αs are connected.

Suppose G_α is P. We want to show that G_α strictly contains $G_{\alpha+1}$, i.e., its generic is not foreign to both G/G_1 and G_1/G_2. Indeed, by the minimality of G_1, the generic of G_α is not foreign to G/G_α, hence there is K normal in G_α such that G_α/K is G/G_α-internal. Now if the generic of G_α and hence G_α/K were foreign to both G/G_1 and G_1/G_2, by Fact 6 and the construction of the G_β, $\beta < \alpha$, this internality would transfer all the way through to make G_α/K even G_α/G_α-internal, giving the desired contradiction.

Now by the chain condition, G_{α_0} is $\neg P$ for some α_0 (note that α_0 cannot be a limit ordinal). Consider $[G_1, G_2]$. As the generic of G_1 is foreign to G/G_1 and the generic of G_2 is foreign to G_1/G_2, by Lemma 19 the generic of $[G_1, G_2]$ is foreign to both of them. But G/G_{α_0} is $\{G/G_1, G_1/G_2\}$-analyzable. Hence $[G_1, G_2]/G_{\alpha_0}$ is finite and trivial by connectedness.

Corollary 21 *A nonsoluble $\mathfrak{R}$-component is nonsoluble $\mathfrak{R}$-connected. A nonhypercentral $\mathfrak{R}$-component is nonhypercentral $\mathfrak{R}$-connected.*

We shall now define an analog of "having monomial U-rank".

Definition 22 A formula $\varphi(x)$ is *Frattini* if for all sets Σ of formulas, if G is $\Sigma \cup \{\varphi\}$-analyzable, then G is Σ-analyzable. We denote the set of Frattini formulas for G by $\Phi(G)$.

Definition 23 A stable group is *Frattini-free* if its generic is foreign to all Frattini formulas.

Note that a group of monomial U-rank is Frattini-free (a formula nonorthogonal to its generic carries nontrivial weight). However, if A_1 and A_2 are two groups with mutually foreign generics and $U(A_1) = \omega^{\alpha_1} \cdot n_1$ and $U(A_2) = \omega^{\alpha_2} \cdot n_2$, then $A_1 \oplus A_2$ has no Frattinis, and U-rank $\omega^{\alpha_1} \cdot n_1 \oplus \omega^{\alpha_2} \cdot n_2$, which need not be monomial. But in a rank-free context, we cannot expect to be able to distinguish between the "size" of A_1 and A_2.

Definition 24 Let G be a stable group. The *Frattini-free component G^Φ of G* is the $\Phi(G)$-analyzable divisor of G.

Proposition 25 *Let G be a stable group. Then G^Φ is nontrivial, connected, normal, and its generic is foreign to G/G^Φ. Furthermore, $\Phi(G^\Phi) = \Phi(G)$. In particular, $(G^\Phi)^\Phi = G^\Phi$, i.e., the generic of G^Φ is foreign to all Frattinis.*

Proof: Nontriviality follows immediately from Fact 4; connectivity, normality, and foreignness from Lemma 14. And if $\varphi \in \Phi(G^\Phi)$ and G is $\Sigma \cup \{\varphi\}$-analyzable, then G^Φ is $\Sigma \cup \{\varphi\}$-analyzable and hence Σ-analyzable. Therefore G is $\Sigma \cup \Phi(G)$-analyzable and so Σ-analyzable. Thus $\varphi \in \Phi(G)$.

Conversely, if $\varphi \in \Phi(G)$ and G^Φ is $\Sigma \cup \{\varphi\}$-analyzable, it follows that G is $\Sigma \cup \{\varphi\} \cup \Phi(G)$-analyzable and hence Σ-analyzable. Thus G is Σ-analyzable and $\varphi \in \Phi(G^\Phi)$.

If G is definable, G^Φ is the intersection of all definable groups H such that G/H is Frattini. (To motivate the terminology, remember that in finite group theory the Frattini subgroup is the intersection of all maximal proper subgroups, and g is an element of it iff g can be left out of any generating set for G.)

Remark 26 *In fact, for any definable superset X of G^Φ, G is X-analyzable.*

Lemma 27 *If G is definable and $Z(G^\Phi) = \{1\}$, then G is G^Φ-analyzable. More generally, if G^Φ is bi-analyzable with some definable set (in particular if T is superstable), then G is G^Φ-analyzable.*

Proof: Let φ be a formula bi-analyzable with G^Φ. Then G is $\{\varphi\} \cup \Phi(G)$-analyzable. By Fact 4, we need only finitely many formulas, and we can leave out those in $\Phi(G)$. Hence G is φ-analyzable and thus G^Φ-analyzable.

