CONVERGENCE THEOREMS FOR GRAPH SEQUENCES

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Abstract. In this paper, we deal with a notion of Banach space-valued mappings defined on a set consisting of finite graphs with uniformly bounded vertex degree. These functions will be endowed with certain boundedness and additivity criteria. We examine their normalized longterm behaviour along a particular class of graph sequences. Using techniques developed by Elek, we show convergence in the topology of the Banach space if the corresponding graph sequence possesses a hyperfinite structure. These considerations extend and complement the corresponding results for amenable groups. As an application, we verify the uniform approximation of the integrated density of states for bounded, finite range operators on discrete structures. Further, we extend results concerning an abstract version of Fekete's Lemma for amenable groups and cancellative semigroups to the geometric situation of convergent graph sequences.

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1. Introduction

The following work is devoted to convergence results for Banach space valued functions with particular (sub-)additivity properties. More precisely, those functions F will be defined on a sequence (G_n) of finite graphs with uniform vertex degree bound d. Our main goal is to study the normalized limit behaviour, i.e. to determine a class of sequences for which the limit

$$
\overline{F} := \lim_{n \to \infty} \frac{F(G_n)}{|V(G_n)|}
$$

exists in the topology of the Banach space. This question is motivated by various mathematical applications:

- For subadditive functions (with $\mathbb R$ as underlying Banach space), convergence theorems of the above kind can be used to show the existence of invariants in topological dynamical systems. Those quantities comprise topological entropy and topological mean dimension, see $\vert 13, 22 \vert$ in the situation of amenable groups and $\vert 4 \vert$ for amenable, left cancellative semigroups.
- The notion of almost-additivity for general Banach space-valued functions F has been introduced in [\[17\]](#page-12-3). Defining F on the set of associated words in a subshift dynamical system, the author characterizes unique ergodicity of the subshift by the normalized Banach space convergence of F . A related result for Banach space valued functions on tilings of Euclidean spaces can also be found in [\[20\]](#page-12-4). Further, it turns out that this class of almostadditive function is also useful for spectral issues for operators on discrete structures. In detail, it allows for the (uniform) approximation of the spectral distribution function (integrated density of states, IDS) via finite volume analogues, cf. [\[20,](#page-12-4) [9\]](#page-12-5). In the context of abstract amenable groups, this has for instance been done in [\[15,](#page-12-6) [32,](#page-12-7) [19,](#page-12-8) [29,](#page-12-9) [28\]](#page-12-10). Related results can also be found in [\[24,](#page-12-11) [25\]](#page-12-12), where the authors prove almost-additive convergence theorems that characterize unique ergodicity of Delone dynamical systems. Those assertions lead to finite-dimensional spectral approximations as well.

However, all of those convergence theorems deal with Følner or van Hove sequences in amenable groups or semigroups. In the present paper, we turn to a very general class of graph sequences including Følner sequences in finitely generated groups. Proving convergence theorems along those objects, we significantly extend the geometric framework of the above results. We do so by identifying subadditive and almost-additive functions as mappings defined on graphs satisfying a continuity property with respect to a particular distance function introduced in [\[9\]](#page-12-5). Precisely, we determine sufficient and necessary conditions for naturally occurring graph sequences to be Cauchy in this distance function. Our proof is based on the groundwork presented in [\[11\]](#page-12-13). Another independent proof via algorithmic techniques has recently given by NEWMAN and SOHLER, cf. [\[26\]](#page-12-14), Theorem 3.1.

This leads to our first key result given in Theorem [3.1.](#page-5-0) As an immediate consequence, we are able to derive strong Banach space-valued convergence results given in Corollary [4.3](#page-6-0) (almostadditive functions) and Theorem [5.3](#page-7-0) (subadditive functions) for so-called hyperfinite (and weakly convergent, see below) graph sequences as introduced in [\[7\]](#page-12-15). Roughly speaking, hyperfiniteness of a family of graphs means that for all elements in the family, the proportion of edges that need to be deleted in order to obtain (edge-)disjoint components with a uniform (vertex) size bound, is small uniformly over the family. As a main example, every Følner sequence in a finitely generated, amenable group is hyperfinite. A more general conjecture of Elek (Conjecture 1 in [\[9\]](#page-12-5)) refers to bounded vertex degree graphs with edges and vertices labeled by finitely many colours. So far, we only have been able to cover the geometric situation ignoring the colours. Thus, within the uncoloured situation, Corollary [4.3](#page-6-0) brings out a unification of all results that concern amenable Cayley graphs, see e.g. [\[18,](#page-12-16) [19,](#page-12-8) [29\]](#page-12-9). In addition to that, the graphs modeling quasicrystals in [\[20,](#page-12-4) [21\]](#page-12-17) belong just as well to the class to which our convergence theorem applies. Theorem [5.3](#page-7-0) can be interpreted as a version of the 'Ornstein-Weiss' lemma which is a crucial tool in the world of group dynamics, see e.g. [\[13,](#page-12-0) [22,](#page-12-1) [16\]](#page-12-18). Our result extends an assertion by Gromov in [\[13\]](#page-12-0). Accepting a monotonicity assumption on the function under consideration, it can be also be understood as an upgrading of [\[16,](#page-12-18) [4\]](#page-12-2) in the situation of sequences (in the latter papers, the authors are also able to deal with Følner nets).

Before we explain the content of this paper in more detail, let us briefly overview the classes of graph sequences which are crucial in our elaborations.

The original notion of convergent graph sequences (G_n) was introduced in the seminal work of BENJAMINI and SCHRAMM, cf. [\[3\]](#page-12-19). It means the existence of the occurrence frequencies for arbitrary geometric patterns as $n \to \infty$. In the present work, we will refer to those sequences as *weakly* convergent. In general, the limit is given by a so-called graphing which is a Borel probability space consisting of countable, uniformly vertex-degree bounded graphs, see e.g. [\[8\]](#page-12-20). In particular situations, graph sequences might also converge to one deterministic, countable graph (almost-surely). This is for instance the case for sofic approximations of finitely generated groups, cf. [\[33,](#page-12-21) [6,](#page-12-22) [9,](#page-12-5) [1\]](#page-11-0).

