

Proof theory of weak compactness

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Abstract

We show that the existence of a weakly compact cardinal over the Zermelo-Fraenkel's set theory ZF is proof-theoretically reducible to iterations of Mostowski collapsings and Mahlo operations.

1 Introduction

It is well known that a cardinal is weakly compact iff it is Π_1^1 -indescribable. From this characterization we see readily that the set of Mahlo cardinals below a weakly compact cardinal is stationary, i.e., every club (closed and unbounded) subset of a weakly compact cardinal contains a Mahlo cardinal. In other word, any weakly compact cardinal is hyper Mahlo. Furthermore any weakly compact cardinal κ is in the diagonal intersection $\kappa \in M^\Delta = \bigcap \{M(M^\alpha) : \alpha < \kappa\}$ for the α -th iterate M^α of the Mahlo operation M : for classes X of ordinals,

$$\kappa \in M(X) :\Leftrightarrow X \cap \kappa \text{ is stationary in } \kappa \Leftrightarrow \forall Y \subset \kappa [(Y \text{ is club}) \rightarrow X \cap Y \neq \emptyset].$$

Note that $\kappa \in M(X)$ is Π_1^1 on V_κ .

On the other side R. Jensen[11] showed under the axiom $V = L$ of constructibility that for regular cardinals κ , κ is weakly compact iff $\forall X \subset \kappa [\kappa \in M(X) \Rightarrow M(X) \cap \kappa \neq \emptyset]$ iff $\forall X \subset \kappa [\kappa \in M(X) \Rightarrow \kappa \in M(M(X))]$.

Jensen's proof in [11] yields a normal form theorem of Π_1^1 -formulae on $L_\kappa = J_\kappa$ uniformly for regular uncountable cardinals κ as follows.

For a first order formula $\varphi[D]$ with unary predicates A, D , let

$$\begin{aligned} \alpha \in S^\varphi(A) :\Leftrightarrow & \text{ there exists a limit } \beta \text{ such that } \alpha < \beta < \alpha^+, A \cap \alpha \in J_\beta, \\ & \langle J_\beta, \in, A \cap \alpha \rangle \models \forall D \subset \alpha \varphi[D], \alpha \text{ is regular in } \beta \text{ and} \\ & \exists p \in J_\beta \forall X [(p \cup \{\alpha\} \subset X \prec J_\beta) \wedge (X \cap \alpha \text{ is transitive}) \Rightarrow X = J_\beta] \end{aligned} \quad (1)$$

where α is regular in β iff there is no cofinal function from a smaller ordinal $< \alpha$ into α , which is definable on J_β .

The following Proposition 1.1 is the Lemma 5.2 in [11].

Proposition 1.1 *Let $\alpha \in S^\varphi(A)$ and β be an ordinal as in the definition of $S^\varphi(A)$. Then α is Σ_1 -singular in $\beta + 1$, i.e., there exists a cofinal function from a smaller ordinal $< \alpha$ into α , which is Σ_1 -definable on $J_{\beta+1}$.*

Fix a regular uncountable cardinal κ , a set $A \subset \kappa$. For a finite set $\{A, \dots\}$ of subsets A, \dots of κ and ordinals $\alpha < \kappa$, let $N_\alpha(A, \dots)$ denote the least Σ_1 -elementary submodel of J_{κ^+} , $N_\alpha(A, \dots) \prec_{\Sigma_1} J_{\kappa^+}$, such that $\alpha \cup \{A, \dots\} \cup \{\kappa\} \subset N_\alpha(A, \dots)$. Namely $N_\alpha(A, \dots)$ is the Σ_1 -Skolem hull $\text{Hull}_{\Sigma_1}^{J_{\kappa^+}}(\alpha \cup \{A, \dots\} \cup \{\kappa\})$ of $\alpha \cup \{A, \dots\} \cup \{\kappa\}$ on J_{κ^+} . Let

$$C(A, \dots) := \{\alpha < \kappa : N_\alpha(A, \dots) \cap \kappa \subset \alpha\}.$$

Then it is easy to see that $C(A, \dots)$ is club in κ , and definable over J_{κ^+} .

Proposition 1.2 *Let κ be a regular uncountable cardinal, $A \subset \kappa$, $\varphi[D]$ a first order formula with parameters A, D .*

1. *Suppose $\langle J_{\kappa^+}, \in, A \rangle \models \forall D \subset \kappa \varphi[D]$, and let C be a club subset of κ . Then the least element of the club set $C(A, C)$ is in $S^\varphi(A)$.*
2. *Suppose $\langle J_{\kappa^+}, \in, A \rangle \not\models \forall D \subset \kappa \varphi[D]$, then $S^\varphi(A) \cap C(A) = \emptyset$.*

Thus $\langle J_{\kappa^+}, \in, A \rangle \models \forall D \subset \kappa \varphi[D]$ iff $S^\varphi(A)$ is stationary in κ . And κ is weakly compact iff for any stationary subset $S \subset \kappa$ there exists an uncountable regular cardinal $\alpha < \kappa$ such that $S \cap \alpha$ is stationary in α .

Proof.

1.2.1. Suppose $\langle J_{\kappa^+}, \in, A \rangle \models \forall D \subset \kappa \varphi[D]$, and let C be a club subset of κ . Consider the club subset $C(A, C)$ of κ . Then $C(A, C) \subset C$. We show that $\alpha \in S^\varphi(A)$ for the least element α of $C(A, C)$. Let $\pi : \langle J_\beta, \in, A \cap \alpha, C \cap \alpha \rangle \cong N_\alpha(A, C) \prec_{\Sigma_1} J_{\kappa^+}$ be the transitive collapse of $N_\alpha(A, C)$. β is a limit ordinal with $\alpha < \beta < \alpha^+$. From $\langle J_{\kappa^+}, \in, A \rangle \models \forall D \subset \kappa \varphi[D]$ we see $\langle J_\beta, \in, A \cap \alpha \rangle \models \forall D \subset \alpha \varphi[D]$, and $A \cap \alpha, C \cap \alpha \in J_\beta$ from $A, C \in N_\alpha(A, C)$. It remains to show (1) for $p = \{A \cap \alpha, C \cap \alpha\}$. Assume $\{A \cap \alpha, C \cap \alpha, \alpha\} \subset X \prec J_\beta$ and $X \cap \alpha = \gamma$ for an ordinal $\gamma \leq \alpha$. Then $\gamma \cup \{A, C, \kappa\} \subset \pi'' X \prec N_\alpha(A, C) \prec_{\Sigma_1} J_{\kappa^+}$. This yields $N_\gamma(A, C) \prec_{\Sigma_1} \pi'' X$, and $N_\gamma(A, C) \cap \kappa \subset (\pi'' X) \cap \kappa = \pi''(X \cap \alpha) = \gamma$ by $N_\alpha(A, C) \cap \kappa \subset \alpha$. This means that $\gamma \in C(A, C)$, and hence $X \cap \alpha = \gamma = \alpha$. Therefore $\pi'' X = N_\alpha(A, C)$, and $X = J_\beta$.

1.2.2. Suppose $\langle J_{\kappa^+}, \in, A \rangle \not\models \forall D \subset \kappa \varphi[D]$. Assume $\alpha \in S^\varphi(A) \cap C(A)$. Let $\langle J_{\bar{\beta}}, \in, A \cap \alpha \rangle \cong N_\alpha(A) \prec_{\Sigma_1} J_{\kappa^+}$ be the transitive collapse of $N_\alpha(A)$. Then $\langle J_{\bar{\beta}}, \in, A \cap \alpha \rangle \not\models \forall D \subset \alpha \varphi[D]$. On the other hand we have by $\alpha \in S^\varphi(A)$, there exists a limit β such that $\langle J_\beta, \in, A \cap \alpha \rangle \models \forall D \subset \alpha \varphi[D]$, and α is Σ_1 -singular in $\beta + 1$ by Proposition 1.1. Hence $\beta < \bar{\beta}$ and α is Σ_1 -singular in $\bar{\beta}$. This means that κ is Σ_1 -singular in κ^+ . However κ is assumed to be regular. A contradiction. \square

In this paper we show that the existence of a weakly compact cardinal over the Zermelo-Fraenkel's set theory ZF is proof-theoretically reducible to iterations of Mostowski collapsings and Mahlo operations.

Let \mathbb{K} denote the formula stating that ‘there exists a weakly compact cardinal \mathcal{K} ’.

For Σ_2^1 -sentences $\varphi \equiv \exists Y \forall X \theta$, let $\varphi^{V\kappa}$ be $\exists Y \subset V_\kappa \forall X \subset V_\kappa \theta^{V\kappa}$ where θ^a denotes the result of restricting any unbounded quantifiers $\exists x, \forall x$ to $\exists x \in a, \forall x \in a$, resp.

Theorem 1.3 *There are Σ_{n+1} -formulae $\theta_n(x)$ for which the following holds:*

1. For each $n < \omega$,

$$\mathbf{ZF} + (V = L) \vdash \forall \mathcal{K}[(\mathcal{K} \text{ is a weakly compact cardinal}) \rightarrow \theta_n(\mathcal{K})]$$

and

$$\mathbf{ZF} + (V = L) \vdash \forall \mathcal{K}[\theta_{n+1}(\mathcal{K}) \rightarrow \mathcal{K} \in M(\{\pi < \mathcal{K} : \theta_n(\pi)\})].$$

2. For any Σ_2^1 -sentences φ , if

$$\mathbf{ZF} \vdash \forall \mathcal{K}[(\mathcal{K} \text{ is a weakly compact cardinal}) \rightarrow \varphi^{V\kappa}],$$

then we can find an $n < \omega$ such that

$$\mathbf{ZF} + (V = L) \vdash \forall \mathcal{K}[\theta_n(\mathcal{K}) \rightarrow \varphi^{V\kappa}].$$

Hence $\mathbf{ZF} + (V = L) + (\mathcal{K} \text{ is weakly compact})$ is $\Sigma_2^1(\mathcal{K})$ -conservative over $\mathbf{ZF} + (V = L) + \{\theta_n(\mathcal{K}) : n < \omega\}$, and $\mathbf{ZF} + (V = L) + \mathbb{K}$ is conservative over $\mathbf{ZF} + (V = L) + \{\exists \mathcal{K} \theta_n(\mathcal{K}) : n < \omega\}$, e.g., with respect to first-order sentences $\varphi^{V_{I_0}}$ for the least weakly inaccessible cardinal I_0 .

Note that $T_n = \mathbf{ZF} + (V = L) + \{\exists \mathcal{K} \theta_n(\mathcal{K})\}$ is weaker than $\mathbf{ZF} + \mathbb{K}$, e.g., $\mathbf{ZF} + \mathbb{K}$ proves the existence of a model of T_n for each $n < \omega$.

The Σ_{n+1} -formulae $\theta_n(x)$ are defined by

$$\theta_n(x) :\Leftrightarrow x \in Mh_n^{\omega_n(I+1)}.$$

The Σ_{n+1} -class Mh_n^ξ for ordinals ξ is defined through iterations of Mostowski collapsings and Mahlo operations, cf. Definition 2.2.

Let us explain some backgrounds of this paper. Π_3 -reflecting ordinals are known to be recursive analogues to weakly compact cardinals. Proof theory (*ordinal analysis*) of Π_3 -reflection has been done by M. Rathjen[12], and [1, 2, 3, 4].

As observed in [2, 5], ordinal analyses of Π_{N+1} -reflection yield a proof-theoretic reduction of Π_{N+1} -reflection in terms of iterations of Π_N -recursively Mahlo operations. Specifically we show the following Theorem 1.4 in [8]. Let $\mathbf{KP}\omega$ denote the Kripke-Platek set theory with the axiom of Infinity, $\Pi_N(a)$ a universal Π_N -formula, and $RM_N(\mathcal{X})$ the Π_N -recursively Mahlo operation for classes of transitive sets \mathcal{X} :

$$P \in RM_N(\mathcal{X}) \quad :\Leftrightarrow \quad \forall b \in P[P \models \Pi_N(b) \rightarrow \exists Q \in \mathcal{X} \cap P(Q \models \Pi_N(b))]$$

(read: P is Π_N -reflecting on \mathcal{X} .)

The iteration of RM_N along a definable relation \prec is defined as follows.

$$P \in RM_N(a; \prec) :\Leftrightarrow a \in P \in \bigcap \{RM_N(RM_N(b; \prec)) : b \in P \models b \prec a\}.$$

Let $Ord \subset V$ denote the class of ordinals, $Ord^\varepsilon \subset V$ and $<^\varepsilon$ be Δ -predicates such that for any transitive and wellfounded model V of $KP\omega$, $<^\varepsilon$ is a well ordering of type ε_{I+1} on Ord^ε for the order type I of the class Ord in V . Specifically let us encode ‘ordinals’ $\alpha < \varepsilon_{I+1}$ by codes $[\alpha] \in Ord^\varepsilon$ as follows. $[\alpha] = \langle 0, \alpha \rangle$ for $\alpha \in Ord$, $[I] = \langle 1, 0 \rangle$, $[\omega^\alpha] = \langle 2, [\alpha] \rangle$ for $\alpha > I$, and $[\alpha] = \langle 3, [\alpha_1], \dots, [\alpha_n] \rangle$ if $\alpha = \alpha_1 + \dots + \alpha_n > I$ with $\alpha_1 \geq \dots \geq \alpha_n$, $n > 1$ and $\exists \beta_i (\alpha_i = \omega^{\beta_i})$ for each α_i . Then $[\omega_n(I+1)] \in Ord^\varepsilon$ denotes the code of the ‘ordinal’ $\omega_n(I+1)$.

$<^\varepsilon$ is assumed to be a canonical ordering such that $KP\omega$ proves the fact that $<^\varepsilon$ is a linear ordering, and for any formula φ and each $n < \omega$,

$$KP\omega \vdash \forall x (\forall y <^\varepsilon x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x <^\varepsilon [\omega_n(I+1)] \varphi(x) \quad (2)$$

For a definition of Δ -predicates Ord^ε and $<^\varepsilon$, and a proof of (2), cf. [7].

Theorem 1.4 *For each $N \geq 2$, $KPII_{N+1}$ is Π_{N+1} -conservative over the theory*

$$KP\omega + \{V \in RM_N([\omega_n(I+1)]; <^\varepsilon) : n \in \omega\}.$$

On the other side, we[7] have lifted up the ordinal analysis of recursively inaccessible ordinals in [10] to one of weakly inaccessible cardinals. This paper aims to lift up [12] and [5] to the weak compactness.

Let us mention the contents of this paper. In the next section 2 iterated Skolem hulls $\mathcal{H}_{\alpha,n}(X)$ of sets X of ordinals, ordinals $\Psi_{\kappa,n}\gamma$ for regular ordinals κ ($\mathcal{K} < \kappa \leq I$), and classes $Mh_n^\alpha[\Theta]$ are defined for finite sets Θ of ordinals. It is shown that for each $n, m < \omega$, (\mathcal{K} is a weakly compact cardinal) $\rightarrow \mathcal{K} \in Mh_n^{\omega_m(I+1)}$ in $ZF + (V = L)$. In the third section 3 we introduce a theory for weakly compact cardinals, which are equivalent to $ZF + (V = L) + (\mathcal{K}$ is a weakly compact cardinal).

In the section 4 cut inferences are eliminated from operator controlled derivations of Σ_2^1 -sentences $\varphi^{V\kappa}$ over \mathcal{K} , and $\varphi^{V\kappa}$ is shown to be true. Everything up to this is seen to be formalizable in $ZF + (V = L) + \{\theta_n(\mathcal{K}) : n \in \omega\}$. Hence the Theorem 1.3 follows in the final section 5.

2 Ordinals for weakly compact cardinals

In this section iterated Skolem hulls $\mathcal{H}_{\alpha,n}(X)$ of sets X of ordinals, ordinals $\Psi_{\kappa,n}\gamma$ for regular ordinals κ ($\mathcal{K} < \kappa \leq I$), and classes $Mh_n^\alpha[\Theta]$ are defined for finite sets Θ of ordinals. It is shown that for each $n, m < \omega$, $\mathcal{K} \in Mh_n^{\omega_m(I+1)}$ in $ZF + (V = L)$ assuming \mathcal{K} is a weakly compact cardinal.

