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# A Note on Generically Stable Measures and fsg Groups

Ehud Hrushovski, Anand Pillay, and Pierre Simon

**Abstract** We prove (Proposition 2.1) that if $\mu$ is a generically stable measure in an NIP (no independence property) theory, and $\mu(\varphi(x,b)) = 0$ for all $b$, then for some $n$, $\mu^{(n)}\big(\exists y(\varphi(x_1,y) \wedge \cdots \wedge \varphi(x_n,y))\big) = 0$. As a consequence we show (Proposition 3.2) that if $G$ is a definable group with fsg (finitely satisfiable generics) in an NIP theory, and $X$ is a definable subset of $G$, then $X$ is generic if and only if every translate of $X$ does not fork over $\emptyset$, precisely as in stable groups, answering positively an earlier problem posed by the first two authors.

## 1 Introduction and Preliminaries

This short paper is a contribution to the generalization of stability theory and stable group theory to NIP theories and also provides another example where we need to resort to measures to prove statements (about definable sets and/or types) which do not explicitly mention measures. The observations in the current paper can and will be used in the future to sharpen existing results around measure and NIP theories (and this is why we wanted to record the observations here). Included in these sharpenings will be the following:

(i) replacing average types by generically stable types in a characterization of strong dependence in terms of measure and weight in Pillay [6], and

(ii) showing the existence of "external generic types" (in the sense of Newelski [5]), over any model, for fsg groups in NIP theories, improving on Lemma 4.14 and related results from [5].

If $p(x) \in S(A)$ is a stationary type in a stable theory and $\varphi(x,b)$ is any formula, then we know that $\varphi(x,b) \in p|\mathfrak{C}$ if and only if $\models \bigwedge_{i=1,\dots,n} \varphi(a_i,b)$ for some independent realizations $a_1,\dots,a_n$ of $p$ (for some $n$ depending on $\varphi(x,y)$). Hence $\varphi(x,b) \notin p|\mathfrak{C}$ for all $b$ implies that (and is clearly implied by) the inconsistency of $\bigwedge_{i=1,\dots,n} \varphi(a_i,y)$ for some (any) independent set $a_1,\dots,a_n$ of realizations of $p$. This also holds for generically stable types in NIP theories (as well as for generically

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stable types in arbitrary theories, as defined in Pillay and Tanovic [7]). In [6], an analogous result was proved for "average measures" in strongly dependent theories. Here we prove it (Proposition 2.1) for generically stable measures in arbitrary NIP theories and give a generalization as well (Remark 2.2).

The fsg condition on a definable group $G$ is a kind of "definable compactness" assumption, and in fact means precisely this in $o$-minimal theories and suitable theories of valued fields (and of course stable groups are fsg). The genericity of a definable subset $X$ of $G$ means that finitely many translates of $X$ cover $G$. Proposition 2.1 is used to show that for $X$ a definable subset of an fsg group $G$, $X$ is generic if and only if every translate of $X$ does not fork over $\emptyset$. This is a somewhat striking extension of stable group theory to the NIP environment.

We work with an NIP theory $T$ and inside some monster model $\mathfrak{C}$. If $A$ is any set of parameters, let $L_x(A)$ denote the Boolean algebra of $A$-definable sets in the variable $x$. A *Keisler measure* over $A$ is a finitely additive probability measure on $L_x(A)$. Equivalently, it is a regular Borel probability measure on the compact space $S_x(A)$. We will denote by $\mathfrak{M}_x(A)$ the space of Keisler measures over $A$ in the variable $x$. We might omit $x$ when it is not needed or when it is included in the notation of the measure itself (e.g., $\mu_x$). If $X$ is a sort, or more generally a definable set, we may also use notation such as $L_X(A)$, $S_X(A)$, $\mathfrak{M}_X(A)$, where, for example, $S_X(A)$ denotes the complete types over $A$ which contain the formula defining $X$ (or which "concentrate on $X$").

