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# TIGHTER LINEAR AND SEMIDEFINITE RELAXATIONS FOR MAX-CUT BASED ON THE LOVÁSZ–SCHRIJVER LIFT-AND-PROJECT PROCEDURE*

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**Abstract.** We study how the lift-and-project method introduced by Lovász and Schrijver [*SIAM J. Optim.,* 1 (1991), pp. 166–190] applies to the cut polytope. We show that the cut polytope of a graph can be found in $k$ iterations if there exist $k$ edges whose contraction produces a graph with no $K_5$-minor. Therefore, for a graph $G$ with $n \geq 4$ nodes with stability number $\alpha(G)$, $n-4$ iterations suffice instead of the $m$ (number of edges) iterations required in general and, under some assumption, $n - \alpha(G) - 3$ iterations suffice. The exact number of needed iterations is determined for small $n \leq 7$ by a detailed analysis of the new relaxations. If positive semidefiniteness is added to the construction, then one finds in one iteration a relaxation of the cut polytope which is tighter than its basic semidefinite relaxation and than another one introduced recently by Anjos and Wolkowicz [*Discrete Appl. Math.,* to appear]. We also show how the Lovász–Schrijver relaxations for the stable set polytope of $G$ can be strengthened using the corresponding relaxations for the cut polytope of the graph $G^\nabla$ obtained from $G$ by adding a node adjacent to all nodes of $G$.

**Key words.** linear relaxation, semidefinite relaxation, lift-and-project, cut polytope, stable set polytope

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## 1. Introduction.

Lovász and Schrijver [22] have introduced a method for constructing a higher dimensional convex set whose projection $N(K)$ approximates the convex hull $P$ of the 0–1 valued points in a polytope $K$ defined by a given system of linear inequalities. If the linear system is in $d$ variables, the convex set consists of symmetric matrices of order $d+1$ satisfying certain linear conditions. A fundamental property of the projection $N(K)$ is that one can optimize over it in polynomial time and thus find an approximate solution to the original problem in polynomial time. Moreover, after $d$ iterations of the operator $N$, one finds the polytope $P$. Lovász and Schrijver [22] also introduce some strengthenings of the basic construction; in particular, adding positive semidefinite constraints leads to the operator $N_+$, and adding stronger linear conditions in the definition of the higher dimensional set of matrices leads to the operators $N'$ and $N'_+$. They study in detail how the method applies to the stable set polytope. Starting with $K = \mathrm{FRAC}(G)$ (the fractional stable set polytope defined by nonnegativity and the edge constraints), they show that in one iteration of the $N$ operator one obtains all odd hole inequalities (and no more), while in one iteration of the $N_+$ operator one obtains many inequalities including odd wheel, clique, and odd antihole inequalities and orthogonality constraints; therefore, the relaxation $N_+(\mathrm{FRAC}(G))$ is tighter than the basic semidefinite relaxation of the stable set polytope by the theta body $\mathrm{TH}(G)$. In particular, this method permits one to solve the maximum stable set problem in a $t$-perfect graph or in a perfect graph in polynomial time. They also show that the stable set polytope of $G$ is found after at most $n - \alpha(G) - 1$ iterations of the $N$ operator (resp., $\alpha(G)$ iterations of the $N_+$ operator) applied to $\mathrm{FRAC}(G)$, if $G$ has at least one edge.

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On the other hand, there exist "easy" polytopes $P$ (meaning that their linear description is known and one can optimize over them in polynomial time) for which the number of iterations of the $N$ or $N_+$ operators needed in order to find $P$ grows linearly with the dimension of $P$. For example, Stephen and Tunçel [29] showed that $n$ iterations are needed for finding the matching polytope of $K_{2n+1}$ (starting with the polytope defined by nonnegativity and the degree constraints) using the $N_+$ operator. Recently, Cook and Dash [8] and Goemans and Tunçel [12] constructed examples where positive semidefiniteness does not help; namely, the same number $d$ of iterations is needed for finding some $d$-dimensional polytope $P$ using the $N$ or the $N_+$ operator. This is the case, for instance, for the polytope $P := \{x \in \mathbf{R}^d \mid \sum_{i=1}^d x_i \geq 1\}$ if we start from its relaxation $K := \{x \in \mathbf{R}^d \mid \sum_{i=1}^d x_i \geq \frac{1}{2}\}$.

In this paper we study how the method applies to the cut polytope when starting with its linear relaxation by the metric polytope $\mathrm{MET}(G)$ (to be defined later). When using the operator $N_+$, one obtains in one iteration a semidefinite relaxation of the cut polytope which is tighter than its basic semidefinite relaxation and also tighter than a refinement of the basic relaxation introduced recently by Anjos and Wolkowicz [2]. One can, in fact, refine the relaxation $N(\mathrm{MET}(G))$ by first applying the $N$ operator to the metric polytope of the complete graph and then projecting on the edge set of the graph; the relaxation denoted as $N(G)$ obtained in this way satisfies $\mathrm{CUT}(G) \subseteq N(G) \subseteq N(\mathrm{MET}(G))$. We consider in this paper both constructions $N(G)$ and $N(\mathrm{MET}(G))$, also for the stronger operators $N_+, N', N'_+$ and their iterates.

We show that $\mathrm{CUT}(G) = N^k(\mathrm{MET}(G))$ if there exist $k$ edges in $G$ whose contraction produces a graph with no $K_5$-minor. In particular, the cut polytope of a graph on $n$ nodes can be found after $n-4$ (resp., $n-5$) iterations of the $N$ (resp., $N'$) operator if $n \geq 4$ (resp., $n \geq 6$) (while the cut polytope has dimension $m$, the number of edges of the graph). Moreover, if $G$ has stability number $\alpha(G)$, then $\mathrm{CUT}(G) = N^k(G)$, where $k := \max(0, n - \alpha(G) - 3)$; equality $\mathrm{CUT}(G) = N^k(\mathrm{MET}(G))$ holds if there exists a maximum stable set in $G$ whose complement induces a graph with at most three connected components. The upper bound $n - \alpha(G) - 3$ is similar to the upper bound in [22] for the stable set polytope. It is well known that the stable set polytope $\mathrm{STAB}(G)$ can be realized as a face of the cut polytope $\mathrm{CUT}(G^\nabla)$, where $G^\nabla$ is obtained by adding a new node to $G$ adjacent to all nodes of $G$; moreover, an analogous relation exists between their basic linear and positive semidefinite relaxations. We study how this fact extends to their relaxations obtained via the Lovász–Schrijver procedure. Namely, we show that $N^k(\mathrm{MET}(G^\nabla))$ (resp., $\nu^k(\mathrm{MET}(G^\nabla)))$ yields a relaxation of $\mathrm{STAB}(G)$ which is tighter than $N^{k+1}(\mathrm{FRAC}(G))$ (resp., $\nu^k(\mathrm{FRAC}(G))$ for $\nu = N_+, N', N'_+$).

Although the inclusion $N_+(\mathrm{MET}(G)) \subseteq N(\mathrm{MET}(G))$ is strict for certain graphs (e.g., for any complete graph on $n \geq 6$ nodes), we do not know of an example of a graph $G$ for which the number of iterations needed for finding $\mathrm{CUT}(G)$ is smaller when using the operator $N_+$ than when using the operator $N$. This contrasts with the case of the stable set polytope where, for instance, $\mathrm{STAB}(K_n)$ is found in one iteration of the $N_+$ operator applied to $\mathrm{FRAC}(G)$, while $n-2$ iterations of the $N$ operator are needed.

The paper is organized as follows. Section 2 gives a general description of the Lovász–Schrijver (LS) procedure, and section 3 contains a presentation of the various relaxations of the cut polytope considered in the paper. In section 4, we study the index of a graph (the smallest number of iterations of the LS procedure needed for finding its cut polytope); upper bounds are proved in sections 4.1 and 4.3, the behavior of the index under taking graph minors and clique sums is investigated in section 4.4, and a number of needed technical tools are provided in section 4.2. We study in section 5 the validity of hypermetric inequalities for the new relaxations, which enables us to determine the exact value of the index of a graph on $n \leq 7$ nodes; some technical proofs are delayed until section 7. Finally, in section 6 we study the links between the LS relaxations for the cut polytope and the original LS relaxations for the stable set polytope.

**2. The LS Procedure.** Let $F \subseteq \{\pm 1\}^d$, let $P := \text{conv}(F)$ be the integral polytope whose linear description one wishes to find, and let

$$K = \{x \in \mathbf{R}^d \mid Ax \geq b\}$$

be a linear relaxation of $P$ such that $K \subseteq [-1,1]^d$ and $K \cap \{\pm 1\}^d = F$ ($K$ is a *linear programming formulation* for $P$).

Starting from $K$, the LS method constructs a hierarchy of linear relaxations for $P$ which in $d$ steps finds the exact description of $P$. The basic idea is as follows. If we multiply an inequality $a^T x \geq \beta$, valid for $F$, by $1 \pm x_i \geq 0$, we obtain two nonlinear inequalities which remain valid for $F$. Applying this to all the inequalities from the system $Ax \geq b$, substituting $x_i^2$ by 1, and linearizing $x_i x_j$ by a new variable $y_{ij}$ for $i \neq j$, we obtain a polyhedron in the $\binom{d+1}{2}$-space whose projection $N(K)$ on the original $d$-space contains $P$ and is contained in $K$. The method was described in [22] in terms of 0–1 variables, but for our application to the max-cut problem it is more convenient to work with $\pm 1$ variables, which is why we present it here in this setting.

It is useful to reformulate the construction in matrix terms. First we introduce some notation. As it is often more convenient to work with homogeneous systems of inequalities, i.e., with cones rather than polytopes, one embeds the $d$-space into $\mathbf{R}^{d+1}$ as the hyperplane: $x_0 = 1$. For a polytope $P$ in $\mathbf{R}^d$, $\tilde{P} := \{\lambda(1,x) \mid x \in P, \lambda \geq 0\}$ denotes the cone in $\mathbf{R}^{d+1}$ obtained by homogenization of $P$; thus $P = \{x \in \mathbf{R}^d \mid (1,x) \in \tilde{P}\}$. Given a cone $K$, its *dual cone* $K^*$ is defined as

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}.$$

Consider the cube $Q := [-1,1]^d$ and its homogenization $\tilde{Q} = \{(x_0, x) \in \mathbf{R}^{d+1} \mid -x_0 \leq x_i \leq x_0 \text{ for all } i = 1, \ldots, d\}$. Thus the dual cone of $\tilde{Q}$ is generated by the $2d$ vectors $e_0 \pm e_i$ ($i = 1, \ldots, d$), where $e_0, e_1, \ldots, e_d$ denote the standard unit vectors in $\mathbf{R}^{d+1}$.

Given two polytopes $K_1 \subseteq K_2 \subseteq Q$, let $M(K_1, K_2)$ denote the set of symmetric matrices $Y = (y_{ij})_{i,j=0}^d$ satisfying the conditions

$$y_{i,i} = y_{0,0} \quad \text{for } i = 1, \ldots, d, \tag{2.1}$$

$$Y\tilde{K}_2^* \subseteq \tilde{K}_1, \tag{2.2}$$

and set

$$N(K_1, K_2) := \{x \in \mathbf{R}^d \mid (1,x) = Ye_0 \text{ for some } Y \in M(K_1, K_2)\}.$$

One can easily verify that

$$K_1 \cap \{\pm 1\}^d \subseteq N(K_1, K_1) \subseteq N(K_1, K_2) \subseteq N(K_1, Q) \subseteq K_1.$$

Therefore, the choice $(K_1, K_2) = (K, K)$ provides the best relaxation $N(K, K)$ for $P$. However, it is also interesting to consider the choice $(K_1, K_2) = (K, Q)$, giving the weaker relaxation $N(K, Q)$, as it behaves better algorithmically. Indeed, as observed in [22], if one can solve in polynomial time the (weak) separation problem over $K$, then the same holds for $M(K, Q)$ and thus also for its projection $N(K, Q)$; this property holds for $N(K, K)$ under the more restrictive assumption that an explicit linear description whose size is polynomial is known for $K$ (details will be given later in this section).