Let Ord^ε and $<^\varepsilon$ are Δ -predicates as described before Theorem 1.4. In the definition of Ord^ε and $<^\varepsilon$, I with its code $[I] = \langle 1, 0 \rangle$ is *intended* to denote the

least weakly inaccessible cardinal above the least weakly compact cardinal \mathcal{K} , though we *do not assume* the existence of weakly inaccessible cardinals above \mathcal{K} anywhere in this paper. We are working in $\text{ZF} + (V = L)$ assuming \mathcal{K} is a weakly compact cardinal.

Reg denotes the set of uncountable regular ordinals above \mathcal{K} , while $R := Reg \cap \{\rho : \mathcal{K} < \rho < I\}$ and $R^+ := R \cup \{I\}$. $\kappa, \lambda, \rho, \pi$ denote elements of R . κ^+ denotes the least regular ordinal above κ . Θ denotes finite sets of ordinals $\leq \mathcal{K}$. $\Theta \subset_{fin} X$ iff Θ is a finite subset of X . Ord denotes the class of ordinals less than I , while Ord^ε the class of codes of ordinals less than the next epsilon number ε_{I+1} to I .

For admissible ordinals σ and $X \subset L_\sigma$, $\text{Hull}_{\Sigma_n}^\sigma(X)$ denotes the Σ_n -Skolem hull of X over L_σ , cf. [7]. $F(y) = F^{\Sigma_n}(y; \sigma, X)$ denotes the Mostowski collapsing $F : \text{Hull}_{\Sigma_n}^\sigma(X) \leftrightarrow L_\gamma$ of $\text{Hull}_{\Sigma_n}^\sigma(X)$ for a γ . Let $F^{\Sigma_n}(\sigma; \sigma, X) := \gamma$. When $\sigma = I$, we write $F_X^{\Sigma_n}(y)$ for $F^{\Sigma_n}(y; I, X)$.

In what follows $n \geq 1$ denotes a fixed positive integer.

$Code^\varepsilon$ denotes the union of codes Ord^ε of ordinals $< \varepsilon_{I+1}$, and codes $L_I := \{\langle 0, x \rangle : x \in L\}$ of sets x in the universe L .

For $\alpha, \beta \in Ord^\varepsilon$, $\alpha \oplus \beta, \tilde{\omega}^\alpha \in Ord^\varepsilon$ denotes the codes of the sum and exponentiation, resp.

Let

$$I := \langle 1, 0 \rangle, \omega_n(I+1) := \tilde{\omega}_n(\langle 3, \langle 1, 0 \rangle, \langle 0, 1 \rangle \rangle), \text{ and } L_I := \{\langle 0, x \rangle : x \in L\}$$

and for codes $X, Y \in Code^\varepsilon$

$$X \subset^\varepsilon Y :\Leftrightarrow \forall x \in^\varepsilon X (x \in^\varepsilon Y).$$

For simplicity let us identify the code $x \in Code^\varepsilon$ with the ‘set’ coded by x , and $\in^\varepsilon [<^\varepsilon]$ is denoted by $\in [<]$, resp. when no confusion likely occurs. For example, the code $\langle 0, x \rangle$ is identified with the set $\{\langle 0, y \rangle : y \in x\}$ of codes.

Define simultaneously the classes $\mathcal{H}_{\alpha, n}(X) \subset L_I \cup \{x \in Ord^\varepsilon : x <^\varepsilon \omega_{n+1}(I+1)\}$, and the ordinals $\Psi_{\kappa, n, \alpha} (\kappa \in R^+)$ for $\alpha <^\varepsilon \omega_{n+1}(I+1)$ and sets $X \subset L_I$ as follows. We see that $\mathcal{H}_{\alpha, n}(X)$ and $\Psi_{\kappa, n, \alpha}$ are (first-order) definable as a fixed point in $\text{ZF} + (V = L)$ cf. Proposition 2.4.

$\mathcal{H}_{\alpha, n}$ is an operator in the sense defined below.

Definition 2.1 By an *operator* we mean a map $\mathcal{H}, \mathcal{H} : \mathcal{P}(L_I) \rightarrow \mathcal{P}(L_I \cup \{x \in Ord^\varepsilon : x <^\varepsilon \omega_{n+1}(I+1)\})$, such that

1. $\forall X \subset L_I [X \subset \mathcal{H}(X)]$.
2. $\forall X, Y \subset L_I [Y \subset \mathcal{H}(X) \Rightarrow \mathcal{H}(Y) \subset \mathcal{H}(X)]$.

For an operator \mathcal{H} and $\Theta, \Lambda \subset L_I$, $\mathcal{H}[\Theta](X) := \mathcal{H}(X \cup \Theta)$, and $\mathcal{H}[\Theta][\Lambda] := (\mathcal{H}[\Theta])[\Lambda]$, i.e., $\mathcal{H}[\Theta][\Lambda](X) = \mathcal{H}(X \cup \Theta \cup \Lambda)$.

Obviously $\mathcal{H}[\Theta]$ is an operator.

Definition 2.2 $\mathcal{H}_{\alpha,n}(X)$ is a Skolem hull of $\{\langle 0, 0 \rangle, \mathcal{K}, I\} \cup X$ under the functions $\oplus, \alpha \mapsto \tilde{\omega}^\alpha, \kappa \mapsto \kappa^+ (\kappa \in R), \Psi_{\kappa,n} \upharpoonright \alpha (\kappa \in R^+)$, the Skolem hullings:

$$X \mapsto \text{Hull}_{\Sigma_n}^I(X \cap I)$$

and the Mostowski collapsing functions

$$x = \Psi_{\kappa,n}\gamma \mapsto F_{x \cup \{\kappa\}}^{\Sigma_1} (\kappa \in R)$$

and

$$x = \Psi_{I,n}\gamma \mapsto F_x^{\Sigma_n}$$

1. (Inductive definition of $\mathcal{H}_{\alpha,n}(X)$).

- (a) $\{\langle 0, 0 \rangle, \mathcal{K}, I\} \cup X \subset \mathcal{H}_{\alpha,n}(X)$.
- (b) $x, y \in \mathcal{H}_\alpha(X) \Rightarrow x \oplus y, \tilde{\omega}^x \in \mathcal{H}_{\alpha,n}(X)$.
- (c) $\kappa \in \mathcal{H}_{\alpha,n}(X) \cap (\{\mathcal{K}\} \cup R) \Rightarrow \kappa^+ \in \mathcal{H}_{\alpha,n}(X)$.
- (d) $\gamma \in \mathcal{H}_{\alpha,n}(X) \cap \alpha \Rightarrow \Psi_{I,n}\gamma \in \mathcal{H}_{\alpha,n}(X)$.
- (e) If $\kappa \in \mathcal{H}_{\alpha,n}(X) \cap R, \gamma \in \mathcal{H}_{\alpha,n}(X) \cap \alpha$ and $\kappa \in \mathcal{H}_{\gamma,n}(\kappa)$, then $\Psi_{\kappa,n}\gamma \in \mathcal{H}_{\alpha,n}(X)$.
- (f)

$$\text{Hull}_{\Sigma_n}^I(\mathcal{H}_{\alpha,n}(X) \cap L_I) \cap \text{Code}^\varepsilon \subset \mathcal{H}_{\alpha,n}(X).$$

Namely for any Σ_n -formula $\varphi[x, \vec{y}]$ in the language $\{\in\}$ and parameters $\vec{a} \subset \mathcal{H}_{\alpha,n}(X) \cap L_I$, if $b \in L_I, (L_I, \in^\varepsilon) \models \varphi[b, \vec{a}]$ and $(L_I, \in^\varepsilon) \models \exists! x \varphi[x, \vec{a}]$, then $b \in \mathcal{H}_{\alpha,n}(X)$.

- (g) If $\kappa \in \mathcal{H}_{\alpha,n}(X) \cap R, \gamma \in \mathcal{H}_{\alpha,n}(X) \cap \alpha, x = \Psi_{\kappa,n}\gamma \in \mathcal{H}_{\alpha,n}(X), \kappa \in \mathcal{H}_{\gamma,n}(\kappa)$ and $\delta \in (\text{Hull}_{\Sigma_1}^I(x \cup \{\kappa\}) \cup \{I\}) \cap \mathcal{H}_{\alpha,n}(X)$, then $F_{x \cup \{\kappa\}}^{\Sigma_1}(\delta) \in \mathcal{H}_{\alpha,n}(X)$.
- (h) If $\gamma \in \mathcal{H}_{\alpha,n}(X) \cap \alpha, x = \Psi_{I,n}\gamma \in \mathcal{H}_{\alpha,n}(X)$, and $\delta \in (\text{Hull}_{\Sigma_n}^I(x) \cup \{I\}) \cap \mathcal{H}_{\alpha,n}(X)$, then $F_x^{\Sigma_n}(\delta) \in \mathcal{H}_{\alpha,n}(X)$.

2. (Definition of $\Psi_{\kappa,n}\alpha$).

Assume $\kappa \in R^+$ and $\kappa \in \mathcal{H}_{\alpha,n}(\kappa)$. Then

$$\Psi_{\kappa,n}\alpha := \min_{\varepsilon} \{\beta <^\varepsilon \kappa : \kappa \in \mathcal{H}_{\alpha,n}(\beta), \mathcal{H}_{\alpha,n}(\beta) \cap \kappa \subset^\varepsilon \beta\}.$$

Definition 2.2 is essentially the same as in [7].

The classes $Mh_n^\alpha[\Theta]$ are defined for $n < \omega, \alpha < \varepsilon_{I+1}$, and $\Theta \subset_{fin} (\mathcal{K} + 1)$.

Definition 2.3 ($Mh_n^\alpha[\Theta]$)

Let $\Theta \subset_{fin} (\mathcal{K} + 1)$ and $\mathcal{K} \geq \pi \in \text{Reg}$. Then

$$\begin{aligned} \pi \in Mh_n^\alpha[\Theta] & :\Leftrightarrow \mathcal{H}_{\alpha,n}(\pi) \cap \mathcal{K} \subset^\varepsilon \pi \ \& \ \alpha \in \mathcal{H}_{\alpha,n}[\Theta](\pi) \\ & \ \& \ \forall \xi \in \mathcal{H}_{\xi,n}[\Theta \cup \{\pi\}](\pi) \cap \alpha [\pi \in M(Mh_n^\xi[\Theta \cup \{\pi\}])] \end{aligned} \quad (3)$$

where $\forall \xi \in \mathcal{H}_{\xi,n}[\Theta \cup \{\pi\}](\pi) \cap \alpha[\cdot \cdot \cdot]$ is a short hand for $\forall \xi <^\varepsilon \alpha[\xi \in \mathcal{H}_{\xi,n}[\Theta \cup \{\pi\}](\pi) \cap \alpha \rightarrow \cdot \cdot \cdot]$.

$$Mh_n^\alpha := Mh_n^\alpha[\{\mathcal{K}\}] = Mh_n^\alpha[\emptyset].$$

The following Propositions 2.4 and 2.5 are easy to see.

Proposition 2.4 *Each of $x = \mathcal{H}_{\alpha,n}(\beta)$ ($\alpha \in Ord^\varepsilon, \beta <^\varepsilon I$), $\beta = \Psi_{\kappa,n}\alpha$ ($\kappa \in R^+$) and $x = Mh_n^\alpha[\Theta]$ is a Σ_{n+1} -predicate as fixed points in $ZF + (V = L)$.*

Proposition 2.5 *$(\alpha, y) \mapsto \mathcal{H}_{\alpha,n}[\Theta](y)$ is weakly monotonic in the sense that*

$$\alpha \leq^\varepsilon \alpha' \wedge y \subset y' \wedge x = \mathcal{H}_{\alpha,n}[\Theta](y) \wedge x' = \mathcal{H}_{\alpha',n}[\Theta](y') \rightarrow x \subset x'.$$

Also $(\alpha, y) \mapsto \mathcal{H}_{\alpha,n}[\Theta](y)$ is continuous in the sense that if $\alpha = \sup_{i \in I} \alpha_i$ is a limit ordinal with an increasing sequence $\{\alpha_i\}_{i \in I}$ and $y = \bigcup_{j \in J} y_j$ with a directed system $\{y_j\}_{j \in J}$, then

$$x = \mathcal{H}_{\alpha,n}[\Theta](\beta) \wedge \forall i \in I \forall j \in J (x_{i,j} = \mathcal{H}_{\alpha_i,n}[\Theta](y_j)) \rightarrow x = \bigcup_{i \in I, j \in J} x_{i,j}.$$

Let $A_n(\alpha)$ denote the conjunction of $\forall \beta <^\varepsilon I \exists! x[x = \mathcal{H}_{\alpha,n}(\beta)]$, $\forall \kappa \in R^+ \forall x[\kappa \in x = \mathcal{H}_{\alpha,n}(\kappa) \rightarrow \exists! \beta(\beta = \Psi_{\kappa,n}\alpha)]$ and $\forall \Theta \subset_{fin} (\mathcal{K} + 1) \exists! x[x = Mh_n^\alpha[\Theta]]$.

The Σ_{n+1} -formula $\theta_n(x)$ in Theorem 1.3 is defined to be

$$\theta_n(x) := \exists y[y = Mh_n^{\omega_n(I+1)} \wedge x \in y].$$

The following Lemma 2.6.3 shows Theorem 1.3.1.

$card(x)$ denotes the cardinality of sets x .

Lemma 2.6 *For each $n, m < \omega$, $ZF + (V = L)$ proves the followings.*

1. $y = \mathcal{H}_{\alpha,n}(x) \rightarrow card(y) \leq \max\{card(x), \aleph_0\}$.
2. $\forall \alpha <^\varepsilon \omega_m(I+1) A_n(\alpha)$.
3. *If \mathcal{K} is weakly compact and $\Theta \subset_{fin} (\mathcal{K} + 1)$, then $\mathcal{K} \in Mh_n^{\omega_m(I+1)}[\Theta] \cap M(Mh_n^{\omega_m(I+1)}[\Theta])$.*

Proof.

2.6.2. We show that $A_n(\alpha)$ is progressive, i.e., $\forall \alpha <^\varepsilon \omega_m(I+1) [\forall \gamma <^\varepsilon \alpha A_n(\gamma) \rightarrow A_n(\alpha)]$.

Assume $\forall \gamma <^\varepsilon \alpha A_n(\gamma)$ and $\alpha <^\varepsilon \omega_m(I+1)$. $\forall \beta <^\varepsilon I \exists! x[x = \mathcal{H}_{\alpha,n}(\beta)]$ follows from IH and the Replacement.

Next assume $\kappa \in R^+$ and $\kappa \in \mathcal{H}_{\alpha,n}(\kappa)$. Then $\exists! \beta(\beta = \Psi_{\kappa,n}\alpha)$ follows from the regularity of κ and Proposition 2.5.

$\exists!x[x = Mh_n^\alpha[\Theta]]$ is easily seen from IH.

2.6.3. Suppose \mathcal{K} is Π_1^1 -inaccessible. We show

$$B_n(\alpha) :\Leftrightarrow \forall \Theta \subset_{fin} (\mathcal{K} + 1)[\alpha \in \mathcal{H}_{\alpha,n}[\Theta](\mathcal{K}) \rightarrow \mathcal{K} \in Mh_n^\alpha[\Theta] \cap M(Mh_n^\alpha[\Theta])]$$

is progressive in α .

Suppose $\forall \xi <^\varepsilon \alpha B_n(\xi)$, $\Theta \subset_{fin} (\mathcal{K} + 1)$ and $\alpha \in \mathcal{H}_{\alpha,n}[\Theta](\mathcal{K})$. We have to show that $Mh_n^\alpha[\Theta]$ meets every club subset C_0 of \mathcal{K} . $\mathcal{K} \in Mh_n^\alpha[\Theta]$ follows from $\mathcal{K} \in M(Mh_n^\alpha[\Theta])$, cf. Proposition 2.9.2. We can assume that $\forall \pi \in C_0[(\mathcal{H}_{\alpha,n}(\pi) \cap \mathcal{K} \subset \pi) \wedge (\alpha \in \mathcal{H}_{\alpha,n}[\Theta](\pi))]$ since both of $\{\pi < \mathcal{K} : \mathcal{H}_{\alpha,n}(\pi) \cap \mathcal{K} \subset \pi\}$ and $\{\pi < \mathcal{K} : \alpha \in \mathcal{H}_{\alpha,n}[\Theta](\pi)\}$ are club in \mathcal{K} .

