

On Independence Atoms and Keys^{*}

Miika Hannula¹, Juha Kontinen¹, and Sebastian Link²

¹ University of Helsinki, Department of Mathematics and Statistics, Helsinki, Finland {miika.hannula, juha.kontinen}@helsinki.fi

² University of Auckland, Department of Computer Science, New Zealand s.link@auckland.ac.nz

Abstract. Uniqueness and independence are two fundamental properties of data. Their enforcement in database systems can lead to higher quality data, faster data service response time, better data-driven decision making and knowledge discovery from data. The applications can be effectively unlocked by providing efficient solutions to the underlying implication problems of keys and independence atoms. Indeed, for the sole class of keys and the sole class of independence atoms the associated finite and general implication problems coincide and enjoy simple axiomatizations. However, the situation changes drastically when keys and independence atoms are combined. We show that the finite and the general implication problems are already different for keys and unary independence atoms. Furthermore, we establish a finite axiomatization for the general implication problem, and show that the finite implication problem does not enjoy a k -ary axiomatization for any k .

1 Introduction

Keys and independence atoms are two classes of data dependencies that enforce the uniqueness and independence of data in database systems. Keys are one of the most important classes of integrity constraints as effective data processing largely depends on the identification of data records. Their importance is manifested in the de-facto industry standard for data management, SQL, and they enjoy native support in every real-world database system. A relation r satisfies the key $k(X)$ for a set X of attributes, if for all tuples $t_1, t_2 \in r$ it is true that $t_1 = t_2$ whenever t_1 and t_2 have matching values on all the attributes in X . Independence atoms also occur naturally in data processing, including query languages. For example, one of the most fundamental operators in relational algebra is the Cartesian product, combining every tuple from one relation with every tuple from a second relation. In SQL, users must specify this database operation in form of the FROM clause. A relation r satisfies the independence atom $X \perp Y$ between two sets X and Y of attributes, if for all tuples $t_1, t_2 \in r$ there is some tuple $t \in r$ which matches the values of t_1 on all attributes in X and matches the values of t_2 on all attributes in Y . In other words, in relations that satisfy $X \perp Y$, the occurrence of X -values is independent of the occurrence

^{*} The first two authors were supported by grant 264917 of the Academy of Finland.

of Y -values. Due to their fundamental importance in everyday data processing in practice, both keys and independence atoms have also received detailed interest from the research community since the 1970s [1,5,6,7,8,13,14,16,17,20]. One of the core problems studied for approximately 100 different classes of relational data dependencies alone are their associated implication problems [18]. Efficient solutions to these problems have their applications in database design, query and update processing, data cleaning, exchange, integration and security to name a few. Both classes of keys and independence atoms in isolation enjoy efficient computational properties: finite and general implication problems coincide, and are axiomatizable by finite sets of Horn rules, respectively [17,8,13,18].

Given their importance for data processing in practice, given that keys and independence atoms naturally co-exist and given the long and fruitful history of research into relational data dependencies, it is rather surprising that keys and independence atoms have not been studied together. For an illustrative example of their interaction consider the SQL query Q

<pre>Query Q: SELECT p.id, COUNT(DISTINCT s.id) FROM PART p, SUPPLIER s GROUP BY p.id</pre>	<pre>Query Q': SELECT p.id, COUNT(s.id) FROM PART p, SUPPLIER s GROUP BY p.id</pre>
--	--

which returns for each part (identified by $p.id$) the number of distinct possible suppliers (identified by $s.id$). Here, the command `DISTINCT` is used to eliminate duplicate suppliers. In data processing duplicate elimination is time-consuming and not executed by default. However, duplicate elimination in query Q is redundant. The `GROUP BY` clause uses $p.id$ values to partition the Cartesian product $PART \times SUPPLIER$, generated by the `FROM` clause, into sub-relations. That is, each sub-relation satisfies the independence atom $PART \perp PART$. As the Cartesian product $PART \times SUPPLIER$ satisfies the key $k(p.id, s.id)$, so does each of its sub-relations. However, the key $k(p.id, s.id)$ and the independence atom $PART \perp PART$ together imply the key $k(s.id)$. Hence, there are no duplicate $s.id$ values in any sub-relation and Q can be replaced by the more efficient query Q' .

Motivated by these strong real-world applications and the lack of previous research we study the interaction of key dependencies and independence atoms. Somewhat surprisingly, the good computational properties that hold for each class in isolation do not carry over to the combined class. In fact, we show that for the combined class of keys and independence atoms:

- The finite and the general implication problem differ from one another,
- For keys and unary independence atoms the general implication problem has a 2-ary axiomatization by Horn rules, but
- Their finite implication problem is not finitely axiomatizable.

Our results are somewhat similar to those known for the combined class of functional dependencies (FDs) and inclusion dependencies (INDs). While both classes in isolation have matching finite and general implication problems and enjoy finite axiomatizations, the finite and the general implication problem differ

for the combined class of FDs and unary INDs already [2]. For FDs and unary INDs the general implication problem has a 2-ary axiomatization by Horn rules [4], while their finite implication problem is not finitely axiomatizable [2]. Interestingly, key dependencies are strictly subsumed by FDs. It is also known that both implication problems are undecidable for FDs and INDs [3,15], but decidable for FDs and unary INDs [4]. We would also like to mention that independence atoms form an efficient fragment of embedded multivalued dependencies whose expressivity results in the non-axiomatizability of its implication problem by a finite set of Horn rules [12] and its undecidability [11].

Our work is further motivated by the recent development of the area of dependence logic constituting a novel approach to the study of various notions of dependence and independence that is intimately linked with databases and their data dependencies [9,19]. It has been shown recently, e.g. that the general implication problem of so-called conditional independence atoms and inclusion atoms can be finitely axiomatized in this context [10]. For databases, this result establishes a finite axiomatization (utilizing implicit existential quantification) of the general implication problem for inclusion, functional, and embedded multivalued dependencies taken together. This result is similar to the axiomatization of the general implication problem for FDs and INDs [15].

2 Preliminaries

2.1 Definitions

A *relation schema* R is a set of symbols A called *attributes*, each equipped with a domain $\text{Dom}(A)$ representing the possible values that can occur in the column named A . A tuple t over R is a mapping $R \rightarrow \bigcup_{A \in R} \text{Dom}(A)$ where $t(A) \in \text{Dom}(A)$ for each $A \in R$. For a tuple t over R and $R' \subseteq R$, $t(R')$ is the restriction of t on R' . A *relation* r over R is a set of tuples t over R . If $R' \subseteq R$ and r is a relation over R , then we write $r(R')$ for $\{t(R') : t \in r\}$. If $A \in R$ is an attribute and r is a relation over R , then we write $r(A = a)$ for $\{t \in r : t(A) = a\}$. For sets of attributes X and Y , we often write XY for $X \cup Y$, and denote singleton sets of attributes $\{A\}$ by A . Also, for a relation schema $A_1 \dots A_n$, a relation $r(A_1 \dots A_n)$ is sometimes identified with the set notation $\{(a_1, \dots, a_n) \mid \exists t \in r : t(A_i) = a_i \text{ for } 1 \leq i \leq n\}$.