One can obtain tighter relaxations for $P$ by iterating the constructions $N(K, Q)$ and $N(K, K)$. One can iterate the construction $N(K, Q)$ by the sequence $N(K, Q)$, $N(N(K, Q), Q)$, etc. A first way in which the construction $N(K, K)$ can be iterated is by considering the sequence $N(K, K)$, $N(N(K, K), N(K, K))$, etc. A major drawback is then that, even if $K$ is given by an explicit linear system of polynomial length, it is not clear whether this holds for the next iterate $N(K, K)$. A more tractable way is to consider the sequence $N(K, K)$, $N(N(K, K), K)$, etc. For simplicity in the notation, for a polytope $H \subseteq K \subseteq Q$ set

$$M(H) := M(H, Q),\ M'(H) := M(H, K),\ N(H) := N(H, Q),\ N'(H) := N(H, K).$$

The sequences $K, N(K, Q), N(N(K, Q), Q), \dots$ and $K, N(K, K), N(N(K, K), K), \dots$ can then be defined iteratively by

$$N^0(K) = (N')^0(K) := K, \qquad N^k(K) := N(N^{k-1}(K), Q),$$
$$(N')^k(K) := N((N')^{k-1}(K), K)$$

for $k \geq 1$. Thus $x \in \nu^k(K)$ if and only if $(1, x) = Ye_0$ for some $Y \in \mu(\nu^{k-1}(K))$, where $\mu = M$ (resp., $M'$) if $\nu = N$ (resp., $N'$).

One can reinforce the operators $N$ and $N'$ by adding positive semidefiniteness constraints. For a polytope $H \subseteq Q$, define $M_+(H)$ (resp., $M'_+(H)$) as the set of *positive semidefinite* matrices $Y \in M(H)$ (resp., $Y \in M'(H)$); the projections $N_+(H)$ and $N'_+(H)$ and their iterates are then defined in the obvious way. The following hierarchy holds:

$$P \subseteq N'_+(K) \subseteq N'(K) \subseteq N(K) \subseteq K, \qquad P \subseteq N'_+(K) \subseteq N_+(K) \subseteq N(K) \subseteq K. \tag{2.3}$$

For membership in $M(K)$, condition (2.2) can be rewritten as

$$Y(e_0 \pm e_i) \in \tilde{K} \quad \text{for } i = 1, \dots, d. \tag{2.4}$$

As $Ye_0 = \frac{1}{2}(Y(e_0 + e_i) + Y(e_0 - e_i))$, we deduce that

$$N(K) \subseteq \mathrm{conv}(K \cap \{x \mid x_i = \pm 1\}) \quad \text{for any } i = 1, \dots, d. \tag{2.5}$$

Using this fact and induction, one can prove that after $d$ iterations of the operator $N$, one finds the polytope $P$.

THEOREM 2.1 (see [22]). $N^d(K) = P$.

Obviously, the same holds for the operators $N_+$, $N'$, or $N'_+$, but the corresponding sequences of relaxations may converge faster to $P$.

**2.1. Comparison with other lift-and-project methods.** Other lift-and-project methods have been proposed in the literature, in particular by Balas, Ceria, and Cornuéjols [3], by Sherali and Adams [28], and, recently, by Lasserre [16, 17].

Each of these methods produces a hierarchy of linear or semidefinite (in the case of Lasserre) relaxations: $P \subseteq K^d \subseteq \cdots \subseteq K^1 \subseteq K$ such that $P = K^d$. For $k \geq 1$, the $k$th iterate $S_k(K)$ in the Sherali–Adams hierarchy is obtained by multiplying the system $Ax \geq b$ by each of the products $\prod_{i \in I}(1+x_i)\prod_{j\in J}(1-x_j)$ for $I, J \subseteq [1,d]$ disjoint with $|I \cup J| = k$ and then replacing each square $x_i^2$ by 1, linearizing each product $\prod_{i\in I} x_i$, and projecting back on $\mathbf{R}^d$; hence, the first step is identical to the first step of the LS method, i.e., $S_1(K) = N(K)$. It is shown in [22] that $S_t(K) \subseteq N^k(K)$ (see [18] for a simple proof).

The first relaxation $P_i(K)$ in the Balas–Ceria–Cornuéjols hierarchy is obtained by multiplying $Ax \geq b$ by $1 \pm x_i$ for some given $i \in [1, d]$ (and then linearizing and projecting back on $\mathbf{R}^d$); the next relaxations are defined iteratively by $P_{i_1\ldots i_k}(K) := P_{i_k}(P_{i_1\ldots i_{k-1}}(K))$. It is shown in [3] that $P_{i_1\ldots i_k}(K) = \mathrm{conv}(K \cap \{x \mid x_{i_1}, \ldots, x_{i_k} = \pm 1\})$. Setting

$$N_0(K) := \bigcap_{i=1}^{d} P_i(K) = \bigcap_{i=1}^{d} \mathrm{conv}(K \cap \{x \mid x_i = \pm 1\}), \tag{2.6}$$

we deduce from (2.5) that

$$N(K) \subseteq N_0(K), \tag{2.7}$$

and thus $N^k(K) \subseteq N_0^k(K) = \bigcap_{i_1\ldots i_k} P_{i_1\ldots i_k}(K)$ for $k \geq 1$. In fact, $N_0(K)$ can be seen as the "noncommutative" analogue of $N(K)$, as $N_0(K) = \{x \in \mathbf{R}^d \mid (1, x) = Ye_0 \text{ for some } Y \in M_0(K)\}$, where $M_0(K)$ is the set of matrices (not necessarily symmetric) satisfying (2.1) and (2.4).

Using facts about moment sequences and representations of positive polynomials as sums of squares, Lasserre [16, 17] introduces a new hierarchy of semidefinite relaxations $Q_k(K)$ of $P$. It is shown in [18] that this new hierarchy refines the LS hierarchy; that is, $Q_k(K) \subseteq N_+^k(K)$, and its relation to the Sherali–Adams hierarchy is explained.

**2.2. Algorithmic aspects.** Given a convex body $B \subseteq \mathbf{R}^d$, the *separation problem* for $B$ is the problem of deciding whether a given point $y \in \mathbf{R}^d$ belongs to $B$ and, if not, of finding a hyperplane separating $y$ from $B$; the *weak separation problem* is the analogous problem where one allows for numerical errors. An important application of the ellipsoid method is that if one can solve in polynomial time the weak separation problem for $B$, then one can optimize any linear objective function over $B$ in polynomial time (with an arbitrary precision), and vice versa. (One should assume some technical information over $B$, like the knowledge of a ball contained in $B$ and of a ball containing $B$.) See [14] for details.

An important property of the LS construction is that if one can solve in polynomial time the weak separation problem for $K$, then the same holds for $M(K)$ and $M_+(K)$, and thus for their projections $N(K)$ and $N_+(K)$. Therefore, for any fixed $k$, one can optimize in polynomial time a linear objective function over the relaxations $N^k(K)$ and $N_+^k(K)$; the same holds for the relaxations $S_k(K)$ and $P_{i_1\ldots i_k}(K)$ of Sherali–Adams and of Balas–Ceria–Cornuéjols. For the operators $N'$ and $N'_+$ and for the Lasserre hierarchy, an analogous result holds under the more restrictive assumption that an explicit linear description is known for $K$ whose size is part of the input data.

**2.3. Identifying valid inequalities for $N(K)$ and $N_+(K)$.** We mention two results from [22] permitting us to construct inequalities valid for $N(K)$ and $N_+(K)$; the first one follows directly from (2.5) and we prove the second one for completeness.

LEMMA 2.2. *Suppose that, for some $i = 1, \dots, d$, the inequality $a^T x \geq \beta$ is valid for $K \cap \{x \mid x_i = \pm 1\}$. Then the inequality $a^T x \geq \beta$ is valid for $P_i(K)$ and thus for $N_0(K)$ and $N(K)$.*

LEMMA 2.3. *Suppose that $a_i \geq 0$ for $i = 1, \dots, d$ and $\beta \leq 0$. If the inequality $a^T x \geq \beta$ is valid for $K \cap \{x \mid x_i = -1\}$ for every $i$ for which $a_i > 0$, then the inequality $a^T x \geq \beta$ is valid for $N_+(K)$.*

*Proof.* Set $b := (-\beta, a) \in \mathbf{R}^{d+1}$; thus $b \geq 0$. Let $Y \in M_+(K)$. We show that $b^T Y e_0 \geq 0$. By the assumption, we know that $b^T Y(e_0 - e_i) \geq 0$ if $a_i > 0$. Multiplying both sides of the inequality by $a_i$ and summing over $i = 1, \dots, d$ yields

$$\left(\sum_{i=1}^{d} a_i\right) b^T Y e_0 \geq b^T Y \left(\sum_{i=1}^{d} a_i e_i\right) = b^T Y(b + \beta e_0),$$

and thus $(\sum_i a_i - \beta) b^T Y e_0 \geq b^T Y b$. The result now follows since $b^T Y b \geq 0$ (as $Y$ is positive semidefinite) and $\sum_i a_i - \beta > 0$ (else, there is nothing to prove). □

**2.4. Comparing $N_+(K)$ with the basic semidefinite relaxation in the equality case.** The relaxation $N_+(K)$ is often stronger than some basic semidefinite relaxation one can think of for the problem at hand; this is the case for the stable set problem and for max-cut (see later) and, as we see now, when $K$ is defined by an equality system. Suppose that $K = \{x \in \mathbf{R}^d \mid Ax = b\}$. The set $\hat{K}$ consisting of the vectors $x \in \mathbf{R}^d$ for which there exists a positive semidefinite matrix $Y = \left(\begin{smallmatrix} 1 & x^T \\ x & X \end{smallmatrix}\right)$, satisfying $X_{ii} = 1$ $(i = 1, \dots, d)$ and $\mathrm{Tr}(A^T A X) = b^T b$, is a natural semidefinite relaxation for $P$ which is contained in $K$. (This relaxation can be obtained by taking the dual of the Lagrange dual of the formulation: $Ax = b$, $x_i^2 = 1$ $(i = 1, \dots, d)$, and $(Ax - b)^T (Ax - b) = 0$; cf. [24], [21]).

PROPOSITION 2.4. $N_+(K) \subseteq \hat{K}$.

*Proof.* Let $x \in N_+(K)$ and $Y \in M_+(K)$ such that $(1, x) = Y e_0$. Then $Y(e_0 \pm e_i) \in \widetilde{K}$, which means that $Ax = b$ and $AXe_i = bx_i$ $(i = 1, \dots, d)$ (setting $X := (Y_{i,j})_{i,j=1}^d$). Since $b = Ax = \sum_{i=1}^d (Ae_i) x_i$, then $\mathrm{Tr}(A^T A X) - b^T b = \sum_{i=1}^d (Ae_i)^T A X e_i - \sum_{i=1}^d (Ae_i)^T b x_i = \sum_{i=1}^d (Ae_i)^T (AXe_i - bx_i) = 0$, implying $x \in \hat{K}$. □

## 3. The cut polytope and some relaxations.

**3.1. The cut polytope and the metric polytope.** Given an integer $n \geq 3$, set $V_n := \{1, \dots, n\}$, $E_n := \{ij \mid 1 \leq i < j \in V_n\}$, and $d_n := |E_n| = \binom{n}{2}$. Let $\mathcal{S}_n$ denote the set of $n \times n$ symmetric matrices. For $X \in \mathcal{S}_n$, $X \succeq 0$ means that $X$ is positive semidefinite (abbreviated as *sdp*). Set

$$\mathcal{S}_n^1 := \{X \in \mathcal{S}_n \mid x_{ii} = 1 \text{ for all } i \in V_n\}, \qquad \mathcal{E}_n := \{X \in \mathcal{S}_n^1 \mid X \succeq 0\}.$$

Given a vector $x \in \mathbf{R}^{E_n}$, let $\mathrm{smat}(x)$ denote the matrix $X \in \mathcal{S}_n^1$ whose off-diagonal entries are given by $x$; conversely, given a symmetric matrix $X = (x_{ij})_{i,j=1}^n$, $\mathrm{svec}(X) := (x_{ij})_{1 \leq i < j \leq n}$ denotes the vector consisting of the upper triangular entries of $X$. Hence, smat and svec are inverse bijections between the sets $\mathbf{R}^{E_n}$ and $\mathcal{S}_n^1$.