Since $\forall \pi \leq \mathcal{K}[card(\mathcal{H}_{\alpha,n}[\Theta \cup \{\pi\}](\pi)) \leq \pi]$, pick an injection $f : \mathcal{H}_{\alpha,n}[\Theta \cup \{\mathcal{K}\}](\mathcal{K}) \rightarrow \mathcal{K}$ so that $f''\mathcal{H}_{\alpha,n}[\Theta \cup \{\pi\}](\pi) \subset \pi$ for any weakly inaccessible $\pi \leq \mathcal{K}$.

Let $R_0 = \{f(\alpha)\}$, $R_1 = C_0$, $R_2 = \{f(\xi) : \xi \in \mathcal{H}_{\xi,n}[\Theta](\mathcal{K}) \cap \alpha\}$, $R_3 = \bigcup\{(Mh_n^\xi[\Theta \cup \{\pi\}] \cap \mathcal{K}) \times \{f(\pi)\} \times \{f(\xi)\} : \xi \in \mathcal{H}_{\xi,n}[\Theta](\mathcal{K}) \cap \alpha, \pi \leq \mathcal{K}\}$, and $R_4 = \{(f(\beta), f(\gamma)) : \{\beta, \gamma\} \subset \mathcal{H}_{\alpha,n}[\Theta \cup \{\mathcal{K}\}](\mathcal{K}), \beta < \gamma\}$.

By IH we have $\forall \xi \in \mathcal{H}_{\xi,n}[\Theta](\mathcal{K}) \cap \alpha[\mathcal{K} \in M(Mh_n^\xi[\Theta])]$. Hence $\langle V_{\mathcal{K}}, \in, R_i \rangle_{i \leq 4}$ enjoys a Π_1^1 -sentence saying that \mathcal{K} is weakly inaccessible, $R_0 \neq \emptyset$, R_1 is a club subset of \mathcal{K} and

$$\varphi :\Leftrightarrow \forall C:\text{club } \forall x, y[R_2(x) \wedge \theta(R_4, y) \rightarrow C \cap \{a : R_3(a, y, x)\} \neq \emptyset]$$

where $\theta(R_4, y)$ is a Σ_1^1 -formula such that for any $\pi \leq \mathcal{K}$

$$V_\pi \models \theta(R_4, y) \Leftrightarrow y = f(\pi)$$

Namely $\theta(R_4, y)$ says that there exists a function G on the class *Ord* of ordinals such that $\forall \beta, \gamma \in \text{Ord}[(\beta < \gamma \leftrightarrow R_4(G(\beta), G(\gamma)) \wedge (G(\beta) < y)]$ and $\forall z(R_4(z, y) \rightarrow \exists \beta \in \text{Ord}(G(\beta) = z))$.

By the Π_1^1 -inaccessibility of \mathcal{K} , pick a $\pi < \mathcal{K}$ such that $\langle V_\pi, \in, R_i \cap V_\pi \rangle_{i \leq 4}$ enjoys the Π_1^1 -sentence.

We claim $\pi \in C_0 \cap Mh_n^\alpha[\Theta]$. π is weakly inaccessible, $f(\alpha) \in V_\pi$ and C_0 is club in π , and hence $\pi \in C_0$. It remains to see $\forall \xi \in \mathcal{H}_{\xi,n}[\Theta \cup \{\pi\}](\pi) \cap \alpha[\pi \in M(Mh_n^\xi[\Theta \cup \{\pi\}])]$. This follows from the fact that φ holds in $\langle V_\pi, \in, R_i \cap V_\pi \rangle_{i \leq 4}$, and $\forall \xi \in \mathcal{H}_{\xi,n}[\Theta \cup \{\pi\}](\pi) \cap \alpha[f(\xi) \in V_\pi]$ by $f''\mathcal{H}_{\alpha,n}[\Theta \cup \{\pi\}](\pi) \subset \pi$ and $\mathcal{H}_{\xi,n}[\Theta \cup \{\pi\}](\pi) \subset \mathcal{H}_{\xi,n}[\Theta](\mathcal{K})$.

Thus $\mathcal{K} \in M(Mh_n^\alpha[\Theta])$. \square

Definition 2.7 $\mathcal{H}(n)$ denotes a subset of $\mathcal{H}_{\omega_n(I+1),n}(\emptyset)$ such that every ordinal is hereditarily less than $\omega_n(I+1)$.

This means $\alpha \in \mathcal{H}(n) \Rightarrow \alpha < \omega_n(I+1)$, etc.

Corollary 2.8 For each $n < \omega$, $\mathcal{H}(n)$ is well-defined in $\text{ZF} + (V = L)$.

Let us see some elementary facts.

Proposition 2.9 1. $\alpha \in \mathcal{H}_{\alpha,n}[\Theta](\pi) \& \pi \in Mh_n^\alpha[\Theta \cup \{\rho\}] \Rightarrow \pi \in Mh_n^\alpha[\Theta]$.

2. $\pi \in M(Mh_n^\alpha[\Theta \cup \{\pi\}]) \Rightarrow \pi \in Mh_n^\alpha[\Theta \cup \{\pi\}]$.

3. $\pi \in Mh_n^\alpha[\Theta] \& \xi \in \mathcal{H}_{\xi,n}[\Theta \cup \{\pi\}](\pi) \cap \alpha \Rightarrow \pi \in Mh_n^\xi[\Theta \cup \{\pi\}]$, and
 $\pi \in Mh_n^\alpha[\Theta] \& \xi \in \mathcal{H}_{\xi,n}[\Theta](\pi) \cap \alpha \Rightarrow \pi \in Mh_n^\xi[\Theta]$.

Proof.

2.9.2. This is seen from Proposition 2.9.1.

2.9.3. This is seen from Proposition 2.9.2. □

2.1 Greatly Mahlo cardinals

Let us compare the class $Mh_n^\alpha[\Theta]$ with Rathjen's class M^α in [12]. The difference lies in augmenting finite sets Θ of ordinals, which are given in advance. Moreover the finite set grows when we step down to previously defined classes, cf. (3). For example if an ordinal $\xi < \alpha$ is Σ_1 -definable from $\{\pi, \pi^+\}$, then $\xi \in \mathcal{H}_{\xi,n}[\Theta \cup \{\pi\}](\pi)$ for $n \geq 1$. Hence $Mh_n^\xi[\Theta \cup \{\pi\}]$ is stationary in π for such an ordinal $\xi < \alpha$ if $\pi \in Mh_n^\alpha[\Theta]$. Cf. **Case 2** in the proof of Lemma 4.26 below.

This yields that any σ with $\sigma \in Mh_n^{\sigma^+}$ is a greatly Mahlo cardinal in the sense of Baumgartner-Taylor-Wagon[9]. Moreover if $\mathcal{K} \in Mh_n^{\mathcal{K}+1}$, then the class of the greatly Mahlo cardinals below \mathcal{K} is stationary in \mathcal{K} as seen in Proposition 2.10.

M^α ($\alpha < \mathcal{K}^+$) denotes the set of α -weakly Mahlo cardinals defined as follows. $M^0 := Reg \cap \mathcal{K}$, $M^{\alpha+1} = M(M^\alpha)$, $M^\lambda = \bigcap \{M(M^\alpha) : \alpha < \lambda\}$ for limit ordinals λ with $cf(\lambda) < \mathcal{K}$, and $M^\lambda := \Delta \{M(M^{\lambda_i}) : i < \mathcal{K}\}$ for limit ordinals λ with $cf(\lambda) = \mathcal{K}$, where $\sup_{i < \mathcal{K}} \lambda_i = \lambda$ and the sequence $\{\lambda_i\}_{i < \mathcal{K}}$ is chosen so that it is the $<_L$ -minimal such sequence.

In the last case for $\pi < \mathcal{K}$, $\pi \in M^\lambda \Leftrightarrow \forall i < \pi (\pi \in M(M^{\lambda_i}))$.

Proposition 2.10 For $n \geq 1$ and $\sigma \leq \mathcal{K}$, the followings are provable in ZF + $(V = L)$.

1. If $\sigma \in \Theta$, $\pi \in Mh_n^\alpha[\Theta] \cap \sigma$, and $\alpha \in \text{Hull}_{\Sigma_1}^I(\{\sigma, \sigma^+\} \cup \pi) \cap \sigma^+$, then $\pi \in M^\alpha$.
2. $\sigma \in Mh_n^{\sigma^+}[\Theta] \rightarrow \forall \alpha < \sigma^+ (\sigma \in M(M^\alpha))$.
3. The class of the greatly Mahlo cardinals below \mathcal{K} is stationary in \mathcal{K} if $\mathcal{K} \in Mh_n^{\mathcal{K}+1}$.

Proof.

2.10.1 by induction on $\alpha < \sigma^+$. Suppose $\sigma \in \Theta$, $\pi \in Mh_n^\alpha[\Theta] \cap \sigma$ and $\alpha \in \text{Hull}_{\Sigma_1}^I(\{\sigma, \sigma^+\} \cup \pi) \cap \sigma^+$.

First consider the case when $cf(\alpha) = \sigma$, and let $\{\alpha_i\}_{i < \sigma}$ be the $<_L$ -minimal sequence such that $\sup_{i < \sigma} \alpha_i = \alpha$. Then $\{\alpha_i\}_{i < \sigma} \in \text{Hull}_{\Sigma_1}^I(\{\alpha, \sigma\}) \subset \text{Hull}_{\Sigma_1}^I(\{\sigma, \sigma^+\} \cup$

π). For $i < \pi$, $\alpha_i \in \text{Hull}_{\Sigma_1}^I(\{\sigma, \sigma^+\} \cup \pi) \cap \alpha \subset \mathcal{H}_{\alpha_i, n}[\Theta \cup \{\pi\}](\pi) \cap \alpha$ by $\sigma \in \Theta$. $\pi \in Mh_n^\alpha[\Theta]$ yields $\pi \in M(Mh_n^{\alpha_i}[\Theta \cup \{\pi\}])$. Now for a club subset C in π , pick a $\rho < \pi$ such that $\rho \in C \cap Mh_n^{\alpha_i}[\Theta \cup \{\pi\}]$. We can assume that $\alpha_i \in \text{Hull}_{\Sigma_1}^I(\{\sigma, \sigma^+\} \cup \rho)$ by $\alpha_i \in \text{Hull}_{\Sigma_1}^I(\{\sigma, \sigma^+\} \cup \pi)$. Thus IH yields $\rho \in M^{\alpha_i}$, and hence $\pi \in M(M^{\alpha_i})$ for any $i < \pi$.

Second consider the case when $cf(\alpha) < \sigma$. Then $cf(\alpha) \in \text{Hull}_{\Sigma_1}^I(\{\alpha\}) \cap \sigma \subset \text{Hull}_{\Sigma_1}^I(\{\sigma, \sigma^+\} \cup \pi) \cap \sigma \subset \mathcal{H}_{\alpha, n}[\{\sigma\}](\pi) \cap \sigma \subset \pi$ by $\pi \in Mh_n^\alpha[\Theta]$ and $\sigma \in \Theta$. Thus $cf(\alpha) < \pi$. Pick a cofinal sequence $\{\alpha_i\}_{i < cf(\alpha)} \in \text{Hull}_{\Sigma_1}^I(\{\sigma, \sigma^+\} \cup \pi)$. Then for any $i < cf(\alpha) < \pi$ we have $\alpha_i \in \text{Hull}_{\Sigma_1}^I(\{\sigma, \sigma^+\} \cup \pi) \cap \alpha$, and hence $\pi \in M(Mh_n^{\alpha_i}[\Theta \cup \{\pi\}])$. As in the first case we see that $\pi \in M(M^{\alpha_i})$ for any $i < cf(\alpha)$.

Finally let $\alpha = \beta + 1$. Then $\beta \in \text{Hull}_{\Sigma_1}^I(\{\sigma, \sigma^+\} \cup \pi)$ together with IH yields $\pi \in M(M^\beta)$.

2.10.2. Suppose $\sigma \in Mh_n^{\sigma^+}[\Theta]$ and $\exists \alpha < \sigma^+$ ($\sigma \notin M(M^\alpha)$). Let $\alpha < \sigma^+$ be the minimal ordinal such that $\sigma \notin M(M^\alpha)$, and C be a club subset of σ such that $C \cap M^\alpha = \emptyset$. Then $\alpha \in \text{Hull}_{\Sigma_1}^I(\{\sigma, \sigma^+\}) \cap \sigma^+ \subset \mathcal{H}_{\alpha, n}[\Theta \cup \{\sigma\}](\sigma) \cap \sigma^+$. By $\sigma \in Mh_n^{\sigma^+}[\Theta]$ we have $\sigma \in M(Mh_n^\alpha[\Theta \cup \{\sigma\}])$. Pick a $\pi \in C \cap Mh_n^\alpha[\Theta \cup \{\sigma\}] \cap \sigma$. Proposition 2.10.1 yields $\pi \in M^\alpha$. A contradiction.

2.10.3. If $\mathcal{K} \in Mh_n^{\mathcal{K}+1}$, then $\mathcal{K} \in M(Mh_n^\mathcal{K})$. Let $\sigma \in Mh_n^\mathcal{K} \cap \mathcal{K}$. Then $\sigma^+ \in \mathcal{H}_{\sigma^+, n}[\{\sigma\}](\sigma) \cap \mathcal{K}$, and hence $\sigma \in M(Mh_n^{\sigma^+}[\{\sigma\}])$. Proposition 2.9.2 yields $\sigma \in Mh_n^{\sigma^+}[\{\sigma\}]$. From Proposition 2.10.2 we see that σ is greatly Mahlo. \square

3 A theory for weakly compact cardinals

In this section the set theory $\text{ZF} + (V = L) + (\mathcal{K} \text{ is weakly compact})$ is paraphrased to another set theory $\text{T}(\mathcal{K}, I)$ as in [7].

Let \mathcal{K} be the least weakly compact cardinal, and $I > \mathcal{K}$ the least weakly inaccessible cardinal above \mathcal{K} . κ, λ, ρ ranges over uncountable regular ordinals such that $\mathcal{K} < \kappa, \lambda, \rho < I$.

In the following Definition 3.2, the predicate P is intended to denote the relation

$$P(\lambda, x, y) \Leftrightarrow x = F_{x \cup \{\lambda\}}^{\Sigma_1}(\lambda) \ \& \ y = F_{x \cup \{\lambda\}}^{\Sigma_1}(I) := \text{rng}(F_{x \cup \{\lambda\}}^{\Sigma_1}) \cap \text{Ord}$$

and the predicate $P_{I, n}(x)$ is intended to denote the relation

$$P_{I, n}(x) \Leftrightarrow x = F_x^{\Sigma_n}(I).$$

Definition 3.1 1. Let $\vec{X} = X_0, \dots, X_{n-1}$ be a list of unary predicates. A *stratified formula with respect to the variables* $\vec{x} = x_0, \dots, x_{n-1}$ is a formula $\varphi[\vec{x}]$ in the language $\{\in\}$ obtained from a (first-order) formula $\varphi[\vec{X}]$ in the language $\{\in\} \cup \vec{X}$ by replacing any atomic formula $X_i(z)$ by $z \in x_i$ for $i < n$.

2. For a formula φ and a set x , φ^x denotes the result of restricting every unbounded quantifier $\exists z, \forall z$ in φ to $\exists z \in x, \forall z \in x$.
3. $\alpha \in Ord : \Leftrightarrow \forall x \in a \forall y \in x (y \in a) \wedge \forall x, y \in a (x \in y \vee x = y \vee y \in x)$, and by $\alpha < \beta$ we tacitly assume that α, β are ordinals, i.e., $\alpha < \beta : \Leftrightarrow \{\alpha, \beta\} \subset Ord \wedge \alpha \in \beta$.