Now suppose $Z(G^\Phi) = \{1\}$. Then $G/C(G^\Phi)$ is definable, G^Φ-analyzable, and contains definably a copy of G^Φ. So we finish by the first part.

Finally, in a superstable theory there is a definable group F with $F^0 = G^\Phi$. And clearly F is F^0-analyzable (this holds for any group).

The following two lemmas tell us what happens in quotients and commutator subgroups:

Lemma 28 *Suppose $H \lhd G$, φ is a formula, and Σ, Σ' are two sets of formulas.*
(1) *If $\varphi \in \Phi(H) \cap \Phi(G/H)$, then $\varphi \in \Phi(G)$.*
(2) *If G/H is $\Sigma \cup \Sigma'$-analyzable, then $[G^{\Sigma'}, G^{\Sigma}) \leq H$.*

Proof: (1) Suppose G is $\Sigma \cup \{\varphi\}$-analyzable. Then both H and G/H are $\Sigma \cup \{\varphi\}$-analyzable and hence already Σ-analyzable. Hence G is Σ-analyzable, implying that $\varphi \in \Phi(G)$.

(2) As $G^{\Sigma'}$ is foreign to Σ' and G^{Σ} is foreign to Σ, by Lemma 19, the generic of $[G^{\Sigma'}, G^{\Sigma}]$ is foreign to $\Sigma \cup \Sigma'$. But G/H is $\Sigma \cup \Sigma'$-analyzable. Hence $[G^{\Sigma'}, G^{\Sigma}]/H$ must be finite and indeed trivial by connectedness.

Lemma 29 *Suppose G and H are normal in some stable group, and $\varphi \in \Phi([G,H])$. Then for any set Σ of formulas such that G is $\Sigma \cup \{\varphi\}$-analyzable, $G/C_G(H)$ is Σ-analyzable. Indeed, $[G^{\Sigma}, H] = 1$.*

Proof: As $\varphi \in \Phi([G,H])$, we know that $[G,H]$ is Σ-analyzable, and hence the generic of G^{Σ} is foreign to $[G,H]$. By Lemma 19, the generic of $[G^{\Sigma}, H]$ is foreign to $[G,H]$, but this can be only if it is trivial.

These lemmas suggest, for nonsoluble groups, a generalization of Frattini-freeness.

Definition 30 Let $\Phi_s(G)$ be the set of all formulas φ with the property that for all sets Σ of formulas, if $G^{\Sigma \cup \{\varphi\}}$ is soluble, then G^{Σ} is soluble.

Again by Fact 4, $G^{\Phi_s(G)}$ is an invariant subgroup of G not contained in the Radical of G (i.e. the subgroup generated by all normal soluble subgroups). Now Lemma 28(2) implies that if the generic of G is foreign to all formulas in $\Phi_s(G)$ and H is a normal subgroup, then $\Phi_s(G) \subseteq \Phi_s(G/H)$. And if the generic of G/H is foreign to all formulas in $\Phi_s(G/H)$, then $\Phi_s(G/H) \subseteq \Phi_s(G)$.

3 Semisimple groups In this section, we shall generalize Theorem 4.2 of [1] about the decomposition of a connected semisimple group of monomial U-rank $\omega^{\alpha} \cdot n$ into a finite direct product of α-semiregular groups.

Definition 31 A group is *semisimple* if it has no nontrivial abelian normal subgroups.

Definition 32 A group is *$\mathfrak{R}$-semisimple* if its generic is foreign to any abelian normal subgroup.

Lemma 33 *Let G be a stable, $\mathfrak{R}$-semisimple group, with centralizer-connected component G_1. Then $G_1/Z(G_1)$ is semisimple.*

Proof: Suppose A is a normal subgroup of G_1 such that $A/Z(G_1)$ is abelian. We claim that the generic of G is foreign to A. So suppose otherwise. A and its finitely many G-conjugates generate a (definable) soluble subgroup of G_1. Let D be its last derivative such that the generic of G is not foreign to D (this exists, as A is contained in that group). Then for $d \in D'$, because $G/C(d)$ is D'-internal, $C(d)$ must have finite index in G and hence contains G_1. Thus $D' \leq Z(G_1)$.