Given $x \in \{\pm 1\}^n$, $xx^T$ is called a *cut matrix* and $\mathrm{svec}(xx^T) \in \mathbf{R}^{E_n}$ is the associated *cut vector* of the complete graph $K_n = (V_n, E_n)$. Thus, $\mathrm{svec}(xx^T)$ is the $(\pm 1)$-incidence vector of the *cut* $\delta(S) := \{ij \in E_n \mid |S \cap \{i,j\}| = 1\}$, where $S := \{i \mid x_i = 1\}$.

Let $G = (V_n, E)$ be a graph where $E \subseteq E_n$. The cut polytope $\mathrm{CUT}(K_n)$ of the complete graph $K_n$ is defined as the convex hull of the cut vectors $\mathrm{svec}(xx^T)$ for $x \in \{\pm 1\}^n$, and the cut polytope $\mathrm{CUT}(G)$ of $G$ is then defined as the projection of

$\mathrm{CUT}(K_n)$ on the subspace $\mathbf{R}^E$ indexed by the edge set of $G$. As linear programming formulation for $\mathrm{CUT}(G)$ we consider the *metric polytope* $\mathrm{MET}(G)$ defined by the conditions $x \in [-1,1]^E$ and the *circuit inequalities*:

$$\sum_{ij\in D} x_{ij} - \sum_{ij\in C\setminus D} x_{ij} \geq 2 - |C| \tag{3.1}$$

for all circuits $C$ of $G$ and all subsets $D \subseteq C$ with $|D|$ odd. It is known that $\mathrm{CUT}(G) = \mathrm{MET}(G)$ if and only if $G$ has no $K_5$-minor [7]. In the linear description of $\mathrm{MET}(G)$, it suffices to consider the circuit inequalities for *chordless* circuits [7]. Therefore, $\mathrm{MET}(K_n)$ is defined by the $4\binom{n}{3}$ *triangle inequalities*:

$$x_{ij} + x_{ik} + x_{jk} \geq -1, \qquad x_{ij} - x_{ik} - x_{jk} \geq -1 \tag{3.2}$$

for all distinct $i, j, k \in V_n$. The polytope $\mathrm{MET}(G)$ coincides with the projection of $\mathrm{MET}(K_n)$ on the subspace $\mathbf{R}^E$ [6]; therefore, one can optimize a linear objective function over $\mathrm{MET}(G)$ in polynomial time and thus solve the separation problem for $\mathrm{MET}(G)$ in polynomial time. For a direct proof of the latter fact, see [7].

**3.2. Semidefinite relaxations.** We present here a number of semidefinite relaxations for the cut polytope.

**The basic sdp relaxation** As every cut matrix $xx^T$ ($x \in \{\pm 1\}^n$) belongs to $\mathcal{E}_n$, we have

$$\mathrm{smat}(\mathrm{CUT}(K_n)) \subseteq \mathcal{E}_n.$$

The set $\mathcal{E}_n$ is the basic semidefinite relaxation of the cut polytope underlying the approximative algorithm for max-cut of Goemans and Williamson [13].

**The Anjos–Wolkowicz sdp relaxation.** In what follows, matrices in $\mathcal{S}_{d_n+1}$ or $\mathcal{E}_{d_n+1}$ are assumed to be indexed by the set $E_n \cup \{0\}$, and $e_0, e_{ij}$ ($ij \in E_n$) denote the standard unit vectors in $\mathbf{R}^{d_n+1}$. For $x \in \{\pm 1\}^n$, let $y := (1, \mathrm{svec}(xx^T))$ be the associated cut vector in $\widetilde{\mathrm{CUT}}(K_n)$ and set $Y := yy^T$. Then $\mathrm{svec}(xx^T) = (Y_{0,ij})_{ij\in E_n}$. Moreover, $Y$ belongs to $\mathcal{E}_{d_n+1}$ and satisfies the equations

$$Y_{ik,jk} = Y_{0,ij} \quad \text{for all distinct } i, j, k \in V_n, \tag{3.3}$$

$$Y_{ij,hk} = Y_{ih,jk} = Y_{ik,jh} \quad \text{for all distinct } i, j, h, k \in V_n. \tag{3.4}$$

Anjos and Wolkowicz [2] used condition (3.3) for defining the following sets $\mathcal{F}_n$ and $F_n$:

$$\mathcal{F}_n := \{Y \in \mathcal{E}_{d_n+1} \mid Y \text{ satisfies (3.3)}\}, \qquad F_n := \{(Y_{0,ij})_{ij\in E_n} \mid Y \in \mathcal{F}_n\}.$$

The set $\mathcal{F}_n$ is obviously contained in the set $\mathcal{G}_n$ of matrices $Y \in \mathcal{E}_{d_n+1}$ satisfying

$$Y_{0,ij} = \frac{1}{n-2} \sum_{k\in V_n,\ k\neq i,j} Y_{ik,jk} \ (ij \in E_n);$$

the relaxation $\mathcal{G}_n$ is introduced in [2] as bidual (dual of the Lagrange dual) of some formulation of max-cut.

PROPOSITION 3.1 (see [2]). $\mathrm{CUT}(K_n) \subseteq F_n \subseteq \mathrm{MET}(K_n) \cap \mathrm{svec}(\mathcal{E}_n)$.

*Proof.* The inclusion $\mathrm{CUT}(K_n) \subseteq F_n$ has already been observed above. The inclusion $F_n \subseteq \mathrm{svec}(\mathcal{E}_n) \cap \mathrm{MET}(K_n)$ can be verified as follows. For $Y \in \mathcal{F}_n$, set $y := (Y_{0,ij})_{ij\in E_n}$ and $X := \mathrm{smat}(y)$. By the relation (3.3), the matrix $X$ coincides with the principal submatrix of $Y$ with row and column indices in the set $\{0, 12, \ldots, 1n\}$. Therefore $X \in \mathcal{E}_n$, and thus $y \in \mathrm{svec}(\mathcal{E}_n)$. In order to show the triangle inequality $y_{12} + y_{13} + y_{23} \geq -1$, consider the principal submatrix $Z$ of $Y$ indexed by the set $\{0, 12, 13, 23\}$ and let $\sigma$ denote the sum of the entries of $Z$. As $Z \succeq 0$, we have $\sigma \geq 0$, which implies that $y_{12} + y_{13} + y_{23} \geq -1$. The other triangle inequalities follow by the same argument after suitably flipping signs in $Z$. ∎

For $n \leq 4$, equality $\mathrm{MET}(K_n) = \mathrm{CUT}(K_n)$ holds. It is shown in [2] that both inclusions in Proposition 3.1 are strict for $n \geq 5$; for instance, the minimum of the linear objective function $\sum_{ij\in E_5} x_{ij}$ over $\mathrm{CUT}(K_5)$ is $-2$, while its minimum over $F_5$ is $-2.5$.

**New sdp relaxations based on the LS procedure.** If we apply the LS construction to the cut polytope $\mathrm{CUT}(G)$ starting with its linear relaxation by the metric polytope $\mathrm{MET}(G)$, we obtain the relaxations $N(\mathrm{MET}(G))$, $N_+(\mathrm{MET}(G))$, $N'(\mathrm{MET}(G))$, and $N'_+(\mathrm{MET}(G))$ satisfying the hierarchy (2.3).

As $G = (V_n, E)$ is a subgraph of the complete graph $K_n = (V_n, E_n)$, we have that $\mathrm{CUT}(G) = \pi_E(\mathrm{CUT}(K_n))$ and $\mathrm{MET}(G) = \pi_E(\mathrm{MET}(K_n))$, where $\pi_E : \mathbf{R}^{E_n} \longrightarrow \mathbf{R}^E$ denotes the projection onto the subspace indexed by the edge set of $G$. Let $\nu$ stand for one of the operators $N$, $N_+$, $N'$, or $N'_+$ and let $\mu$ denote the corresponding operator $M$, $M_+$, $M'$, $M'_+$ (i.e., $\mu = M$ if $\nu = N$, etc.). Taking projections at both sides of the inclusion $\mathrm{CUT}(K_n) \subseteq \nu(\mathrm{MET}(K_n))$, we obtain

$$\mathrm{CUT}(G) \subseteq \pi_E(\nu(\mathrm{MET}(K_n))).$$

LEMMA 3.2. $\pi_E(\nu(\mathrm{MET}(K_n))) \subseteq \nu(\mathrm{MET}(G))$.

*Proof.* Let $y \in \pi_E(\nu(\mathrm{MET}(K_n)))$. Then $(1, y) = \pi_E(Ye_0)$, where $Y \in \mu(\mathrm{MET}(K_n))$. Let $X$ denote the principal submatrix of $Y$ indexed by the set $\{0\} \cup E$. Then $X \in \mu(\mathrm{MET}(G))$. (This follows from the fact that each column of $X$ is the projection on $\mathbf{R}^{\{0\}\cup E}$ of the corresponding column of $Y$ and $\mathrm{MET}(G) = \pi_E(\mathrm{MET}(K_n))$.) Therefore $y = ((Xe_0)_f)_{f\in E}$ belongs to $\nu(\mathrm{MET}(G))$. ∎

Equality holds obviously in the inclusion of Lemma 3.2 when $G = K_n$. We do not know whether equality holds in general, i.e., whether the two operators $\nu$ and $\pi_E$ commute. Note that not every matrix $Y \in M(\mathrm{MET}(G))$ can be extended to a matrix of $M(\mathrm{MET}(K_n))$; for example, the matrix

$$Y := \begin{array}{c} \\ 0 \\ 12 \\ 23 \\ 34 \\ 14 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 12 & 23 & 34 & 14 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

belongs to $M(\mathrm{MET}(G))$, where $G$ is the circuit $(1, 2, 3, 4)$, but $Y$ cannot be extended to a matrix of $M(\mathrm{MET}(K_4))$ (because $Y_{12,23} \neq Y_{14,34}$; cf. Proposition 3.4(i) below). For simplicity in the notation, we set

$$\nu(G) := \pi_E(\nu(\mathrm{MET}(G))).$$

Iterates are defined in the obvious manner: $\nu^k(G) := \pi_E(\nu^k(\mathrm{MET}(K_n)))$. The inclusion from Lemma 3.2 will be extended to higher iterates in Corollary 4.13.

It seems preferable to work with the relaxation $\nu(G)$ rather than $\nu(\mathrm{MET}(G))$, as it provides a better relaxation for CUT$(G)$. Moreover, one can optimize a linear objective function over $\nu(G)$ in polynomial time for any graph and $\nu = N, \dots, N'_+$. In contrast, this is true for $\nu(\mathrm{MET}(G))$ for any graph $G$ if $\nu = N, N_+$ and, if $\nu = N', N'_+$, for any graph $G$ for which the list of circuit inequalities (3.1) (for chordless circuits) has a polynomial length (thus, for instance, if $G$ is a complete graph or more generally a chordal graph). One more attractive feature of the relaxation $\nu(G)$ is that the class of graphs $G$ for which $\mathrm{CUT}(G) = \nu(G)$ is well behaved; e.g., it is closed under taking deletion minors while it is not clear whether this property holds for the relaxation $\nu(\mathrm{MET}(G))$ (cf. section 4.4). On the other hand, it will be convenient to work with the relaxation $\nu(\mathrm{MET}(G))$ in order to establish results about valid inequalities (cf. section 4.2).