2.2 Independence Atoms and Key Dependencies

Let R be a relation schema and $X \subseteq R$. Then $k(X)$ is a *R-key*, given the following semantic rule for a relation r over R :

S-K $r \models k(X)$ if and only if $\forall t, t' \in r : t(X) = t'(X) \Rightarrow t = t'$.

Let R be a relation schema and $X, Y \subseteq R$. Then $X \perp Y$ is a *R-independence atom*, given the following semantic rule for a relation r over R :

S-I $r \models X \perp Y$ if and only if $\forall t, t' \in r \exists t'' \in r : t''(X) = t(X) \wedge t''(Y) = t'(Y)$.

An independence atom $X \perp Y$ is called unary if X and Y are single attributes. R -keys and R -independence atoms are together called R -constraints. If Σ is a set of R -constraints and $R' \subseteq R$, then we write $\Sigma \upharpoonright R'$ for the subset of all R' -constraints of Σ .

2.3 Implication Problems

For a set $\Sigma \cup \{\phi\}$ of independence atoms and keys we say that Σ *implies* ϕ , written $\Sigma \models \phi$, if every relation that satisfies every element in Σ also satisfies ϕ . We write $\Sigma \models_{\text{FIN}} \phi$, if every finite relation that satisfies every element in Σ also satisfies ϕ . We say that ϕ is a k -ary (finite) *implication* of Σ , if there exists $\Sigma' \subseteq \Sigma$ such that $|\Sigma'| \leq k$ and $\Sigma' \models \phi$ ($\Sigma' \models_{\text{FIN}} \phi$).

In this article we consider the axiomatizability of the so-called finite and the general implication problem for unary independence atoms and keys. The general implication problem for independence atoms and keys is defined as follows.

PROBLEM: General implication problem for independence atoms and keys	
INPUT:	Relation schema R , Set $\Sigma \cup \{\phi\}$ of independence atoms and keys over R
OUTPUT:	Yes, if $\Sigma \models \phi$; No, otherwise

The finite implication problem is defined analogously by replacing $\Sigma \models \phi$ with $\Sigma \models_{\text{FIN}} \phi$.

For a set \mathcal{J} of inference rules, we denote by $\Sigma \vdash_{\mathcal{J}} \phi$ the *inference* of ϕ from Σ . That is, there is some sequence $\gamma = [\sigma_1, \dots, \sigma_n]$ of independence atoms and keys such that $\sigma_n = \phi$ and every σ_i is an element of Σ or results from an application of an inference rule in \mathcal{J} to some elements in $\{\sigma_1, \dots, \sigma_{i-1}\}$. A set \mathcal{J} of inference rules is said to be *sound* for the general implication problem of independence atoms and keys, if for every R and for every set Σ , $\Sigma \vdash_{\mathcal{J}} \phi$ implies that $\Sigma \models \phi$. A set \mathcal{J} is called *complete* for the general implication problem if $\Sigma \models \phi$ implies that $\Sigma \vdash_{\mathcal{J}} \phi$. The (finite) set \mathcal{R} is said to be a (finite) *axiomatization* of the general implication for independence atoms and keys if \mathcal{R} is both sound and complete. These notions are defined analogously for the finite implication problem. For $k \geq 1$, an axiomatization \mathcal{R} is called k -ary if all the rules of \mathcal{R} are of the form

$$\frac{A_1 \quad A_2 \quad \dots \quad A_{l-1} \quad A_l}{B}$$

where $l \leq k$.

3 The General Implication Problem

In this section we shown that the below set of axioms \mathcal{J} is complete for the general implication problem of unary independence atoms and arbitrary keys taken together.

It is straightforward to check the soundness of the axioms \mathcal{J} .

$\frac{}{\emptyset \perp X}$ (trivial independence, $\mathcal{R}1$)	$\frac{X \perp Y}{Y \perp X}$ (symmetry, $\mathcal{R}2$)	$\frac{X \perp X \quad Y \perp Z}{XY \perp Z}$ (constancy, $\mathcal{R}3$)
$\frac{X \perp YZ}{X \perp Y}$ (decomposition, $\mathcal{R}4$)	$\frac{X \perp Y \quad XY \perp Z}{X \perp YZ}$ (exchange, $\mathcal{R}5$)	$\frac{}{k(R)}$ (trivial key, $\mathcal{R}6$)
$\frac{k(X)}{k(XY)}$ (upward closure, $\mathcal{R}7$)	$\frac{X \perp X \quad k(XY)}{k(Y)}$ (1st composition, $\mathcal{R}8$)	$\frac{X \perp Y \quad k(X)}{Y \perp Y}$ (2nd composition, $\mathcal{R}9$)

Table 1: Axiomatization \mathfrak{J} of Independence Atoms and Keys in Database Relations

Theorem 1. *The axioms \mathfrak{J} are sound for the general implication problem of independence atoms and keys.*

Next we will show that the set of axioms \mathfrak{J} is complete for the general implication problem of unary independence atoms and arbitrary keys.

Theorem 2. *Assume that R is a relation schema and $\Sigma \cup \{\phi\}$ consists of R -keys and unary R -independence atoms. Then $\Sigma \vdash_{\mathfrak{J}} \phi$ iff $\Sigma \models \phi$.*

Proof. Assume to the contrary that $\Sigma \not\vdash_{\mathfrak{J}} \phi$. We will construct a countably infinite relation witnessing $\Sigma \not\models \phi$. Let $\Sigma_i \cup \Sigma_k$ be the partition of Σ to independence atoms and keys, respectively. Let $X_1 \perp Y_1, \dots, X_N \perp Y_N$ be an enumeration of Σ_i , and let A_1, \dots, A_M be an enumeration of R . Moreover, let $R' := \{A \in R : \Sigma \vdash_{\mathfrak{J}} A \perp A\}$. We will construct an increasing chain (with respect to \subseteq) of finite relations r_n , for $n \geq 0$, such that

1. $r_n(R') = \{\mathbf{0}\}$,
2. $r_n \models \Sigma_k$, and $r_n \models X_l \perp Y_l$ if $1 \leq n = l$ modulo N .

Then letting $r := \bigcup_{n \geq 0} r_n$, we obtain that $r \models \Sigma$. Regarding ϕ , we also have two cases: ϕ is either of the form

- (i) $k(D)$ or
- (ii) $X \perp Y$.