Now let $d \in D$ be arbitrary. There is a map from D to $Z(G_1)$ with kernel $C(d)$, namely $x \mapsto [x,d]$. Hence the generic of G is foreign to $D/C_D(d)$, and indeed to $D/Z(D)$, as $d \in D$ was arbitrary. But it is also foreign to $Z(D)$, this being an abelian normal subgroup of G_1. Now the generic of G is not foreign to D, so G/K is D-internal for some $K < G^0$, and foreign to $D/Z(D)$, hence $Z(D)$-internal by Fact 5, contradicting $\mathfrak{R}$-semisimplicity. Thus there is no such D, and the generic of G is foreign to A.

But now by the same reasoning as above for $D', A \le Z(G_1)$.

For the rest of this section, let G be a stable semisimple group.

Lemma 34 *A normal subgroup N of G is semisimple as well.*

Proof: Let A be a nontrivial abelian normal subgroup of N and X the group generated by all the G-conjugates of A. X is locally nilpotent; by the chain condition on centralizers, it has a nontrivial center Z. But then Z is a normal abelian subgroup of G, a contradiction.

Observation 35 *A nonsoluble $\mathfrak{R}$-component is the same as an infinite $\mathfrak{R}$-component.*

Corollary 36 *Any infinite $\mathfrak{R}$-component is $\mathfrak{R}$-connected.*

Proof: By Corollary 21, $(G^s)^s = G^s$. Now Lemma 34 and Observation 35 imply that $(G^\infty)^\infty = G^\infty$.

Theorem 37 *Let $H_1 \neq H_2$ be two infinite $\mathfrak{R}$-components, with generic types p_1 and p_2, respectively. Then p_1 is foreign to H_2, and $H_1 \cap H_2 = \{1\}$. In particular, H_1 and H_2 commute.*

Proof: Put $I = H_1 \cap H_2$. By $\mathfrak{R}$-connectedness, I is a proper subgroup of both H_1 and H_2, and H_2 is H_2/I-analyzable. So if p_1 were not foreign to H_2, it would be H_2- and hence H_2/I-analyzable by $\mathfrak{R}$-primarity. But then it could not be foreign to $G/H_1 \ge H_2/I$, a contradiction.

Hence p_1 is foreign to H_2. But now by Lemma 19, the generic of $[H_1, H_2]$ is foreign to H_2. Therefore this commutator subgroup is finite and trivial by connectedness. So I is abelian and trivial by semisimplicity.

Theorem 38 *Let G be a definable (stable semisimple) group. Then G^Φ is a direct product of finitely many $\mathfrak{R}$-connected groups.*

Proof: By Lemma 33, G^Φ is also semisimple, and by Theorem 37, different infinite $\mathfrak{R}$-components of G are pairwise foreign and have trivial intersection. They are nonabelian, so the chain condition on centralizers gives us a finite bound n on their number. Let them be $H_1, H_2, \ldots, H_n$ with generic types $p_1, p_2, \ldots, p_n$, and set $H = \sum_{i=1}^n H_i$, with generic type p. This sum is direct, since for $j \neq i$, $H_j \le C(H_i)$ and $C(H_i) \cap H_i = \{1\}$. Hence p is bialgebraic with $p_1 \otimes \cdots \otimes p_n$ and foreign to G^Φ/H.

Claim *G is p-analyzable.*

Proof of Claim: If G^p is nontrivial, it contains an infinite $\mathfrak{R}$-component H_i by Lemma 15. But G^p is contained in $C(H)$ (as $G/C(H)$ is H-internal), so H_i is abelian. Contradiction.

Claim *p is foreign to the Frattinis.*

Proof of Claim: Suppose not. Then there is $1 \leq i \leq n$ and a Frattini φ such that p_i is not foreign to φ. By $\mathfrak{R}$-primarity, p_i is φ-analyzable. But $G/C_G(H_i)$ is H_i-and hence p_i-internal and thus φ-analyzable. This means that $G^\Phi \leq C(H_i)$ cannot contain H_i, clearly a contradiction.

We now want to show that $G^\Phi = H$. So let H_1 be a definable supergroup of H. p is foreign to G/G^Φ and to G^Φ/H. Hence it is foreign to G/H and in particular to G/H_1. We claim that G/H_1 is a Frattini. So let Σ be a set of formulas such that G is $\Sigma \cup \{G/H_1\}$-analyzable. Then all the p_i are $\Sigma \cup \{G/H_1\}$-analyzable; as they are foreign to G/H_1, they cannot be foreign to Σ. By $\mathfrak{R}$-primarity, they are Σ-analyzable. Hence p and also G is Σ-analyzable.