**Definition 1.1** A type $p \in S_x(A)$ is *weakly random* for $\mu_x$ if $\mu(\varphi(x)) > 0$ for any $\varphi(x) \in L(A)$ such that $p \vdash \varphi(x)$. A point $b$ is weakly random for $\mu$ over $A$ if $\operatorname{tp}(b/A)$ is weakly random for $\mu$.

We briefly recall some definitions and properties of Keisler measures, referring the reader to Hrushovski, Pillay, and Simon [4] for more details.

If $\mu \in \mathfrak{M}_x(\mathfrak{C})$ is a global measure and $M$ is a small model, then we say that $\mu$ is $M$-invariant if $\mu(\varphi(x,a) \triangle \varphi(x,a')) = 0$ for every formula $\varphi(x,y)$ and $a, a' \in \mathfrak{C}$ having the same type over $M$. Such a measure admits a Borel defining scheme over $M$. For every formula $\varphi(x,y)$, the value $\mu(\varphi(x,b))$ depends only on $\operatorname{tp}(b/M)$ and for any Borel $B \subset [0,1]$, the set $\{p \in S_y(M) : \mu(\varphi(x,b)) \in B$ for some $b \models p\}$ is a Borel subset of $S_y(M)$.

Let $\mu_x \in \mathfrak{M}(\mathfrak{C})$ be $M$-invariant. If $\lambda_y \in \mathfrak{M}(\mathfrak{C})$ is any measure, then we can define the *invariant extension* of $\mu_x$ over $\lambda_y$, denoted by $\mu_x \otimes \lambda_y$. It is a measure in the two variables $x, y$ defined in the following way. Let $\varphi(x,y) \in L(\mathfrak{C})$. Take a small model $N$ containing $M$ and the parameters of $\varphi$. Define $\mu_x \otimes \lambda_y(\varphi(x,y)) = \int f(p)\, d\lambda_y$, the integral ranging over $S_y(N)$ where $f(p) = \mu_x(\varphi(x,b))$ for $b \in \mathfrak{C}$, $b \models p$. (This function is Borel by Borel definability.) It is easy to check that this does not depend on the choice of $N$.

If $\lambda_y$ is also invariant, then we can also form the product $\lambda_y \otimes \mu_x$. In general it will not be the case that $\lambda_y \otimes \mu_x = \mu_x \otimes \lambda_y$.

If $\mu_x$ is a global $M$-invariant measure, then we define by induction: $\mu^{(n)}_{x_1,\dots,x_n}$ by $\mu^{(1)}_{x_1} = \mu_{x_1}$ and $\mu^{n+1}_{x_1,\dots,x_{n+1}} = \mu_{x_{n+1}} \otimes \mu^{(n)}_{x_1,\dots,x_n}$. We let $\mu^{(\omega)}_{x_1 x_2 \dots}$ be the union and call it the *Morley sequence* of $\mu_x$.

Special cases of $M$-invariant measures include definable and finitely satisfiable measures. A global measure $\mu_x$ is *definable* over $M$ if it is $M$-invariant and for every formula $\varphi(x,y)$ and open interval $I \subset [0,1]$ the set $\{p \in S_y(M) : \mu(\varphi(x,b)) \in I$

for some $b \models p\}$ is open in $S_y(M)$. The measure $\mu$ is *finitely satisfiable* in $M$ if $\mu(\varphi(x,b)) > 0$ implies that $\varphi(x,b)$ is satisfied in $M$. Equivalently, any weakly random type for $\mu$ is finitely satisfiable in $M$.

**Lemma 1.2** *Let $\mu \in \mathfrak{M}_x(\mathfrak{C})$ be definable over $M$, and let $p(x) \in S_x(\mathfrak{C})$ be weakly random for $\mu$. Let $\varphi(x_1, \dots, x_n)$ be a formula over $\mathfrak{C}$. Suppose that $\varphi(x_1, \dots, x_n) \in p^{(n)}$. Then $\mu^{(n)}(\varphi(x_1, \dots, x_n)) > 0$.*