The notion of weakly convergent graph sequences is widely used in the context of various mathematical problems. Let us mention some examples:

- \bullet In [\[3\]](#page-12-19), the authors prove that in the situation of planar graph sequences, almost all elements in the graphing are recurrent.
- Further, graph sequences turn out to be useful in order to show that certain geometric quantities can be obtained as a limit of their finite-dimensional restrictions. A very prominent question in this field is given by the so-called Lück conjecture concerning the approximation of the von Neumann dimensions on finitely generated groups. The Lück conjecture was proven to be true for very large classes of geometries, see e.g. [\[23,](#page-12-23) [5,](#page-12-24) [6,](#page-12-22) [31,](#page-12-25) [2\]](#page-11-1).
- Moreover, the theory of weakly convergent graph sequences has been used before to draw connections between the geometry and the spectral properties of discrete structures. In this context, in [\[1\]](#page-11-0), the authors characterize amenability of subgroups in a finitely generated group via the limit behaviour of the spectral radii evolving from a convergent sequence of Schreier graphs.
- It is also well-known that the distribution function of the spectrum of selfadjoint operators on graphs can be approximated by finite-volume restrictions. In [\[9\]](#page-12-5), the author defines

a class of operator sequences defined over the elements of a weakly convergent graph sequence. It turns out that the normalized spectral distributions converge in the sense of weak convergence of measures. For the approximations of sofic, finitely generated groups, a related result can be found in [\[30\]](#page-12-26). In fact, a group is sofic if it contains a weakly convergent graph sequence approximating the local structures of the corresponding Cayley graph.

Considering further kinds of convergence for graph sequences, there is no unified mathematical theory so far. However, for certain applications, it is essential to find refinements of the Benjamini-Schramm convergence, see e.g. [\[14\]](#page-12-27). As it is our goal to prove uniform spectral approximation results for operators on discrete structures, we will have to deal with a notion which strongly takes into account the overall geometry of the graphs under consideration. It turns out that the 'right' criterion relies on a metric introduced by Elek in [\[9\]](#page-12-5). We show that convergence in this metric (so-called strong convergence) is equivalent to the sequence being weakly convergent and hyperfinite. As a consequence, we are able to derive the uniform convergence of the spectral distributions of finite range operators on weakly convergent, hyperfinite graph sequences. As mentioned before, this widely opens the range of geometries that can be considered. However, in the geometrically more restricted situation of amenable groups (Følner sequences), one may deal with colourings allowing for more general dynamical systems, see e.g. [\[19,](#page-12-8) [29\]](#page-12-9).

Let us briefly summarize the organization of this paper. In Section [2,](#page-2-0) we give the necessary preliminaries and we introduce the notions of weak and strong convergence for graph sequences. Further, we define hyperfiniteness for graph sequences and we provide some examples. Next, we prove in Section [3](#page-5-1) that for a weakly convergent sequence, hyperfiniteness is a necessary and sufficient condition for its strong convergence, cf. Theorem [3.1.](#page-5-0) The proof relies on the so-called Equipartition Theorem of Elek, cf. [\[11\]](#page-12-13). The following two Sections [4](#page-5-2) and [5](#page-7-1) are devoted to the proofs of the announced Banach space valued convergence theorems. In Corollary [4.3,](#page-6-0) we show the validity of the above limit relation for almost-additive functions. The corresponding result for subadditive functions can be found in Theorem [5.3.](#page-7-0) Moreover, we provide an operator theoretic framework for an application of the almost-additive convergence thorem, in Section [6.](#page-8-0) In detail, just as well as in [\[9\]](#page-12-5), we obtain the uniform convergence of the spectral distribution functions for bounded, self-adjoint, finite hopping range operators towards an abstract notion of integrated density of states (IDS) of the limit object (Theorem [6.7\)](#page-10-0).

2. Graph convergence

For the main purposes of this work, we introduce in this section the notions of graph isomorphism classes and explain the concept of weakly convergent and strongly convergent graph sequences. Before doing so, we clarify what we mean by graphs and we give the basic definitions related to these objects.

In the following, we will always deal with countable (in most cases in fact finite) graphs. These latter elements can be represented as a pair $G := (V(G), E(G))$, where $V(G)$ is a countable set called vertex set and $E(G) \subseteq \{ \{x, y\} \mid x, y \in V(G), x \neq y \}$ is the so-called *edge set* describing abstract connections between pairs of vertices. If $\{x, y\} \in E$, we say that y is a neighbour of x and vice versa. The number of neighbours of a vertex $x \in V(G)$ is called the *vertex degree* in x. Note that each countable graph can be endowed in a natural way with the path distance metric d_G . Thus, if for $x, y \in V(G)$, there is a finite number of distinct vertices $z_i \in V(G)$, $i = 1, \ldots n$ such that $\{x, z_1\}, \{z_j, z_{j+1}\}, \{z_n, y\} \in E(G)$, $(j = 1, ..., n-1)$, we say that there is a path of length $(n + 1)$ in G joining x and y (and vice versa). In this case, the value $d_G(x, y)$ is given by the minimum of the lengths of paths joining x with y. As a convention, we set $d_G(x, y) = \infty$ if there is no such path. Next, we define first the notion of subgraphs which will become essential in the considerations below.

Definition 2.1 (subgraphs). Let G be a countable graph with vertex set $V(G)$ and edge set $E(G)$. We call $H = (V(H), E(H))$ a subgraph of G if

- $V(H) \subseteq V(G)$ and
- $E(H) \subseteq \{ \{x, y\} \in E(G) \mid x, y \in V(H) \}.$

A special class of subgraphs of a graph G is given by graphs induced by a subset of the vertex set $V(G)$.

Definition 2.2 (induced graph). Let G be a countable graph with vertex set $V(G)$ and edge set $E(G)$. Then, for each set $T \subseteq V(G)$, we shall define the graph on T induced by G as $G(T) :=$ (V_T, E_T) , where

$$
V_T := T,
$$

\n
$$
E_T := \{ \{x, y\} \in E(G) \mid x, y \in T \}.
$$

Note that induced subgraphs can be interpreted as subgraphs in the sense of Definition [2.1](#page-2-1) with maximal possible number of edges (for a given vertex set).

Following a concept introduced by Elek, see [\[11\]](#page-12-13), we now turn to convergent graph sequences. To do so, let $d \in \mathbb{N}$ and denote by \mathcal{G}_d a set of finite graphs of vertex degree bounded by d. We assume further that the graphs in \mathcal{A}_d do not have loops and that there are no multiple edges. We say that $G \in \mathcal{G}_d$ is a rooted graph of radius bounded by $r \in \mathbb{N}$ if there is one distinguished vertex $x := \text{rt}(G) \in V(G)$ (the root of G) such that $d_G(x, y) \leq r$ for every $y \in V(G)$, where d_G is the canonical shortest path metric in G , see above. In this sense we say that two graphs G_1 and G_2 are rooted isomorphic if there is a bijection $\varphi: V(G_1) \to V(G_2)$ mapping root to root and where $\{\varphi(x), \varphi(y)\}\in E(G_2)$ if and only if $\{x, y\}\in E(G_1)$. We will write $G_1 \sim G_2$ if G_1 and G_2 are rooted isomorphic.