Definition 3.2 $T(\mathcal{K}, I, n)$ denotes the set theory defined as follows.

1. Its language is $\{\in, P, P_{I,n}, Reg, \mathcal{K}\}$ for a ternary predicate P , unary predicates $P_{I,n}$ and Reg , and an individual constant \mathcal{K} .
2. Its axioms are obtained from those of Kripke-Platek set theory with the axiom of infinity $KP\omega$ in the expanded language, the axiom of constructibility, $V = L$ together with the axiom schemata saying that
 - (a) the ordinals κ with $Reg(\kappa)$ is an uncountable regular ordinal $> \mathcal{K}$ ($Reg(\kappa) \rightarrow \mathcal{K} < \kappa \in Ord$) and ($Reg(\kappa) \rightarrow a \in Ord \cap \kappa \rightarrow \exists x, y \in Ord \cap \kappa [a < x \wedge P(\kappa, x, y)]$), and the ordinal x with $P(\kappa, x, y)$ is a critical point of the Σ_1 elementary embedding from an $L_y \cong \text{Hull}_{\Sigma_1}^I(x \cup \{\kappa\})$ to the universe L_I ($P(\kappa, x, y) \rightarrow \{x, y\} \subset Ord \wedge x < y < \kappa \wedge Reg(\kappa)$ and $P(\kappa, x, y) \rightarrow a \in Ord \cap x \rightarrow \varphi[\kappa, a] \rightarrow \varphi^y[x, a]$ for any Σ_1 -formula φ in the language $\{\in\}$),
 - (b) there are cofinally many regular ordinals ($\forall x \in Ord \exists y [x \geq \mathcal{K} \rightarrow y > x \wedge Reg(y)]$),
 - (c) the ordinal x with $P_{I,n}(x)$ is a critical point of the Σ_n elementary embedding from $L_x \cong \text{Hull}_{\Sigma_n}^I(x)$ to the universe L_I ($P_{I,n}(x) \rightarrow x \in Ord$ and $P_{I,n}(x) \rightarrow a \in Ord \cap x \rightarrow \varphi[a] \rightarrow \varphi^x[a]$ for any Σ_n -formula φ in the language $\{\in\}$), and there are cofinally many such ordinals x ($\mathcal{K} < a \in Ord \rightarrow \exists x \in Ord [a < x \wedge P_{I,n}(x)]$),
 - (d) the axiom ‘ \mathcal{K} is uncountable regular’ is:

$$(\mathcal{K} > \omega) \wedge \forall \alpha < \mathcal{K} \forall f \in {}^\alpha \mathcal{K} \exists \beta < \mathcal{K} (f'' \alpha \subset \beta)$$

and the axiom saying that $\forall B \subset \mathcal{K} [\mathcal{K} \in M(B) \rightarrow \exists \rho < \mathcal{K} (\rho \in M(B) \wedge Reg(\rho))]$, which is codified by the following (4).

$$\forall B \in L_{\mathcal{K}^+} [B \subset \mathcal{K} \rightarrow \neg \tau(B, \mathcal{K}) \rightarrow \exists \rho < \mathcal{K} (\neg \tau(B, \rho) \wedge Reg(\rho))] \quad (4)$$

where

$$\tau(B, \rho) : \Leftrightarrow \exists C \subset \rho [(C \text{ is club})^\rho \wedge (B \cap C = \emptyset)] \quad (5)$$

and $(C \text{ is club})^\rho$ is a formula saying that C is a club subset of ρ . Namely $\tau(B, \rho)$ says that the set B is thin, i.e., non-stationary in ρ . Note that $(C \text{ is club})^\rho \wedge (B \cap C = \emptyset)$ is stratified with respect to B, C , and $\tau(B, \rho)$ is stratified with respect to B .

The following Lemma 3.3 is seen as in [7].

Lemma 3.3 $\mathsf{T}(\mathcal{K}, I) := \bigcup_{n \in \omega} \mathsf{T}(\mathcal{K}, I, n)$ is equivalent to the set theory $\mathsf{ZF} + (V = L) + (\mathcal{K} \text{ is weakly compact})$.

4 Operator controlled derivations for weakly compact cardinals

In this section, operator controlled derivations are first introduced, and inferences $(\mathbf{Ref}_{\mathcal{K}})$ for Π_1^1 -indescribability are then eliminated from operator controlled derivations of Σ_2^1 -sentences $\varphi^{V_{\mathcal{K}}}$ over \mathcal{K} .

In what follows n denotes a fixed positive integer. We tacitly assume that any ordinal is in $\mathcal{H}(n)$.

For $\alpha <^\varepsilon I = \langle 1, 0 \rangle$, $L_\alpha = \{\langle 0, x \rangle : x \in L_{(\alpha)_1}\}$. $L_I = \{\langle 0, x \rangle : x \in L\} = \bigcup_{\alpha <^\varepsilon I} L_\alpha$ denotes the universe. Both $(L_I, \in^\varepsilon) \models A$ and ‘ A is true’ are synonymous with A .

4.1 An intuitionistic fixed point theory $\mathsf{FiX}^i(\mathsf{ZFLK}_n)$

For the fixed positive integer n , ZFLK_n denotes the set theory $\mathsf{ZF} + (V = L) + (\mathcal{K} \in \mathit{Mh}_n^{\omega_n(I+1)})$ in the language $\{\in, \mathcal{K}\}$ with an individual constant \mathcal{K} . Let us also denote the set theory $\mathsf{ZF} + (V = L) + (\mathcal{K} \text{ is weakly compact})$ in the language $\{\in, \mathcal{K}\}$ by ZFLK .

To analyze the theory ZFLK , we need to handle the relation $(\mathcal{H}_\gamma[\Theta_0], \Theta, \kappa, n) \vdash_b^a \Gamma$ defined in subsection 4.3, where n is the fixed integer, γ, κ, a, b are codes of ordinals with $a <^\varepsilon \omega_n(I+1)$, $b <^\varepsilon I \oplus \omega$ and $\kappa \leq^\varepsilon I$ the code of a regular ordinal, Θ_0, Θ are finite subsets of L_I and Γ a sequent, i.e., a finite set of sentences. Usually the relation is defined by recursion on ‘ordinals’ a , but such a recursion is not available in ZFLK_n since a may be larger than I . Instead of the recursion, the relation is defined for each $n < \omega$, as a fixed point,

$$H_n(\gamma, \Theta_0, \Theta, \kappa, a, b, \Gamma) \Leftrightarrow (\mathcal{H}_{\gamma, n}[\Theta_0], \Theta, \kappa, n) \vdash_b^a \Gamma \quad (6)$$

In this way the whole proof in this section is formalizable in an intuitionistic fixed point theory $\mathsf{FiX}^i(\mathsf{ZFLK}_n)$ over ZFLK_n .

Throughout this section we work in an intuitionistic fixed point theory $\mathsf{FiX}^i(\mathsf{ZFLK}_n)$ over ZFLK_n . The intuitionistic theory $\mathsf{FiX}^i(\mathsf{ZFLK}_n)$ is introduced in [7], and shown to be a conservative extension of ZFLK_n . Let us reproduce definitions and results on $\mathsf{FiX}^i(\mathsf{ZFLK}_n)$ here.

Fix an X -strictly positive formula $\mathcal{Q}(X, x)$ in the language $\{\in, \mathcal{K}, =, X\}$ with an extra unary predicate symbol X . In $\mathcal{Q}(X, x)$ the predicate symbol X occurs only strictly positive. This means that the predicate symbol X does not occur in the antecedent φ of implications $\varphi \rightarrow \psi$ nor in the scope of negations \neg in $\mathcal{Q}(X, x)$. The language of $\mathsf{FiX}^i(\mathsf{ZFLK}_n)$ is $\{\in, \mathcal{K}, =, \mathcal{Q}\}$ with a fresh unary predicate symbol \mathcal{Q} . The axioms in $\mathsf{FiX}^i(\mathsf{ZFLK}_n)$ consist of the following:

1. All provable sentences in ZFLK_n (in the language $\{\in, \mathcal{K}, =\}$).
2. Induction schema for any formula φ in $\{\in, \mathcal{K}, =, Q\}$:

$$\forall x(\forall y \in x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x) \quad (7)$$

3. Fixed point axiom:

$$\forall x[Q(x) \leftrightarrow \mathcal{Q}(Q, x)].$$

The underlying logic in $\text{FiX}^i(\text{ZFLK}_n)$ is defined to be the intuitionistic (first-order predicate) logic (with equality).

(7) yields the following Lemma 4.1.

Lemma 4.1 *Let $<^\varepsilon$ denote a Δ_1 -predicate as described before Theorem 1.4. For each $n < \omega$ and each formula φ in $\{\in, \mathcal{K}, =, Q\}$,*

$$\text{FiX}^i(\text{ZFLK}_n) \vdash \forall x(\forall y <^\varepsilon x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x <^\varepsilon \omega_n(I+1)\varphi(x).$$

The following Theorem 4.2 is seen as in [6, 7].

Theorem 4.2 *$\text{FiX}^i(\text{ZFLK}_n)$ is a conservative extension of ZFLK_n .*

In what follows we work in $\text{FiX}^i(\text{ZFLK}_n)$ for a fixed integer n .

4.2 Classes of sentences

$\mathcal{K} \in L = L_I = \bigcup_{\alpha \in \text{Ord}} L_\alpha$ denotes a transitive and wellfounded model of $\text{ZF} + (V = L)$, where $L_{\alpha+1}$ is the set of L_α -definable subsets of L_α . Ord denotes the class of all ordinals in L , and I the least ordinal not in L , while Ord^ε denotes the codes of ordinals less than $\omega_n(I+1)$.

Definition 4.3 For $a \in L$, $\text{rk}_L(a)$ denotes the L -rank of a .

$$\text{rk}_L(a) := \min\{\alpha \in \text{Ord} : a \in L_{\alpha+1}\}.$$

If $a \in b \in L$, then $a \in b \subset L_\beta$ for $\beta = \text{rk}_L(b)$ and $a \in L_\beta$. Hence $\text{rk}_L(a) < \beta = \text{rk}_L(b)$.

The language \mathcal{L}_c is obtained from the language $\{\in, P, P_{I,n}, \text{Reg}, \mathcal{K}\}$ by adding names (individual constants) c_a of each set $a \in L$. c_a is identified with a .

Then *formulae* in \mathcal{L}_c is defined as usual. Unbounded quantifiers $\exists x, \forall x$ are denoted by $\exists x \in L_I, \forall x \in L_I$, resp.

For formulae A in \mathcal{L}_c , $\text{qk}(A)$ denotes the finite set of L -ranks $\text{rk}_L(a)$ of sets a which are bounds of ‘bounded’ quantifiers $\exists x \in a, \forall x \in a$ occurring in A . Moreover $\text{k}(A)$ denotes the set of L -ranks of sets occurring in A , while $\text{k}^E(A)$ denotes the set of L -ranks of sets occurring in an unstratified position in A . Both $\text{k}(A)$ and $\text{k}^E(A)$ are defined to include L -ranks of bounds of ‘bounded’ quantifiers. Thus $\text{qk}(A) \subset \text{k}^E(A) \subset \text{k}(A) \leq I$. By definition we set $0 \in \text{qk}(A)$.

In the following definition, Var denotes the set of variables and set $\text{rk}_L(x) := 0$ for variables $x \in \text{Var}$.

Definition 4.4 1. $k(\neg A) = k(A)$ and similarly for k^E, \mathbf{qk} .

2. $\mathbf{qk}(M) = \{0\}$ for any literal M .
3. $k^E(M) = k(M) = \{\text{rk}_L(t) : t \in \vec{t}\} \cup \{0\}$ for literals $Q(\vec{t})$ with predicates $Q \in \{P, P_{I,n}, \text{Reg}\}$.
4. $k(t \in s) = \{\text{rk}_L(t), \text{rk}_L(s), 0\}$ and $k^E(t \in s) = \{\text{rk}_L(t), 0\}$.
5. $k(A_0 \vee A_1) = k(A_0) \cup k(A_1)$ and similarly for k^E, \mathbf{qk} .
6. For $t \in L_I \cup \{L_I\} \cup \text{Var}$, $k(\exists x \in t A(x)) = \{\text{rk}_L(t)\} \cup k(A(x))$ and similarly for k^E, \mathbf{qk} .

For example $k^E(a \in b) = \{\text{rk}_L(a), 0\}$, and $\mathbf{qk}(\exists x \in a A(x)) = \{\text{rk}_L(a)\} \cup \mathbf{qk}(A(x))$.

Definition 4.5 1. $A \in \Delta_0$ iff there exists a Δ_0 -formula $\theta[\vec{x}]$ in the language $\{\in\}$ and terms \vec{t} such that $A \equiv \theta[\vec{t}]$. This means that A is bounded, and the predicates $P, P_{I,n}, \text{Reg}$ do not occur in A .

2. Putting $\Sigma_0 := \Pi_0 := \Delta_0$, the classes Σ_m and Π_m of formulae in the language $\{\in\}$ with terms are defined as usual using quantifiers $\exists x \in L_I, \forall x \in L_I$, where by definition $\Sigma_m \cup \Pi_m \subset \Sigma_{m+1} \cap \Pi_{m+1}$.

Each formula in $\Sigma_m \cup \Pi_m$ is in prenex normal form with alternating unbounded quantifiers and Δ_0 -matrix.

3. $A \in \Delta_0(\lambda)$ iff there exists a Δ_0 -formula $\theta[\vec{x}]$ in the language $\{\in\}$ and terms \vec{t} such that $A \equiv \theta[\vec{t}]$ and $k(A) < \lambda$.
4. $A \in \Sigma_1(\lambda)$ iff either $A \in \Delta_0(\lambda)$ or $A \equiv \exists x \in L_\lambda B$ with $B \in \Delta_0(\lambda)$.
Note that $\Sigma(\lambda) \subset \Delta_0$ for any $\lambda < I$.
5. The class of sentences $\Sigma_m(\lambda), \Pi_m(\lambda)$ ($m < \omega$) are defined as usual.
6. $\Sigma_0^1(\lambda)$ denotes the set of first-order formulae on L_λ , i.e., $\Sigma_0^1(\lambda) := \bigcup_{m \in \omega} \Sigma_m(\lambda)$.

Note that the predicates $P, P_{I,n}, \text{Reg}$ do not occur in Σ_m -formulae nor in $\Sigma_0^1(\lambda)$ -formulae.

Definition 4.6 A set $\Sigma^{\Sigma_{n+1}}(\lambda)$ of sentences is defined recursively as follows.

1. $\Sigma_{n+1} \subset \Sigma^{\Sigma_{n+1}}(\lambda)$.
2. Each literal including $\text{Reg}(a), P(a, b, c), P_{I,n}(a)$ and their negations is in $\Sigma^{\Sigma_{n+1}}(\lambda)$.
3. $\Sigma^{\Sigma_{n+1}}(\lambda)$ is closed under propositional connectives \vee, \wedge .
4. Suppose $\forall x \in b A(x) \notin \Delta_0$. Then $\forall x \in b A(x) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ iff $A(\emptyset) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ and $\text{rk}_L(b) < \lambda$.

5. Suppose $\exists x \in b A(x) \notin \Delta_0$. Then $\exists x \in b A(x) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ iff $A(\emptyset) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ and $\text{rk}_L(b) \leq \lambda$.

Definition 4.7 Let us extend the domain $\text{dom}(F_{x \cup \{\kappa\}}^{\Sigma_1}) = \text{Hull}_{\Sigma_1}^I(x \cup \{\kappa\})$ of Mostowski collapse to formulae.

$$\text{dom}(F_{x \cup \{\kappa\}}^{\Sigma_1}) = \{A \in \Sigma_1 \cup \Pi_1 : \text{k}(A) \subset \text{Hull}_{\Sigma_1}^I(x \cup \{\kappa\})\}.$$

For $A \in \text{dom}(F_{x \cup \{\kappa\}}^{\Sigma_1})$, $F_{x \cup \{\kappa\}}^{\Sigma_1} \ulcorner A$ denotes the result of replacing each constant γ by $F_{x \cup \{\kappa\}}^{\Sigma_1}(\gamma)$, each unbounded existential quantifier $\exists z \in L_I$ by $\exists z \in L_{F_{x \cup \{\kappa\}}^{\Sigma_1}}(I)$, and each unbounded universal quantifier $\forall z \in L_I$ by $\forall z \in L_{F_{x \cup \{\kappa\}}^{\Sigma_1}}(I)$.