But this shows that $H_1 \geq G^\Phi$, thus completing the proof of Theorem 38.

4 Soluble $\mathfrak{R}$-groups

In this section, we shall prove that a soluble Frattini-free $\mathfrak{R}$-group is nilpotent-by-abelian.

Theorem 39 *Let G be a definable $\mathfrak{R}$-group. Then no derivative of G^Φ can have an infinite strictly increasing ascending central series.*

Proof: Suppose otherwise and let $Z_1 < Z_2 < \cdots < \bigcup_{i<\omega} Z_i = Z$ be the strictly ascending central series of $(G^\Phi)^{(n)}$. (Z is not infinitely definable but a union of infinitely definable groups.)

Claim *$(G^\Phi)^{(n)}/C(Z)$ is soluble.*

Proof of Claim: Let $x \in Z_{i+1} - Z_i$, and consider the map $g \mapsto [x, g]$. This is a homomorphism from $C(Z_i)$ to $C(Z_i) \cap Z_i$ with kernel $C(x)$ and abelian image. Hence $C(x) \geq C(Z_i)'$, and consequently $C(Z_{i+1}) \geq C(Z_i)'$. By the chain condition on centralizers, $C(Z) = C(Z_{i_0})$ for some $i_0 < \omega$, and then $(G^\Phi)^{(n)}/C(Z)$ is i_0-step soluble.

But now let $H = (G^\Phi)^{(k)}$ $(n \leq k < n + i_0)$ be the last derivative of G^Φ such that $Z \not\leq C_j(H)$ for any $j < \omega$. Clearly we may assume that we work in a saturated model.

As $Z/C(Z)$ and hence Z is soluble, Z is contained in a minimal G-invariant infinitely definable soluble group S. By our choice of $H, Z \leq C_i(H')$ for some $i < \omega$, so there is a minimal $i_1 < \omega$ such that $C_{i_1}(H') \cap Z$ is not contained in $C_j(H)$ for any $j < \omega$. As $Z_j \leq C_j(H)$, this means that the sequence of $C_j(H) \cap C_{i_1}(H') \cap S$ is also infinitely strictly increasing. By our choice of i_1 and minimality of S, $C_{i_1-1}(H') \cap S \leq C_j(H)$ for some $j < \omega$, and modulo $C_j(H)$, $C_{i_1}(H') \cap S$ is centralized by H'. We assume $C_j(H)$ to be trivial.

Let $B \leq C_{i_1}(H') \cap S$ be a minimal infinitely definable G-invariant group containing elements in infinitely many of the $C_{i+1}(H) - C_i(H)$, $i < \omega$. Note that any such group contains elements in all of them, so this is a type-definable property preserved under intersections. Put $C = B \cap \bigcup_{i<\omega} C_i(H)$. $B \leq S$ is soluble and $[B, C] < B$ is contained in some $C_j(H)$ by minimality. Again by minimality, $B \leq C(B/C_j(H))$. Hence (dividing out by $C_j(H)$), we may assume that B is in fact abelian. As H is connected, if $B^0 \cap C \leq C_i(H)$ for any $i < \omega$, then $B \cap C \leq C_{i+1}(H)$. Hence B is connected.

Let $A \leq B$ be a minimal H-invariant group such that $A \cap C \not\leq C_i(H)$ for any $i < \omega$. As before, A is connected. Put $M = H/C(A)$, an abelian group ($C(A) \geq C(B) \geq H'$).

Consider $\mathrm{End}_H(A/C \cap A)$, the ring of endomorphisms of $A/C \cap A$ induced by H. Our aim will be to show that this is definable. To this end, put $R = \mathrm{End}_M(A)$ and $I = \{r \in R : rA < A\}$. Note that the elements of R are definable, and R is the union of an increasing sequence of definable sets.

Claim *For generic $g \in M$, $(g-1)A = A$. Furthermore, $I = (0)$.*

Proof of Claim: Indeed, if $(g-1)A < A$, then $(g-1)A \cap C \leq C_i(H)$ for some $i < \omega$. But every element is the product of two generics and $(gh-1) = (g-1)(h-1) + (g-1) + (h-1)$. Hence $(H-1)A \cap C \leq C_i(H)$, and thus $C \leq C_{i+1}(H)$, a contradiction.

But now take any $r \in I$. $rA \cap C \leq C_i(H)$ for some $i < \omega$, hence for any generic $g \in M$ we have $(g-1)^i r(C \cap A) = 0$. By minimality, $A = \ker(g-1)^i r$. On the other hand, R is commutative and $(g-1)A = A$. Hence $rA = 0$.