**Permutation and switching.** Every permutation $\sigma$ acts in a natural way on an $n \times n$ symmetric matrix $X$ and on a vector $x \in \mathbf{R}^{E_n}$, producing the vector $x^\sigma := (x_{\sigma(i)\sigma(j)})_{ij \in E_n}$. As $\sigma$ induces a permutation of $E_n$, it also acts on a matrix $Y \in \mathcal{S}_{d_n+1}$, producing the matrix $Y^\sigma \in \mathcal{S}_{d_n+1}$ defined by

$$(3.5) \qquad Y^\sigma_{0,ij} := Y_{0,\sigma(i)\sigma(j)}, \quad Y^\sigma_{ij,rs} := Y_{\sigma(i)\sigma(j),\sigma(r)\sigma(s)} \quad \text{for } ij, rs \in E_n.$$

Permutation preserves the cut polytope of the complete graph $K_n$ and all its relaxations considered in the paper.

Given a subset $S \subseteq V_n$ and $X \in \mathcal{S}_n$, let $X^S$ denote the matrix obtained from $X$ by changing the signs of its rows and columns indexed by $S$; in other words, one switches the signs of the entries of $X$ indexed by edges in the cut $\delta(S)$. Switching extends naturally to matrices $Y \in \mathcal{S}_{d_n+1}$ and produces $Y^{\delta(S)}$ obtained from $Y$ by changing signs of its rows and columns indexed by the set $\delta(S)$. Switching also applies to vectors $x \in \mathbf{R}^E$ ($E \subseteq E_n$): simply change the signs of the entries of $x$ indexed by the set $\delta(S) \cap E$.

Clearly, $X^S \in \mathrm{smat}(\mathrm{CUT}(K_n))$ (resp., $X^S \in \mathcal{E}_n$) if and only if $X \in \mathrm{smat}(\mathrm{CUT}(K_n))$ (resp., $X \in \mathcal{E}_n$). For $X, Y \in \mathcal{S}_n$, one has $\langle X, Y\rangle = \langle X^S, Y^S\rangle$. (Here $\langle X, Y\rangle = \sum_{i,j=1}^n x_{ij}y_{ij}$ denotes the usual inner product in $\mathcal{S}_n$.) Therefore, if an inequality $\langle A, X\rangle \geq \beta$ is valid for $\mathrm{smat}(\mathrm{CUT}(K_n))$, its switching $\langle A^S, X\rangle \geq \beta$ remains valid for $\mathrm{smat}(\mathrm{CUT}(K_n))$. Note that the classes of triangle inequalities and of circuit inequalities are closed under switching. Switching preserves all the relaxations of the cut polytope considered in the paper.

**3.3. Basic properties of the new relaxations.** The following is an easy but important property of the metric polytope that will be repeatedly used in this paper.

PROPOSITION 3.3. *If $y \in \mathrm{MET}(G)$ satisfies $y_{uv} = \epsilon$ for some edge $uv \in E$ and $\epsilon \in \{\pm 1\}$, then*

$$(3.6) \qquad y_{ui} = \epsilon y_{vi} \quad \textit{for every node } i \textit{ adjacent to both } u \textit{ and } v.$$

*Proof.* Apply the triangle inequalities (3.2) to the triple $uvi$. ∎

As a first application, we find that (3.3) and (3.4) are valid for $M(\mathrm{MET}(G))$ and $M'(\mathrm{MET}(G))$, respectively.

PROPOSITION 3.4.
- (i) *If $Y \in M(\mathrm{MET}(G))$, then $Y_{ik,jk} = Y_{0,ij}$ for all distinct pairwise adjacent $i, j, k \in V_n$.*
- (ii) *If $Y \in M'(\mathrm{MET}(G))$, then $Y_{ij,hk} = Y_{ih,jk} = Y_{ik,jh}$ for all distinct pairwise adjacent $i, j, h, k \in V_n$.*

*Proof.* (i) Let $1, 2, 3$ be pairwise adjacent nodes and $Y \in M(\mathrm{MET}(G))$. By assumption, the vector $y := Y(e_0 - e_{12})$ belongs to $\widetilde{\mathrm{MET}(G)}$. As $y_0 = -y_{12}$, we have from (3.6) that $y_{13} = -y_{23}$, which implies

$$Y_{0,13} + Y_{0,23} = Y_{12,13} + Y_{12,23}.$$

Similarly, using the fact that $Y(e_0 - e_{13}), Y(e_0 - e_{23}) \in \widetilde{\mathrm{MET}(G)}$, we obtain

$$Y_{0,12} + Y_{0,23} = Y_{13,12} + Y_{13,23} \quad \text{and} \quad Y_{0,12} + Y_{0,13} = Y_{23,12} + Y_{23,13}.$$

From this it follows that $Y_{0,12} = Y_{23,13}$, which shows (i).

(ii) Let $1, 2, 3, 4$ be pairwise adjacent nodes in $G$ and $Y \in M'(\mathrm{MET}(G))$. By assumption, the vector $y := Y(e_0 + e_{12} + e_{13} + e_{23})$ belongs to $\widetilde{\mathrm{MET}(G)}$ and thus satisfies the triangle inequalities $-y_{12} + y_{14} - y_{24} \geq -y_0$ and $-y_{12} - y_{14} + y_{24} \geq -y_0$. Using the above result (i), we find that $y_{12} = y_0$. Now (3.6) implies that $y_{14} = y_{24}$, which, using (i) again, yields $Y_{14,23} = Y_{13,24}$. ∎

Corollary 3.5. $N_+(K_n) \subseteq F_n$.

We will see later that $N_+(K_5) = \mathrm{CUT}(K_5)$; therefore, the inclusion $N_+(K_n) \subseteq F_n$ is strict for $n \geq 5$.

**4. The index of a graph.** The *$N$-index* $\eta_N(G)$ of a graph $G$ is defined as the smallest integer $k$ for which $\mathrm{CUT}(G) = N^k(\mathrm{MET}(G))$, and its *projected $N$-index* $\eta_N^\pi(G)$ is the smallest $k$ for which $\mathrm{CUT}(G) = N^k(G)$; the indexes $\eta_\nu$ and $\eta_\nu^\pi$ are defined analogously with respect to the other operators $\nu = N_+,\ N',$ or $N'_+$. Obviously, $\eta_\nu^\pi(G) \leq \eta_\nu(G)$. By Theorem 2.1, the $N$-index of $G$ is bounded by the number of edges of $G$; in section 4.1, we show some sharper upper bounds which, in fact, remain valid for the $N_0$-index since they are obtained using Lemma 2.2. In particular, we show that $\eta_N(G) \leq n - 4$ for a graph $G$ on $n \geq 4$ nodes, and in section 4.3 we prove the upper bound $n - 5$ for the $N'$-index of a graph on $n \geq 6$ nodes. In section 4.4, we study how the index of a graph behaves with respect to the graph operations of taking minors and clique sums. Section 4.2 contains some technical results needed for establishing the upper bounds on the $N'$-index and for proving the minor monotonicity of the index of a graph.

**4.1. Upper bounds for the $N$-index of a graph.** Let $G = (V_n, E)$ be a graph. We show here a linear upper bound in $O(n)$ for the $N$-index of $G$ (in place of the bound $|E|$). The basic idea is to use Lemma 2.2 and to reformulate the validity of an inequality $a^T x \geq \beta$ for $\mathrm{MET}(G) \cap \{x \mid x_{uv} = \epsilon\}$ in terms of the validity of a transformed inequality for $\mathrm{MET}(G/uv)$, the metric polytope of the contracted graph $G/uv$.

We need some definitions. For $u \in V_n$, $N_G(u)$ denotes the set of nodes adjacent to $u$ in $G$. Given an edge $uv \in E$, let $H := G/uv$ denote the graph obtained from $G$ by contracting $uv$; its node set is $V_n \setminus \{u, v\} \cup \{w\}$, where $w$ is the new node created by contraction of edge $uv$, and we denote by $F$ its edge set (multiple edges are erased). Clearly $F$ is in bijection with the subset $\hat{F} := \{\hat{f} \mid f \in F\}$ of $E$ where, for $f \in F$,

$$\begin{aligned} &\hat{f} := f \text{ if } w \notin f, \qquad \hat{f} := ui \text{ if } f = wi \text{ with } i \in N_G(u), \\ &\hat{f} := vi \text{ if } f = wi \text{ with } i \in N_G(v) \setminus N_G(u). \end{aligned} \tag{4.1}$$

Given $y \in \mathbf{R}^E$ satisfying $y_{uv} = \epsilon \in \{\pm 1\}$ and (3.6), its *$\epsilon$-restriction* $y^{F,\epsilon} \in \mathbf{R}^F$ is defined by

$$y_f^{F,\epsilon} := y_{\hat{f}} \text{ for all } f \in F \text{ except } y_{wi}^{F,\epsilon} := \epsilon y_{vi} \text{ for } i \in N_G(v) \setminus N_G(u). \tag{4.2}$$

Conversely, relation (4.2) permits us to define for any vector $x \in \mathbf{R}^F$ its $\epsilon$-*extension* $y \in \mathbf{R}^E$ in such a way that $y_{uv} = \epsilon$ and $y^{F,\epsilon} = x$. Note that for $\epsilon = -1$, $y^{F,-1}$ coincides with the 1-restriction of the vector $y'$ obtained from $y$ by switching the signs of its entries indexed by edges in the cut $\delta(v)$. Our objective is to show that membership of $y$ in some iterate $\nu^k(\mathrm{MET}(G))$ is equivalent to membership of its $\epsilon$-restriction in the corresponding iterate $\nu^k(\mathrm{MET}(G/uv))$ of the contracted graph ($\nu$ being any of the operators $N, \ldots, N'_+$). We treat here the case $k = 0$, and the general case will be treated in the next subsection. It will be convenient to use the following correspondence between the circuits of $G$ and those of $H = G/uv$:

$$\begin{array}{ll} & \text{To any circuit } C \text{ of } H \text{ there corresponds a circuit } C' \text{ of } G, \text{ where} \\ (4.3) & C' := \hat{C} \cup \{uv\} \text{ if } w \in C \text{ and its neighbors } a, b \text{ on } C \text{ satisfy} \\ & a \in N_G(u),\ b \in N_G(v) \setminus N_G(u), \text{ and } C' := \hat{C} \text{ otherwise} \end{array}$$

(setting $\hat{C} := \{\hat{f} \mid f \in C\}$, where $\hat{f}$ is defined by (4.1)).

LEMMA 4.1. *Let $x \in \mathbf{R}^F$ and let $y \in \mathbf{R}^E$ be its $\epsilon$-extension, where $\epsilon = \pm 1$. Then*
(i) $x \in \mathrm{MET}(G/uv) \Longleftrightarrow y \in \mathrm{MET}(G)$,
(ii) $x \in \mathrm{CUT}(G/uv) \Longleftrightarrow y \in \mathrm{CUT}(G)$.