We denote by \mathcal{A}_d the set of all rooted isomorphism classes of graphs of vertex degree bounded by d and we define the radius of the classes $\alpha \in \mathcal{A}_d$

$$
rad(\alpha) := \min\{t \in \mathbb{N} \, | \, d_G(\text{rt}(\alpha), y) \le t, \, y \in V(\alpha) \}.
$$

For $r \in \mathbb{N}$, we write \mathcal{A}_d^r for the elements in \mathcal{A}_d with radius exactly r. Note that it follows from the fact that $\mathcal{A}_d = \bigcup_{r=1}^{\infty} \mathcal{A}_d^r$ that \mathcal{A}_d is a countable set.

Now for $G \in \mathcal{G}_d$, $r \in \mathbb{N}$ and $\alpha \in \mathcal{A}_d^r$, we set

$$
T_r(G, \alpha) := \{ x \in V(G) \mid B_r^G(x) \sim \alpha \},
$$

where $B_r^G(x) := G(B_r(x))$ is the induced subgraph of G with vertex set $B_r(x) := \{y \in V(G) \mid d_G(x, y) \leq 1\}$ r}, i.e. the r-ball centered $x \in V(G)$. Further, for each $\alpha \in \mathcal{A}_d$, the number

$$
p(G, \alpha) := \frac{|T_{\text{rad}(\alpha)}(G, \alpha)|}{|V(G)|}
$$

can be interpreted as the probability that the rad(α)-ball around a random vertex of G is rootedisomorphic to the element α . If we enumerate the elements of the (countable) set \mathcal{A}_d , we get a map

$$
\mathcal{L}: \mathcal{G}_d \to [0,1]^{\mathbb{N}}: \mathcal{L}(G) := (p(G,\alpha_i))_{i \in \mathbb{N}},
$$

cf. [\[11\]](#page-12-13). Further, we equip $[0, 1]^{\mathbb{N}}$ with a metric d_{π} generating the usual product topology. With no loss of generality, we represent d_{π} as

$$
d_{\pi}(\mathcal{L}(G), \mathcal{L}(H)) := \sum_{k=1}^{\infty} 2^{-k} \frac{|p(G, \alpha_k) - p(H, \alpha_k)|}{1 + |p(G, \alpha_k) - p(H, \alpha_k)|}
$$

for $G, H \in \mathcal{G}_d$. Note that the map $\mathcal L$ is 'almost' injective in the sense that $\mathcal L(G) = \mathcal L(H)$ implies that there must be a graph $K \in \mathcal{G}_d$ such that both G and H are disjoint unions of K-copies, cf. [\[11\]](#page-12-13).

We are now in position to introduce notions of convergence for sequences $(G_n)_{n=1}^{\infty} \subseteq \mathcal{G}_d$.

Definition 2.3 (Weak convergence of graphs). We say that a sequence $(G_n) \subseteq \mathcal{G}_d$ converges weakly if for all $r \in \mathbb{N}$ and every $\alpha \in \mathcal{A}_d^r$, the limit

$$
p(\alpha) := \lim_{n \to \infty} p(G_n, \alpha)
$$

exists. So $(G_n)_{n=1}^{\infty}$ is weakly convergent if and only if $(\mathcal{L}(G_n))_{n=1}^{\infty}$ is convergent pointwise, i.e. it is a Cauchy sequence with respect to the the distance d_{π} .

In fact, the sequence (G_n) consisting of *finite* graphs converges to a *probability distribution* of possibly infinite graphs. This notion of convergence for graphs has been introduced in the influential work of Benjamini and Schramm, cf. [\[3\]](#page-12-19). So we will equivalently refer to weak convergence as Benjamini-Schramm convergence.

By defining a particular metric δ_{ρ} on \mathcal{G}_d , ELEK introduced the notion of *strongly convergent* graph sequences in [\[9\]](#page-12-5). We briefly explain the construction of δ_{ρ} . Assume first that the graphs $G, H \in \mathcal{G}_d$ can be represented on the same vertex set V. Let

$$
\delta(G, H) := \frac{|\{v \in V \mid S^G(v) \neq S^H(v)\}|}{|V|},
$$

where $S^{G}(v)$ stands for the 1-ball (the *star*) around $v \in V$ in the graph G. In this context, the \neq -sign must be understood not only in the sense of rooted isomorphism classes, but it also takes into account the differences in the vertex numberings (one can assume that all vertices in G and H are marked by numbers $\{1, \ldots, |V|\}$ of both stars. Then, δ defines a metric on the graphs $G \in \mathcal{G}_d$ which are defined over the vertex set V (cf. [\[9\]](#page-12-5), Lemma 2.1). As a convention, we set $\delta(G, \epsilon) = 1$ for all $G \in \mathcal{G}_d$, where ϵ shall be the empty graph. In a next step, one can come up with a metric which is invariant under permutations of the vertex numbering. In Lemma 2.2 of [\[9\]](#page-12-5), it is shown that for the graphs defined on the same vertex set V , the expression

$$
\delta_S(G, H) := \min_{\sigma \in S(V)} \delta(G, H^{\sigma})
$$

is a well-defined metric, where $S(V)$ is the set of all possible permutations of the vertex numbering in V and H^{σ} is the graph obtained by renumbering the vertices, also making sure that the edge relations are preserved. Finally, for graphs G, H which are not necessarily defined on the same vertex set, the so-called geometric distance

$$
\delta_\rho(G,H):=\inf_{\{q,p\in\mathbb{N}\,|\,q|V(G)|=p|V(H)|\}}\delta_S(qG,pH)
$$

has been introduced in [\[9\]](#page-12-5), where the graph qG is represented by q disjoint copies of the graph G. By Proposition 2.1 of the same work, δ_{ρ} defines a metric on the set of isomorphism classes in \mathcal{A}_d . For some finite graph $G \in \mathcal{G}_d$, we let $\alpha(G)$ be the corresponding isomorphism class in the set \mathcal{A}_d .

Definition 2.4 (Strong convergence of graphs). We say that a sequence $(G_n) \subseteq \mathcal{G}_d$ converges strongly, if $(\alpha(G_n))_{n=1}^{\infty}$ is a Cauchy sequence in the δ_{ρ} -metric.

It is a well-known fact that strong convergence of graphs implies weak convergence, see e.g. [\[9\]](#page-12-5), Proposition 2.2. We will show in the next section that under the additional assumption of hyperfiniteness, the converse is also true.

Definition 2.5. A family $P \subseteq \mathcal{G}_d$ is called *hyperfinite* or *amenable* if for every $\varepsilon > 0$, one can find a number $K_{\varepsilon} \in \mathbb{N}$ such that we can remove from every $G \in \mathcal{P}$ less than $\varepsilon |E(G)|$ edges such that the resulting graph consists of disjoint components containing at most K_{ε} vertices.

Remark 2.6. In the literature, hyperfinite graphs are also referred to as so-called *anti-expanders*.