For sequent, i.e., finite set of sentences $\Gamma \subset \text{dom}(F_{x \cup \{\kappa\}}^{\Sigma_1})$, put $F_{x \cup \{\kappa\}}^{\Sigma_1} \ulcorner \Gamma = \{F_{x \cup \{\kappa\}}^{\Sigma_1} \ulcorner A : A \in \Gamma\}$.

Likewise the domain $\text{dom}(F_x^{\Sigma_n}) = \text{Hull}_{\Sigma_n}^I(x)$ is extended to

$$\text{dom}(F_x^{\Sigma_n}) = \{A \in \Sigma_n \cup \Pi_n : \text{k}(A) \subset \text{Hull}_{\Sigma_n}^I(x)\}$$

and for formula $A \in \text{dom}(F_x^{\Sigma_n})$, $F_x^{\Sigma_n} \ulcorner A$, and sequent $\Gamma \subset \text{dom}(F_x^{\Sigma_n})$, $F_x^{\Sigma_n} \ulcorner \Gamma$ are defined similarly.

Proposition 4.8 For $F = F_{x \cup \{\kappa\}}^{\Sigma_1}, F_x^{\Sigma_n}$ and $A \in \text{dom}(F)$

$$L_I \models A \leftrightarrow F \ulcorner A.$$

The assignment of disjunctions and conjunctions to sentences is defined as in [7].

Definition 4.9 1. If M is one of the literals $a \in b, a \notin b$, then for $J := 0$

$$M := \begin{cases} \bigvee (A_i)_{i \in J} & \text{if } M \text{ is false (in } L_I) \\ \bigwedge (A_i)_{i \in J} & \text{if } M \text{ is true} \end{cases}$$

2. $(A_0 \vee A_1) := \bigvee (A_i)_{i \in J}$ and $(A_0 \wedge A_1) := \bigwedge (A_i)_{i \in J}$ for $J := 2$.
3.

$$\text{Reg}(a) := \bigvee (a = a)_{i \in J} \text{ and } \neg \text{Reg}(a) := \bigwedge (a \neq a)_{i \in 1}$$

with

$$J := \begin{cases} 1 & \text{if } a \in R \\ 0 & \text{otherwise} \end{cases}.$$

- 4.

$$P(a, b, c) := \bigvee (a = a)_{i \in J} \text{ and } \neg P(a, b, c) := \bigwedge (a \neq a)_{i \in J}$$

with

$$J := \begin{cases} 1 & \text{if } a \in R \ \& \ \exists \alpha \in \text{Ord}_\varepsilon [b = \Psi_{a,n} \alpha \ \& \ \alpha \in \mathcal{H}_\alpha(b) \ \& \ c = F_{b \cup \{a\}}^{\Sigma_1}(I)] \\ 0 & \text{otherwise} \end{cases}.$$

5.

$$P_{I,n}(a) := \bigvee (a = a)_{i \in J} \text{ and } \neg P_{I,n}(a) := \bigwedge (a \neq a)_{i \in J}$$

with

$$J := \begin{cases} 1 & \text{if } \exists \alpha \in \text{Ord}_\varepsilon [a = \Psi_{I,n} \alpha \ \& \ \alpha \in \mathcal{H}_\alpha(a)] \\ 0 & \text{otherwise} \end{cases} .$$

6. Let $(\exists z \in b \theta[z]) \in \Sigma_n$ for $b \in L_I \cup \{L_I\}$, and $(\exists z \in b \theta[z]) \notin \Sigma_0^1(\mathcal{K}^+)$. Then for the set

$$\mu z \in b \theta[z] := \min_{<_L} \{d : (d \in b \wedge \theta[d]) \vee (\neg \exists z \in b \theta[z] \wedge d = 0)\} \quad (8)$$

with a canonical well ordering $<_L$ on L , and $J = \{d\}$

$$\begin{aligned} \exists z \in b \theta[z] &:= \bigvee (d \in b \wedge \theta[d])_{d \in J} \\ \forall z \in b \neg \theta[z] &:= \bigwedge (d \in b \rightarrow \neg \theta[d])_{d \in J} \end{aligned} \quad (9)$$

where $d \in b$ denotes a true literal, e.g., $d \notin d$ when $b = L_I$.

This case is applied only when $\exists z \in b \theta[z]$ is a formula in $\{\in\} \cup L_I$, and $(\exists z \in b \theta[z]) \in \Sigma_n$ but $(\exists z \in b \theta[z]) \notin \Sigma_0^1(\mathcal{K}^+)$.

7. Otherwise set for $a \in L_I \cup \{L_I\}$

$$\exists x \in a A(x) := \bigvee (A(b))_{b \in J} \text{ and } \forall x \in a A(x) := \bigwedge (A(b))_{b \in J}$$

for

$$J := \{b : b \in a\}.$$

This case is applied if one of the predicates $P, P_{I,n}, \text{Reg}$ occurs in $\exists x \in a A(x)$, or $(\exists x \in a A(x)) \notin \Sigma_n$, or $(\exists x \in a A(x)) \in \Sigma_0^1(\mathcal{K}^+)$.

In particular we have

$$\begin{aligned} \neg \tau(B, \mathcal{K}) &:= \bigwedge \{(C \not\subset \mathcal{K}) \vee \neg(C \text{ is club})^\mathcal{K} \vee (B \cap C \neq \emptyset) : C \in L_{\mathcal{K}^+}\} \\ \tau(B, \mathcal{K}) &:= \bigvee \{(C \subset \mathcal{K}) \wedge (C \text{ is club})^\mathcal{K} \wedge (B \cap C = \emptyset) : C \in L_{\mathcal{K}^+}\} \end{aligned}$$

where

$$\tau(B, \rho) := \Leftrightarrow \exists C \subset \rho [(C \text{ is club})^\rho \wedge (B \cap C = \emptyset)] \quad (5)$$

The definition of the rank $\text{rk}(A)$ of sentences A in [7] is slightly changed as follows. The rank $\text{rk}(A)$ of sentences A is defined by recursion on the number of symbols occurring in A .

Definition 4.10 1. $\text{rk}(\neg A) := \text{rk}(A)$.

2. $\text{rk}(a \in b) := \text{rk}(a \notin b) := 0$.
3. $\text{rk}(\text{Reg}(\alpha)) := \text{rk}(P(\alpha, \beta, \gamma)) := \text{rk}(P_{I,n}(\alpha)) := 1$.
4. $\text{rk}(A_0 \vee A_1) := \max\{\text{rk}(A_0), \text{rk}(A_1)\} + 1$.
5. $\text{rk}(\exists x \in a A(x)) := \max\{\omega\alpha, \text{rk}(A(\emptyset)) + 2\}$ for $\alpha = \text{rk}_L(a)$.

Proposition 4.11 *Let $A \simeq \bigvee (A_\iota)_{\iota \in J}$ or $A \simeq \bigwedge (A_\iota)_{\iota \in J}$.*

1. $A \in \Sigma^{\Sigma_{n+1}}(\lambda) \Rightarrow \forall \iota \in J (A_\iota \in \Sigma^{\Sigma_{n+1}}(\lambda))$.
2. For an ordinal $\lambda \leq I$ with $\omega\lambda = \lambda$, $\text{rk}(A) < \lambda \Rightarrow A \in \Sigma^{\Sigma_{n+1}}(\lambda)$.
3. $\text{rk}(A) < I + \omega$.
4. $\text{rk}(A)$ is in the Skolem hull of $\omega\text{qk}(A) \cup \{0, 1\}$ under the addition with $\omega\text{qk}(A) = \{\omega\alpha : \alpha \in \text{qk}(A)\}$.
5. $\forall \iota \in J (\text{rk}(A_\iota) < \text{rk}(A))$.

Proof.

4.11.5. This is seen from the fact that $a \in b \in L \Rightarrow \text{rk}_L(a) < \text{rk}_L(b)$. \square

4.3 Operator controlled derivations

$\kappa, \lambda, \sigma, \pi$ ranges over R^+ .

Let \mathcal{H} be an operator, Θ a finite set of ordinals, $\kappa \in R^+$, Γ a sequent, $a \in \text{Ord}^\varepsilon$ and $b < I + \omega$. We define a relation $(\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Gamma$, which is read ‘there exists an infinitary derivation of Γ which is (κ, n) -controlled by \mathcal{H} and Θ , and whose height is at most a and its cut rank is less than b ’.

Recall that R denotes the set of uncountable cardinals ρ such that $\mathcal{K} < \rho < I$, and $\lambda > \mathcal{K}$ in the inference rules (\mathbf{P}_λ) and $(\mathbf{F}_{x \cup \{\lambda\}}^{\Sigma_1})$.

Sequents are finite sets of sentences, and inference rules are formulated in one-sided sequent calculus.

Definition 4.12

$$\mathbf{k}_{\mathcal{K}}^E(A) := \begin{cases} \mathbf{k}^E(A) & \text{if } A \in \Sigma_0^1(\mathcal{K}^+) \\ \mathbf{k}(A) & \text{otherwise} \end{cases}$$

Definition 4.13 $(\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Gamma$ holds if

$$\mathbf{k}_{\mathcal{K}}^E(\Gamma) := \bigcup \{\mathbf{k}_{\mathcal{K}}^E(A) : A \in \Gamma\} \subset \mathcal{H} := \mathcal{H}(\emptyset) \& a \in \mathcal{H}[\Theta] \quad (10)$$

and one of the following cases holds:

1. $A \simeq \bigvee \{A_\iota : \iota \in J\}$, $A \in \Gamma$ and for an $\iota \in J$, $a(\iota) < a$ and $\text{rk}_L(\iota) < \kappa \Rightarrow \text{rk}_L(\iota) < a$

$$\frac{(\mathcal{H}, \Theta, \kappa, n) \vdash_b^{a(\iota)} \Gamma, A_\iota}{(\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Gamma} (\vee)$$

2. $A \simeq \bigwedge \{A_\iota : \iota \in J\}$, $A \in \Gamma$ and $a(\iota) < a$ for any $\iota \in J$

$$\frac{\{(\mathcal{H}[\{\text{rk}_L(\iota)\}], \Theta, \kappa, n) \vdash_b^{a(\iota)} \Gamma, A_\iota : \iota \in J\}}{(\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Gamma} (\wedge)$$

3. $\text{rk}(C) < b$ and an $a_0 < a$

$$\frac{(\mathcal{H}, \Theta, \kappa, n) \vdash_b^{a_0} \Gamma, \neg C \quad (\mathcal{H}, \Theta, \kappa, n) \vdash_b^{a_0} C, \Gamma}{(\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Gamma} (\text{cut})$$

4. $\alpha < \lambda \in R$ and $\{\exists x < \lambda \exists y < \lambda [\alpha < x \wedge P(\lambda, x, y)]\} \cup \Gamma_0 = \Gamma$

$$\frac{}{\exists x < \lambda \exists y < \lambda [\alpha < x \wedge P(\lambda, x, y)], \Gamma_0} (\mathbf{P}_\lambda)$$

5. Let $\lambda \in R$ and $x \in \mathcal{H}[\Theta]$ where for some b

$$x = \Psi_{\lambda, n} b.$$

If $\Gamma = \Lambda \cup (F_{x \cup \{\lambda\}}^{\Sigma_1} \text{''} \Gamma_0)$, $\Gamma_0 \subset \Sigma_1$, $a_0 < a$ and

$$\mathbf{k}(\Gamma_0) \subset \text{Hull}_{\Sigma_1}^I((\mathcal{H} \cap x) \cup \{\lambda\})$$

then

$$\frac{(\mathcal{H}, \Theta, \kappa, n) \vdash_b^{a_0} \Lambda, \Gamma_0}{(\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Lambda, F_{x \cup \{\lambda\}}^{\Sigma_1} \text{''} \Gamma_0} (\mathbf{F}_{x \cup \{\lambda\}}^{\Sigma_1})$$

where $F_{x \cup \{\lambda\}}^{\Sigma_1}$ denotes the Mostowski collapse $F_{x \cup \{\lambda\}}^{\Sigma_1} : \text{Hull}_{\Sigma_1}^I(x \cup \{\lambda\}) \leftrightarrow L_{F_{x \cup \{\lambda\}}^{\Sigma_1}}(I)$.

6. $\alpha < I$ and $\{\exists x < I [\alpha < x \wedge P_{I, n}(x)]\} \cup \Gamma_0 = \Gamma$

$$\frac{}{\exists x < I [\alpha < x \wedge P_{I, n}(x)], \Gamma_0} (\mathbf{P}_{I, n})$$

7. Let

$$x = \Psi_{I, n} b \in \mathcal{H}[\Theta].$$

If $\Gamma = \Lambda \cup (F_x^{\Sigma_n} \text{''} \Gamma_0)$, $\Gamma_0 \subset \Sigma_n$, $a_0 < a$ and

$$\mathbf{k}(\Gamma_0) \subset \text{Hull}_{\Sigma_n}^I(\mathcal{H} \cap x)$$

then

$$\frac{(\mathcal{H}, \Theta, \kappa, n) \vdash_b^{a_0} \Lambda, \Gamma_0}{(\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Lambda, F_x^{\Sigma_n} \text{''} \Gamma_0} (\mathbf{F}_x^{\Sigma_n})$$

where $F_x^{\Sigma_n}$ denotes the Mostowski collapse $F_x^{\Sigma_n} : \text{Hull}_{\Sigma_n}^I(x) \leftrightarrow L_{F_x^{\Sigma_n}}(I)$.

8. If $\max\{a_\ell, a_r\} < a$, and $B \subset \mathcal{K}$, $B \in \text{Hull}_{\Sigma_1}^I(\{\mathcal{K}, \mathcal{K}^+\})$, then

$$\frac{(\mathcal{H}, \Theta, \kappa, n) \vdash_b^{a_\ell} \Gamma, \neg\tau(B, \mathcal{K}) \quad (\mathcal{H}, \Theta, \kappa, n) \vdash_b^{a_r} \Gamma, \forall \rho < \mathcal{K} \tau(B, \rho)}{(\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Gamma} \text{ (Ref}_{\mathcal{K}})$$

where

$$\tau(B, \rho) := \exists C \subset \rho [(C \text{ is club})^\rho \wedge (B \cap C = \emptyset)] \quad (5)$$

which is stratified with respect to B .

An inspection to Definition 4.13 shows that there exists a strictly positive formula H_n such that the relation $(\mathcal{H}_{\gamma, n}[\Theta_0], \Theta, \kappa, n) \vdash_b^a \Gamma$ is a fixed point of H_n as in (6).

In what follows the relation should be understood as a fixed point of H_n , and recall that we are working in the intuitionistic fixed point theory $\text{Fix}^i(\text{ZFLK}_n)$ over ZFLK_n defined in subsection 4.1.

Proposition 4.14 $(\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Gamma \ \& \ \lambda \leq \kappa \Rightarrow (\mathcal{H}, \Theta, \lambda, n) \vdash_b^a \Gamma$.

We will state some lemmata for the operator controlled derivations with sketches of their proofs since these can be shown as in [10] and [7].