For showing that $r \not\models \phi$, it suffices to define relations r_n so that $r_0 := \{t_0, t_1\}$ where

3. $\begin{cases} r_0 \not\models k(D) & \text{in case (i),} \\ \text{for no } t \in r_n : t(XY) = t_0(X)t_1(Y) & \text{in case (ii).} \end{cases}$

Relations r_n are now constructed inductively as follows:

– Assume first that $n = 0$. We let $r_0 := \{t_0, t_1\}$ where, for $1 \leq i \leq M$,

$$\begin{aligned} \bullet t_0(A_i) &:= \begin{cases} 0 & \text{if } A_i \in R'D \text{ in case (i) or } A_i \in R' \text{ in case (ii),} \\ i & \text{otherwise,} \end{cases} \\ \bullet t_1(A_i) &:= \begin{cases} 0 & \text{if } A_i \in R'D \text{ in case (i) or } A_i \in R' \text{ in case (ii),} \\ M + i & \text{otherwise.} \end{cases} \end{aligned}$$

Then item 1 follows from the definition. For showing that $r_0 \models \Sigma_k$, let $k(B) \in \Sigma_k$. Assume to the contrary that $r_0 \not\models k(B)$. Then we have two cases:

- In case (i), $B \subseteq R'D$ when we obtain that $\Sigma \vdash_{\gamma} B \cap R' \perp B \cap R'$ using repeatedly $\mathcal{R}2$ and $\mathcal{R}3$. From this, since $k(B) \in \Sigma$, we obtain $k(B \setminus R')$ with $\mathcal{R}8$. Since $B \setminus R' \subseteq D$, we then obtain ϕ with $\mathcal{R}7$. This again contradicts with the assumption $\Sigma \not\vdash_{\gamma} \phi$.
- In case (ii), $B \subseteq R'$, when we obtain that $\Sigma \vdash_{\gamma} B \perp X$ using first $\mathcal{R}1$ and then repeatedly $\mathcal{R}3$. From this, since $k(B) \in \Sigma$, we then obtain $X \perp X$ by $\mathcal{R}9$. From $X \perp X$ we obtain ϕ with $\mathcal{R}1$ and $\mathcal{R}3$ which contradicts with the assumption $\Sigma \not\vdash_{\gamma} \phi$.

Hence $r_0 \models k(B)$ when we obtain that $r_0 \models \Sigma_k$. For item 3, note that in case (i), $r_0 \not\models k(D)$ by the definition of r_0 . Also in case (ii) where ϕ is $X \perp Y$, we must have $XY \subseteq R \setminus R'$, since otherwise we would obtain that $\Sigma \vdash_{\gamma} \phi$ using $\mathcal{R}3$ and $\mathcal{R}2$. Thus by the definition of r_0 , we conclude that for no $t \in r_0 : t(XY) = t_0(X)t_1(Y)$.

– Assume then that r_n is a finite relation satisfying the induction assumption; we will construct a finite relation r_{n+1} also satisfying the induction assumption. Assume that $l = n + 1$ modulo N . If $r_n \models X_l \perp Y_l$, then we let $r_{n+1} := r_n$. Otherwise, let $(a_1, b_1), \dots, (a_k, b_k)$ be an enumeration of $r_n(X_l) \times r_n(Y_l) \setminus r_n(X_l Y_l)$, and assume that m is the maximal number occurring in r_n . We then let r_{n+1} be obtained by extending r_n with tuples s_i , for $1 \leq i \leq k$, such that $s_i(A_j)$, for $1 \leq j \leq M$, is defined as follows:

$$\bullet s_i(A_j) = \begin{cases} 0 & \text{if } A_j \in R', \\ a_i & \text{if } A_j = X_l, \\ b_i & \text{if } A_j = Y_l, \\ m + iM + j & \text{otherwise.} \end{cases}$$

Note that r_{n+1} is well defined: from the assumption $r_n \not\models X_l \perp Y_l$ and the induction assumption $r_n(R') = \{\bar{0}\}$ we obtain that

$$X_l, Y_l \notin R' \tag{1}$$

from which it also follows that X_l and Y_l are two distinct attributes. Now item 1 of the claim and $r_{n+1} \models X_l \perp Y_l$ of item 2 follow from the definition. For showing that $r_{n+1} \models \Sigma_k$, let $k(B) \in \Sigma_k$. Assume to the contrary that $r_{n+1} \not\models k(B)$. Then, by the definition of r_{n+1} , and since $r_n \models k(B)$ by the induction assumption, we obtain that $B \subseteq R'X_l$ or $B \subseteq R'Y_l$. Assume first that $B \subseteq R'X_l$. Since $k(B) \in \Sigma$, we then obtain $k(R'X_l)$ by $\mathcal{R}7$. By the definition of R' , we obtain $R' \perp R'$ using repeatedly $\mathcal{R}2$ and $\mathcal{R}3$. Then

from $R' \perp R'$ and $k(R'X_l)$, we obtain $k(X_l)$ by $\mathcal{R}8$. From this and $X_l \perp Y_l$ we would then, by $\mathcal{R}9$, obtain $Y_l \perp Y_l$ when $Y_l \in R'$ contradicting with (1). The case where $B \subseteq R'Y_l$ is analogous. Therefore the counter-assumption $r_{n+1} \not\models k(B)$ is false, and hence $r_{n+1} \models \Sigma_k$.

For item 3 of the claim, assume that ϕ is $X \perp Y$. Assume to the contrary that for some $t \in r_{n+1} \setminus r_n : t(XY) = t_0(X)t_1(Y)$. First recall that $XY \subseteq R \setminus R'$ because $\Sigma \not\vdash_{\mathcal{J}} \phi$ when by the definition of r_{n+1} , we obtain that $XY \subseteq X_l Y_l$. Moreover, by the assumption and the definition of t_0 and t_1 , it follows that X and Y are two distinct attributes. Hence $X \perp Y$ is either $X_l \perp Y_l$ or $Y_l \perp X_l$. Since $X_l \perp Y_l \in \Sigma$, we then, by $\mathcal{R}2$, obtain that $\Sigma \vdash_{\mathcal{J}} \phi$ which contradicts with the assumption. Hence item 3 of the induction assumption also holds. This concludes the construction of the relations r_n .

By the above construction, taking $r := \bigcup_{n \geq 0} r_n$, we obtain that $r \models \Sigma$ and $r \not\models \phi$. This concludes the proof of Theorem 2. \square

4 The Finite Implication Problem

In Subsect. 4.1 we will show that the general and the finite implication do not coincide for keys and unary independence atoms. Using these results, we will show in Subsect. 4.2 that for no k , there exists a k -ary axiomatization of the corresponding finite implication problem.

4.1 Separation of the Finite and the General Implication Problems

For $n \geq 2$, let $R_n := \{A_i, B_i : 1 \leq i \leq n\}$ be a relation schema, and let $\Sigma_n := \{A_i \perp B_i : 1 \leq i \leq n\} \cup \{k(B_i A_{i+1}) : 1 \leq i \leq n, i \text{ modulo } n\}$.³ In this subsection we will show in Lemma 1 and 2 that $\Sigma_n \models_{\text{FIN}} k(A_1 B_1)$, for $n \geq 2$, and $\Sigma_2 \not\models k(A_1 B_1)$. Hence we will obtain the main result of this subsection.

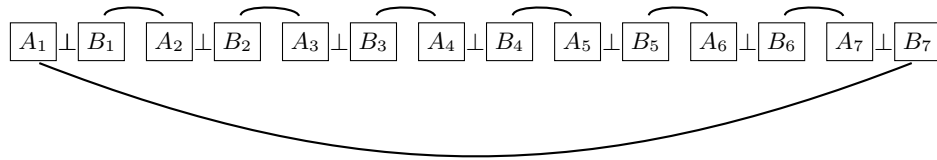


Fig. 1: Σ_7

Lemma 1. For $n \geq 2$, $\Sigma_n \models_{\text{FIN}} k(A_1 B_1)$.