It now follows that I is a prime ideal, i.e. there are no zero-divisors.

Claim *Let $A_0 < A$ be M-invariant and $a \in A$ be generic. Then for any $r \in R$, $ra \in A_0$ implies $r = 0$.*

Proof of Claim: Suppose not, and consider another generic a' and $0 \neq r' \in R$, defined the same way as r, with $r'a' \in A_0$. Then (varying nonzero $r, r' \in R$) $A(r,r') = \{a \in A : rr'a \in A_0\}$ can contain elements in $C_{i+1}(H) - C_i(H)$ for arbitrarily big $i < \omega$ (every element in $C \cap A$ is product of two generics), but these groups are uniformly definable. It follows that there are nonzero r, r' such that $A(r,r') \cap C$ is not contained in any $C_j(H)$. But then it must equal A by minimality. So $rr'A \leq A_0$, and $rr' \in I = (0)$. Hence $r = 0$ or $r' = 0$, a contradiction.

We now claim that M is uniformly A/A_0-internal for any M-invariant $A_0 < A$. So consider $A_0 < A$ and generic $a, b \in A$ such that $aA_0 = bA_0$ and $maA_0 = nbA_0$ for some $m, n \in M$. Then $(m-n)a \in A_0$ and hence $(m-n) = 0$. Thus m is definable over $aA_0{}^\frown maA_0$.

Claim *The generic of B is not foreign to M.*

Proof of Claim: Suppose otherwise. As $H \not\leq C(A)$, H is not foreign to M, so by Lemma 19, G^Φ is not foreign to M. Hence there is a definable subgroup $D \not\geq G^\Phi$ such that $G^\Phi/(G^\Phi \cap D)$ is M-internal. We claim that G/D is a Frattini formula.

So let Σ be a set of formulas such that G is $\Sigma \cup \{G/D\}$-analyzable. Then B is also $\Sigma \cup \{G/D\}$-analyzable; as its generic is foreign to M and hence to $G^\Phi/(G^\Phi \cap D)$, it cannot be foreign to $\Sigma \cup \{G/G^\Phi\}$, nor to $\Sigma \cup \Phi(G)$. By Fact 5 there is a G-invariant subgroup B_0 such that B/B_0 is $\Sigma \cup \Phi(G)$-internal, and by minimality $B_0 \cap C \leq C_j(H)$ for some $j < \omega$. Now M is $A/(B_0 \cap A)$-internal, hence B/B_0-internal and thus $\Sigma \cup \Phi(G)$-internal; and so is $G^\Phi/(G^\Phi \cap D)$ (by M-internality). As G/G^Φ is $\Phi(G)$-analyzable, G/D is $\Sigma \cup \Phi(G)$-analyzable, whence G is $\Sigma \cup \Phi(G)$-analyzable and thus Σ-analyzable.

But this clearly implies $D \geq G^\Phi$, a contradiction.

Hence the generic of B is not foreign to M. By Fact 5, there is a G-invariant $B_0 < B$ such that B/B_0 is M-internal. Again $B_0 \cap C \leq C_j(H)$ for some $j < \omega$. Note that B_0 is definable relative to B, hence $C_G(B/B_0)$ is definable, and $H \nleq C(B/B_0)$. So if we quotient out by B_0, we retain A and $B(\mathrm{mod}\, B_0)$ and see as above that there is $B_1 < B$ such that B/B_1 is $M/C(B/B_0)$-internal.

For $r_i \neq 0$ the groups $A_i = \{a \in A : r_i a \in B_1\}$ are proper subgroups of $A (i = 1,2)$, and $A_1 + A_2 \leq \{a \in A : r_1 r_2 a \in B_1\} < A$ is again proper. Put $A_0 = \bigcup_{r \in R} \{a \in A : ra \in B_1\}$, a proper union of M-invariant definable groups, which by the second claim does not contain any generics (over the parameters needed for the definition of B_1). The proof now proceeds as that of Theorem 4.2 in [8]. R embeds into the field K of automorphisms of A/A_0 which are generated by M. For $\bar{a} \in A/A_0$, define $G(\bar{a}) = \langle K\bar{a} \rangle = K\bar{a}$ (again, not infinitely definable but the union of definable sets). Suppose that for some nonzero $\bar{a}, \bar{b} \in A/A_0$, $G(\bar{a}) < G(\bar{b})$. Then there is nonzero $\sigma \in K$ mapping $\bar{b}$ to $\bar{a}$, so $\sigma^{-1} \in K$ maps $\bar{a}$ to $\bar{b}$. Hence $\bar{b} \in G(\bar{a})$, a contradiction. Therefore the $G(\bar{a})$ are minimal K-invariant. So for any $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n \in A/A_0$, if $\bar{a}_j \notin \sum_{i \neq j} G(\bar{a}_i)$, then the sum is direct:

$$\sum_{i=1}^{n} G(\bar{a}_i) = \bigoplus_{i=1}^{n} G(\bar{a}_i).$$

However, $G/C(B/B_0)$ is definable, B/B_0-internal and hence M-internal, and for generic $a \in A$, M is in bijection with $M(aA_0)$ and thus uniformly $G(aA_0)$-internal. If the number n of direct summands in $\bigoplus_{i=1}^{n} G(\bar{a}_i)$ were not bounded, then an element $a \in A$ could algebraicize (over suitable parameters $a_i \in A$) arbitrarily many elements $m_i \in M$ via $aA_0 = \sum m_i(a_i A_0)$ and hence arbitrarily many elements in $G/C(B/B_0)$. But $aA_1 \in \mathrm{dcl}(aB_1)$ and aB_1 is algebraic over a bounded number of elements $\bar{m} \in H/C(B/B_0)$ by the $H/C(B/B_0)$-internality of B/B_1. This contradicts the $\mathfrak{R}$-ness of $G/C(B/B_0)$. So $A/A_1 = \sum_{i=1}^{n} G(\bar{a}_i)$ for some $n < \omega$. By compactness this means that there is a definable subset $R' \subseteq R$ and some definable M-invariant $A_1 < A_0$ such that for any $a_1, \ldots, a_n$ with $R'a_i \cap \sum_{j \neq i} R'a_j = \{0\}$, we have $A = (R')^{-1}(A_1 + \sum_{i=1}^{n} R'a_i)$. However, we now can replace the a_i by $r_i a$ for any generic $a \in A$ and suitable $r_i \in R$, to see that for any $q \in R$ there is $0 \neq r \in R'$ satisfying

$$(*) \qquad rqa \in A_1 + \sum_{i=1}^{n} R'(r_i a).$$

Put $R'' = \sum_{i=1}^{n} R' r_i$. Then (*) implies that for any q in R there is $r \in R'$ and $r' \in R''$ with $(rq - r')a \in A_1$. As a was generic over A_1, $rq = r'$. Now if for arbitrarily big j there were $0 \neq r \in R''$ and $z \in A \cap (C_j(H) - C_{j-1}(H))$ such that $rz \in A_1$, by compactness we would get $0 \neq r \in R''$ such that $\{x \in A : rx \in A_1\} \cap C \nleq C_j(H)$ for any $j < \omega$. By minimality this subgroup equals A, a contradiction. So there is $j_0 < \omega$ such that for any $0 \neq r \in R''$ we have $\{a \in A \cap C : ra \in A_1\} \leq C_{j_0}(H)$. But there is $c \in A \cap (C_{j_0+1}(H) - C_{j_0}(H))$, and for some $q \in R$, $qc = 0$. However, for some $r \in R'$ and $r' \in R''$ we have $rq = r'$ and hence $r'c \in A_1$, a contradiction.

This finishes the proof of Theorem 39.

Theorem 40 *Let G be a definable $\mathfrak{R}$-group such that G^Φ is soluble. Then $(G^\Phi)'$ is nilpotent.*

Proof: Let $H = (G^\Phi)^{(n)}$ be the last nonnilpotent derivative. Assume for a contradiction that $n > 0$. We shall successively quotient out by $Z(H)$ and claim that after each step $Z(H)$ still remains nontrivial. This will clearly contradict Theorem 39, and we finish.

Denote the centralizer-connected component of G by G^{cc}, and let $A \leq Z(H')$ be G^{cc}-minimal. If A is finite, it is central and we are done. Otherwise, it must be connected, and we consider the action of $G^{cc}/C(A)$ on A.

Let M be some definable centralizer-connected abelian supergroup of $H/C(A)$ in $G^{cc}/C(A)$. Suppose first that the generic of A is not foreign to M. By Fact 9, A is definably isomorphic to an n-dimensional vector space (for some $n < \omega$), on which the centralizer-connected component G_0 of $G^{cc}/C(A)$ acts linearly and M scalarly. But for any $m \in M$, $\det(m) = 1$ (this is a commutator) and hence $m = \rho \cdot I$ for an n-th root of unity ρ. By connectedness, M centralizes A, i.e., $M = \{1\}$, a contradiction.