In what follows by an operator we mean an $\mathcal{H}_\gamma[\Theta]$ for a finite set Θ of ordinals.

$$(\mathcal{H}, \kappa, n) \vdash_b^a \Gamma := (\mathcal{H}, \emptyset, \kappa, n) \vdash_b^a \Gamma$$

Lemma 4.15 (Tautology)

$$(\mathcal{H}[\mathbf{k}_{\mathcal{K}}^E(A)], I, n) \vdash_0^{I+2\text{rk}(A)} \Gamma, \neg A, A.$$

Lemma 4.16 ($\Delta_0(I)$ -completeness) *If $\Gamma \subset \Delta_0(I)$ and $\bigvee \Gamma$ is true, then*

$$(\mathcal{H}[\mathbf{k}_{\mathcal{K}}^E(\Gamma)], I, n) \vdash_0^{I+2\text{rk}(\Gamma)} \Gamma$$

where $\text{rk}(\Gamma) = \text{rk}(A_0) \# \dots \# \text{rk}(A_n)$ for $\Gamma = \{A_0, \dots, A_n\}$.

Lemma 4.17 (Elimination of false sentences)

Let A be a false sentence, i.e., $L_I \not\models A$, such that $\mathbf{k}(A) \subset \text{Hull}_{\Sigma_1}^I((\mathcal{K} + 1) \cup \{\mathcal{K}^+\}) \cap \mathcal{K}^+$. Then

$$(\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Gamma, A \Rightarrow (\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Gamma.$$

Proof.

Consider the case when A is a main formula of an $(\mathbf{F}_{x \cup \{\mathcal{K}^+\}}^{\Sigma_1})$ with $x > \mathcal{K}$. We have $F_{x \cup \{\mathcal{K}^+\}}^{\Sigma_1}(a) = a$ for any a with $\text{rk}_L(a) < x$.

We claim $F_{x \cup \{\mathcal{K}^+\}}^{\Sigma_1} A \equiv A$. Let $b \in \mathbf{k}(A)$. Then $\text{rk}_L(b) \in \text{Hull}_{\Sigma_1}^I((\mathcal{K} + 1) \cup \{\mathcal{K}^+\}) \cap \mathcal{K}^+ \subset \text{Hull}_{\Sigma_1}^I(x \cup \{\mathcal{K}^+\}) \cap \mathcal{K}^+ \subset x$. Hence $F_{x \cup \{\mathcal{K}^+\}}^{\Sigma_1}(b) = b$. \square

Lemma 4.18 (Embedding)

For each axiom A in $\mathsf{T}(\mathcal{K}, I, n)$, there is an $m < \omega$ such that for any operator \mathcal{H}

$$(\mathcal{H}[\{\mathcal{K}\}], I, n) \vdash_I^{I \cdot m} \text{'}\mathcal{K} \text{ is uncountable regular'} \rightarrow A.$$

Proof.

The axiom for Π_1^1 -indescribability

$$\forall B \in L_{\mathcal{K}^+} [B \subset \mathcal{K} \rightarrow \neg\tau(B, \mathcal{K}) \rightarrow \exists \rho < \mathcal{K} (\neg\tau(B, \rho) \wedge \text{Reg}(\rho))] \quad (4)$$

follows from the inference rule (**Ref** $_{\mathcal{K}}$) and (4) $\simeq \bigwedge (B \subset \mathcal{K} \rightarrow \neg\tau(B, \mathcal{K}) \rightarrow \exists \rho < \mathcal{K} (\neg\tau(B, \rho) \wedge \text{Reg}(\rho)))_{B \in L_{\mathcal{K}^+}}$ for $B := \mu B \in L_{\mathcal{K}^+} (B \subset \mathcal{K} \wedge \neg\tau(B, \mathcal{K}) \wedge \forall \rho < \mathcal{K} (\text{Reg}(\rho) \rightarrow \tau(B, \rho))) \in \text{Hull}_{\Sigma_1}^I(\{\mathcal{K}, \mathcal{K}^+\})$. \square

Lemma 4.19 (Inversion)

Let $d = \mu z \in b A[\vec{c}, z]$ for $(\exists z \in b A) \in \Sigma_n \setminus \Sigma_0^1(\mathcal{K}^+)$.

$$(\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Gamma, \exists z \in b A[\vec{c}, z] \Rightarrow (\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Gamma, d \in b \wedge A[\vec{c}, d]$$

and

$$(\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Gamma, \forall z \in b \neg A[\vec{c}, z] \Rightarrow (\mathcal{H}, \Theta, \kappa, n) \vdash_b^a \Gamma, d \in b \rightarrow \neg A[\vec{c}, d]$$

Lemma 4.20 (Reduction)

Let $C \simeq \bigvee (C_\iota)_{\iota \in J}$.

1. Suppose $C \notin \{\exists x < \lambda \exists y < \lambda [\alpha < x \wedge P(\lambda, x, y)] : \alpha < \lambda \in R\} \cup \{\exists x < I [\alpha < x \wedge P_{I,n}(x)] : \alpha < I\}$.

Then

$$(\mathcal{H}, \Theta, \kappa, n) \vdash_c^a \Delta, \neg C \ \& \ (\mathcal{H}, \kappa, n) \vdash_c^b C, \Gamma \ \& \ \mathcal{K} \leq \text{rk}(C) \leq c \Rightarrow (\mathcal{H}, \Theta, \kappa, n) \vdash_c^{a+b} \Delta, \Gamma$$

2. Assume $C \equiv (\exists x < \lambda \exists y < \lambda [\alpha < x \wedge P(\lambda, x, y)])$ for an $\alpha < \lambda \in R$ and $\beta \in \mathcal{H}_\beta$.

Then

$$(\mathcal{H}_\beta, \kappa, n) \vdash_b^a \Gamma, \neg C \Rightarrow (\mathcal{H}_{\beta+1}, \kappa, n) \vdash_b^a \Gamma$$

3. Assume $C \equiv (\exists x < I [\alpha < x \wedge P_{I,n}(x)])$ for an $\alpha < I$ and $\beta \in \mathcal{H}_\beta$.

Then

$$(\mathcal{H}_\beta, \kappa, n) \vdash_b^a \Gamma, \neg C \Rightarrow (\mathcal{H}_{\beta+1}, \kappa, n) \vdash_b^a \Gamma$$

Lemma 4.21 (Predicative Cut-elimination)

1. $(\mathcal{H}, \kappa, n) \vdash_{c+\omega^a}^b \Gamma \ \& \ [c, c + \omega^a [\cap (\{\lambda + 1 : \lambda \in R\} \cup \{I\})] = \emptyset \ \& \ a \in \mathcal{H} \Rightarrow (\mathcal{H}, \kappa, n) \vdash_c^{\varphi^{ab}} \Gamma$.
2. For $\lambda \in R$, $(\mathcal{H}_\gamma, \kappa, n) \vdash_{\lambda+2}^b \Gamma \ \& \ \gamma \in \mathcal{H}_\gamma \ \& \Rightarrow (\mathcal{H}_{\gamma+b}, \kappa, n) \vdash_{\lambda+1}^{\omega^b} \Gamma$.

$$3. (\mathcal{H}_\gamma, \kappa, n) \vdash_{I+1}^b \Gamma \& \gamma \in \mathcal{H}_\gamma \& \Rightarrow (\mathcal{H}_{\gamma+b}, \kappa, n) \vdash_I^{\omega^b} \Gamma.$$

$$4. (\mathcal{H}_\gamma, \kappa, n) \vdash_{c+\omega^a}^b \Gamma \& \max\{a, b, c\} < I \& a \in \mathcal{H}_\gamma \Rightarrow (\mathcal{H}_{\gamma+\varphi ab}, \kappa, n) \vdash_c^{\varphi ab} \Gamma.$$

Definition 4.22 For a formula $\exists x \in dA$ and ordinals $\lambda = \text{rk}_L(d) \in R^+$, $(\exists x \in dA)^{(\exists \lambda \alpha)}$ denotes the result of restricting the *outermost existential quantifier* $\exists x \in d$ to $\exists x \in L_\alpha$, $(\exists x \in dA)^{(\exists \lambda \alpha)} \equiv (\exists x \in L_\alpha A)$.

In what follows $F_{x,\lambda}$ denotes $F_{x,\lambda}^{\Sigma_1}$ when $\lambda \in R$, and $F_x^{\Sigma_n}$ when $\lambda = I$.

Lemma 4.23 (Boundedness)

Let $\lambda \in R^+$, $C \equiv (\exists x \in dA)$ and $C \notin \{\exists x < \lambda \exists y < \lambda [\alpha < x \wedge P(\lambda, x, y)] : \alpha < \lambda \in R\} \cup \{\exists x < I [\alpha < x \wedge P_{I,n}(x)] : \alpha < I\}$. Assume that $\text{rk}(C) = \lambda = \text{rk}_L(d)$.

1.

$$(\mathcal{H}, \Theta, \lambda, n) \vdash_c^a \Lambda, C \& a \leq b \in \mathcal{H} \cap \lambda \Rightarrow (\mathcal{H}, \Theta, \lambda, n) \vdash_c^a \Lambda, C^{(\exists \lambda^b)}.$$

2.

$$(\mathcal{H}, \Theta, \kappa, n) \vdash_c^a \Lambda, \neg C \& b \in \mathcal{H} \cap \lambda \Rightarrow (\mathcal{H}, \Theta, \kappa, n) \vdash_c^a \Lambda, \neg(C^{(\exists \lambda^b)}).$$

Though the following Lemma 4.24(Collapsing down to I) is seen as in Lemma 5.22(Collapsing) of [7], we reproduce a proof of it since [7] has not yet been published.

Recall that

$$(\mathcal{H}, \kappa, n) \vdash_b^a \Gamma : \Leftrightarrow (\mathcal{H}, \emptyset, \kappa, n) \vdash_b^a \Gamma$$

Lemma 4.24 (Collapsing down to I)

Suppose $\gamma \in \mathcal{H}_{\gamma,n}[\Theta]$ with $\Theta \subset \mathcal{H}_{\gamma,n}(\Psi_{I,n}\gamma)$, and

$$\Gamma \subset \Sigma^{\Sigma_{n+1}}(I)$$

Then for $\hat{a} = \gamma + \omega^{I+a}$

$$(\mathcal{H}_{\gamma,n}[\Theta], I, n) \vdash_{I+1}^a \Gamma \Rightarrow (\mathcal{H}_{\hat{a}+1,n}[\Theta], I, n) \vdash_{\Psi_{I,n}\hat{a}}^{\Psi_{I,n}\hat{a}} \Gamma.$$

Proof.

By induction on a .

First note that $\Psi_{I,n}\hat{a} \in \mathcal{H}_{\hat{a}+1,n}[\Theta] = \mathcal{H}_{\hat{a}+1,n}(\Theta)$ since $\hat{a} = \gamma + \omega^{I+a} \in \mathcal{H}_{\gamma,n}[\Theta] \subset \mathcal{H}_{\hat{a}+1,n}[\Theta]$ by the assumption, $\{\gamma, a\} \subset \mathcal{H}_{\gamma,n}[\Theta]$.

Assume $(\mathcal{H}_{\gamma,n}[\Theta][\Lambda], I, n) \vdash_{I+1}^{a_0} \Gamma_0$ with $\Lambda \subset \mathcal{H}_{\gamma,n}(\Psi_{I,n}\gamma)$. Then by $\gamma \leq \hat{a}$, we have $\hat{a}_0 \in \mathcal{H}_{\gamma,n}[\Theta][\Lambda] \subset \mathcal{H}_{\gamma,n}(\Psi_{I,n}\gamma) \subset \mathcal{H}_{\hat{a},n}(\Psi_{I,n}\hat{a})$. This yields that

$$a_0 < a \Rightarrow \Psi_{I,n}\hat{a}_0 < \Psi_{I,n}\hat{a} \tag{11}$$

Second observe that $\text{k}_K^E(\Gamma) \subset \mathcal{H}_{\gamma,n}[\Theta] \subset \mathcal{H}_{\hat{a}+1,n}[\Theta]$ by $\gamma \leq \hat{a} + 1$.

Third we have

$$\mathfrak{k}_{\mathcal{K}}^E(\Gamma) \subset \mathcal{H}_{\gamma,n}(\Psi_{I,n}\gamma) \quad (12)$$

Case 1. First consider the case: $\Gamma \ni A \simeq \bigwedge\{A_\iota : \iota \in J\}$

$$\frac{\{(\mathcal{H}_{\gamma,n}[\Theta \cup \{\text{rk}_L(\iota)\}], I, n) \vdash_{I+1}^{a(\iota)} \Gamma, A_\iota : \iota \in J\}}{(\mathcal{H}_{\gamma,n}[\Theta], I, n) \vdash_{I+1}^a \Gamma} \quad (\wedge)$$

where $a(\iota) < a$ for any $\iota \in J$.

We claim that

$$\forall \iota \in J (\text{rk}_L(\iota) \in \mathcal{H}_{\gamma,n}(\Psi_{I,n}\gamma)) \quad (13)$$

Consider the case when $A \equiv \forall x \in b \neg A'$. There are two cases to consider. First consider the case when $J = \{d\}$ for the set $d = \mu x \in b A'$. Then $\mathfrak{k}_{\mathcal{K}}^E(A) = \mathfrak{k}(A)$, and $\iota = d = (\mu x \in b A') \in \text{Hull}_{\Sigma_n}^I(\mathfrak{k}(A))$, and $\text{rk}_L(\iota) \in \text{Hull}_{\Sigma_n}^I(\mathfrak{k}(A)) \subset \mathcal{H}_{\gamma,n}(\Psi_{I,n}\gamma)$ by (12). Otherwise we have $J = b$ and either $A \in \Sigma_0^1(\mathcal{K}^+)$ and $b \in L_{\mathcal{K}^+} \cup \{L_{\mathcal{K}^+}\}$, or $\text{rk}_L(b) < I$. In the second case we have $b \in \mathfrak{k}(A) = \mathfrak{k}_{\mathcal{K}}^E(A) \subset \mathcal{H}_{\gamma,n}[\Theta]$. In the first case each $\iota \in b$ has L -rank $\text{rk}_L(\iota) < \mathcal{K}^+$. On the other hand we have $\mathcal{K}^+ \in \mathcal{H}_{\gamma,n}(\Psi_{I,n}\gamma) \cap I \subset \Psi_{I,n}\gamma$ by $I > \mathcal{K}^+$. Thus $\text{rk}_L(\iota) < \Psi_{I,n}\gamma$. In the second case we have $\text{rk}_L(\iota) \leq \text{rk}_L(b) \in \mathcal{H}_{\gamma,n}(\Psi_{I,n}\gamma) \cap I \subset \Psi_{I,n}\gamma$ by $\text{rk}_L(b) < I$.

Hence (13) was shown.

SIH yields

$$\frac{\{(\mathcal{H}_{\widehat{a(\iota)+1,n}}[\Theta \cup \{\text{rk}_L(\iota)\}], I, n) \vdash_{\Psi_{I,n}\widehat{a(\iota)}}^{\Psi_{I,n}\widehat{a(\iota)}} \Gamma, A_\iota : \iota \in J\}}{(\mathcal{H}_{\widehat{a}+1,n}[\Theta], I, n) \vdash_{\Psi_{I,n}\widehat{a}}^{\Psi_{I,n}\widehat{a}} \Gamma} \quad (\wedge)$$

for $\widehat{a(\iota)} = \gamma + \omega^{I+a(\iota)}$, since $\Psi_{I,n}\widehat{a(\iota)} < \Psi_{I,n}\widehat{a}$ by (11).

Case 2. Next consider the case for an $A \simeq \bigvee\{A_\iota : \iota \in J\} \in \Gamma$ and an $\iota \in J$ with $a(\iota) < a$ and $\text{rk}_L(\iota) < I \Rightarrow \text{rk}_L(\iota) < a$

$$\frac{(\mathcal{H}_{\gamma,n}[\Theta], I, n) \vdash_{I+1}^{a(\iota)} \Gamma, A_\iota}{(\mathcal{H}_{\gamma,n}[\Theta], I, n) \vdash_{I+1}^a \Gamma} \quad (\vee)$$

Assume $\text{rk}_L(\iota) < I$. We show $\text{rk}_L(\iota) < \Psi_{I,n}\widehat{a}$. By $\Psi_{I,n}\gamma \leq \Psi_{I,n}\widehat{a}$, it suffices to show $\text{rk}_L(\iota) < \Psi_{I,n}\gamma$.