³ Σ_n forms a smiley face of $n - 1$ eyes. For instance, Σ_7 is illustrated in Figure 1 where each pair of attributes connected by an edge represents a key of Σ_7 .

Proof. Let $n \geq 2$, and let r be a finite relation over R_n such that $r \models \Sigma_n$. We show that $r \models k(A_1 B_1)$. First note that since $r \models k(B_n A_1)$, we obtain that

$$|r| = |r(B_n A_1)| \leq |r(B_n)| \cdot |r(A_1)|. \quad (2)$$

Let then $2 \leq i \leq n$, and assume that $|r(B_i)| = m$. Then since $r \models A_i \perp B_i$, each member of $r(A_i)$ has at least m repetitions in r , that is, $|r(A_i = b)| \geq m$ for each $b \in r(A_i)$. Since $r \models k(B_{i-1} A_i)$, we hence obtain that $|r(B_{i-1})| \geq m$ when $|r(B_i)| \leq |r(B_{i-1})|$. Therefore we conclude that $|r(B_n)| \leq |r(B_1)|$ when $|r| \leq |r(B_1)| \cdot |r(A_1)|$ by (2). But now since $r \models A_1 \perp B_1$, we obtain that $|r(B_1)| \cdot |r(A_1)| = |r(B_1 A_1)|$ from which the claim follows. \square

The following lemma can be proved by constructing a counter example for $\Sigma_2 \models k(A_1 B_1)$, similar to the one presented in the proof of Theorem 2.

Lemma 2. $\Sigma_2 \not\models k(A_1 B_1)$.

Proof. We will construct a countably infinite relation r over R_2 witnessing $\Sigma \not\models \phi$. For this we will inductively define an increasing chain (with respect to \subseteq) of finite relations r_n over R_2 such that $r_1 \not\models k(A_1 B_1)$ and, for $n \geq 1$,

1. $r_n \models \begin{cases} k(B_2 A_1), \\ k(B_1 A_2), \end{cases}$
2. $r_n \models \begin{cases} A_1 \perp B_1 \text{ if } n \text{ is odd,} \\ A_2 \perp B_2 \text{ if } n \text{ is even.} \end{cases}$

Then, letting $r := \bigcup_{n \geq 1} r_n$, we obtain that $r \models \Sigma$ and $r \not\models \phi$. The construction of relations r_n is done as follows:

- For $n = 1$, we let $r_1(A_1 B_1 A_2 B_2) := \{(0, 0, 1, 2), (0, 0, 3, 4)\}$. Then $r_1 \models k(B_2 A_1)$, $r_1 \models k(B_1 A_2)$ and $r_1 \models A_1 \perp B_1$.
- Assume that $r_n(A_1 B_1 A_2 B_2)$ is a finite relation satisfying the induction assumption; we will construct a finite relation r_{n+1} also satisfying the induction assumption. Without loss of generality we may assume that $n + 1$ is even. Let m be the maximal number occurring in r_n , and let $(a_1, b_1), \dots, (a_k, b_k)$ enumerate the set $(r_n(A_2) \times r_n(B_2)) \setminus r_n(A_2 B_2)$. Note that this set is non-empty because otherwise, by the induction assumption, we would obtain a finite relation r witnessing $\Sigma_2 \models k(A_1 B_1)$, contrary to Lemma 1. We then let

$$r_{n+1}(A_1 B_1 A_2 B_2) := r_n \cup \{(a_i, b_i, m + 2i - 1, m + 2i) : 1 \leq i \leq k\}.$$

By the construction and the induction assumption, it is straightforward to check that items 1 and 2 hold. This concludes the construction and the proof. \square

Hence, from Lemma 1 and 2, we directly obtain the following corollary.

Corollary 1. *For keys and unary independence atoms taken together, the finite implication problem and the general implication problem do not coincide.*

4.2 Non-axiomatizability of the Finite Implication

In this subsection we will show that for no k there exists a k -ary axiomatization of the finite implication problem for unary independence atoms and keys taken together. For this, we first define, for $n \geq 2$, an upward closure of Σ_n with respect to keys as follows:

$$\text{Cl}_\uparrow(\Sigma_n) := \Sigma_n \cup \{k(D) : C \subseteq D \subseteq R_n, k(C) \in \Sigma_n\}.$$

Then we will show that $\text{Cl}_\uparrow(\Sigma_n)$ is closed under $2n - 1$ -ary finite implication. Hence, and since $k(A_1B_1) \notin \text{Cl}_\uparrow(\Sigma_n)$, it follows that the rule

$$\frac{\Sigma_n}{k(A_1B_1)}$$

for finite relations, is irreducible. That is, we cannot hope to deduce $k(A_1B_1)$ from Σ_n with a set of sound $2n - 1$ -ary rules.

Next we will show that $\text{Cl}_\uparrow(\Sigma_n)$ is closed under $2n - 1$ -ary finite implication. For this, since $\text{Cl}_\uparrow(\Sigma_n)$ is the closure of Σ_n under the unary rule $\mathcal{R}7$, it suffices to show that all $2n - 1$ -ary finite implications of Σ_n are included in $\text{Cl}_\uparrow(\Sigma_n)$. Namely, we will prove the following theorem.

Theorem 3. *Let $n \geq 2$, $\Sigma' := \Sigma_n \setminus \{\psi\}$ where $\psi \in \Sigma_n$, and let ϕ be a R_n -key or a unary R_n -independence atom such that $\Sigma' \models_{\text{FIN}} \phi$. Then $\phi \in \text{Cl}_\uparrow(\Sigma_n)$.*

This will be done in Lemma 3, 4, 5 and 6 where in each case we consider one of the four different scenarios.

In the first case, Lemma 3, ψ and ϕ are both keys (without loss of generality $\psi = k(B_nA_1)$). In the proof of the lemma, we assume that $\phi \notin \text{Cl}_\uparrow(\Sigma_n)$ and show that $\Sigma' \not\models_{\text{FIN}} \phi$ by constructing a finite relation r such that $r \models \Sigma'$ and $r \not\models \phi$. Note that then, as in the proof of Theorem 1, $|r(B_i)| \leq |r(B_{i-1})|$ and $|r(A_{i-1})| \leq |r(A_i)|$, for $2 \leq i \leq n$. Hence we must construct r so that $|r(A_1)|, |r(A_2)|, \dots$ is increasing and $|r(B_1)|, |r(B_2)|, \dots$ is decreasing. Moreover, if $\phi = k(D)$ for some $D \subseteq R_n$, and $A_iB_i \subseteq D$ for some $2 \leq i \leq n$, then we must have $|r(B_i)| + 1 \leq |r(B_{i-1})|$. Since, from $r \not\models k(A_iB_i)$ it follows that there are two distinct $t, t' \in r$ with $t(A_iB_i) = t'(A_iB_i)$. Because $r \models A_i \perp B_i$, then $t(A_i)$ must have at least $|r(B_i)| + 1$ occurrences in column A_i of r . Hence, by $r \models k(B_{i-1}A_i)$, $r(B_{i-1})$ is at least of size $|r(B_i)| + 1$. Analogously, if $A_iB_i \subseteq D$ for some $1 \leq i \leq n - 1$, then we obtain that $|r(A_i)| + 1 \leq |r(A_{i+1})|$.