- **Examples 2.7.** Let $P = \{P_n\}_{n=1}^{\infty}$, where P_n is a path of length n. Then P is hyperfinite. One may e.g. choose $K_{\varepsilon} = 2/\varepsilon$ for $\varepsilon > 0$.
	- Let $P = \{F_n\}_{n=1}^{\infty}$, where $(F_n)_{n=1}^{\infty}$ are the Cayley graphs of the sets in a Følner sequence in a finitely generated, amenable group. Using the ORNSTEIN/WEISS ε -quasi tiling theory (cf. [\[27,](#page-12-28) [29\]](#page-12-9)), it can be seen that P is hyperfinite.
	- Let $\mathcal{P} = \{G_n\}_{n=1}^{\infty}$ be a graph sequence of *subexponential growth*, i.e. for all $n \in \mathbb{N}$, for each $x \in V(G_n)$ and every $r \in \mathbb{N}$, we have $|B_r^{G_n}(x)| \leq f(r)$, where $f : \mathbb{N} \to \mathbb{N}$ is a function which has *subexponential growth*. Precisely, this latter property means that for each $\beta > 0$, there exists a number $r_\beta > 0$ such that $f(r) \leq \exp(\beta r)$ whenever $r \geq r_\beta$, cf. [\[9\]](#page-12-5). Then, $\mathcal P$ is a hyperfinite family.

In fact, one can verify that strongly convergent graph sequences are hyperfinite.

Proposition 2.8 (cf. [\[9\]](#page-12-5), Proposition 2.3). If $(G_n) \subseteq \mathcal{G}_d$ is strongly convergent, then the family $P = \{G_n\}_{n=1}^{\infty}$ is hyperfinite.

3. Hyperfiniteness and weak convergence

The fundamental insight that two statistically alike graphs with the same number of vertices are also close in their geometry (i.e. w.r.t. δ_{ρ} as introduced before) has been obtained in [\[11\]](#page-12-13) (Theorem 5) and independently in [\[26\]](#page-12-14) (Theorem 3.1). We deduce from this that hyperfinite, weakly convergent graph sequences are in fact strongly convergent. This confirms a conjecture of Elek in [\[9\]](#page-12-5) for the case of graphs with uncoloured vertices and colourless edges. This section is devoted to a proof which relies on techniques of ELEK's work, cf. [\[11\]](#page-12-13).

Theorem 3.1. Let $\mathcal{P} := (G_n) \subseteq \mathcal{G}_d$ be a weakly convergent sequence of graphs which is also hyperfinite. Then, (G_n) is in fact strongly convergent.

For the proof of Theorem [3.1,](#page-5-0) we need the so-called *Equipartition Theorem* stating that statistically similar graphs in a hyperfinite family can be partitioned similarly, cf. [\[11\]](#page-12-13), Theorem 4.

Theorem 3.2 (ELEK, Equipartition Theorem, cf. [\[11\]](#page-12-13), Theorem 4). Let $P \subseteq \mathcal{G}_d$ be a hyperfinite family. Then, for any $\varepsilon > 0$, there exists $K_{\varepsilon} \in \mathbb{N}$ with the following property: for any $\beta > 0$, there exists $\delta > 0$ such that if $G \in \mathcal{P}$ and $H \in \mathcal{G}_d$ with $d_\pi(\mathcal{L}(G), \mathcal{L}(H)) \leq \delta$, then there is a way to remove less than $2\varepsilon|E(G)|$ egdes of G and less than $2\varepsilon|E(H)|$ edges of H such that

- in the remaining graphs G' and H' , all connected components have size at most K_{ε} ,
- $\sum_{\alpha \in \mathcal{A}_d : |V(\alpha)| \leq K_{\varepsilon}} |c_{\alpha}^{G'} c_{\alpha}^{H'}| < \beta,$

where $C_{\alpha}^{G'}$ is the set of points that are in a component of G' isomorphic to α , and $c_{\alpha}^{G'}$:= $|C_{\alpha}^{G'}|/|V(G')|.$

We now prove Theorem [3.1.](#page-5-0)

Proof of Theorem [3.1.](#page-5-0) Let $\varepsilon > 0$. Further, choose $\varepsilon_1 := \varepsilon/(6d)$. For any $n \in \mathbb{N}$, we remove $\varepsilon_1|E(G_n)|$ edges from G_n such that in the remaining graphs ${G}'_n$ $\binom{n}{n}$ ∞ , all connected components have at most K_{ε_1} vertices, where the number K_{ε_1} is chosen according to the Equipartition The-orem [3.2.](#page-5-3) For $n \in \mathbb{N}$, we call a vertex $x \in G'_n$ n_n exceptional if we removed at least one of the edges incident to x. Hence, there can be at most $2d\varepsilon_1|V(G_n)|$ exceptional vertices in G_n . We denote by $\alpha_1, \alpha_2, \ldots, \alpha_{M_{\varepsilon_1}}$ ($M_{\varepsilon_1} \in \mathbb{N}$) the isomorphism classes in \mathcal{A}_d consisting of at most K_{ε_1} vertices. So for $n \in \mathbb{N}$ and $1 \leq i \leq M_{\varepsilon_1}$, we write $\kappa_i^n \in \mathbb{N}$ for the number of connected components in G'_n which are rooted isomorphic to α_i . We further set $\gamma_i^n := \kappa_i^n / |V(G_n)|$ for $n \in \mathbb{N}$ and $1 \le i \le M_{\varepsilon_1}$. Note that due to the weak convergence, the sequence (G_n) is Cauchy in the d_{π} -distance and the Equipar-tition Theorem [3.2](#page-5-3) is applicable. Therefore, using that $\kappa_i^n := |C_{\alpha_i}^{G'_n}|/|V(\alpha_i)| \leq |C_{\alpha_i}^{G'_n}|$, where $C_{\alpha_i}^{G'_n}$

is defined as in the Equipartition Theorem, we find for $\beta := \varepsilon_1/(M_{\varepsilon_1} K_{\varepsilon_1})$ some number $L \in \mathbb{N}$ such that

$$
|\gamma_i^n-\gamma_i^m|<\beta=\frac{\varepsilon}{6d\,M_{\varepsilon_1}\,K_{\varepsilon_1}}
$$

for all $n, m \geq L$ and all $1 \leq i \leq M_{\varepsilon_1}$. Next, using Lemma 2.3 of [\[9\]](#page-12-5), we conclude that indeed,

$$
d_{\rho}(G_n, G_m) \le 4d\,\varepsilon_1 + 2\,M_{\varepsilon_1} K_{\varepsilon_1} \beta \le \varepsilon
$$

for all $n, m \geq L$.