Consider the case when $A \equiv \exists x \in b A'$. There are two cases to consider. First consider the case when $J = \{d\}$ for the set $d = \mu x \in b A'$. Then $\mathfrak{k}_{\mathcal{K}}^E(A) = \mathfrak{k}(A)$, and $\iota = d = (\mu x \in b A') \in \text{Hull}_{\Sigma_n}^I(\mathfrak{k}(A))$, and $\text{rk}_L(\iota) \in \text{Hull}_{\Sigma_n}^I(\mathfrak{k}(A)) \subset \mathcal{H}_{\gamma,n}(\Psi_{I,n}\gamma)$ by (12). If $\text{rk}_L(\iota) < I$, then $\text{rk}_L(\iota) \in \mathcal{H}_{\gamma,n}(\Psi_{I,n}\gamma) \cap I \subset \Psi_{I,n}\gamma$.

Otherwise we have $J = b$, and either $A \in \Sigma_0^1(\mathcal{K}^+)$ and $b \in L_{\mathcal{K}^+} \cup \{L_{\mathcal{K}^+}\}$, or $b \in \mathfrak{k}(A) = \mathfrak{k}_{\mathcal{K}}^E(A) \subset \mathcal{H}_{\gamma,n}[\Theta]$. In the second case we can assume that $\iota \in \mathfrak{k}(A_\iota) = \mathfrak{k}_{\mathcal{K}}^E(A_\iota) \subset \mathcal{H}_{\gamma,n}[\Theta]$. Otherwise set $\iota = 0$.

In the first case each $\iota \in b$ has L -rank $\text{rk}_L(\iota) < \mathcal{K}^+$. On the other hand we have $\mathcal{K}^+ \in \mathcal{H}_{\gamma,n}(\Psi_{I,n}\gamma) \cap I \subset \Psi_{I,n}\gamma$ by $I > \mathcal{K}^+$. Thus $\text{rk}_L(\iota) < \Psi_{I,n}\gamma$. In the second case we have $\text{rk}_L(\iota) < \text{rk}_L(b) \leq I$, and $\text{rk}_L(\iota) \in \mathcal{H}_{\gamma,n}(\Psi_{I,n}\gamma) \cap I \subset \Psi_{I,n}\gamma$.

SIH yields for $\widehat{a(\iota)} = \gamma + \omega^{I+a(\iota)}$

$$\frac{(\mathcal{H}_{\widehat{a(\iota)}+1,n}[\Theta], I, n) \vdash_{\Psi_{I,n}\widehat{a(\iota)}}^{\Psi_{I,n}\widehat{a(\iota)}} \Gamma, A_\iota}{(\mathcal{H}_{\widehat{a}+1,n}[\Theta], I, n) \vdash_{\Psi_{I,n}\widehat{a}}^{\Psi_{I,n}\widehat{a}}} \quad (\vee)$$

Case 3. Third consider the case for an $a_0 < a$ and a C with $\text{rk}(C) < I + 1$.

$$\frac{(\mathcal{H}_{\gamma,n}[\Theta], I, n) \vdash_{I+1}^{a_0} \Gamma, \neg C \quad (\mathcal{H}_{\gamma,n}[\Theta], I, n) \vdash_{I+1}^{a_0} C, \Gamma}{(\mathcal{H}_{\gamma,n}[\Theta], I, n) \vdash_{I+1}^a \Gamma} \quad (cut)$$

Case 3.1. $\text{rk}(C) < I$.

We have by (12) $\mathbf{k}_{\mathcal{K}}^E(C) \subset \mathcal{H}_{\gamma,n}(\Psi_{I,n}\gamma)$. Proposition 4.11.4 yields $\text{rk}(C) \in \mathcal{H}_{\gamma,n}(\Psi_{I,n}\gamma) \cap I \subset \Psi_{I,n}\gamma \leq \Psi_{I,n}\widehat{a}$. By Proposition 4.11.2 we see that $\{-C, C\} \subset \Sigma^{\Sigma_{n+1}}(I)$.

SIH yields for $\widehat{a}_0 = \gamma + \omega^{I+a_0}$

$$\frac{(\mathcal{H}_{\widehat{a}_0+1,n}[\Theta], I, n) \vdash_{\Psi_{I,n}\widehat{a}_0}^{\Psi_{I,n}\widehat{a}_0} \Gamma, \neg C \quad (\mathcal{H}_{\widehat{a}_0+1,n}[\Theta], I, n) \vdash_{\Psi_{I,n}\widehat{a}_0}^{\Psi_{I,n}\widehat{a}_0} C, \Gamma}{(\mathcal{H}_{\widehat{a}+1,n}[\Theta], I, n) \vdash_{\Psi_{I,n}\widehat{a}}^{\Psi_{I,n}\widehat{a}}} \quad (cut)$$

Case 3.2. $\text{rk}(C) = I$.

Then $C \in \Sigma^{\Sigma_{n+1}}(I)$. C is either a sentence $\exists x < I[\alpha < x \wedge P_{I,n}(x)]$, or a sentence $\exists x \in L_I A(x)$ with $\mathbf{qk}(A) < I$.

In the first case we have $(\mathcal{H}_{\gamma+1,n}[\Theta], I, n) \vdash_{I+1}^{a_0} \Gamma$ by Reduction 4.20.3, and IH yields the lemma.

Consider the second case. From the right uppersequent, SIH yields for $\widehat{a}_0 = \gamma + \omega^{I+a_0}$ and $\beta_0 = \Psi_{I,n}\widehat{a}_0 \in \mathcal{H}_{\widehat{a}_0+1,n}[\Theta]$

$$(\mathcal{H}_{\widehat{a}_0+1,n}[\Theta], I, n) \vdash_{\beta_0}^{\beta_0} C, \Gamma$$

Then by Boundedness 4.23.1 and $\beta_0 \in \mathcal{H}_{\widehat{a}_0+1,n}[\Theta]$, we have

$$(\mathcal{H}_{\widehat{a}_0+1,n}[\Theta], I, n) \vdash_{\beta_0}^{\beta_0} C^{(\exists I\beta_0)}, \Gamma$$

On the other hand we have by Boundedness 4.23.2 from the left uppersequent

$$(\mathcal{H}_{\widehat{a}_0+1,n}[\Theta], I, n) \vdash_{\mu}^{a_0} \Gamma, \neg(C^{(\exists I\beta_0)})$$

Moreover we have $\neg(C^{(\exists I\beta_0)}) \in \Sigma^{\Sigma_{n+1}}(I)$. SIH yields for $\widehat{a}_0 < \widehat{a}_1 = \widehat{a}_0 + 1 + \omega^{I+a_0} = \gamma + \omega^{I+a_0} + 1 + \omega^{I+a_0} < \gamma + \omega^{I+a} = \widehat{a}$ and $\beta_1 = \Psi_{I,n}\widehat{a}_1$

$$(\mathcal{H}_{\widehat{a}_1+1,n}[\Theta], I, n) \vdash_{\beta_1}^{\beta_1} \Gamma, \neg(C^{(\exists I\beta_0)})$$

Now we have $\hat{a}_i \in \mathcal{H}_{\hat{a}_i, n}(\Psi_{I, n} \hat{a})$ and $\hat{a}_i < \hat{a}$ for $i < 2$, and hence $\beta_0 = \Psi_{I, n} \hat{a}_0 < \beta_1 = \Psi_{I, n} \hat{a}_1 < \Psi_{I, n} \hat{a}$. Therefore $\text{rk}(C^{(\exists I \beta_0)}) < \beta_1 < \Psi_{I, n} \hat{a}$.

Consequently

$$\frac{(\mathcal{H}_{\hat{a}_1+1, n}[\Theta], I, n) \vdash_{\beta_1}^{\beta_1} \Gamma, \neg C^{(\exists I \beta_0)} \quad (\mathcal{H}_{\hat{a}_0+1, n}[\Theta], I, n) \vdash_{\beta_0}^{\beta_0} C^{(\exists I \beta_0)}, \Gamma}{(\mathcal{H}_{\hat{a}_1+1, n}[\Theta], I, n) \vdash_{\beta_1}^{\beta_1+1} \Gamma} \text{ (cut)}$$

Hence $(\mathcal{H}_{\hat{a}+1, n}, I, n) \vdash_{\Psi_{I, n} \hat{a}}^{\Psi_{I, n} \hat{a}} \Gamma$.

Case 4. Fourth consider the case for an $a_0 < a$

$$\frac{(\mathcal{H}_{\gamma, n}[\Theta], I, n) \vdash_{I+1}^{a_0} \Lambda, \Gamma_0}{(\mathcal{H}_{\gamma, n}[\Theta], I, n) \vdash_{I+1}^a \Gamma} \text{ (F)}$$

where $\Gamma = \Lambda \cup F \cup \Gamma_0$ and either $F = F_{x \cup \{\rho\}}^{\Sigma_1}$, $\Gamma_0 \subset \Sigma_1$ for some x and ρ , or $F = F_x^{\Sigma_n}$, $\Gamma_0 \subset \Sigma_n$ for an x . Then $\Lambda \cup \Gamma_0 \subset \Sigma_n$. SIH yields the lemma. \square

4.4 Elimination of Π_1^1 -indescribability

In the subsection we eliminate inferences (**Ref \mathcal{K}**) for Π_1^1 -indescribability.

For second-order sentences φ on L_π with parameters $A \subset L_\pi$ and ordinals $\alpha < \pi$, $\varphi^{(\alpha, \pi)}$ denotes the result of replacing second-order quantifiers $\exists X \subset L_\pi, \forall X \subset L_\pi$ by $\exists X \subset L_\alpha, \forall X \subset L_\alpha$, resp., first-order quantifiers $\exists x \in L_\pi, \forall x \in L_\pi$ by $\exists x \in L_\alpha, \forall x \in L_\alpha$, resp. and the parameters A by $A \cap L_\alpha$. For sequents $\Gamma, \Gamma^{(\alpha, \pi)} := \{\varphi^{(\alpha, \pi)} : \varphi \in \Gamma\}$.

Proposition 4.25 *Let $\Gamma \subset \Pi_1^1(\pi)$ for $\pi \in \text{Mh}_n^\alpha[\Theta]$. Assume*

$$\exists \xi \in \mathcal{H}_{\xi, n}[\Theta \cup \{\pi\}](\pi) \cap \alpha \forall \rho \in \text{Mh}_n^\xi[\Theta \cup \{\pi\}] \bigvee (\Gamma^{(\rho, \pi)}).$$

Then $\bigvee(\Gamma)$ is true.

Proof.

By $\pi \in \text{Mh}_n^\alpha[\Theta]$ we have $\pi \in M(\text{Mh}_n^\xi[\Theta \cup \{\pi\}])$ for any $\xi \in \mathcal{H}_{\xi, n}[\Theta \cup \{\pi\}](\pi) \cap \alpha$, cf. (3).

Suppose the $\Sigma_1^1(\pi)$ -sentence $\varphi := \bigwedge(\neg \Gamma) := \bigwedge\{-\theta : \theta \in \Gamma\}$ is true. Then the set $\{\rho < \pi : \varphi^{(\rho, \pi)}\}$ is club in π .

Hence for any $\xi \in \mathcal{H}_{\xi, n}[\Theta \cup \{\pi\}](\pi) \cap \alpha$ we can pick a $\rho \in \text{Mh}_n^\xi[\Theta \cup \{\pi\}]$ such that $\varphi^{(\rho, \pi)}$. \square

$$\mathcal{H}_{\gamma, n}[\Theta] \vdash_b^a \Gamma \Leftrightarrow (\mathcal{H}_{\gamma, n}, \Theta, I, n) \vdash_b^a \Gamma.$$

Lemma 4.26 (Collapsing down to \mathcal{K})

Let γ be an ordinal such that $\gamma \in \mathcal{H}_{\gamma, n}$.

Suppose for a finite set Θ of ordinals and an ordinal a

$$\mathcal{H}_{\gamma,n}[\Theta] \vdash_0^a \Gamma$$

where Γ consists of sentences $\neg\tau(B, \mathcal{K})$, $(B \cap C \neq \emptyset)$, $\forall \rho < \mathcal{K} \tau(B, \rho)$ for a $B \subset \mathcal{K}$ with $B \in \text{Hull}_{\Sigma_1}^I(\{\mathcal{K}, \mathcal{K}^+\})$ and sets $C \in L_{\mathcal{K}+1}$ such that C is a club subset of \mathcal{K} , and their subformulas:

$$\tau(B, \rho) := \exists C \subset \rho [(C \text{ is club})^\rho \wedge (B \cap C = \emptyset)] \quad (5)$$

Then for $\xi = \gamma + a$

$$\forall \pi \in \text{Mh}_n^\xi[\Theta] \{ \models \Gamma^{(\pi, \mathcal{K})} \}.$$

which means that $\bigvee (\Gamma^{(\pi, \mathcal{K})})$ is true for any $\pi \in \text{Mh}_n^\xi[\Theta]$.

Proof.

By induction on a . Let $\pi \in \text{Mh}_n^\xi[\Theta]$ and $\xi = \gamma + a$.

Case 1. First consider the case when the last inference is a (**Ref $_{\mathcal{K}}$**): we have $\{a_\ell, a_r\} \subset \mathcal{H}_{\gamma,n}[\Theta] \cap a$ and $B \subset \mathcal{K}$ with $B \in \text{Hull}_{\Sigma_1}^I(\{\mathcal{K}, \mathcal{K}^+\})$.

$$\frac{\mathcal{H}_{\gamma,n}[\Theta] \vdash_0^{a_\ell} \Gamma, \neg\tau(B, \mathcal{K}) \quad \mathcal{H}_{\gamma,n}[\Theta] \vdash_0^{a_r} \Gamma, \forall \rho < \mathcal{K} \tau(B, \rho)}{\mathcal{H}_{\gamma,n}[\Theta] \vdash_0^a \Gamma} \quad (\mathbf{Ref}_{\mathcal{K}})$$

We have $\xi_r := \gamma + a_r \in \mathcal{H}_{\xi_r, n}[\Theta](\pi) \cap \xi$ by $\xi_r \geq \gamma$ and $a_r < a$. By Proposition 2.9.3 with $\xi_r \in \mathcal{H}_{\xi_r, n}[\Theta](\pi)$ we have $\pi \in \text{Mh}_n^{\xi_r}[\Theta]$. IH yields $\bigvee (\Gamma^{(\pi, \mathcal{K})}) \vee \forall \rho < \pi \tau(B, \rho)$.

On the other hand we have $\xi_\ell := \gamma + a_\ell \in \mathcal{H}_{\xi_\ell, n}[\Theta](\pi) \cap \xi$. By IH we have for any $\rho \in \text{Mh}_n^{\xi_\ell}[\Theta \cup \{\pi\}] \cap \pi$, $\bigvee (\Gamma^{(\rho, \mathcal{K})}) \vee \neg\tau(B, \rho)$. Hence we have $\forall \rho \in \text{Mh}_n^{\xi_\ell}[\Theta \cup \{\pi\}] \cap \pi \{ \bigvee (\Gamma^{(\rho, \mathcal{K})}) \vee \bigvee (\Gamma^{(\pi, \mathcal{K})}) \}$. Proposition 4.25 yields $\bigvee (\Gamma^{(\pi, \mathcal{K})})$.

Case 2. Second consider the case when the last inference introduces a $\Pi_1^1(\mathcal{K})$ -sentence $\neg\tau(B, \mathcal{K})$ with a $B \subset \mathcal{K}$ such that $B \in \text{Hull}_{\Sigma_1}^I(\{\mathcal{K}, \mathcal{K}^+\})$.

$$\frac{\{ \mathcal{H}_{\gamma,n}[\Theta \cup \{\text{rk}_L(C)\}] \vdash_0^{a(C)} \Gamma, (C \not\subset \mathcal{K}) \vee \neg(C \text{ is club})^\mathcal{K} \vee (B \cap C \neq \emptyset) : C \in L_{\mathcal{K}+} \}}{\mathcal{H}_{\gamma,n}[\Theta] \vdash_0^a \Gamma, \neg\tau(B, \mathcal{K})} \quad (\wedge)$$

where $\forall C \in L_{\mathcal{K}+}(a(C) \in \mathcal{H}_{\gamma,n}[\Theta \cup \{\text{rk}_L(C)\}] \cap a)$ and $\neg\tau(B, \mathcal{K}) \simeq \bigwedge \{ (C \not\subset \mathcal{K}) \vee \neg(C \text{ is club})^\mathcal{K} \vee (B \cap C \neq \emptyset) : C \in L_{\mathcal{K}+} \}$. For each C , $(C \not\subset \mathcal{K}) \vee \neg(C \text{ is club})^\mathcal{K} \vee (B \cap C \neq \emptyset)$ is stratified with respect to C .