*Proof.* (i) We let $\epsilon = 1$, as the case $\epsilon = -1$ can be derived from it by applying switching. Obviously, $y \in [-1,1]^E$ if and only if $x \in [-1,1]^F$. Suppose first that $y \in \mathrm{MET}(G)$; we show that $x \in \mathrm{MET}(H)$. For this let $C$ be a circuit in $H$ and let $D \subseteq C$ be a subset of odd cardinality; we show that $x(D) - x(C \setminus D) \geq 2 - |C|$. Let $\hat{D} := \{\hat{f} \mid f \in D\}$ and let $C'$ be the circuit in $G$ derived from $C$ as indicated in (4.3). Then, $x(D) - x(C \setminus D) = y(\hat{D}) - y(\hat{C} \setminus \hat{D})$. If $C' = \hat{C}$, then $y(\hat{D}) - y(\hat{C} \setminus \hat{D}) \geq 2 - |C'| = 2 - |C|$; if $C' = \hat{C} \cup \{uv\}$, then $y(\hat{D}) - y(\hat{C} \setminus \hat{D}) \geq 2 - |C'| + y_{uv} = 2 - |C|$, using the assumption $y_{uv} = 1$. We omit the proof for the reverse implication which is similar. Assertion (ii) follows from the fact that the extension/restriction operation maps the cut vectors of $H$ to cut vectors of $G$. ◻

Given $a \in \mathbf{R}^E$ and $\epsilon = \pm 1$, let $a_\epsilon \in \mathbf{R}^F$ be defined by

(4.4)
$$\begin{array}{l} (a_\epsilon)_{wi} := a_{ui} \text{ for } i \in N_G(u) \setminus N_G(v), \quad (a_\epsilon)_{wi} := \epsilon a_{vi} \text{ for } i \in N_G(v) \setminus N_G(u), \\ (a_\epsilon)_{wi} := a_{ui} + \epsilon a_{vi} \text{ for } i \in N_G(u) \cap N_G(v), \quad (a_\epsilon)_{ij} := a_{ij} \text{ for } ij \in E, i, j \neq u, v. \end{array}$$

It follows from these definitions that

$$a^T y = a_\epsilon^T x + \epsilon a_{uv} \quad \text{for } x \in \mathbf{R}^F \text{ and its } \epsilon\text{-extension } y \in \mathbf{R}^E. \tag{4.5}$$

LEMMA 4.2. *Let $a \in \mathbf{R}^E$, $\epsilon \in \{\pm 1\}$, $a_\epsilon \in \mathbf{R}^F$ as in (4.4), and $\beta \in \mathbf{R}$ be given. Then*

$$\begin{array}{l} a^T y \geq \beta \textit{ is valid for } \mathrm{MET}(G) \cap \{y \mid y_{uv} = \epsilon\} \\ \qquad \Longleftrightarrow a_\epsilon^T x \geq \beta - \epsilon\ a_{uv} \textit{ is valid for } \mathrm{MET}(G/uv), \\ a^T y \geq \beta \textit{ is valid for } \mathrm{CUT}(G) \cap \{y \mid y_{uv} = \epsilon\} \\ \qquad \Longleftrightarrow a_\epsilon^T x \geq \beta - \epsilon\ a_{uv} \textit{ is valid for } \mathrm{CUT}(G/uv). \end{array}$$

*Proof.* Apply Lemma 4.1 and (4.5). ◻

THEOREM 4.3. *Let $G$ be a graph and $e_1, \ldots, e_k$ be distinct edges in $G$. Then*

$$\mathrm{CUT}(G) = \mathrm{conv}(\mathrm{MET}(G) \cap \{x \mid x_{e_1}, \ldots, x_{e_k} = \pm 1\})$$

*if and only if the graph $G/\{e_1, \ldots, e_k\}$ has no $K_5$-minor.*

*Proof.* The proof is by induction on $k \geq 0$. The result holds for $k = 0$ since it is shown in [7] that $\mathrm{CUT}(G) = \mathrm{MET}(G)$ if and only if $G$ has no $K_5$-minor. Let $k \geq 1$ and suppose that the result from Theorem 4.3 holds for $k-1$; we show that it also holds for $k$. Applying the induction assumption to the graph $G/e_k$, we obtain that $\mathrm{CUT}(G/e_k) = \mathrm{conv}(\mathrm{MET}(G/e_k) \cap \{x \mid x_{e_1}, \ldots, x_{e_{k-1}} = \pm 1\})$ if and only if $G/\{e_1, \ldots, e_k\}$ has no $K_5$-minor. Therefore, it remains to show that the two statements

$$\mathrm{CUT}(G/e_k) = \mathrm{conv}(\mathrm{MET}(G/e_k) \cap \{x \mid x_{e_1}, \ldots, x_{e_{k-1}} = \pm 1\}),$$
$$\mathrm{CUT}(G) = \mathrm{conv}(\mathrm{MET}(G) \cap \{x \mid x_{e_1}, \ldots, x_{e_k} = \pm 1\})$$

are equivalent, which is a simple verification using Lemma 4.1. □

COROLLARY 4.4. *If a graph $G$ has a set of $k$ edges whose contraction produces a graph with no $K_5$-minor, then* $\mathrm{CUT}(G) = N_0^k(G) = N^k(G)$. *In particular,* $\mathrm{CUT}(G) = N^{n-4}(\mathrm{MET}(G))$ *if $G$ has $n \geq 4$ nodes.*

*Proof.* The first statement is a direct application of Theorem 4.3 and (2.6), (2.7). We now show that in a graph $G$ on $n$ nodes there exist at most $n-4$ edges whose contraction produces a graph with no $K_5$-minor. If $G$ is connected, let $T$ be a spanning tree in $G$ and let $u, v, w \in V_n$ for which $T' := T \backslash \{u, v, w\}$ is still a tree. (Such nodes can be easily found if $T$ is a path, and otherwise choose three leaves of $T$.) Then the graph obtained from $G$ by contracting the $n-4$ edges of $T'$ has no $K_5$-minor. If $G$ is not connected, apply the same reasoning to each connected component of $G$. □

Given an integer $r \geq 1$, let $\alpha_r(G)$ denote the maximum cardinality of a subset $S \subseteq V_n$ for which the induced subgraph $G[S]$ has no $K_{r+1}$ minor; thus $\alpha_1(G)$ is the stability number $\alpha(G)$ of $G$, and $\alpha_{r+1}(G) \geq \alpha_r(G) + 1$ if $\alpha_r(G) \leq n-1$. As a consequence of Corollary 4.4, we can show the following.

COROLLARY 4.5. *Let $r \in \{1, 2, 3\}$ and $G = (V_n, E)$ be a graph on $n$ nodes. Then,*

$$\eta_N^{\pi}(G) \leq \max(0, n - \alpha_r(G) + r - 4). \tag{4.6}$$

*If there exists a subset $S \subseteq V_n$ for which $G[S]$ has no $K_{r+1}$ minor, $G[V_n \setminus S]$ has at most $4-r$ connected components, and $|S| = \alpha_r(G)$, then*

$$\eta_N(G) \leq \max(0, n - \alpha_r(G) + r - 4). \tag{4.7}$$

*Proof.* We use the following observation: The graph $G^*$, obtained from $G[S]$ by adding to it $4-r$ pairwise adjacent nodes that are adjacent to all nodes of $S$, has no $K_5$-minor, and thus the same holds for any subgraph of $G^*$. We first verify that (4.7) holds. For this, suppose that $S \subseteq V_n$ with $|S| = \alpha_r(G)$, $G[S]$ has no $K_{r+1}$ minor, and $G[V_n \setminus S]$ has at most $4-r$ connected components; we show that a graph with no $K_5$-minor can be obtained from $G$ by contracting at most $k_r := \max(0, n - \alpha_r(G) + r - 4)$ edges. Indeed, using the assumption that $G[V_n \setminus S]$ has at most $4-r$ components, one can find at most $k_r$ edges in $G[V_n \setminus S]$ whose contraction transforms $G[V_n \setminus S]$ into a graph on at most $4-r$ nodes. We now verify (4.6). If $G[V_n \setminus S]$ has $t$ components, let $G'$ be the graph obtained from $G$ by adding $t-1$ edges between the components of $G[V_n \setminus S]$ so as to make $G'[V_n \setminus S]$ connected. We just saw that $\eta_N(G') \leq k_r$ and thus $\mathrm{CUT}(G') = N^{k_r}(G')$. By projecting out the added edges, we obtain that $\mathrm{CUT}(G) = N^{k_r}(G)$, that is, $\eta_N^{\pi}(G) \leq k_r$. □

In particular, the $N$-index of the graph $G^{\nabla}$, obtained from $G$ by adding a new node adjacent to all nodes of $G$, is at most $n - \alpha(G) - 2$. Some rationale for the similarity between this upper bound and the known upper bound $n - \alpha(G) - 1$ for the $N$-index of the stable set polytope of $G$ will be given in section 6.

Consider, for example, the complete bipartite graph $K_{4,5}$: then $\eta_N^\pi(K_{4,5}) = 1$ (by (4.6)) but the upper bound from (4.7) does not apply (since the complement of a maximum stable set induces a graph with four connected components). It would be interesting to determine whether $\eta_N(K_{4,5}) = 1$. If not, then $K_{4,5}$ would be an example of a graph for which the inclusion $N(G) \subseteq N(\mathrm{MET}(G))$ is strict; moreover, this would show that the $N$-index is not monotone with respect to deletion of edges, since the $N$-index of the graph obtained from $K_{4,5}$ by adding one edge is equal to 1.

As another consequence of Corollary 4.4, we have found a compact representation for the cut polytope of a graph having $k$ edges whose contraction produces a graph with no $K_5$-minor. Therefore, the max-cut problem can be solved in polynomial time for such graphs (for fixed $k$). This result can, however, be checked directly using a branching strategy. For instance, if $G/uv$ has no $K_5$-minor and one wishes to find the maximum weight $W$ of a cut in $G$ with respect to some weight function $a$, then $W = \max(W_1, W_{-1} + a(\delta_G(v)))$, where, for $\epsilon = \pm 1$, $W_\epsilon$ is the maximum weight of a cut in $G/uv$ with respect to the weight function $a_\epsilon$ (defined as in (4.4)). (This idea is also present, e.g., in [23].)

**4.2. Validity for the new relaxations via contraction.** We saw in Lemma 4.2 that the validity of an inequality $a^T x \geq \beta$ for $\mathrm{MET}(G) \cap \{x \mid x_{uv} = \epsilon\}$ can be reformulated in terms of the validity of the transformed inequality $a_\epsilon^T x \geq \beta - \epsilon\, a_{uv}$ for $\mathrm{MET}(G/uv)$. We here extend this result for any iterate $\nu^k(\mathrm{MET}(G))$, where $\nu = N, \ldots, N'_+$ and $k \geq 1$. For this we need to extend the notions of $\epsilon$-extension and restriction to matrices. We begin with an application of (3.6) to matrices in $M(\mathrm{MET}(G))$.

PROPOSITION 4.6. *Let $Y \in M(\mathrm{MET}(G))$ and assume that $Y_{0,uv} = \epsilon Y_{0,0}$ for some edge $uv \in E$ and $\epsilon = \pm 1$. Then $Y$ satisfies*

$$Ye_0 = \epsilon Y e_{uv},\ Ye_{ui} = \epsilon Y e_{vi} \quad \textit{for every node } i \in N_G(u) \cap N_G(v); \tag{4.8}$$

*that is, $Y$ has the following block decomposition:*

$$Y = \begin{array}{c} \\ I \\ K \\ J \end{array} \!\!\begin{array}{c} \begin{array}{ccc} I & K & J \end{array} \\ \begin{pmatrix} A & B^T & \epsilon A \\ B & C & \epsilon B \\ \epsilon A & \epsilon B^T & A \end{pmatrix} \end{array}, \tag{4.9}$$

*setting $I := \{0\} \cup \{ui \mid i \in N_G(u) \cap N_G(v)\}$, $J := \{uv\} \cup \{vi \mid i \in N_G(u) \cap N_G(v)\}$, and $K := E \setminus (I \cup J)$.*

*Proof.* As $y := Y(e_0 - \epsilon e_{uv}) \in \widetilde{\mathrm{MET}(G)}$ with $y_0 = 0$, we have that $-y_0 \leq y_f \leq y_0$, which yields $y_f = 0$ for all $f \in E$, and thus $Ye_0 = \epsilon Y e_{uv}$. Let $i$ be a node adjacent to both $u$ and $v$. As $x := Ye_0 \in \widetilde{\mathrm{MET}(G)}$ with $x_0 = \epsilon x_{uv}$, we have from (3.6) that $x_{ui} = \epsilon x_{vi}$, i.e., $Y_{0,ui} = \epsilon Y_{0,vi}$. Given $f \in E$, set $z := Y(e_0 - e_f)$; then $z \in \widetilde{\mathrm{MET}(G)}$ and $z_0 = \epsilon z_{uv}$ by the above. By (3.6) this implies that $z_{ui} = \epsilon z_{vi}$ and thus $Y_{ui,f} = \epsilon Y_{vi,f}$. This shows that $Ye_{ui} = \epsilon Y e_{vi}$. $\square$