4. A Banach-space valued convergence theorem

This section is devoted to the proof of a general convergence result for so-called *almost additive* functions on \mathcal{G}_d which take their values in an arbitrary Banach space. Precisely, we show that for strongly convergent graph sequences $(G_n) \subseteq \mathcal{G}_d$, the expression $F(G_n)/|V(G_n)|$ converges in the Banach space topology as n tends to infinity, where F belongs to the class of functions under consideration. Similar results in the situation of general (not necessarily finitely generated) amenable

groups can already be found in [\[19,](#page-12-8) [29\]](#page-12-9) and [\[28\]](#page-12-10).

We start with the definition of almost-additive functions on \mathcal{G}_d . Let X be a Banach space equipped with a norm $\|\cdot\|$.

Definition 4.1. A mapping

$$
F: \mathcal{G}_d \to (X, \|\cdot\|)
$$

is called *almost-additive* (on \mathcal{G}_d) if $F(\epsilon) = 0$ (in case the empty graph $\epsilon \in \mathcal{G}_d$) and if there is a constant $D > 0$ depending on d such that if $G, H \in \mathcal{G}_d$ and $p, q \in \mathbb{N}$ are such that $p |V(G)| =$ $q|V(H)|$, then

$$
||p F(G) - q F(H)|| \le D \delta_S(pG, qH) p |V(G)|.
$$

We call this the *almost-additivity* property of F.

The following statements are straight forward consequences from Definition [4.1](#page-6-1) and the considerations of the previous Section [3.](#page-5-1)

Theorem 4.2. Assume that F is an almost-additive mapping on \mathcal{G}_d with values in a Banach space $(X, \|\cdot\|)$. Then, if $(G_n) \subseteq \mathcal{G}_d$ is a strongly convergent graph sequence, then there is an element $\overline{F} \in X$ such that

$$
\lim_{n \to \infty} \left\| \frac{F(G_n)}{|V(G_n)|} - \overline{F} \right\| = 0.
$$

Proof. Choose $\varepsilon > 0$ and find $L \in \mathbb{N}$ such that $d_{\rho}(G_n, G_m) \leq \varepsilon$ whenever $n, m \geq L$. By the definition of d_{ρ} , for each choice of integers n and m, we find $p, q \in \mathbb{N}$ with $p|V(G_n)| = q|V(G_m)|$ such that $d_S(pG_n, qG_m) \leq 2\varepsilon$. Consequently, we can use the almost-additivity property of F to compute

$$
\left\| \frac{F(G_n)}{|V(G_n)|} - \frac{F(G_m)}{|V(G_m)|} \right\| = \left\| \frac{p F(G_n)}{p |V(G_n)|} - \frac{q F(G_m)}{q |V(G_m)|} \right\|
$$

$$
\leq 2D \varepsilon
$$

for all $n, m \geq L$. It follows that $(F(G_n)/|V(G_n)|)_{n=1}^{\infty}$ is a Cauchy sequence in the Banach space X and hence convergent to some element $\overline{F} \in X$.

The following corollary demonstrates that in the situation of translation invariance of the almost-additive function under consideration, the convergence result of [\[29\]](#page-12-9) along Følner sequences in a finitely generated, amenable group can be extended to graph sequences. In fact, Definition [4.1](#page-6-1) includes periodicity of the function F in the sense that $F(G) = F(H)$ if G is rooted isomorphic to H. Introducing graph colourings in [\[29\]](#page-12-9), the authors are able to work with a slightly weaker notion of invariance. However, the corresponding geometric situation is far more restricted as the graph sequences are given as subgraphs of the Cayley graph of the group induced by the elements of the Følner sequence under consideration. This latter graph sequence is in fact weakly convergent with limit probabilities 0 or 1. Moreover, as we have mentioned in the Examples 2.7 above, the sequence is also hyperfinite. It follows from this that in the translation invariant situation, the setting in [\[29\]](#page-12-9) is a special case of the next corollary.

Corollary 4.3. Assume that F is an almost-additive mapping on \mathcal{G}_d with values in a Banach space $(X, \|\cdot\|)$. Then, if $(G_n) \subseteq \mathcal{G}_d$ is a weakly convergent, hyperfinite graph sequence, then there is an element $\overline{F} \in X$ such that

$$
\lim_{n \to \infty} \left\| \frac{F(G_n)}{|V(G_n)|} - \overline{F} \right\| = 0.
$$

Proof. This follows directly from the previous Theorem [4.2](#page-6-2) and Theorem [3.1.](#page-5-0)

5. Subadditive convergence

Note that Fekete's Lemma is a well known, elementary statement providing essential applications in various mathematical areas. It reads as follows.

Lemma 5.1. Let $(a_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be a subadditive sequence, i.e.

$$
a_{m+n} \le a_m + a_n
$$

for all $n, m \in \mathbb{N}$. Then the sequence $\left(\frac{a_n}{n}\right)$ converges to its infimum (which might be $-\infty$).

For the sake of applications, one may raise the question whether the integer index set can be replaced by more complicated structures, such as sets or graphs. Quite recently, a corresponding result was proven in [\[4\]](#page-12-2), where the authors prove subadditive convergence for amenable, cancellative semigroups in order to apply it to problems concerning the entropy of measure preserving dynamical systems. Including a natural additional condition on the function h under consideration, we prove convergence along hyperfinite graph sequences. Precisely, we need the subadditive function to be non-decreasing with respect to the subgraph ordering. This extends the geometric situation in results obtained by LINDENSTRAUSS and WEISS, cf. $[22]$ as well as by GROMOV, cf. [\[13\]](#page-12-0) in the context of amenable groups. Let us clarify first what we mean by subadditive functions. Assume that $h : \mathcal{G}_d \to \mathbb{R}$ is a *subadditive function*, i.e. it satisfies the following properties:

- there exists a constant $C > 0$ such that $h(G) \leq C |V(G)|$ for all $G \in \mathcal{G}_d$ (boundedness);
- if $G \in \mathcal{G}_d$ is a subgraph of $\tilde{G} \in \mathcal{G}_d$, then

$$
h(G) \le h(\tilde{G}) \qquad \text{(monotonicity)};
$$

• if G and G' are \mathcal{G}_d -subgraphs of some $\tilde{G} \in \mathcal{G}_d$ with $V(G) \sqcup V(G') = V(\tilde{G})$,

$$
h(\tilde{G}) \le h(G) + h(G') \qquad \text{(subadditivity)};
$$

in the edge-disjoint situation, i.e. if we also have and $e \in E(\tilde{G})$ if and only if either $e \in E(G)$ or $e \in E(G')$, then we have in fact

$$
h(\tilde{G}) = h(G) + h(G') \qquad \text{(special additivity)};
$$

• it is true that $h(G) = h(G')$ if G is isomorphic to G' in \mathcal{G}_d (pattern-invariance).