**Proof** Note that $p^{(m)}$ is $M$-invariant for all $m$. The proof of the lemma is by induction on $n$. For $n = 1$ it is just the definition of weakly random. Assume this to be true for $n$, and we prove it for $n+1$. So suppose that $\varphi(x_1, \dots, x_n, x_{n+1}) \in p^{(n+1)}$. This means that for $(a_1, \dots, a_n)$ realizing $p^{(n)}|M$, $\varphi(a_1, \dots, a_n, x) \in p$. So as $p$ is weakly random for $\mu$, $\mu(\varphi(a_1, \dots, a_n, x)) = r > 0$. So as $\mu$ is $M$-invariant, $\mathrm{tp}(a'_1, \dots, a'_n/M) = \mathrm{tp}(a_1, \dots, a_n/M)$ implies that $\mu(\varphi(a'_1, \dots, a'_n, x)) = r$ and thus also that $r - \epsilon < \mu(\varphi(a'_1, \dots, a'_n, x))$ for any small positive $\epsilon$. By definability of $\mu$ and compactness there is a formula $\psi(x_1, \dots, x_n) \in \mathrm{tp}(a_1, \dots, a_n/A)$ such that $\models \psi(a'_1, \dots, a'_n)$ implies that $0 < r - \epsilon < \mu(\varphi(a'_1, \dots, a'_n, x))$. By the induction hypothesis, $\mu^{(n)}(\psi(x_1, \dots, x_n)) > 0$. So by definition of $\mu^{(n+1)}$ we have that $\mu^{(n+1)}(\varphi(x_1, \dots, x_n, x_{n+1})) > 0$, as required. □

A measure $\mu_{x_1,\dots,x_n}$ is *symmetric* if for any permutation $\sigma$ of $\{1, \dots, n\}$ and any formula $\varphi(x_1, \dots, x_n)$, we have $\mu(\varphi(x_1, \dots, x_n)) = \mu(\varphi(x_{\sigma.1}, \dots, x_{\sigma.n}))$. A special case of a symmetric measure is given by powers of a generically stable measure as we recall now. The following is Theorem 3.2 of [4].

**Fact 1.3** Let $\mu_x$ be a global $M$-invariant measure. Then the following are equivalent:

(1) $\mu_x$ is both definable and finitely satisfiable (necessarily over $M$);
(2) $\mu^{(n)}_{x_1,\dots,x_n}|_M$ is symmetric for all $n < \omega$;
(3) for any global $M$-invariant Keisler measure $\lambda_y$, $\mu_x \otimes \lambda_y = \lambda_y \otimes \mu_x$;
(4) $\mu$ commutes with itself: $\mu_x \otimes \mu_y = \mu_y \otimes \mu_x$.

If $\mu_x$ satisfies one of those properties, we say that it is *generically stable*.

If $\mu \in \mathfrak{M}_x(A)$ and $D$ is a definable set such that $\mu(D) > 0$, we can consider the *localization* of $\mu$ at $D$ which is a Keisler measure $\mu_D$ over $A$ defined by $\mu_D(X) = \mu(X \cap D)/\mu(D)$ for any definable set $X$.

We will use the notation $\mathrm{Fr}(\theta(x), x_1, \dots, x_n)$ to mean

$$\frac{1}{n}\left|\{i \in \{1, \dots, n\} : \models \theta(x_i)\}\right|.$$

The following is a special case of Lemma 3.4 of [4].

**Proposition 1.4** *Let $\varphi(x,y)$ be a formula over $M$, and fix $r \in (0,1)$ and $\epsilon > 0$. Then there is $n$ such that for any symmetric measure $\mu_{x_1,\dots,x_{2n}}$, we have*

$$\mu_{x_1,\dots,x_{2n}}\big(\exists y \big(\big|\mathrm{Fr}\big(\varphi(x,y), x_1, \dots, x_n\big) - \mathrm{Fr}\big(\varphi(x,y), x_{n+1}, \dots, x_{2n}\big)\big| > r\big)\big) \le \epsilon.$$