Let $Y$ be a symmetric matrix indexed by $\{0\} \cup E$ and satisfying (4.8) for some $\epsilon = \pm 1$ and $uv \in E$; then, $Y$ has the form (4.9). We define its *$\epsilon$-restriction* $Y^{F,\epsilon}$ in the following manner: If $\epsilon = 1$, then $Y^{F,1}$ is the principal submatrix $\left(\begin{smallmatrix} A & B^T \\ B & C \end{smallmatrix}\right)$ of $Y$ indexed by the subset $\{0\} \cup \hat{F} = I \cup K$. If $\epsilon = -1$, let $Y'$ be the matrix obtained from $Y$ by switching the signs of its rows/columns indexed by edges in the cut $\delta(v)$; then $Y^{F,-1}$ is the principal submatrix of $Y'$ indexed by $I \cup K$. As $F$ is in bijection with $\hat{F}$ we can view $Y^{F,\epsilon}$ as being indexed by $\{0\} \cup F$. Conversely, one can define the *$\epsilon$-extension* $Y$ of a matrix $X$ indexed by $\{0\} \cup F$ in such a way that $Y_{0,0} = \epsilon Y_{0,uv}$ and $Y^{F,\epsilon} = X$. Clearly,

$$Y \succeq 0 \Longleftrightarrow Y^{F,\epsilon} \succeq 0. \tag{4.10}$$

Recall that the dual cone of the cone $\widetilde{\mathrm{MET}(G)}$ is spanned by the vectors $e_0 \pm e_f$ ($f \in E$) and

$$\xi^{C,D} := (|C| - 2)e_0 + \sum_{f \in D} e_f - \sum_{f \in C \setminus D} e_f$$

for all chordless circuits $C$ of $G$ and all odd subsets $D \subseteq C$.

Proposition 4.7. *Let $k \geq 0$ be an integer, let $Y$ be a symmetric matrix indexed by $\{0\} \cup E$ satisfying (4.8) for some $\epsilon = \pm 1$ and $uv \in E$, and let $Y^{F,\epsilon}$ be its $\epsilon$-restriction. Let $\nu$ be one of the operators $N, \ldots, N'_+$ and $\mu$ the corresponding operator from $M, \ldots, M'_+$. Then*

$$Y \in \mu(\nu^k(\mathrm{MET}(G))) \Longleftrightarrow Y^{F,\epsilon} \in \mu(\nu^k(\mathrm{MET}(G/uv))).$$

*Proof.* Let $\epsilon = 1$, as the case $\epsilon = -1$ can be derived from it by applying switching. In view of relation (4.10) it suffices to show the result for the operators $\nu = N,\ N'$. The proof is by induction on $k \geq 0$ and uses Lemma 4.1 together with the following observation: For $f \in F$, the $\hat{f}$th column of $Y$ is the 1-extension of the corresponding $f$th column of $Y^{F,1}$, while the remaining columns of $Y$ are duplicates of some of those. We first consider the case $k = 0$. The statement for the case $\nu = N$ follows as a direct application of the above observation. Suppose now that $Y \in M'(\mathrm{MET}(G))$; we show that $Y^{F,1} \in M'(\mathrm{MET}(H))$. For this let $C$ be a circuit in $H$ and let $D \subseteq C$ with an odd cardinality; we show that $x := Y^{F,1}\xi^{C,D} \in \widetilde{\mathrm{MET}(H)}$. Set $\hat{D} := \{\hat{f} \mid f \in D\}$ and let $C'$ be the circuit in $G$ obtained from $C$ as indicated in (4.3). By assumption, $y := Y\xi^{C',\hat{D}} \in \widetilde{\mathrm{MET}(G)}$ and $y_0 = y_{uv}$. Thus, by Lemma 4.1, its 1-restriction $y^{F,1}$ belongs to $\widetilde{\mathrm{MET}(H)}$. It suffices now to observe that $Y^{F,1}\xi^{C,D}$ coincides with $y^{F,1}$ (using the fact that $Ye_0 = Ye_{uv}$ in the case in which $C' = \hat{C} \cup \{uv\}$). The proof for the implication $Y^{F,1} \in M(\mathrm{MET}(H)) \Longrightarrow Y \in M(\mathrm{MET}(G))$ is analogous and thus omitted.

Let $k \geq 1$ and suppose that the result from Proposition 4.7 holds for $k-1$; we show that it holds for $k$. We treat only the case when $\nu = N$, as the proof is analogous for $N'$. Suppose first that $Y \in M(N^k(\mathrm{MET}(G)))$; we show that $Y^{F,1} \in M(N^k(\mathrm{MET}(H)))$. For this, let $f \in F$, $\epsilon' = \pm 1$, and $x := Y^{F,1}(e_0 + \epsilon' e_f)$; we show that $x \in N^k(\widetilde{\mathrm{MET}(H)})$. By assumption, the vector $y := Y(e_0 + \epsilon' e_{\hat{f}})$ belongs to $N^k(\widetilde{\mathrm{MET}(G)})$ and satisfies $y_0 = y_{uv}$. Hence there exists a matrix $A \in M(N^{k-1}(\mathrm{MET}(G)))$ such that $y = Ae_0$. As $A_{0,0} = A_{0,uv}$, $A$ satisfies (4.8) by Proposition 4.6, and we deduce from the induction assumption that $A^{F,1} \in M(N^{k-1}(\mathrm{MET}(H)))$. Thus $y^{F,1} = A^{F,1}e_0$ belongs to $N^k(\widetilde{\mathrm{MET}(H)})$. The result now follows since $x = y^{F,1}$. We omit the details of the proof for the converse implication: $Y^{F,1} \in M(N^k(\mathrm{MET}(H))) \Longrightarrow Y \in M(N^k(\mathrm{MET}(G)))$. ◻

Corollary 4.8. *Let $k \geq 0$ be an integer, let $y \in \mathbf{R}^{\{0\} \cup E}$ satisfying $y_{uv} = \epsilon y_0$ and (3.6) for some $\epsilon = \pm 1$ and $uv \in E$, and let $y^{F,\epsilon} \in \mathbf{R}^{\{0\} \cup F}$ be its $\epsilon$-restriction* *defined by* (4.2). *Let* $\nu$ *be one of the operators* $N,\dots,N'_+$. *Then*

$$y \in \nu^k(\widetilde{\mathrm{MET}(G)}) \Longleftrightarrow y^{F,\epsilon} \in \nu^k(\widetilde{\mathrm{MET}(G/uv)}).$$

*Proof.* For $k = 0$ the result holds by Lemma 4.1, and for $k \geq 1$ it follows from Proposition 4.7. ∎

Relation (4.5) together with Corollary 4.8 imply the following.

PROPOSITION 4.9. *Let* $k \geq 0$ *be an integer,* $\epsilon = \pm 1$, $uv \in E$, $a \in \mathbf{R}^E$, $\beta \in \mathbf{R}$, *and* $\nu$ *one of* $N,\dots,N'_+$. *The inequality* $a^T x \geq \beta$ *is valid for* $\nu^k(\mathrm{MET}(G)) \cap \{x \mid x_{uv} = \epsilon\}$ *if and only if the inequality* $a_\epsilon^T x \geq \beta - \epsilon a_{uv}$ *is valid for* $\nu^k(\mathrm{MET}(G/uv))$.

Let us say that the inequality $a_\epsilon^T x \geq \beta - \epsilon a_{uv}$ is obtained from the inequality $a^T x \geq \beta$ by *collapsing* ($\epsilon = 1$) or *anticollapsing* ($\epsilon = -1$) nodes $u$ and $v$. Recall that anticollapsing amounts to first switching the signs of entries of $a$ indexed by the cut $\delta(v)$ and then collapsing $u$ and $v$. The following reformulations of Lemmas 2.2 and 2.3 will be used later in the paper.

PROPOSITION 4.10. *Let* $\nu = N,\dots,N'_+$. *The inequality* $a^T x \geq \beta$ *is valid for* $\nu^{k+1}(\mathrm{MET}(G))$ *if there is an edge* $uv \in E$ *for which both inequalities obtained from it by collapsing and anticollapsing nodes* $u$ *and* $v$ *are valid for* $\nu^k(\mathrm{MET}(G/uv))$.

PROPOSITION 4.11. *Suppose that* $a_f \geq 0$ *for all* $f \in E$ *and* $\beta \leq 0$. *The inequality* $a^T x \geq \beta$ *is valid for* $N_+^{k+1}(\mathrm{MET}(G))$ *if, for every edge* $uv \in E$ *for which* $a_{uv} > 0$, *the inequality obtained from* $a^T x \geq \beta$ *by anticollapsing nodes* $u$ *and* $v$ *is valid for* $N_+^k(\mathrm{MET}(G/uv)$.

It is obvious that $\mathrm{CUT}(K_n)$ is equal to the projection of $\mathrm{CUT}(K_{n+1})$ on the subspace $\mathbf{R}^{E_n}$ indexed by the edge set of $K_n$; similarly for $\mathrm{MET}(K_n)$. The same can be verified for $F_n$ and for any iterate $\nu^k(\mathrm{MET}(K_n))$. (In the latter case, use Corollaries 4.8 and 4.13.)

PROPOSITION 4.12. *Let* $G = (V_n, E)$ *be a graph,* $F \subseteq E$, *and* $H := (V_n, F)$ *the corresponding subgraph of* $G$. *Let* $\nu$ *be one of the operators* $N,\dots,N'_+$, $\mu$ *the associated operator from* $M,\dots,M'_+$, *and let* $k \geq 0$ *be an integer. If* $Y \in \mu(\nu^k(\mathrm{MET}(G)))$, *then its principal submatrix* $X$ *indexed by the set* $\{0\} \cup F$ *belongs to* $\mu(\nu^k(\mathrm{MET}(H)))$.

*Proof.* It suffices to consider the case when $\nu = N, N'$ as $Y \succeq 0$ implies $X \succeq 0$. We use the following facts in the proof: $\mathrm{MET}(H)$ is the projection on $\mathbf{R}^F$ of $\mathrm{MET}(G)$; if $\xi$ belongs to the dual cone of $\mathrm{MET}(H)$, then its extension $\xi' := (\xi, 0, \dots, 0) \in \mathbf{R}^E$ belongs to the dual of $\mathrm{MET}(G)$; and $X\xi$ is the projection on $\mathbf{R}^{\{0\}\cup F}$ of $Y\xi'$.

The proof is by induction for $k \geq 0$. The case $k = 0$ is obvious in view of the above observations. Let $k \geq 1$ and suppose that the result holds for $k-1$. Assume that $Y \in \mu(\nu^k(\mathrm{MET}(G)))$; we show that $X \in \mu(\nu^k(\mathrm{MET}(H)))$. For this, consider $\xi \in \mathrm{MET}(H)^*$ and its extension $\xi' \in \mathrm{MET}(G)^*$. We show that $x := X\xi \in \nu^k(\widetilde{\mathrm{MET}(H)})$. By assumption, $y := Y\xi' \in \nu^k(\widetilde{\mathrm{MET}(G)})$. Therefore, $y = Ae_0$ for some $A \in \mu(\nu^{k-1}(\mathrm{MET}(G)))$. Using the induction assumption, the principal submatrix $B$ of $A$ indexed by $\{0\}\cup F$ belongs to $\mu(\nu^{k-1}(\mathrm{MET}(H)))$, and thus $Be_0 \in \nu^k(\widetilde{\mathrm{MET}(H)})$. Note now that $x$, being the projection on $\mathbf{R}^{\{0\}\cup F}$ of $y$, is equal to $Be_0$. This shows the result; indeed, for $\nu = N$, restrict the above argument to $\xi$ of the form $e_0 \pm e_f$ ($f \in F$). ∎

COROLLARY 4.13. *Let* $G = (V_n, E)$ *be a graph,* $H = (V_n, F)$ *a subgraph of* $G$, $\pi_F$ *the projection from* $\mathbf{R}^E$ *onto* $\mathbf{R}^F$, $k \geq 0$ *an integer, and* $\nu = N,\dots,N'_+$. *Then* $\pi_F(\nu^k(\mathrm{MET}(G))) \subseteq \nu^k(\mathrm{MET}(H))$. *In particular,* $\mathrm{CUT}(G) \subseteq \nu^k(G) \subseteq \nu^k(\mathrm{MET}(G))$.