Remark 5.2. We need the special additivity assumption for the edge-disjoint case as a technical feature of the more general subadditivity poperty. Note that in many settings, this is a pathological situation, e.g. if one considers sequences consisting of connected graphs. However, this property will become relevant in the proofs below, where the strong graph metric δ_{ρ} forces us to work with q-fold $(q \in \mathbb{N})$ vertex- and edge-disjoint copies of finite graphs $G \in \mathcal{G}_d$. For those objects, the special additivity assumption and the pattern-invariance condition guarantee that $h(qG) = q h(G)$.

Theorem 5.3. Assume that $h : \mathcal{G} \to \mathbb{R}$ is a subadditive function. Then, for every weakly convergent, hyperfinite graph sequence $(G_n) \subseteq \mathcal{G}_d$, there is a number $\lambda \in \mathbb{R} \cup \{-\infty\}$ such that

$$
\lim_{n \to \infty} \frac{h(G_n)}{|V(G_n)|} = \lambda.
$$

Proof. By Theorem [3.1,](#page-5-0) the sequence (G_n) is in fact strongly convergent. Further, we set

$$
\lambda := \liminf_{n \to \infty} \frac{h(G_n)}{|V(G_n)|},
$$

which is an element in the infinite interval $[-\infty, C]$ due to the boundedness of h from above. We denote by $K \subseteq \mathbb{N}$ an infinite set containing the indices of a subsequence of $(h(G_n)/|V(G_n)|)_n$ that converges to λ . We fix an arbitrary positive number $\varepsilon > 0$ and we choose $k_0 \in K$ such that for all $n, k \geq k_0$, we have $\delta_{\rho}(G_n, G_k) < \varepsilon$. We fix such n and k, where we also make sure that $k \in K$. By definition of δ_{ρ} , there exist integer numbers $q_n, q_k \in \mathbb{N}$ such that $q_n|V(G_n)| = q_k|V(G_k)|$ and

$$
\delta_S(q_n G_n, q_k G_k) \le 2 \,\varepsilon. \tag{5.1}
$$

Assuming that both q_nG_n and q_kG_k are defined on a common vertex set $V_{n,k}$ we can find a subset $\tilde{V}_{n,k} \subseteq V_{n,k}$ containing those vertices such that the 1-balls (including vertex numbering) coincide

in both graphs. By Inequality [\(5.1\)](#page-7-2), we have $|\tilde{V}_{n,k}| \geq (1-2\varepsilon)|V_{n,k}|$. For this set $\tilde{V}_{n,k}$, we denote the associated induced subgraph in q_nG_n (in q_kG_k respecively) by $\tilde{G}_{n,k}$. Thus, using the special additivity property, as well as subadditivity, boundedness and pattern-invariance of h , we obtain

$$
\frac{h(G_n)}{|V(G_n)|} = \frac{h(q_n G_n)}{q_n |V(G_n)|} \le \frac{h(\tilde{G}_{n,k})}{|V_{n,k}|} + 2C\varepsilon.
$$
\n(5.2)

Since h is non-decreasing, it follows that $h(\tilde{G}_{n,k}) \leq h(q_kG_k)$. Further, the special additivity property and the pattern-invariance condition of h yield $h(\tilde{G}_{n,k}) \le q_k h(G_k)$. Continueing Inequality [\(5.2\)](#page-8-1), we arrive at

$$
\frac{h(G_n)}{|V(G_n)|} \le \frac{q_k h(G_k)}{|V_{n,k}|} + 2C \,\varepsilon = \frac{h(G_k)}{|V(G_k)|} + 2C \,\varepsilon.
$$

Note that this latter relation holds true for all large enough $k \in K$ and all large enough $n \in \mathbb{N}$. Since $\varepsilon > 0$ was arbitrary, we have

$$
\limsup_{n \to \infty} \frac{h(G_n)}{|V(G_n)|} \le \liminf_{n \to \infty} \frac{h(G_n)}{|V(G_n)|} + 2C \varepsilon = \lambda + 2C \varepsilon
$$

for all $\varepsilon > 0$. Hence, sending $\varepsilon \to 0$ concludes the proof.

6. Integrated density of states

In this section, we demonstrate that the notion of almost-additive, Banach space valued mappings is the right one in order to verify uniform approximation results concerning the integrated density of states for certain operators. More precisely, in the situation of a strongly convergent graph sequence, we show that the spectral distribution functions of the associated operators determined by the local patterns of the graph converge uniformly. As the results of this section have already been proven in [\[9\]](#page-12-5), we do not claim originality, but emphasize the application of the Banach space convergence theorem, Corollary [4.3.](#page-6-0) A related IDS result for random Schrödinger operators has been proven in [\[10\]](#page-12-29). To complete this section, we add a short discussion concerning an alternative method to derive uniform convergence in general geometric situations. The corresponding result is not new and detailed elaborations can be found in [\[31,](#page-12-25) [2\]](#page-11-1).

6.1. The model. The following setting is deduced from $[9]$. Suppose that G is a graph with countable vertex set $V(G)$ and with vertex degree bound $d \in \mathbb{N}$. Assume further that a function

$$
h: V(G) \times V(G) \to \mathbb{R}
$$

is given and possesses the following properties.

- symmetry: $h(x, y) = h(y, x)$ for all $x, y \in V(G)$.
- finite hopping range: there exists some constant $R > 0$ such that $h(x, y) = 0$ whenever $d_G(x, y) > R$.
- pattern invariance: it holds true that for the same constant $R > 0$, for each $x \in V(G)$, the function $h_x = h(x, \cdot) : V(G) \to \mathbb{R}$ is determined by the R-neighborhood of x. Precisely, this means that if $B_R^G(x) \sim^\varphi \alpha$ for some $\alpha \in \mathcal{A}_d^R$, then there is a map $h_\alpha: V(\alpha) \to \mathbb{R}$ with

$$
h_x(y) = h(x, y) = h_\alpha(\varphi(y))
$$

for every $y \in V(B_R^G(x))$.

In this situation, we call the number R the *overall range* of the function h . Note that the pattern invariance condition makes sure that h is a translation invariant function. Precisely, if there is another $x' \in V(G)$ such that $B_R^G(x') \sim B_R^G(x)$ with associated isomorphism $\varphi_{x,x'} : V(B_R^G(x)) \to V(G_R^G(x))$ $V(B_R^G(x'))$, $\varphi_{x,x'}(x) = x'$, then

$$
h(x, y) = h(x', \varphi_{x, x'}(y))
$$

for all $y \in V(B_R^G(x))$. Hence, indeed the function h only depends on the local patterns of the graph G.

In the following, we will refer to functions of the kind as h as admissible kernel functions.