Let

$$C_\pi := \mu C \in L_{\pi+} [(C \subset \pi) \wedge (C \text{ is club})^\pi \wedge (B \cap C = \emptyset)]$$

Then $\neg[(C_\pi \subset \pi) \wedge (C_\pi \text{ is club})^\pi \wedge (B \cap C_\pi = \emptyset)] \Rightarrow \neg\tau(B, \pi) \equiv (\neg\tau(B, \mathcal{K}))^{(\pi, \mathcal{K})}$.

We can assume that $(C_\pi \subset \pi) \wedge (C_\pi \text{ is club})^\pi$. Otherwise $(\neg\tau(B, \mathcal{K}))^{(\pi, \mathcal{K})}$ and hence $\bigvee (\Gamma^{(\pi, \mathcal{K})}) \vee (\neg\tau(B, \mathcal{K}))^{(\pi, \mathcal{K})}$.

Let

$$C = \{\gamma \in \mathcal{K} : \exists x, y < \mathcal{K} (\gamma = \pi \cdot x + y \wedge y \in C_\pi \cup \{0\})\}$$

Then C is an $L_{\mathcal{K}}$ -definable club subset of \mathcal{K} , $C \in L_{\mathcal{K}+1}$, and $C \in J \cap \text{Hull}_{\Sigma_1}^I(\{\pi, \pi^+, \mathcal{K}, B\}) \subset \text{Hull}_{\Sigma_1}^I(\{\pi, \pi^+, \mathcal{K}, \mathcal{K}^+\}) \subset \mathcal{H}_{\gamma, n}[\Theta \cup \{\pi\}]$. Hence $\text{rk}_L(C) \in \mathcal{H}_{\gamma, n}[\Theta \cup \{\pi\}]$ and $a(C) \in \mathcal{H}_{\gamma, n}[\Theta \cup \{\pi\}]$. By inversion

$$\mathcal{H}_{\gamma, n}[\Theta \cup \{\pi\}] \vdash_0^{a(C)} \Gamma, C \not\subset \mathcal{K}, \neg(C \text{ is club})^{\mathcal{K}}, B \cap C \neq \emptyset$$

Eliminate false sentences $C \not\subset \mathcal{K}$ and $\neg(C \text{ is club})^{\mathcal{K}}$ by Lemma 4.17.

$$\mathcal{H}_{\gamma, n}[\Theta \cup \{\pi\}] \vdash_0^{a(C)} \Gamma, B \cap C \neq \emptyset$$

IH yields for $\xi(C) = \gamma + a(C)$, $\forall \rho \in \text{Mh}_n^{\xi(C)}[\Theta \cup \{\pi\}] \cap \pi \{ \vee (\Gamma^{(\rho, \mathcal{K})}) \vee (B \cap C \neq \emptyset)^{(\rho, \mathcal{K})} \}$, where $(B \cap C \neq \emptyset)^{(\rho, \mathcal{K})} \equiv (B \cap C_\pi \cap \rho \neq \emptyset) \equiv ((B \cap C \neq \emptyset)^{(\pi, \mathcal{K})})^{(\rho, \pi)}$. Proposition 4.25 with $\xi(C) \in \mathcal{H}_{\xi(C), n}[\Theta \cup \{\pi\}] \cap \xi$ yields $\vee (\Gamma^{(\pi, \mathcal{K})}) \vee (B \cap C \neq \emptyset)^{(\pi, \mathcal{K})}$, and hence $\vee (\Gamma^{(\pi, \mathcal{K})}) \vee (\neg \tau(B, \mathcal{K}))^{(\pi, \mathcal{K})}$.

Case 3. Third consider the case : $\Gamma \ni (B \cap C \neq \emptyset)$ with $B \subset \mathcal{K}$, $B \in \text{Hull}_{\Sigma_1}^I(\{\mathcal{K}, \mathcal{K}^+\})$ and a club subset C of \mathcal{K} .

$$\frac{\mathcal{H}_{\gamma, n}[\Theta] \vdash_0^{a_0} \Gamma, (d \in B) \wedge (d \in C)}{\mathcal{H}_{\gamma, n}[\Theta] \vdash_0^a \Gamma} (\vee)$$

where $a_0 < a$ and $d \in \mathcal{K}$.

Then $(B \cap C \neq \emptyset)^{(\pi, \mathcal{K})} \leftrightarrow (B \cap C \cap \pi \neq \emptyset)$ and $((d \in B) \wedge (d \in C))^{(\pi, \mathcal{K})} \leftrightarrow (d \in (B \cap \pi)) \wedge (d \in (C \cap \pi))$. IH with Proposition 2.9.3 yields the lemma.

Case 4. Fourth consider the case : $\Gamma \ni ((d \in B) \wedge (d \in C))$ with $B \subset \mathcal{K}$, $B \in \text{Hull}_{\Sigma_1}^I(\{\mathcal{K}, \mathcal{K}^+\})$ and a club subset C of \mathcal{K} .

$$\frac{\mathcal{H}_{\gamma, n}[\Theta] \vdash_0^{a_0} \Gamma, d \in B \quad \mathcal{H}_{\gamma, n}[\Theta] \vdash_0^{a_1} \Gamma, d \in C}{\mathcal{H}_{\gamma, n}[\Theta] \vdash_0^a \Gamma} (\wedge)$$

where $a_0, a_1 < a$.

IH with Proposition 2.9.3 yields the lemma.

Case 5. Fifth consider the case: for a true literal $M \equiv (d \in B)$, $M \in \Gamma$, where $B \subset \mathcal{K}$ such that either $B \in \text{Hull}_{\Sigma_1}^I(\{\mathcal{K}, \mathcal{K}^+\})$, or B is a club subset of \mathcal{K} , and $d \in \mathcal{K}$.

$$\overline{\mathcal{H}_{\gamma, n}[\Theta] \vdash_0^a \Gamma} (\wedge)$$

Then $M^{(\pi, \mathcal{K})} \equiv (d \in (B \cap \pi)) \in \Gamma^{(\pi, \mathcal{K})}$.

It suffices to show $d = \text{rk}_L(d) < \pi$. We have by (10) $d \in \text{k}^E(d \in B) \cap \mathcal{K} \subset \mathcal{H}_{\gamma, n} \cap \mathcal{K} \subset \pi$ by $\pi \in \text{Mh}_n^{\xi}[\Theta]$, i.e., by $\mathcal{H}_{\xi, n}(\pi) \cap \mathcal{K} \subset \pi$.

Case 6. Sixth consider the case when the last inference introduces a sentence $\forall \rho < \mathcal{K} \tau(B, \rho)$.

$$\frac{\{\mathcal{H}_{\gamma,n}[\{\rho\}][\Theta] \vdash_0^{a(\rho)} \Gamma, \tau(B, \rho) : \rho < \mathcal{K}\}}{\mathcal{H}_{\gamma,n}[\Theta] \vdash_0^a \Gamma, \forall \rho < \mathcal{K} \tau(B, \rho)} (\wedge)$$

We have for any $\rho < \pi$ and $\xi(\rho) = \gamma + a(\rho)$, $\xi(\rho) \in \mathcal{H}_{\xi(\rho),n}[\Theta](\pi)$. Proposition 2.9.3 yields $\pi \in Mh_n^{\xi(\rho)}[\Theta]$. By IH we have $\forall \rho < \pi \{\forall(\Gamma^{(\pi, \mathcal{K})}) \vee \tau(B, \rho)\}$, and hence $(\forall(\Gamma) \vee \forall \rho < \mathcal{K} \tau(B, \rho))^{(\pi, \mathcal{K})}$ with $(\forall \rho < \mathcal{K} \tau(B, \rho))^{(\pi, \mathcal{K})} \equiv \forall \rho < \pi \tau(B, \rho)$.

Case 7. Seventh consider the case when the last inference introduces a sentence $\forall x \in c \varphi(x) \in \Gamma$ for $c \in L_{\mathcal{K}}$ and $k^E(\varphi(x)) < \mathcal{K} \& k(\varphi(x)) < \mathcal{K}^+$.

$$\frac{\{\mathcal{H}_{\gamma,n}[\{\text{rk}_L(b)\}][\Theta] \vdash_0^{a(\rho)} \Gamma, \varphi(b) : b \in c\}}{\mathcal{H}_{\gamma,n}[\Theta] \vdash_0^a \Gamma} (\wedge)$$

Then $\gamma = \text{rk}_L(c) \in k^E(\Gamma) \cap \mathcal{K}$ and hence $\gamma < \pi$ as in **Case 5**. As in **Case 6** we have by IH $\forall b \in c(\forall(\Gamma^{(\pi, \mathcal{K})}) \vee \varphi(b))$ where $\varphi(b) \equiv (\varphi(b))^{(\pi, \mathcal{K})}$. Hence $\forall(\Gamma^{(\pi, \mathcal{K})})$.

Case 8. Eighth consider the case when the last inference introduces a sentence $\exists x \in c \varphi(c) \in \Gamma$ for $c \in L_{\mathcal{K}}$, $b \in c$ and $k^E(\varphi(x)) < \mathcal{K} \& k(\varphi(x)) < \mathcal{K}^+$.

$$\frac{\mathcal{H}_{\gamma,n}[\Theta] \vdash_0^{a_0} \Gamma, \varphi(b)}{\mathcal{H}_{\gamma,n}[\Theta] \vdash_0^a \Gamma} (\vee)$$

As in **Case 7** we see $\text{rk}_L(c) < \pi$. IH with Proposition 2.9.3 yields $\forall(\Gamma^{(\pi, \mathcal{K})}) \vee \varphi(b)$, and $\forall(\Gamma^{(\pi, \mathcal{K})})$.

Case 9. Ninth consider the case when the last inference is an **(F)** where either $F = F_{x \cup \{\lambda\}}^{\Sigma_1}$ for a $\lambda \in R$ or $F = F_x^{\Sigma_n}$.

In each case if $A \in \text{rng}(F)$ for an $A \in \Gamma$, then we claim $F'' A \equiv A$. Suppose $x = F_{x \cup \{\mathcal{K}^+\}}^{\Sigma_1}(\mathcal{K}^+) \leq \text{rk}_L(B) < \mathcal{K}^+$ for the set $B \in \text{Hull}_{\Sigma_1}^I(\{\mathcal{K}, \mathcal{K}^+\})$. However by $x > \mathcal{K}$ we have $\text{rk}_L(B) \in \text{Hull}_{\Sigma_1}^I(\{\mathcal{K}, \mathcal{K}^+\}) \cap \mathcal{K}^+ \subset \text{Hull}_{\Sigma_1}^I(x \cup \{\mathcal{K}^+\}) \cap \mathcal{K}^+ \subset x$. Hence this is not the case.

IH yields the assertion. \square

Collapsing down to \mathcal{K} 4.26 yields the following Theorem 4.27.

Theorem 4.27 (Elimination of **(Ref $_{\mathcal{K}}$)**)

Let $\gamma \in \mathcal{H}_{\gamma,n}$, $B \subset \mathcal{K}$, and $B \in \text{Hull}_{\Sigma_1}^I(\{\mathcal{K}, \mathcal{K}^+\})$.

$$[\mathcal{H}_{\gamma,n} \vdash_0^a \neg \tau(B, \mathcal{K})] \Rightarrow [\neg \tau(B, \pi) \text{ is true}]$$

for any $\pi \in Mh_n^{\xi}$ with $\xi = \gamma + a$.

5 Proof of Theorem 1.3

Let φ be a Σ_2^1 -sentence, and assume that ZF proves the sentence

$$\forall \mathcal{K}[(\mathcal{K} \text{ is a weakly compact cardinal}) \rightarrow \varphi^{V_{\mathcal{K}}}] .$$

Under $V = L$, $V_{\sigma} = L_{\sigma}$ for any inaccessible cardinals σ , and we have $\forall \mathcal{K}[(\mathcal{K} \text{ is a weakly compact cardinal}) \rightarrow \varphi^{L_{\mathcal{K}}}]$. Hence $\text{T}(\mathcal{K}, I) \vdash \varphi^{L_{\mathcal{K}}}$. By Proposition 1.2 we can assume that the sentence ‘ \mathcal{K} is uncountable regular’ $\rightarrow \varphi^{L_{\mathcal{K}}}$ is of the form ‘ $\exists B \subset \mathcal{K}(S^{\varphi}(B) \cap \mathcal{K} \text{ is stationary in } \mathcal{K})$ ’.

Let $B := \mu B \subset \mathcal{K}(S^{\varphi}(B) \cap \mathcal{K} \text{ is stationary in } \mathcal{K}) \in \text{Hull}_{\Sigma_1}^I(\{\mathcal{K}, \mathcal{K}^+\})$.

In what follows work in an intuitionistic fixed point theory $\text{FiX}^i(\text{ZFLK}_n)$ over $\text{ZFLK}_n = \text{ZF} + (V = L) + (\mathcal{K} \in \text{Mh}_n^{\omega_n(I+1)})$ for a sufficiently large $n < \omega$. By Embedding 4.18 pick an $m < \omega$ such that $(\mathcal{H}_{0,n}, I, n) \vdash_{I+m-1}^{I \cdot (m-1)} \neg \tau(B, \mathcal{K})$. By Predicative Cut-elimination 4.21 we have $(\mathcal{H}_{0,n}, I, n) \vdash_{I+1}^{\omega_{m-2}(I \cdot (m-1))} \neg \tau(B, \mathcal{K})$.

Then by Collapsing down to I 4.24 we have for $a = \omega_m(I+1)$ and $b = \Psi_{I,n,a}$, $(\mathcal{H}_{a,n}, I, n) \vdash_b^b \neg \tau(B, \mathcal{K})$. Again by Predicative Cut-elimination 4.21 we have $(\mathcal{H}_{a,n}, I, n) \vdash_0^{\varphi^{bb}} \neg \tau(B, \mathcal{K})$.

Elimination of **(Ref) $_{\mathcal{K}}$** 4.27 yields $\neg \tau(B, \pi)$ for any $\pi \in \text{Mh}_n^{\xi}$ with $\xi = a + \varphi^{bb} \in \mathcal{H}_{\xi,n}(\mathcal{K}) \cap \omega_{m+1}(I+1)$.

Proposition 4.25 with $\mathcal{K} \in \text{Mh}_n^{\omega_{m+1}(I+1)}$ yields $\neg \tau(B, \mathcal{K})$, and hence $S^{\varphi}(B) \cap \mathcal{K}$ is stationary in \mathcal{K} . Since the whole proof is formalizable in $\text{FiX}^i(\text{ZFLK}_n)$, we conclude $\text{FiX}^i(\text{ZFLK}_n) \vdash \varphi^{V_{\mathcal{K}}}$. Finally Theorem 4.2 yields $\text{ZFLK}_n \vdash \varphi^{V_{\mathcal{K}}}$. Therefore $\varphi^{V_{\mathcal{K}}}$ follows from $\theta_n(\mathcal{K}) : \Leftrightarrow \mathcal{K} \in \text{Mh}_n^{\omega_n(I+1)}$ over $\text{ZF} + (V = L)$. Thus Theorem 1.3.2 was shown.

Since the least weakly inaccessible cardinal I_0 is below the least weakly Mahlo cardinal,

$$\text{ZF} + \mathbb{K} \vdash \varphi^{V_{I_0}} \Rightarrow \text{ZF} + \{\exists \mathcal{K} \theta_n(\mathcal{K}) : n < \omega\} \vdash \varphi^{V_{I_0}}$$

for any first-order sentence φ , etc.

This completes a proof of Theorem 1.3.

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