**4.3. Upper bound for the $N'$-index of a graph.** We showed in section 4.1 the upper bound $n-4$ for the $N$-index of a graph on $n \geq 4$ nodes. We will see in section 5 that

$$\eta_N(K_6) = \eta_{N_+}(K_6) = 2, \quad \eta_{N'}(K_6) = 1, \quad \eta_{N'_+}(K_n) \geq 2 \quad \text{for } n \geq 7.$$

Thus $\eta_{N'}(G) \leq 1$ for a graph on $n \leq 6$ nodes. Based on this fact, one can show the slightly better upper bound $n-5$ for the $N'$-index of a graph on $n \geq 6$ nodes.

THEOREM 4.14. *Let $\nu$ be one of $N, \ldots, N'_+$ and let $h, k \geq 0$ be integers. If there exist $k$ edges $e_1, \ldots, e_k$ in $G$ for which* $\mathrm{CUT}(G/\{e_1, \ldots, e_k\}) = \nu^h(\mathrm{MET}(G/\{e_1, \ldots, e_k\}))$, *then* $\mathrm{CUT}(G) = \nu^{h+k}(\mathrm{MET}(G))$.

*Proof.* The proof is by induction for $k \geq 0$. The result holds trivially for $k = 0$. Let $k \geq 1$ and suppose that the result holds for $k-1$. Let $a^T x \geq \beta$ be an inequality valid for $\mathrm{CUT}(G)$. By Lemma 4.2, the inequalities obtained from it by collapsing and anticollapsing the end nodes of $e_k$ are valid for $\mathrm{CUT}(G/e_k)$, which is equal to $\nu^{h+k-1}(\mathrm{MET}(G/e_k))$ by the induction assumption. By Proposition 4.10, this implies that $a^T x \geq \beta$ is valid for $\nu^{h+k}(\mathrm{MET}(G))$. □

COROLLARY 4.15. *The $N'$-index of a graph on $n \geq 6$ nodes is at most $n-5$.*

*Proof.* If $G$ is connected, one can find a set $F$ of $n-6$ edges whose contraction produces a graph on six nodes; as $\mathrm{CUT}(G/F) = N'(\mathrm{MET}(G/F))$, we deduce from Theorem 4.14 that $\mathrm{CUT}(G) = (N')^{|F|+1}(\mathrm{MET}(G)) = (N')^{n-5}(\mathrm{MET}(G))$. If $G$ is not connected, then, by the above, $\mathrm{CUT}(G_i) = (N')^{n-5}(\mathrm{MET}(G_i))$ for each connected component $G_i$ of $G$; using Proposition 4.17, this implies that $\mathrm{CUT}(G) = (N')^{n-5}(\mathrm{MET}(G))$. □

**4.4. Behavior of the index under taking graph minors and clique sums.** An important motivation for the study of the LS relaxations is that one can solve the max-cut problem in polynomial time over the class of graphs having bounded $\nu$-index ($\nu = N, N_+$) or bounded projected $\nu$-index ($\nu = N, \ldots, N'_+$). It is therefore of great interest to understand which graphs have small index, e.g., $\leq 1$. This is, however, a difficult question. As a first step, we study here whether these graph classes are closed under taking minors and clique sums.

Let $G = (V_n, E)$ be a graph with edge set $E \subseteq E_n$. Given an edge $e = uv \in E$, recall that $G \backslash e$ is the graph obtained from $G$ by *deleting* edge $e$, and $G/e$ is the graph obtained from $G$ by contracting $e$; a *minor* of $G$ is then a graph obtained from $G$ by a sequence of deletions and/or contractions. Let $G_i(V_i, E_i)$ $(i = 1, 2)$ be two graphs such that the set $V_1 \cap V_2$ induces a clique in both $G_1$ and $G_2$. Then the graph $G := (V_1 \cup V_2, E_1 \cup E_2)$ is called the *clique $t$-sum* of $G_1$ and $G_2$, where $t := |V_1 \cap V_2|$.

PROPOSITION 4.16. *For $\nu = N, \ldots, N'_+$, $\eta^\pi_\nu(H) \leq \eta^\pi_\nu(G)$ if $H$ is a minor of $G$, and $\eta_\nu(H) \leq \eta_\nu(G)$ if $H$ is a contraction minor of $G$.*

*Proof.* Monotonicity of the projected index under taking deletion minors follows directly from the definitions. Suppose now that $H$ is a contraction minor of $G$; say, $G = (V_n, E)$, $e := uv \in E$, and $H = G/uv = (V_n \setminus \{u, v\} \cup \{w\}, F)$. We show that $\eta_\nu(H) \leq \eta_\nu(G)$. For this, suppose that $\widetilde{\mathrm{CUT}}(G) = \nu^k(\mathrm{MET}(G))$; we show that $\mathrm{CUT}(H) = \nu^k(\mathrm{MET}(H))$. Let $x \in \nu^k(\widetilde{\mathrm{MET}}(H))$; then $x = Xe_0$ for some $X \in \mu(\nu^{k-1}(\mathrm{MET}(H)))$. By Proposition 4.7, the 1-extension $Y$ of $X$ belongs to $\mu(\nu^{k-1}(\mathrm{MET}(G)))$, and $Y_{0,0} = Y_{0,uv}$. Thus $y := Ye_0 \in \nu^k(\widetilde{\mathrm{MET}}(G)) = \widetilde{\mathrm{CUT}}(G)$. By Lemma 4.1(ii), this implies that $x = y^{F,1} \in \widetilde{\mathrm{CUT}}(H)$.

We now show that $\eta^\pi_\nu(H) \leq \eta^\pi_N(G)$. Suppose that $\mathrm{CUT}(G) = \nu^k(G)$; we show that $\mathrm{CUT}(H) = \nu^k(H)$. For this, let $x \in \nu^k(H)$. Thus $x = \pi_F(Xe_0)$ for some $X \in \mu(\nu^{k-1}(\mathrm{MET}(K_{n-1})))$ with $X_{0,0} = 1$. Viewing $K_{n-1}$ as $K_n/uv$, we have from Proposition 4.7 that the 1-extension $Y$ of $X$ belongs to $\mu(\nu^{k-1}(\mathrm{MET}(K_n)))$, and

$Y_{0,0} = Y_{0,uv} = 1$. Thus $y := \pi_E(Ye_0) \in \nu^k(G) = \mathrm{CUT}(G)$, implying $x = y^{F,1} \in \mathrm{CUT}(H)$. $\square$

PROPOSITION 4.17. *Let $G$ be the clique $t$-sum of two graphs $G_1$ and $G_2$, where $t = 0, 1, 2, 3$. Then $\eta^\pi_\nu(G) \leq \max(\eta^\pi_\nu(G_1), \eta^\pi_\nu(G_2))$ and $\eta_\nu(G) \leq \max(\eta_\nu(G_1), \eta_\nu(G_2))$.*

*Proof.* Let $G = (V_n, E)$ be the clique $t$-sum of two graphs $G_i = (V_i, E_i)$ for $i = 1, 2$ with $t \leq 3$; thus $V_n = V_1 \cup V_2$ and $E = E_1 \cup E_2$. We use the following fact shown in [4]: Given $y \in \mathbf{R}^{E_1 \cup E_2}$ and its projections $y_i := (y(e))_{e \in E_i}$ for $i = 1, 2$, we then have $y \in \mathrm{CUT}(G) \iff y_i \in \mathrm{CUT}(G_i)$ for $i = 1, 2$. Let $k \geq 0$ be an integer. Suppose first that $\mathrm{CUT}(G_i) = \nu^k(G_i)$ for $i = 1, 2$ and let $y \in \nu^k(G)$; we show that $y \in \mathrm{CUT}(G)$. For this it suffices to show that $y_i \in \nu^k(G_i)$ for $i = 1, 2$. There exists $Y \in \mu(\nu^{k-1}(\mathrm{MET}(K_n)))$ such that $y = \pi_E(Ye_0)$. By Proposition 4.12, the principal submatrix $Y_i$ of $Y$ indexed by $\{0\} \cup F_i$, where $F_i$ is the edge set of the complete graph on $V_i$, belongs to $\mu(\nu^{k-1}(\mathrm{MET}(K_{V_i})))$. Thus $y_i = \pi_{E_i}(Y_i e_0) \in \nu^k(G_i)$ for $i = 1, 2$.

Suppose now that $\mathrm{CUT}(G_i) = \nu^k(\mathrm{MET}(G_i))$ for $i = 1, 2$ and let $y \in \nu^k(\mathrm{MET}(G))$; we show that $y_i \in \nu^k(\mathrm{MET}(G_i))$. There exists $Y \in \mu(\nu^{k-1}(\mathrm{MET}(G)))$ such that $y = Ye_0$. By Proposition 4.12, the principal submatrix $Y_i$ of $Y$ indexed by $\{0\} \cup E_i$ belongs to $\mu(\nu^{k-1}(\mathrm{MET}(G_i)))$, and thus $y_i = Y_i e_0 \in \nu^k(\mathrm{MET}(G_i))$. $\square$

As the class of graphs $G$ with $\eta^\pi_\nu(G) \leq 1$ is closed under taking minors, we know from the theory of Robertson and Seymour [26] that there exists a finite list of *minimal forbidden minors* characterizing membership in that class; that is, $\eta^\pi_\nu(G) \leq 1$ if and only if $G$ does not contain any member of the list as a minor. For $\nu = N, N_+$, $\eta^\pi_\nu(K_6 \backslash e) = 1$ while $\eta^\pi_\nu(K_6) = 2$; hence the graph $K_6$ is a minimal forbidden minor for both properties $\eta^\pi_N(G) \leq 1$ and $\eta^\pi_{N_+}(G) \leq 1$. There are necessarily other minimal forbidden minors. Indeed, the max-cut problem is known to be NP-hard for the class of graphs having no $K_6$-minor (in fact, also for the class of apex graphs; that is, the graphs having a node whose deletion results in a planar graph) (cf. [5]).

Let $G_0$ denote the graph obtained from $K_7$ by removing a matching of size 3. We have verified that, for a graph $G$ on 7 nodes distinct from $G_0$, $\eta^\pi_N(G) \leq 1$ if and only if it does not contain $K_6$ as a minor. It would be interesting to compute $\eta^\pi_N(G_0)$; if its value is $\geq 2$, then $G_0$ is another minimal forbidden minor.

In view of Propositions 4.16 and 4.17, the property $\nu^\pi_\nu(G) \leq 1$ is preserved under the $\Delta Y$ operation (which consists of replacing a triangle by a claw $K_{1,3}$). However, it is not preserved under the converse $Y\Delta$ operation. Indeed, if $G$ is the graph obtained from $K_6$ by applying one $\Delta Y$ transformation, then $\eta_N(G) = \eta^\pi_N(G) = 1$ (by (4.7)) while $\eta_N(K_6) = 2$. We have verified that all the graphs in the Petersen family (consisting of the graphs that can be obtained from $K_6$ by $Y\Delta$ and $\Delta Y$ transformations) except $K_6$ have projected $N$-index equal to 1.