Lemma 3. *Let $n \geq 2$, $\Sigma' := \Sigma_n \setminus \{\psi\}$ where $\psi \in \Sigma_n$ is a key, and let ϕ be a R_n -key such that $\Sigma' \models_{\text{FIN}} \phi$. Then $\phi \in \text{Cl}_\uparrow(\Sigma_n)$.*

Proof. By symmetry, we may assume that $\psi = k(B_nA_1)$. Let us assume to the contrary that $\phi \notin \text{Cl}_\uparrow(\Sigma_n)$ where $\phi = k(D)$ for some $D \subseteq R_n$. We will show that $\Sigma' \not\models_{\text{FIN}} \phi$ by constructing a finite relation r over R_n such that $r \models \Sigma'$ and $r \not\models \phi$. For the construction of r , we will first associate each $1 \leq i \leq n$ with natural numbers a_i and b_i . Later r will be defined inductively so that $|r(A_i)| = a_i$ and $|r(B_i)| = b_i$.

For defining a_i and b_i , first let m be the number of indices $1 \leq i \leq n$ such that $A_i B_i \subseteq D$, and let $M := (m + 3)!$. For $1 \leq i \leq n$, we define $a_i, b_i \geq 2$ as follows:⁴ We let $a_1 := 2$, and if a_i is defined, then we let

$$b_i := \begin{cases} \frac{M}{a_i} & \text{if } A_i B_i \not\subseteq D, \\ \frac{M}{a_{i+1}} & \text{if } A_i B_i \subseteq D, \end{cases}$$

and $a_{i+1} := \frac{M}{b_i}$. It is straightforward to check that with this definition, for all $1 \leq i \leq n$, $a_i, b_i \in \mathbb{N} \setminus \{0, 1\}$ and

$$M = \begin{cases} b_i \cdot a_{i+1} & \text{if } i \leq n - 1, \\ a_i \cdot b_i & \text{if } A_i B_i \not\subseteq D, \\ (a_i + 1) \cdot b_i & \text{if } A_i B_i \subseteq D. \end{cases} \quad (3)$$

a ₁	b ₁	a ₂	b ₂	a ₃	b ₃	a ₄	b ₄	a ₅	b ₅	a ₆	b ₆	a ₇	b ₇
2	240	3	240	3	180	4	180	4	144	5	144	5	144

Fig. 2

We are now ready to define r . First we define two tuples t and t' as follows:⁵

- $t(A) = 0$ for all $A \in R_n$,
- $t'(A) = \begin{cases} 0 & \text{if } A \in D, \\ 1 & \text{if } A \in R_n \setminus D. \end{cases}$

	A ₁	B ₁	A ₂	B ₂	A ₃	B ₃	A ₄	B ₄	A ₅	B ₅	A ₆	B ₆	A ₇	B ₇
t	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t'	0	0	1	1	0	0	1	1	0	0	1	1	0	1

Fig. 3

Since $\{t, t'\} \not\models k(D)$, it suffices to embed $\{t, t'\}$ to a finite relation r such that $r \models \Sigma'$. The relation r will be defined inductively over columns. Namely, for $1 \leq i \leq n$, we will define a relation $r_i = \{t_0, \dots, t_{M-1}\}$ over R_i so that, for $i > 1$,

1. $r_i \models \Sigma' \upharpoonright R_i$,

⁴ The following definition of a_i, b_i , for $1 \leq i \leq 7$, is illustrated in Figure 2 in case $D := \{A_1 B_1 A_3 B_3 A_5 B_5 A_7\}$. Note that in the example $m = 3$ and $M = 720$.

⁵ In our example, t and t' are defined as in Figure 3.

2. $r_i(B_i) = \{0, \dots, b_i - 1\}$ and $|r_i(B_i = l)| = \frac{M}{b_i}$, for each $0 \leq l \leq b_i - 1$,
3. $t \upharpoonright R_i \setminus A_i = t_0$ and $t' \upharpoonright R_i \setminus A_i = t_1$.

For $i = 1$, we will introduce one new extra symbol $*$ that appears in column A_1 . In the end of the construction, we let r be obtained from $\bigcup_{1 \leq i \leq n} r_i$ by replacing, in column A_1 , $*$ with 0. Then we will obtain that $r \models \Sigma'$ and $\{t, t'\} = \{t_0, t_1\}$.

- Assume first that $i = 1$. If $A_1 B_1 \not\subseteq D$, then we let $r_1 = \{t_0, \dots, t_{M-1}\}$ be a relation where $t_0(A_1 B_1), \dots, t_{M-1}(A_1 B_1)$ is an enumeration of $\{0, 1\} \times \{0, \dots, b_1 - 1\}$ such that $t_0(A_1 B_1) = t(A_1 B_1)$ and $t_1(A_1 B_1) = t'(A_1 B_1)$. Assume then that $A_1 B_1 \subseteq D$. Then we let $r_1 = \{t_0, \dots, t_{M-1}\}$ be a relation where $t_0(A_1 B_1), \dots, t_{M-1}(A_1 B_1)$ is an enumeration of $\{0, 1, *\} \times \{0, \dots, b_1 - 1\}$ where $t_0(A_1 B_1) = 00$ and $t_1(A_1 B_1) = *0$. Since $\Sigma' \upharpoonright R_1 = \{A_1 \perp B_1\}$, it is straightforward to check that items 1-3 hold.⁶

	A₁	B₁	A₂	B₂	A₃	B₃	A₄	B₄	A₅	B₅	A₆	B₆	A₇	B₇
t_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_1	*	0	1	1	0	0	1	1	0	0	1	1	0	0
t_2	1	0	2	0										
t_3	0	1	0	1										
t_4	*	1	1	0										
t_5	1	1	2	1										
t_6	0	2	0	2										
t_7	*	2	1	2										
t_8	1	2	2	2										
\vdots	\vdots	\vdots	\vdots	\vdots										
t_{717}	0	239	0	239										
t_{718}	*	239	1	239										
t_{719}	1	239	2	239										

Fig. 4

- Assume that $1 \leq i < n$ and $r_i(R_i) = \{t_0, \dots, t_{M-1}\}$ satisfies items 1-3. We will first extend r_i to a relation $r^*(R_i A_{i+1})$ of size M satisfying $k(B_i A_{i+1})$. First note that by the assumption $k(D) \notin \text{Cl}_\uparrow(\Sigma_n)$, $B_i A_{i+1} \not\subseteq D$ when

$$t(B_i A_{i+1}) \neq t'(B_i A_{i+1}). \quad (4)$$

Also by (3), $M = b_i \cdot a_{i+1}$ when by item 2 of the induction assumption, $r_i(B_i) = \{0, \dots, b_i - 1\}$ and $|r_i(B_i = l)| = a_{i+1}$, for each $0 \leq l \leq b_i - 1$. Hence and by (4) we can define r^* as a relation obtained from r_i by extending each $t \in r_i$ with a value $t(A_{i+1}) \in \{0, \dots, a_{i+1} - 1\}$ where $r^*(B_i A_{i+1})$ is an

⁶ The construction of r_i up to $i = 2$ is illustrated in Figure 4 in our example.

enumeration of $\{0, \dots, b_i - 1\} \times \{0, \dots, a_{i+1} - 1\}$ such that $t_0(B_i A_{i+1}) = t(B_i A_{i+1})$ and $t_1(B_i A_{i+1}) = t'(B_i A_{i+1})$. Since no repetitions occur in the enumeration, we obtain that $r^* \models k(B_i A_{i+1})$.