## 2 Main Result

**Proposition 2.1** *Let $\mu_x$ be a global generically stable measure. Let $\varphi(x,y)$ be any formula in $L(\mathfrak{C})$. Suppose that $\mu(\varphi(x,b)) = 0$ for all $b \in \mathfrak{C}$. Then there is $n$ such that $\mu^{(n)}\big(\exists y (\varphi(x_1, y) \wedge \cdots \wedge \varphi(x_n, y))\big) = 0$.*

*Moreover, $n$ depends only on $\varphi(x,y)$ and not on $\mu$.*

**Proof** Let $\mu_x$ be a global generically stable measure, and let $M$ be a small model over which $\varphi(x, y)$ is defined and such that $\mu_x$ is $M$-invariant. Assume that $\mu(\varphi(x, b)) = 0$ for all $b \in \mathfrak{C}$. For any $k$, define

$$W_k = \Big\{(x_1, \dots, x_k) :\models \exists y\Big(\bigwedge_{i=1,\dots,k} \varphi(x_i, y)\Big)\Big\}.$$

This is a definable set. We want to show that $\mu^{(n)}(W_n) = 0$ for $n$ big enough. Assume, for a contradiction, that this is not the case.

Let $n$ be given by Proposition 1.4 for $r = 1/2$ and $\epsilon = 1/2$. Consider the measure $\lambda_{x_1,\dots,x_{2n}}$ over $M$ defined as being equal to $\mu^{(2n)}$ localized on the set $W_{2n}$. (By our assumption, this is well defined.) As the measure $\mu^{(2n)}$ is symmetric and the set $W_{2n}$ is symmetric in the $2n$ variables, the measure $\lambda = \lambda_{x_1,\dots,x_{2n}}$ mentioned above is symmetric. Let $\chi(x_1, \dots, x_{2n})$ be the formula $(x_1, \dots, x_{2n}) \in W_{2n} \wedge \forall y\big(|\operatorname{Fr}(\varphi(x, y), x_1, \dots, x_n) - \operatorname{Fr}(\varphi(x, y), x_{n+1}, \dots, x_{2n})| \leq 1/2\big)$. By definition of $n$, we have $\lambda\big(\exists y(|\operatorname{Fr}(\varphi(x, y), x_1, \dots, x_n) - \operatorname{Fr}(\varphi(x, y), x_{n+1}, \dots, x_{2n})| > 1/2)\big) \leq 1/2$. Therefore $\mu^{(2n)}(\chi(x_1, \dots, x_{2n})) > 0$.

As $\mu$ is $M$-invariant, we can write

$$\mu^{(2n)}\big(\chi(x_1, \dots, x_{2n})\big) = \int_{q \in S_{x_1,\dots,x_n}(M)} \mu^{(n)}\big(\chi(q, x_{n+1}, \dots, x_{2n})\big)\, d\mu^{(n)},$$

where $\mu^{(n)}(\chi(q, x_{n+1}, \dots, x_{2n}))$ stands for $\mu^{(n)}(\chi(a_1, \dots, a_n, x_{n+1}, \dots, x_{2n}))$ for some (any) realization $(a_1, \dots, a_n)$ of $q$. As $\mu^{(2n)}(\chi(x_1, \dots, x_{2n})) > 0$, there is $q \in S_{x_1,\dots,x_n}(M)$ such that

$$\mu^{(n)}\big(\chi(q, x_{n+1}, \dots, x_{2n})\big) > 0. \tag{*}$$

Fix some $(a_1, \dots, a_n) \models q$. By (*), we have $(a_1, \dots, a_n) \in W_n$. So let $b \in \mathfrak{C}$ be such that $\models \bigwedge_{i=1,\dots,n} \varphi(a_i, b)$. Again by (*), we can find some $(a_{n+1}, \dots, a_{2n})$ weakly random for $\mu^{(n)}$ over $Ma_1, \dots, a_n b$ and such that

$$\models \chi(a_1, \dots, a_n, a_{n+1}, \dots, a_{2n}). \tag{**}$$

In particular, for $j = n+1, \dots, 2n$, $a_j$ is weakly random for $\mu$ over $Mb$, and hence $\models \neg\varphi(a_j, b)$. But then $|\operatorname{Fr}(\varphi(x, b); a_1, \dots, a_n) - \operatorname{Fr}(\varphi(x, b); a_{n+1}, \dots, a_{2n})| = 1$. This contradicts (**). □