Hence the generic of A is foreign to M. We claim that M is a Frattini. So let Σ be a set of formulas such that G is $\Sigma \cup \{M\}$-analyzable. Then A is $\Sigma \cup \{M\}$-analyzable, as its generic is foreign to M, so it cannot be foreign to Σ. Now A is G-minimal, so by Fact 5 it is Σ-internal. But obviously $M \leq G/C(A)$ is A-internal (even the latter is), whence G is Σ-analyzable, and M is a Frattini. By Lemma 19, the generic of H is foreign to M. This can only be if $C(A) \geq H$. Hence $A \leq Z(H)$.

Dividing out by $Z_1 = Z(H)$ (which is nontrivial) and repeating this construction, we get an ascending central sequence $Z_1, Z_2, \ldots, Z_n, \ldots$ for H, and it is strictly ascending as long as it does not cover H. By Theorem 39, we finish.

5 The structure theorem for $\mathfrak{R}$-groups Here, we shall prove that the Frattini-free component of a definable $\mathfrak{R}$-group G has a relatively definable Radical $R(G^\Phi)$ (i.e. a maximal normal soluble subgroup). Furthermore, $G^\Phi/R(G^\Phi)$ decomposes into a finite direct product of $\mathfrak{R}$-connected groups.

Let us notice first that Theorem 39 already says that the hypercenter of G^Φ (or any derivative) is infinitely definable and reached after finitely many steps.

The following is a slight elaboration of Theorem 6 of Wagner [9]:

Lemma 41 *Let G be a stable group and $\mathbf{S} = \{S_i : i \in I\}$ a family of uniformly definable normal soluble subgroups. Then $\mathbf{S}$ generates a normal soluble subgroup.*

Proof: We use induction on the solubility class of the elements of $\mathbf{S}$ (this is bounded by uniformity). So suppose they are abelian. Consider, for some elements $a_i \in \bigcup \mathbf{S}$, the commutators defined as follows: $b_1 = a_1$, $b_{i+1} = [b_i, a_{i+1}]$. If b_i and a_i lie in the same element of $\mathbf{S}$, $b_{i+1} = 1$. Otherwise, b_{i+1} lies in all S_j's which contain either b_i or a_{i+1}, and hence $\bigcap\{S_j : b_i \in S_j\} > \bigcap\{S_j : b_{i+1} \in S_j\}$. By Baldwin–Saxl, there is a finite bound n on how often this sequence can descend, and $\mathbf{S}$ generates a nilpotent group of class $n+1$. And this will be contained in a definable normal nilpotent group of class $n+1$.

Suppose now the theorem is true for families of n-soluble groups, and that $\mathbf{S}$ is a family of uniformly definable $(n+1)$-soluble groups. Then the derivatives of the elements of $\mathbf{S}$ generate a uniformly definable family $\mathbf{S}'$ of n-soluble groups, which by induction hypothesis is contained in a definable soluble group S'. But modulo S', $\mathbf{S}$ is a family of abelian groups and hence generates a soluble group, whose preimage S is still soluble and contains $\bigcup\mathbf{S}$.

Theorem 42 *Let G be a definable $\mathfrak{R}$-group. Then $R(G^{\Phi})$ exists and is definable relative to G^{Φ}.*

Proof: Let $N < G$ be a soluble subgroup. By Theorem 40, $(N^{\Phi})'$ is nilpotent. But then, as in Theorem 39, for $x \in Z_2((N^{\Phi})') - Z_1((N^{\Phi})')$, $(N^{\Phi})'/C(x)$ is abelian. It follows from the chain condition on centralizers that there is a uniform bound on the solubility class of $(N^{\Phi})'$ and hence also of N^{Φ}, independent of the choice of N, say k.

There is a family $\mathbf{S}$ of uniformly definable k-soluble normal subgroups of G such that every k-soluble normal subgroup of G is contained in some element of $\mathbf{S}$. By Lemma 41, $\bigcup\mathbf{S}$ is contained in a definable normal soluble subgroup S.

Now let $N > S$ be a definable normal soluble subgroup of G.