Next we will extend r^* to r_{i+1} satisfying items 1-3 of the induction claim. We have two cases:

- First assume that $A_{i+1} B_{i+1} \not\subseteq D$ when

$$t(A_{i+1} B_{i+1}) \neq t'(A_{i+1} B_{i+1}). \quad (5)$$

Also by the previous construction and since $b_i = b_{i+1}$ by (3), $r^*(A_{i+1}) = \{0, \dots, a_{i+1} - 1\}$ and $|r^*(A_{i+1} = l)| = b_{i+1}$ for $0 \leq l \leq a_{i+1} - 1$. Hence and by (5), we can define r_{i+1} as a relation obtained from r^* by extending each $t \in r^*$ with a value $t(B_{i+1}) \in \{0, \dots, b_{i+1} - 1\}$ where $r_{i+1}(A_{i+1} B_{i+1})$ is an enumeration of $\{0, \dots, a_{i+1} - 1\} \times \{0, \dots, b_{i+1} - 1\}$ such that $t_0(A_{i+1} B_{i+1}) = t(A_{i+1} B_{i+1})$ and $t_1(A_{i+1} B_{i+1}) = t'(A_{i+1} B_{i+1})$. By (3) and the construction it is straightforward to check that r_{i+1} satisfies items 1-3 of the induction claim.

- Assume then that $A_{i+1} B_{i+1} \subseteq D$. Then

$$t(A_{i+1} B_{i+1}) = 00 = t'(A_{i+1} B_{i+1}) \quad (6)$$

and by (3),

$$M = b_i \cdot a_{i+1} = (a_{i+1} + 1) \cdot b_{i+1}. \quad (7)$$

Recall also that by (6) and the previous construction, $r^* = \{t_0, \dots, t_{M-1}\}$ is such that $t_0(A_{i+1}) = t_1(A_{i+1}) = 0$, $r^*(A_{i+1}) = \{0, \dots, a_{i+1} - 1\}$ and $|r^*(A_{i+1} = l)| = b_i$ for $0 \leq l \leq a_{i+1} - 1$. Hence, and since $b_{i+1} < b_i$ by (7), we can also enumerate $r^*(A_{i+1})$ by pairs $(k, l) \in \{0, \dots, a_{i+1} - 1\} \times \{0, \dots, b_i - 1\}$ such that

- * $t_{(k,l)}(A_{i+1}) = k$ for all $(k, l) \in \{0, \dots, a_{i+1} - 1\} \times \{0, \dots, b_i - 1\}$,
- * $t_{(0,0)} = t_0$,
- * $t_{(0,b_{i+1})} = t_1$.

By (6) r_{i+1} should now be defined so that $r_{i+1}(A_{i+1} B_{i+1})$ has repetitions in the first two rows. Therefore, unlike in the first case, we cannot define r_{i+1} as the relation extending r^* with the values of B_{i+1} that are obtained directly from the binary enumeration presented above. Instead, we let r_{i+1} be obtained from r^* by extending each $t_{(k,l)} \in r^*$ with

$$t_{(k,l)}(B_{i+1}) = \begin{cases} l & \text{if } 0 \leq l \leq b_{i+1} - 1, \\ N - 1 & \text{if } b_{i+1} \leq l \leq b_i - 1 \text{ and } (k, l) \text{ is the } N\text{th} \\ & \text{member of } \{0, \dots, a_{i+1} - 1\} \times \{b_{i+1}, \dots, b_i - 1\} \\ & \text{in lexicographic order.} \end{cases}$$

Then we obtain that $t_0(B_{i+1}) = t_1(B_{i+1}) = 0$. Moreover by (7),

$$b_{i+1} = a_{i+1}(b_i - b_{i+1}),$$

and therefore $\{0, \dots, a_{i+1} - 1\} \times \{b_{i+1}, \dots, b_i - 1\}$ is of size b_{i+1} . Hence by the definition of r_{i+1} , we obtain that $r_{i+1}(B_{i+1}) = \{0, \dots, b_{i+1} - 1\}$ and

$$|r_{i+1}(B_{i+1} = l)| = a_{i+1} + 1 = \frac{M}{b_{i+1}},$$

for each $0 \leq l \leq b_{i+1} - 1$. Finally, since $r_{i+1}(A_i B_i) = \{0, \dots, a_{i+1} - 1\} \times \{0, \dots, b_{i+1} - 1\}$, we obtain that $r_i \models A_{i+1} \perp B_{i+1}$ when $r_{i+1} \models \Sigma' \upharpoonright R_{i+1}$. Hence r_{i+1} satisfies the induction claim. This concludes the case $A_{i+1} B_{i+1} \subseteq D$ and the construction.

We then let r be obtained from $\bigcup_{1 \leq i \leq n} r_i$ by replacing, in column A_1 , $*$ with 0. Clearly $r \models A_1 \perp B_1$ and $\{t, t'\} = \{t_0, t_1\}$. Hence we obtain that $r \models \Sigma'$ and $r \not\models k(D)$. This concludes the proof of Lemma 3. \square

The remaining cases are stated in the following lemmata. In the next case ψ is an independence atom and ϕ is a key.

Lemma 4. *Let $n \geq 2$, $\Sigma' := \Sigma_n \setminus \{\psi\}$ where $\psi \in \Sigma_n$ is a unary independence atom, and let ϕ be a R_n -key such that $\Sigma' \models_{\text{FIN}} \phi$. Then $\phi \in \text{Cl}_{\uparrow}(\Sigma_n)$.*

Proof. By symmetry, we may assume that $\psi = A_1 \perp B_1$. Let us assume to the contrary that $\phi \notin \text{Cl}_{\uparrow}(\Sigma_n)$ where $\phi = k(D)$ for some $D \subseteq R_n$. We will show that $\Sigma' \not\models_{\text{FIN}} \phi$. First we define $\Sigma^* := \Sigma_n \setminus \{k(B_n A_1)\}$. Then by the proof of Lemma 3, there exists a finite relation $r^* = \{t_0, \dots, t_{M-1}\}$ such that $r^* \models \Sigma^*$, $\{t_0, t_1\} \not\models k(D)$, $t_0(X) = 0$ for all $X \in R_n$, and

$$t_1(X) = \begin{cases} 0 & \text{if } X \in D, \\ 1 & \text{if } X \in R_n \setminus D. \end{cases}$$

We let r be obtained⁷ from r^* by replacing, for $0 \leq i \leq M - 1$, $t_i(A_1)$ with

$$\begin{aligned} & - i \quad \text{if } i \neq 1, \\ & - \begin{cases} 0 & \text{if } i = 1 \text{ and } B_n \notin D, \\ 1 & \text{if } i = 1 \text{ and } B_n \in D. \end{cases} \end{aligned}$$

From the definition of r and the fact that $A_1 B_n \not\subseteq D$ it follows that $r \not\models k(D)$ and $r \models \Sigma^* \setminus \{A_1 \perp B_1\}$. For $r \models \Sigma'$, we still need to show that $r \models k(B_n A_1)$. Because of the definition of $t_i(A_1)$ in r , $k(B_n A_1)$ could be violated only in $\{t_0, t_1\}$. In that case we would have $t_1(A_1 B_1) = 00$ in r which contradicts with the definitions. Hence we obtain that $r \not\models k(D)$ which concludes the proof. \square

In the third case ψ is a key and ϕ is an independence atom.