**Remark 2.2** The proof above adapts to showing the following generalization. Let $\mu_x$ be a global generically stable measure, and let $\varphi(x, y)$ be a formula in $L(\mathfrak{C})$. Let $\Sigma(x)$ be the partial type (over the parameters in $\varphi$ together with a small model over which $\mu$ is definable) defining $\{b : \mu(\varphi(x, b)) = 0\}$. Then for some $n$, $\mu^{(n)}\big(\exists y(\Sigma(y) \wedge \varphi(x_1, y) \wedge \cdots \wedge \varphi(x_n, y))\big) = 0$.

## 3 Generics in fsg Groups

Let $G$ be a definable group, without loss defined over $\emptyset$. We call a definable subset $X$ of $G$ *left (right) generic* if finitely many left (right) translates of $X$ cover $G$, and a type $p(x) \in S_G(A)$ is left (right) generic if every formula in $p$ is. In Hrushovski, Peterzil, and Pillay [2], we originally defined $G$ to have "finitely satisfiable generics," or to be fsg, if there is some global complete type $p(x) \in S_G(\mathfrak{C})$ of $G$, every left $G$-translate of which is finitely satisfiable in some fixed small model $M$.

The following summarizes the situation, where the reader is referred to [2, Proposition 4.2] for (i) and [3, Theorem 7.7] and [4, Theorem 4.3] for (ii), (iii), and (iv).

**Fact 3.1** Suppose that $G$ is an fsg group. Then we have the following.

(i) A definable subset $X$ of $G$ is left generic iff it is right generic, and the family of nongeneric definable sets is a (proper) ideal of the Boolean algebra of definable subsets of $G$.

(ii) There is a left $G$-invariant Keisler measure $\mu \in \mathfrak{M}_G(\mathfrak{C})$ which is generically stable.

(iii) Moreover, $\mu$ from (ii) is the unique left $G$-invariant global Keisler measure on $G$ as well as the unique right $G$-invariant global Keisler measure on $G$.

(iv) Moreover, $\mu$ from (ii) is *generic* in the sense that for any definable set $X$, $\mu(X) > 0$ iff $X$ is generic.

Remember that a definable set $X$ (or rather a formula $\varphi(x, b)$ defining it) forks over a set $A$ if $\varphi(x, b)$ implies a finite disjunction of formulas $\psi(x, c)$, each of which divides over $A$; and $\psi(x, c)$ is said to divide over $A$ if for some $A$-indiscernible sequence $(c_i : i < \omega)$ with $c_0 = c$, $\{\varphi(x, c_i) : i < \omega\}$ is inconsistent.

**Proposition 3.2** *Suppose that $G$ is fsg and that $X \subseteq G$ a definable set. Then $X$ is generic if and only if for all $g \in X$, $g \cdot X$ does not fork over $\emptyset$ (if and only if for all $g \in G$, $X \cdot g$ does not fork over $\emptyset$).*

**Proof** Left to right: It suffices to prove that any generic definable set $X$ does not fork over $\emptyset$, and as the set of nongenerics forms an ideal it is enough to prove that any generic definable set does not divide over $\emptyset$. This is carried out in (the proof of) [3, Proposition 5.12].

Right to left: Assume that $X$ is nongeneric. We will prove that for some $g \in G$, $g \cdot X$ divides over $\emptyset$ (so also forks over $\emptyset$).