Definition 6.1. Let $(G_n) \subseteq \mathcal{G}_d$ be a weakly convergent graph sequence. Then, we call $\mathcal{H} := (H_n)$ a corresponding weakly convergent operator sequence if

- there is a sequence of admissible kernel functions $(h_n), h_n : V(G_n) \times V(G_n) \to \mathbb{R}$, where the parameter R, as well as the values of h_{α} ($\alpha \in A_d^R$) for the h_n do not change with $n \in \mathbb{N}$.
- for all $n \in \mathbb{N}$, the operator $H_n : \ell^2(V(G_n)) \to \ell^2(V(G_n))$ is given by

$$
(H_nu)(x):=\sum_{y\in B_R^{G_n}(x)}h_n(x,y)\,u(y).
$$

Example 6.2. A simple example for a weakly convergent operator sequence are the discrete Laplacians $\Delta := (\Delta_n)$,

$$
\Delta_n: \ell^2(F_n) \to \ell^2(F_n): (\Delta_n u)(x) := \sum_{y \sim x} (u(y) - u(x)),
$$

where (F_n) is a sequence of Cayley graphs associated with a Følner sequence in a finitely generated, amenable group Γ , cf. [\[9\]](#page-12-5) and [\[29\]](#page-12-9).

Definition 6.3. We call (H_n) a *null operator sequence* if (h_n) is a weakly convergent operator sequence such that for all $\alpha \in \mathcal{A}_d^R$ with $p(\alpha) \neq 0$, we have $h_\alpha = 0$.

Further, it is well-known that for a fixed, weakly convergent graph sequence $\mathcal{G} := (G_n)$, the set $\mathcal{O}_{\mathcal{G}}$ of corresponding convergent operator sequences possess an algebra structure, i.e. if H^1 and H^2 are in $\mathcal{O}_{\mathcal{G}}$, then $H^1 + H^2$ and $H^1 \cdot H^2$ are as well. For a more detailed discussion, the interested reader may e.g. refer to [\[9\]](#page-12-5). In the following, we will write $\mathcal{N}_{\mathcal{G}}$ for the set of all nulloperator sequences associated with G . The following proposition is an easy observation which can be proven in the same manner as in [\[9\]](#page-12-5).

Proposition 6.4 ([\[9\]](#page-12-5), Proposition 3.1). Let $\mathcal{H} = (H_n)$ be a convergent operator sequence on a weakly convergent graph sequence $\mathcal{G} = (G_n)$. Then, the limit

$$
\lim_{n \to \infty} \frac{1}{|V(G_n)|} \sum_{x \in V(G_n)} h_n(x, x)
$$

exists and is equal to

$$
\operatorname{tr}_{\mathcal{G}}(\mathcal{H}) := \sum_{\alpha \in \mathcal{A}_d^R} p(\alpha) \, h_\alpha(\operatorname{rt}(\alpha)),
$$

where the parameters R, h_{α} are as in Definition [6.1.](#page-9-0)

We will call tr_G(H) the trace of H in the algebra $\mathcal{O}_\mathcal{G}$. The next proposition shows that this latter notion contains valuable information on the structure of the quotient space $\mathcal{Q}_G := \mathcal{O}_G/\mathcal{N}_G$, as well as on its closure with respect to a particular (weak) topology.

Proposition 6.5 (ELEK, cf. [\[9\]](#page-12-5), Prop. 3.1, Lemmas 3.2, 3.3, 3.4). Let $\mathcal{G} := (G_n)$ be a weakly convergent graph sequence and assume that $\mathcal{H}^1 := (H_n^1)$ and $\mathcal{H}^2 := (H_n^2)$ are arbitrary corresponding convergent operator sequences in $\mathcal{O}_\mathcal{G}$. Then the following statements hold true.

- (*i*) $\text{tr}_{\mathcal{G}}(\mathcal{H}^1 \cdot \mathcal{H}^2) = \text{tr}_{\mathcal{G}}(\mathcal{H}^2 \cdot \mathcal{H}^1).$
- (ii) $\text{tr}_{\mathcal{G}}$ is faithful on $\mathcal{Q}_{\mathcal{G}}$, i.e. $\text{tr}_{\mathcal{G}}(\mathcal{H}^1 \cdot \mathcal{H}^1) > 0$ if $\mathcal{H}^1 \notin \mathcal{N}_{\mathcal{G}}$ and $\text{tr}_{\mathcal{G}}(\mathcal{H}^1) = 0$ for $\mathcal{H}^1 \in \mathcal{N}_{\mathcal{G}}$.
- (iii) the algebra $\mathcal{Q}_{\mathcal{G}}$ is a pre Hilbert space with inner product

$$
\langle [\mathcal{H}^1], [\mathcal{H}^2] \rangle = \text{tr}_{\mathcal{G}}(\mathcal{H}^2 \cdot \mathcal{H}^1),
$$

where $[\mathcal{H}^i]$ stands for the class of \mathcal{H}^i in $\mathcal{Q}_{\mathcal{G}}$.

- (iv) the representation $\mathcal{L}_{[\mathcal{H}^1]}[\mathcal{H}] := [\mathcal{H}^1 \cdot \mathcal{H}]$ $(\mathcal{H} \in \mathcal{O}_{\mathcal{G}})$ for $[\mathcal{H}^1] \in \mathcal{Q}_{\mathcal{G}}$ is a bounded (multiplication) operator on the quotient space $\mathcal{Q}_{\mathcal{G}}$.
- (v) The weak closure $\overline{Q_G}$ of Q_G is a von-Neumann algebra. Consequently, the trace

$$
\mathrm{tr}_{\mathcal{G}}([\mathcal{H}^1]) = \langle \mathcal{L}_{[\mathcal{H}^1]} \, \mathbf{1}, \mathbf{1} \rangle
$$

extends to $\overline{Q_{\mathcal{G}}}$ as a ultraweakly continuous, faithful trace.