Lemma 5. *Let $n \geq 2$, $\Sigma' := \Sigma_n \setminus \{\psi\}$ where $\psi \in \Sigma_n$ is a key, and let ϕ be a unary R_n -independence atom such that $\Sigma' \models_{\text{FIN}} \phi$. Then $\phi \in \text{Cl}_{\uparrow}(\Sigma_n)$.*

⁷ See Fig. 5

	A_1	B_1	A_2	B_2	A_3	B_3	\dots	\dots	A_{n-1}	B_{n-1}	A_n	B_n
t_0	0											0
t_1	0											1
t_2	2											y_2
t_3	3											y_3
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
t_{M-2}	$M-2$											y_{M-2}
t_{M-1}	$M-1$											y_{M-1}

Fig. 5: r in case $B_n \notin D$

Proof. By symmetry, we may assume that $\psi = k(B_n A_1)$. Assume to the contrary that $\phi \notin \text{Cl}_\uparrow(\Sigma_n)$. We will show that $\Sigma' \not\models_{\text{FIN}} \phi$. Due to $\mathcal{R}2$ and by symmetry of Σ' , it suffices to consider only the cases where $\phi = A_i \perp Y$, for some $1 \leq i \leq n$ and $Y \in R_n \setminus \{B_i\}$.

So let $1 \leq i \leq n$. We will construct two finite relations r and r' such that

1. $r \models \Sigma'$ and $r' \models \Sigma'$,
2. $r \not\models \begin{cases} A_i \perp A_j & \text{for } j \leq i, \\ A_i \perp B_j & \text{for } j > i, \end{cases}$
3. $r' \not\models \begin{cases} A_i \perp A_j & \text{for } j > i, \\ A_i \perp B_j & \text{for } j < i. \end{cases}$

We let $r := \{t_0, t_1, t_2, t_3\}$ where we define, for $X \in R_n$,

- $t_0(X) = 0$,
- $t_1(X) = \begin{cases} 0 & \text{if } X = A_j \text{ for } j \leq i, \text{ or } X = B_j \text{ for } j > i, \\ 1 & \text{otherwise,} \end{cases}$
- $t_2(X) = \begin{cases} 0 & \text{if } X = B_i, \\ 1 & \text{otherwise,} \end{cases}$
- $t_3(X) = \begin{cases} 0 & \text{if } X = B_j \text{ for } j < i, \text{ or } X = A_j \text{ for } j > i, \\ 1 & \text{otherwise.} \end{cases}$

	A_1	B_1	\dots	\dots	A_{i-1}	B_{i-1}	A_i	B_i	A_{i+1}	B_{i+1}	\dots	\dots	A_n	B_n
t_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_1	0	1	0	1	0	1	0	1	1	0	1	0	1	0
t_2	1	1	1	1	1	1	1	0	1	1	1	1	1	1
t_3	1	0	1	0	1	0	1	1	0	1	0	1	0	1

Fig. 6: r

Then we let $r' := \{t_0, t_4\}$ where we define, for $X \in R_n$,

$$- t_4(X) = \begin{cases} 0 & \text{if } X = B_j \text{ for } j < i, \text{ or } X = A_j \text{ for } j \geq i, \\ 1 & \text{otherwise.} \end{cases}$$

	A_1	B_1	\dots	\dots	A_{i-1}	B_{i-1}	A_i	B_i	A_{i+1}	B_{i+1}	\dots	\dots	A_n	B_n
t_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_4	0	1	0	1	0	1	1	0	1	0	1	0	1	0

Fig. 7: r'

It is straightforward to check that items 1-3 hold. This concludes the proof of Lemma 5. \square

In the last case both ψ and ϕ are independence atoms.

Lemma 6. *Let $n \geq 2$, $\Sigma' := \Sigma_n \setminus \{\psi\}$ where $\psi \in \Sigma_n$ is a unary independence atom, and let ϕ be a unary R_n -independence atom such that $\Sigma' \models_{\text{FIN}} \phi$. Then $\phi \in \text{Cl}_{\uparrow}(\Sigma_n)$.*

Proof. By symmetry, we may assume that $\psi = A_1 \perp B_1$. Let us assume to the contrary that $\phi \notin \text{Cl}_{\uparrow}(\Sigma_n)$. We will show that $\Sigma' \not\models_{\text{FIN}} \phi$. Analogously to the proof of Lemma 5, it suffices to consider only the cases where $\phi = A_i \perp Y$, for some $1 \leq i \leq n$ and $Y \in R_n \setminus \{B_i\}$. Let $1 \leq i \leq n$. We will construct four relations r_0, r_1, r_2, r_3 such that

1. $r_i \models \Sigma'$ for $i = 0, 1, 2, 3$,
2. $r_0 \not\models A_i \perp A_j$ for $1 \leq j \leq n$,
3. $r_1 \not\models A_1 \perp B_j$ for $1 < j$,

and if $1 < i$,

4. $r_2 \not\models A_i \perp B_j$ for $j < i$,
5. $r_3 \not\models A_i \perp B_j$ for $i < j$,

We let $r_0 := \{t_0, t_1\}$ where we define, for $X \in R_n$,

$$- t_0(X) = 0, \\ - t_1(X) = \begin{cases} 0 & \text{if } X = B_j \text{ for } j > 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then we let $r_1 := \{t_0, t_2\}$ where we define, for $X \in R_n$,

$$- t_2(X) = \begin{cases} 0 & \text{if } X = A_j \text{ for } j > 1, \\ 1 & \text{otherwise.} \end{cases}$$

	A_1	B_1	\dots	\dots	A_{i-1}	B_{i-1}	A_i	B_i	A_{i+1}	B_{i+1}	\dots	\dots	A_n	B_n
t_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_1	1	1	1	0	1	0	1	0	1	0	1	0	1	0

Fig. 8: r_0

	A_1	B_1	\dots	\dots	A_{i-1}	B_{i-1}	A_i	B_i	A_{i+1}	B_{i+1}	\dots	\dots	A_n	B_n
t_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_2	1	1	0	1	0	1	0	1	0	1	0	1	0	1

Fig. 9: r_1

Assume then that $1 < i$. We now let $r_2 := \{t_0, t_3\}$ where we define, for $X \in R_n$,

$$- t_3(X) = \begin{cases} 0 & \text{if } X = A_j \text{ for } 1 < j < i, \text{ or } X = B_j \text{ for } i \leq j \leq n, \\ 1 & \text{otherwise.} \end{cases}$$