Claim *N/S is a Frattini.*

Proof of Claim: Suppose Σ is a set of formulas such that G is $\Sigma \cup \{N/S\}$-analyzable. Then in particular N is $\Sigma \cup \{N/S\}$-analyzable. But $N^{\Phi} \leq S$, so N/S is a Frattini of N, and N is already Σ-analyzable. But then G is Σ-analyzable.

Hence the generic of G^{Φ} is foreign to N/S, so N/S is centralized by G^{Φ}/S. But again by Lemma 41, any normal soluble subgroup of G^{Φ} is contained in a definable normal soluble subgroup of G. Hence $R(G^{\Phi}) = Z(G^{\Phi}/S) = C(G^{\Phi}/S) \cap G^{\Phi}$.

We can now iterate the process and consider $G_1 = C_G(G^{\Phi}/S)/R$, where R is any definable normal soluble group containing $R(G^{\Phi})$. If we had descending chain condition (even on connected groups), this would eventually force the radical of G to be definable.

Theorem 43 *Let G be a definable $\mathfrak{R}$-group. Then $G^{\Phi}/R(G^{\Phi})$ is semisimple. It is the direct sum of finitely many $\mathfrak{R}$-connected groups.*

Proof: This is similar to the proof of Theorem 38. Put $R = R(G^{\Phi})$ and consider G^{Φ}/R. (Note that R is the intersection of a definable group with G^{Φ}.) This quotient is semisimple and its infinite $\mathfrak{R}$-components are the images of the non-soluble $\mathfrak{R}$-components H_i of G^{Φ}. As in the proof of Theorem 38, there are only finitely many different ones (say n), and their sum H is direct modulo R. Let p be the generic type of H/R and p_i the generic type of H_i/R for $1 \leq i \leq n$.

As before, the p_i are $\mathfrak{R}$-primary, p is bialgebraic with $p_1 \otimes \cdots \otimes p_n$, and G/R is p-analyzable. Furthermore, if p_i were not foreign to some Frattini φ, then $G/C_G(H_i/R)$ would be φ-analyzable and thus $G^{\Phi} \leq C_G(H_i/R)$, a contradiction. So p is foreign to $\Phi(G)$.

Let H_1 be a definable supergroup of H. We claim that G/H_1 is a Frattini. Indeed, p is foreign to G/G^{Φ} and G^{Φ}/H and hence to G/H_1. Let Σ be a set of formulas such that G is $\Sigma \cup \{G/H_1\}$-analyzable. Then the p_i are not foreign to Σ (as they are foreign to G/H_1) and hence Σ-analyzable. Hence G/R is Σ-analyzable and so is G/H_1. Thus G is Σ-analyzable and G/H_1 is a Frattini. Therefore $H_1 \geq G^{\Phi}$.

Proposition 44 *If G is a stable, connected $|T|^+$-saturated group such that $R(G)$ is definable, then $R(G)$ is also the maximal normal locally soluble subgroup of G.*

Proof: Dividing out by $R(G)$, we may assume that G is semisimple. So suppose that there is a nontrivial locally soluble normal subgroup S. Take any $1 \neq x \in S$. There is a number n such that every abelian group defined by centralizers which is normalized by $x^{g_1}, \ldots, x^{g_n}$ for n independent generics $g_1, \ldots, g_n$ is actually normalized by x^g for generic $g \in G$. (Being normalized is expressible by a formula φ, so only $n-1$ independent generics can fork via φ with that group.) Let S_1 be the group generated by $x, x^{g_1}, \ldots, x^{g_n}$ for n independent generics. S_1 has a characteristic abelian subgroup A. Now if A_1 is the intersection of all centralizers containing A, A_1 will be normalized by $x^{g_1}, \ldots, x^{g_n}$ and hence by x^g for generic $g \in G$. In particular, $N = \bigcap_{g \in G} N(A_1)^g$ is a nontrivial normal subgroup of G containing x^g for generic g, and hence the whole of $\langle x^G \rangle$. So $A_2 = A_1 \cap N$ contains A and is normalized by N, as are all its G-conjugates. By Lemma 41, they generate a soluble G-invariant group S, but this contradicts semisimplicity of G.

6 Open problems We close with a list of questions we have been unable to settle:

Problem 45 *Is the radical of an $\mathfrak{R}$-group definable?*

Problem 46 *Is the Fitting subgroup (maximal normal nilpotent subgroup) of the Frattini-free component of an $\mathfrak{R}$-group definable? What about the Fitting subgroup of the whole group?*

Obviously, in these two problems, definability is equivalent with existence.

Problem 47 *Can we say something about the radical and the Fitting subgroup of an arbitrary stable group?*

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