Let $\mu_x$ be the generically stable $G$-invariant global Keisler measure given by Fact 3.1. Let $M_0$ be a small model such that $\mu$ does not fork over $M_0$ (namely, as $\mu$ is generic, every generic formula does not fork over $M_0$) and $X$ is definable over $M_0$. Let $\varphi(x, y)$ denote the formula defining $\{(x, y) \in G \times G : y \in x \cdot X\}$. So $\varphi$ has additional (suppressed) parameters from $M_0$. Note that for $b \in G$, $\varphi(x, b)$ defines the set $b \cdot X^{-1}$. As $X$ is nongeneric, so is $X^{-1}$, and so also is $b \cdot X^{-1}$ for all $b \in G$. Hence, as $\mu$ is generic, $\mu(\varphi(x, b)) = 0$ for all $b$. By Proposition 2.1, for some $n$ $\mu^{(n)}\big(\exists y(\varphi(x_1, y) \wedge \cdots \wedge \varphi(x_n, y))\big) = 0$. Let $p$ be any weakly random type for $\mu$ (which in this case amounts to a global generic type, which we note is $M_0$-invariant). So by Lemma 1.2 the formula $\exists y(\varphi(x_1, y) \wedge \cdots \wedge \varphi(x_n, y)) \notin p^{(n)}$. Let $(a_1, \ldots, a_n)$ realize $p^{(n)}|M_0$. Then $(a_1, \ldots, a_n)$ extends to an $M_0$-indiscernible sequence $(a_i : i = 1, 2, \ldots)$, a Morley sequence in $p$ over $M_0$, and $\models \neg\exists y(\varphi(a_1, y) \wedge \cdots \wedge \varphi(a_n, y))$. So, in particular, $\{\varphi(a_i, y) : i = 1, 2, \ldots\}$ is inconsistent. Hence the formula $\varphi(a_i, y)$ divides over $M_0$, and so also divides over $\emptyset$. But $\varphi(a_1, y)$ defines the set $a_1 \cdot X$, so $a_1 \cdot X$ divides over $\emptyset$, as required. □

Recall that we called a global type $p(x)$ of a $\emptyset$-definable group $G$ *left f-generic* if every left $G$-translate of $p$ does not fork over $\emptyset$.

We conclude the following (answering positively [3, Problem 5.5] as well as strengthening Conversano and Pillay [1, Lemma 4.14]).

**Corollary 3.3** *Suppose that $G$ is fsg and that $p(x) \in S_G(\mathfrak{C})$. Then the following are equivalent:*

(i) *$p$ is generic;*

(ii) *$p$ is left (right) $f$-generic;*

(iii) *(left or right)* $\mathrm{Stab}(p)$ *has bounded index in G (where left* $\mathrm{Stab}(p) = \{g \in G : g \cdot p = p\}$*).*

**Proof** The equivalence of (i) and (ii) is given by Proposition 3.2 and the definitions. We know from [2, Corollary 4.3] that if $p$ is generic, then $\mathrm{Stab}(p)$ is precisely $G^{00}$. Now suppose that $p$ is nongeneric. Hence there is a definable set $X \in p$ such that $X$ is nongeneric. Let $M$ be a small model over which $X$ is defined. Note that the fsg property is invariant under naming parameters. Hence $G$ is fsg in $\mathrm{Th}(\mathfrak{C}, m)_{m \in M}$. By Proposition 3.2 (as well as what is proved in "right to left" there), for some $g \in G$, $g \cdot X$ divides over $M$. As $X$ is defined over $M$, this means that there is an $M$-indiscernible sequence $(g_\alpha : \alpha < \bar{\kappa})$ (where $\bar{\kappa}$ is the cardinality of the monster model) and some $n$ such that $g_{\alpha_1} \cdot X \cap \cdots \cap g_{\alpha_n} \cdot X = \emptyset$ whenever $\alpha_1 < \cdots < \alpha_n$. This clearly implies that among $\{g_\alpha \cdot p : \alpha < \bar{\kappa}\}$, there are $\bar{\kappa}$ many types, whereby $\mathrm{Stab}(p)$ has unbounded index. □

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Hrushovski
Institute of Mathematics
Hebrew University (Giv'at Ram)
Jerusalem 91904
Israel
ehud@math.huji.ac.il

Pillay
School of Mathematics
University of Leeds
Leeds LS2 9JT
UK
pillay@maths.leeds.ac.uk

Simon
Département de mathématiques et applications
École normale supérieure
45 rue d'Ulm
F-75230 Paris Cedex 05
France
Pierre.simon@ens.fr