For item 5, note that since $i < j \leq n$, we have that $i < n$. We let $r_3 := \{t_0, t_4, t_5, t_6\}$ where we define, for $X \in R_n$,

$$- t_4(X) = \begin{cases} 0 & \text{if } X = A_1, \text{ or } X = B_j \text{ for } j \leq i, \\ 1 & \text{otherwise,} \end{cases}$$

$$- t_5(X) = \begin{cases} 0 & \text{if } X = A_j \text{ for } 1 < j \leq i, \text{ or } X = B_j \text{ for } i < j, \\ 1 & \text{otherwise,} \end{cases}$$

$$- t_6(X) = \begin{cases} 0 & \text{if } X = A_j \text{ for } i < j, \\ 1 & \text{otherwise.} \end{cases}$$

Again, it is straightforward to check that items 1-5 hold. This concludes the proof of Lemma 6. \square

From Lemma 3, 4, 5 and 6 we obtain Theorem 3. Using this we can prove the following theorem.

Theorem 4. *For no natural number k , there exists a sound and complete k -ary axiomatization of the finite implication problem for unary independence atoms and keys taken together.*

Proof. Let k be a natural number, and let n be such that $2n > k$. Then $\Sigma_n \models_{\text{FIN}} k(A_1 B_1)$ by Theorem 1. However, by the unary rule $\mathcal{R}7$ and Theorem 3, the closure of Σ_n under k -ary finite implication is $\text{Cl}_\uparrow(\Sigma_n)$. Since $k(A_1 B_1) \notin \text{Cl}_\uparrow(\Sigma_n)$, the claim follows. \square

Note that due to $\mathcal{R}4$ and $\mathcal{R}2$, for any non-unary R_n -independence atom $X \perp Y$ there exists a unary $A \perp B \notin \text{Cl}_\uparrow(\Sigma_n)$ such that $\{X \perp Y\} \models A \perp B$. Hence Theorem 3 can be extended to the case where ϕ is an independence atom of any arity. Therefore we obtain the following corollary.

	A_1	B_1	A_{i-1}	B_{i-1}	A_i	B_i	A_{i+1}	B_{i+1}	A_n	B_n
t_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_3	1	1	0	1	0	1	1	0	1	0	1	0	1	0

Fig. 10: r_2

	A_1	B_1	A_{i-1}	B_{i-1}	A_i	B_i	A_{i+1}	B_{i+1}	A_n	B_n
t_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_4	0	0	1	0	1	0	1	0	1	1	1	1	1	1
t_5	1	1	0	1	0	1	0	1	1	0	1	0	1	0
t_6	1	1	1	1	1	1	1	1	0	1	0	1	0	1

Fig. 11: r_3

Corollary 2. *For no natural number k , there exists a sound and complete k -ary axiomatization of the finite implication problem for independence atoms and keys taken together.*

5 Conclusion

We have studied the implication problem of unary independence atoms and keys taken together, both in the general and in the finite case. We gave a finite axiomatization of the general implication problem and showed that the finite implication problem has no finite axiomatization. The non-axiomatizability result holds also in case the arity of independence atoms is not restricted to one. It remains open whether the general implication problem for arbitrary independence atoms and keys enjoys a finite axiomatization, and whether the finite implication problem is undecidable.

References

1. Bojanczyk, M., Muscholl, A., Schwentick, T., Segoufin, L.: Two-variable logic on data trees and XML reasoning. *J. ACM* 56(3) (2009)
2. Casanova, M.A., Fagin, R., Papadimitriou, C.H.: Inclusion dependencies and their interaction with functional dependencies. *J. Comput. Syst. Sci.* 28(1), 29–59 (1984)
3. Chandra, A.K., Vardi, M.Y.: The implication problem for functional and inclusion dependencies is undecidable. *SIAM Journal on Computing* 14(3), 671–677 (1985)
4. Cosmadakis, S.S., Kanellakis, P.C., Vardi, M.Y.: Polynomial-time implication problems for unary inclusion dependencies. *J. ACM* 37(1), 15–46 (1990)
5. Demetrovics, J.: On the number of candidate keys. *Inf. Process. Lett.* 7(6), 266–269 (1978)
6. Demetrovics, J., Katona, G.O.H., Miklós, D., Seleznev, O., Thalheim, B.: Asymptotic properties of keys and functional dependencies in random databases. *Theor. Comput. Sci.* 190(2), 151–166 (1998)

7. Fagin, R.: A normal form for relational databases that is based on domains and keys. *ACM Trans. Database Syst.* 6(3), 387–415 (1981)
8. Geiger, D., Paz, A., Pearl, J.: Axioms and algorithms for inferences involving probabilistic independence. *Information and Computation* 91(1), 128–141 (1991)
9. Grädel, E., Väänänen, J.: Dependence and independence. *Studia Logica* 101(2), 399–410 (2013), <http://dx.doi.org/10.1007/s11225-013-9479-2>
10. Hammula, M., Kontinen, J.: A finite axiomatization of conditional independence and inclusion dependencies. In: Beierle, C., Meghini, C. (eds.) *Foundations of Information and Knowledge Systems - 8th International Symposium, FoIKS 2014, Bordeaux, France, March 3-7, 2014. Proceedings. Lecture Notes in Computer Science*, vol. 8367, pp. 211–229. Springer (2014)
11. Herrmann, C.: On the undecidability of implications between embedded multivalued database dependencies. *Information and Computation* 122(2), 221 – 235 (1995)
12. Jr., D.S.P., Parsaye-Ghomi, K.: Inferences involving embedded multivalued dependencies and transitive dependencies. In: Chen, P.P., Sprowls, R.C. (eds.) *Proceedings of the 1980 ACM SIGMOD International Conference on Management of Data, Santa Monica, California, May 14-16, 1980.* pp. 52–57. ACM Press (1980)
13. Kontinen, J., Link, S., Väänänen, J.A.: Independence in database relations. In: Libkin, L., Kohlenbach, U., de Queiroz, R.J.G.B. (eds.) *WoLLIC. Lecture Notes in Computer Science*, vol. 8071, pp. 179–193. Springer (2013)
14. Lucchesi, C.L., Osborn, S.L.: Candidate keys for relations. *J. Comput. Syst. Sci.* 17(2), 270–279 (1978)
15. Mitchell, J.C.: The implication problem for functional and inclusion dependencies. *Information and Control* 56(3), 154–173 (1983)
16. Niewerth, M., Schwentick, T.: Two-variable logic and key constraints on data words. In: *ICDT*. pp. 138–149 (2011)
17. Paredaens, J.: The interaction of integrity constraints in an information system. *J. Comput. Syst. Sci.* 20(3), 310–329 (1980)
18. Thalheim, B.: *Dependencies in relational databases.* Teubner (1991)
19. Väänänen, J.: *Dependence Logic.* Cambridge University Press (2007)
20. Wijsen, J.: On the consistent rewriting of conjunctive queries under primary key constraints. *Inf. Syst.* 34(7), 578–601 (2009)