(vi) $\mathcal{L}_{[\mathcal{H}^1]}$ is a bounded, self-adjoint operator on $\overline{\mathcal{Q}_{\mathcal{G}}}$ with $\sup_{n\in\mathbb{N}}||H_n^1|| \leq K(\mathcal{H}^1)$ for some constant $K(\mathcal{H}^1) > 0.$

For convenience, we simply write $[\mathcal{H}]$ instead of $\mathcal{L}_{[\mathcal{H}]}([\mathcal{H}] \in \overline{Q_{\mathcal{G}}})$ and we call $[\mathcal{H}] := (H_n)$ the limit operator (which actually is defined on $\overline{Q_G}$) of the sequence $\mathcal{H} \in \mathcal{O}_G$. We have seen in the above proposition that $[\mathcal{H}]$ is a bounded, self-adjoint operator. It follows from the spectral theorem that there must be a decomposition

$$
[\mathcal{H}] = \int_{\mathbb{R}} \lambda \, dE_{\lambda}^{[\mathcal{H}]},
$$

where $E_{\lambda}^{[\mathcal{H}]} \in \overline{\mathcal{Q}_{\mathcal{G}}}$ is the spectral projection $\mathbf{1}_{]-\infty,\lambda]}([\mathcal{H}])$ for $\lambda \in \mathbb{R}$. Further, the *integrated density* of states of $[\mathcal{H}]$ can be defined by

$$
N_{[\mathcal{H}]}(\lambda) := \text{tr}_{\mathcal{G}}(E_{\lambda}^{[\mathcal{H}]}).
$$

Given a convergent operator sequence $[\mathcal{H}] = (H_n)$, then for each $n \in \mathbb{N}$, we set

$$
n_{H_n}(\lambda) := \# \{ E \le \lambda \, | \, E \text{ is an eigenvalue of } H_n \} \quad (\lambda \in \mathbb{R}),
$$

as the cumulative eigenvalue counting function for the (matrix) operator H_n . The symbol $\#$ shall express that the eigenvalues are counted with multiplicities. The corresponding spectral distribution functions are just the cumulative eigenvalue distributions (normalized eigenvalue counting functions) given by

$$
N_{H_n}(\lambda) := \frac{n_{H_n}(\lambda)}{|V(G_n)|}, \quad n \in \mathbb{N}, \quad \lambda \in \mathbb{R}.
$$

With these notions at hand, it is not hard to show the weak convergence of the eigenvalue counting functions towards the integrated density of states.

Theorem 6.6 (c.f. e.g. [\[9\]](#page-12-5), Theorem 1). Let $\mathcal{H} = (H_n)$ be a convergent operator sequence on a weakly convergent graph sequence $\mathcal{G} := (G_n)$. Then, for any continuity point $\lambda \in \mathbb{R}$, we have

$$
\lim_{n\to\infty}N_{H_n}(\lambda)=N_{[\mathcal{H}]}(\lambda)
$$

Proof. See e.g. [\[9\]](#page-12-5), Theorem 1.

6.2. Uniform approximation of the IDS. We will now outline how the uniform convergence of the IDS can be obtained by using the Banach space convergence theorem, Corollary [4.3.](#page-6-0)

Theorem 6.7. Let $\mathcal{H} = (H_n)$ be a convergent operator sequence on a weakly convergent, hyperfinite graph sequence $\mathcal{G} := (G_n)$. Then, we have in fact uniform convergence of the spectral distributions, *i.e.*

$$
\lim_{n\to\infty}||N_{H_n}(\cdot)-N_{[\mathcal{H}]}(\cdot)||_{\infty}=0.
$$

Remark 6.8. The validity of Theorem [6.7](#page-10-0) has already been verified in [\[9\]](#page-12-5) (cf. Proposition 3.2). However, we would like to emphasize that it is a different approach is to represent the functions $N_{H_n}(\cdot)$ as abstract, almost-additive mappings on $\mathcal{G}_d = \{G_n\}$, where the Banach space under consideration is $(C_{rb}(\mathbb{R}), \|\cdot\|_{\infty})$, i.e. the collection of right-continuous functions endowed with supremum norm. Thus, using Corollary [4.3](#page-6-0) for the proof of uniform convergence towards the integrated density of states, we also show that the notion of almost-additive mappings is an appropriate abstract description for this spectral issue.

The validity of Theorem [6.7](#page-10-0) follows essentially from the following proposition.

Proposition 6.9. Let $\mathcal{H} = (H_n)$ be a convergent operator sequence on a weakly convergent graph sequence $\mathcal{G} := (G_n)$. Then for $\mathcal{G}_d := \{G_n\}$, the mapping

$$
F: \mathcal{G}_d \to \mathcal{C}_{rb}(\mathbb{R}) \quad F(G_n) := n_{H_n}(\cdot)
$$

is almost-additive.

We will not give a proof of the above proposition as it follows essentially from the definition of the metric δ_S in combination with a uniform rank estimate for the difference of the eigenvalue counting functions of self-adjoint, finite dimensional operators, see e.g. [\[19\]](#page-12-8), Proposition 7.2. Very similar computations can also be found in $[21]$, $[29]$, $[28]$ and $[9]$, $[10]$. Combining the above proposition with Corollary [4.3,](#page-6-0) we have indeed verified the validity of Theorem [6.7.](#page-10-0)

It can already be noticed in the group situation that one cannot expect the Banach space convergence of Theorem [4.2](#page-6-2) to hold for non-amenable structures. Indeed, it has been shown in [\[12\]](#page-12-30) (Proposition 4.1) that sofic approximations of groups are hyperfinite if and only if the group under consideration is amenable. Thus, we infer from Proposition [2.8](#page-5-4) that our notion of almostadditive functions will not allow for normalized convergence along approximating sequences in non-amenable groups. However, as far as the question of pointwise (or uniform) convergence of spectral distribution functions is concerned, one can bypass this problem by extending the algebraic eigenvalue conjecture (see e.g. [\[5\]](#page-12-24)) to the general geometric case of graph sequences. This issue has been addressed by ABÉRT, THOM and VIRÁG in $[2]$. The method is to use the number theoretic techniques developed in [\[31\]](#page-12-25) for the proof of the conjecture in the case of sofic groups. Precisely, if ν_n and ν are the positive measures associated with N_{H_n} and $N_{\lbrack H \rbrack}$ respectively, then the following assertion holds true.

Theorem 6.10 (cf. [\[2\]](#page-11-1)). Let $(G_n) \subseteq \mathcal{G}_d$ be a weakly convergent graph sequence. Assume that $[\mathcal{H}] :=$ (H_n) is an associated weakly convergent operator sequence with integer coefficients (admissible kernel functions), i.e. for all $\alpha \in A_d$, we have that h_α takes its values in Z. Then,

$$
\lim_{n \to \infty} \nu_n(\{\lambda\}) = \nu(\{\lambda\})
$$
\n(6.1)

for every $\lambda \in \mathbb{R}$.

Now well-known arguments can be used to derive from the limit relation [\(6.1\)](#page-11-2) the uniform convergence of (N_{H_n}) , see e.g. [\[21\]](#page-12-17), Lemma 6.3. Thus, one obtains the uniform existence of the integrated density of states for bounded, integer coefficient operators on all approximable, discrete structures with vertex degree bound. To the knowledge of the author, this is still an open question for arbitrary coefficients, see the remark following Theorem 4.4 in [\[31\]](#page-12-25).

These elaborations show that strong spectral approximation results hold true even if one does not have a Banach space valued convergence theorem at hand. However, one may raise the question whether one can find a class of Banch space valued functions which comprises spectral distributions and which are endowed with suitable conditions for a convergence theorem to hold. This will be a subject of future investigations.

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