## 5. Valid inequalities for the new relaxations.

We saw above that the $N$-index of $K_n$ is at most $n - 4$, with equality for $n = 4, 5$. We conjecture that equality holds for any $n$. In order to show this conjecture, one has to find an inequality valid for $\mathrm{CUT}(K_n)$ which is not valid for $N^{n-5}(K_n)$. A possible candidate is the inequality

$$\sum_{1 \leq i < j \leq n} x_{ij} \geq -\left\lfloor \frac{n}{2} \right\rfloor. \tag{5.1}$$

Note that (5.1) is not valid for $N^{n-5}(K_n)$ if and only if there exists $a < -\frac{1}{n}$ ($n$ odd) or $a < -\frac{1}{n-1}$ ($n$ even) for which $(a, \dots, a) \in N^{n-5}(K_n)$. We will show in Proposition 5.3 that inequality (5.1) is not valid for $N^{n-5}(K_n)$ if $n = 7$; we conjecture that this remains true for any odd $n$. However, for $n$ even, inequality (5.1) *is* valid for $N^{n-5}(K_n)$. (Indeed, for $n$ even, inequality (5.1) follows by summation from the inequalities (5.1) for $n-1$; as the latter inequalities are valid for $N^{n-5}(K_{n-1})$, we deduce that (5.1) too is valid for $N^{n-5}(K_n)$.) Therefore, for $n$ even, one should use some more complicated inequality. We will show in Proposition 5.2 that the inequality

$$(n-4)\sum_{i=2}^{n} x_{1i} + \sum_{2\le i<j\le n} x_{ij} \ge -\frac{1}{2}(n^2-7n+14) \tag{5.2}$$

is not valid for $N^{n-5}(K_n)$ if $n=6$, and we conjecture that this holds for any even $n \ge 6$. The inequalities (5.1) and (5.2) are special instances of gap inequalities that we now introduce.

**5.1. Gap inequalities.** Given an integer vector $b=(b_1,\dots,b_n)\in \mathbf{Z}^n$, its *gap* $\gamma(b)$ is defined as

$$\gamma(b) := \min_{S\subseteq V_n} \left| \sum_{i\in S} b_i - \sum_{i\in V_n\setminus S} b_i \right|,$$

and the inequality

$$\sum_{1\le i<j\le n} b_i b_j x_{ij} \ge \frac{1}{2}\left(\gamma(b)^2 - \sum_{i=1}^{n} b_i^2\right) \tag{5.3}$$

in the variable $x\in \mathbf{R}^{E_n}$ is called the *gap inequality* associated with $b$. The analogue of (5.3) in the matrix variable $X\in \mathcal{S}_n^1$ takes the simpler form

$$b^T X b \ge \gamma(b)^2. \tag{5.4}$$

Inequality (5.4) is obviously valid for any cut matrix $xx^T$ ($x\in\{\pm1\}^n$); that is, inequality (5.3) is valid for the cut polytope $\mathrm{CUT}(K_n)$. The gap inequalities are introduced in [19] as a generalization of negative-type inequalities (case $\gamma(b)=0$, [27]) and hypermetric inequalities (case $\gamma(b)=1$, [9]); see [10] for a detailed survey.

The class of gap inequalities is closed under switching; indeed, switching the gap inequality for $b\in\mathbf{Z}^n$ along the cut $\delta(S)$ amounts to flipping the signs of the components of $b$ on $S$. (Anti)collapsing specializes to gap inequalities in the following manner. Given $b=(b_1,b_2,\dots,b_n)\in\mathbf{Z}^n$, set $b' := (b_1+b_2,b_3,\dots,b_n)\in\mathbf{Z}^{n-1}$ and $b'' := (b_1-b_2,b_3,\dots,b_n)\in\mathbf{Z}^{n-1}$. As $\gamma(b'),\gamma(b'')\ge\gamma(b)$, we have that $\frac{1}{2}(\gamma(b')^2-\sum_{i=2}^n b_i'^2) \ge \frac{1}{2}(\gamma(b)^2-\sum_{i=1}^n b_i^2)-b_1b_2$ and $\frac{1}{2}(\gamma(b'')^2-\sum_{i=2}^n b_i''^2) \ge \frac{1}{2}(\gamma(b)^2-\sum_{i=1}^n b_i^2)+b_1b_2$. Therefore, if the gap inequality for $b'$ (resp., $b''$) is valid for $\nu^k(K_{n-1})$, then the inequality obtained from the gap inequality for $b$ by collapsing (resp., anticollapsing) nodes 1 and 2 is valid for $\nu^k(K_{n-1})$. This fact will be useful when applying Propositions 4.10 and 4.11 to gap inequalities.

The negative-type inequalities do not induce facets of $\mathrm{CUT}(K_n)$ (since they are implied by the hypermetric inequalities); moreover, they are implied by the condition $X\succeq 0$. In fact, no gap inequality for $b\in\mathbf{Z}^n$ with gap $\gamma(b)\ge 2$ and inducing a facet of the cut polytope is known (cf. [19]). On the other hand, hypermetric inequalities include large classes of facets for the cut polytope. This is the case, for instance, for the following vectors $b$:

$$b=(1,\dots,1)\in\mathbf{Z}^n \text{ for } n \text{ odd}, \qquad b=(n-4,1,\dots,1)\in\mathbf{Z}^n \text{ for } n\ge 4.$$

The hypermetric inequality for $b = (1,1,1)$ is a triangle inequality (occurring in case (5.1) for $n = 3$, and in case (5.2) for $n = 4$); the hypermetric inequality for $b = (1,1,1,1,1)$ is called the *pentagonal inequality* (occurring in cases (5.1) and (5.2) for $n = 5$). Moreover, for $n \le 6$, all facets of $\mathrm{CUT}(K_n)$ are induced by hypermetric inequalities. More precisely, $\mathrm{CUT}(K_n) = \mathrm{MET}(K_n)$ for $n \le 4$; up to switching, all facets of $\mathrm{CUT}(K_5)$ arise from the triangle inequality and the pentagonal inequality; up to switching and permutation, all facets of $\mathrm{CUT}(K_6)$ arise from the triangle inequality, the pentagonal inequality, and the hypermetric inequality for $b = (2,1,1,1,1,1)$ (case $n = 6$ of (5.2)).

**5.2. Valid hypermetric inequalities for the new relaxations.** By construction, the triangle inequalities are valid for $N(K_n)$. As $\mathrm{CUT}(K_5) = N(K_5)$ (by Corollary 4.4), the pentagonal inequality (that is, the gap inequality for $b = (1,1,1,1,1,0,\ldots,0)$) is also valid for $N(K_n)$. We now examine the validity of the gap inequalities for $(1,\ldots,1) \in \mathbf{Z}^n$ ($n \ge 7$, odd) and $(n-4,1,\ldots,1)$ ($n \ge 6$).

PROPOSITION 5.1. *Let $k \ge 1$ be an integer and $n := 2k+3$. The gap inequality for $(1,\ldots,1) \in \mathbf{Z}^n$ is valid for $N_+^k(K_n)$.*

*Proof.* We proceed by induction for $k \ge 1$. The result holds for $k = 1$. Let $k \ge 2$ and assume that the result holds for $k-1$. By the induction assumption, the gap inequality for $b'' := (0,1,\ldots,1) \in \mathbf{Z}^{n-1}$ is valid for $N_+^{k-1}(K_{n-1})$. Therefore, using Proposition 4.11, we deduce that the gap inequality for $b$ is valid for $N_+^k(K_n)$. ☐

One cannot hope to improve the above result and show validity for $N^k(K_n)$ with the help of Proposition 4.10; indeed, collapsing of the gap inequality for $(1,1,1,1,1,1,1) \in \mathbf{Z}^7$ gives the gap inequality for $(2,1,1,1,1,1) \in \mathbf{Z}^6$ which, as we see below, is *not* valid for $N(K_6)$. In fact, the gap inequality for $(1,1,1,1,1,1,1)$ is *not* valid for $N^2(K_7)$ (cf. Proposition 5.3). The proofs of Propositions 5.2–5.4 below, being quite technical, are delayed until section 7.

PROPOSITION 5.2. *The gap inequality for $(n-4,1,\ldots,1) \in \mathbf{Z}^n$ is valid for $N'(K_n)$ if $n = 6,7$, it is not valid for $N'(K_n)$ if $n \ge 8$, it is not valid for $N_+(K_n)$ if $n \ge 6$, and it is valid for $N^{n-5}(K_n)$ for $n \ge 7$.*

PROPOSITION 5.3. *The hypermetric inequality for $(1,\ldots,1) \in \mathbf{Z}^n$ ($n \ge 7$, odd) is not valid for $N'_+(K_n)$ nor for $N^2(K_n)$.*

PROPOSITION 5.4. $\mathrm{CUT}(K_n) = N(K_n)$ *if $n \le 5$,* $\mathrm{CUT}(K_6) = N'(K_6) \subset N_+(K_6)$*, $N'_+(K_n) \subset N_+(K_n) \subset N(K_n)$ for $n \ge 6$, and* $\mathrm{CUT}(K_n) \subset N'_+(K_n)$ *for $n \ge 7$.*

Let $a^T x \ge \beta$ be an inequality valid for $\mathrm{CUT}(K_n)$ and let $G$ denote its *support graph*, whose edges are the pairs $ij$ for which $a_{ij} \ne 0$. Obviously, the inequality $a^T x \ge \beta$ is valid for $N(\mathrm{MET}(G))$ if $\eta_N(G) \le 1$. This is the case, for instance, for parachute inequalities (cf. section 30.4 in [10]) and for bicycle odd wheel inequalities, that is, the inequalities

$$x_{uv} + \sum_{ij \in E(C)} x_{ij} + \sum_{i \in V(C)} (x_{ui} + x_{vi}) \ge 1 - |C|,$$

where $C$ is an odd circuit and $u, v$ two adjacent nodes that are adjacent to all nodes of $C$.

**6. Application to the stable set polytope.** We explain here how the LS relaxations $\nu(\mathrm{MET}(G))$ for the cut polytope permit us to tighten the corresponding LS relaxations for the stable set polytope. Given a graph $G = (V_n, E)$, its *fractional* *stable set polytope* is

$$\mathrm{FRAC}(G) := \{d \in \mathbf{R}^n \mid d \geq 0,\ d_i + d_j \leq 1 \text{ for all } ij \in E\},$$

and its *stable set polytope* is

$$\mathrm{STAB}(G) := \mathrm{conv}(x \in \{0,1\}^n \mid x \in \mathrm{FRAC}(G)).$$

Lovász and Schrijver [22] studied the relaxations $N(\mathrm{FRAC}(G))$ and $N_+(\mathrm{FRAC}(G))$ in detail. (As $\mathrm{FRAC}(G)$ lives in the unit cube $Q = [0,1]^d$, the operators $N, N_+$ are now defined in the context of $0,1$ variables, which means that condition (2.1) is replaced by $y_{i,i} = y_{0,i}$ for $i = 1, \ldots, d$, while condition (2.4) is replaced by $Y(e_i), Y(e_0 - e_i) \in \tilde{K}$ $(i = 1, \ldots, d)$.) In particular, they have shown the following results. The relaxation $N(\mathrm{FRAC}(G))$ is equal to the polytope $\mathrm{ODD}(G)$ defined by nonnegativity, the edge inequalities $d_i + d_j \leq 1$ $(ij \in E)$, and the odd hole inequalities $\sum_{i \in V(C)} d_i \leq \frac{|C|-1}{2}$ ($C$ being an odd circuit in $G$). Any clique inequality $\sum_{i \in V(K)} d_i \leq 1$ ($K$ a clique in $G$) is valid for $N_+(\mathrm{FRAC}(G))$ and $N^{|K|-2}(\mathrm{FRAC}(G))$ but not for $N^{|K|-3}(\mathrm{FRAC}(G))$; odd wheel inequalities, odd antihole inequalities, orthogonality constraints are valid for $N_+(\mathrm{FRAC}(G))$.

Let $G^\nabla$ denote the graph obtained from $G$ by adding a new node $a$ (the *apex* node) adjacent to all nodes of $G$ and set

$$\mathcal{L}_G := \{x \in \mathbf{R}^{E(G^\nabla)} \mid x_{ij} - x_{ai} - x_{aj} = -1 \text{ for all } ij \in E\}.$$

For $d \in \mathbf{R}^{V_n}$ define $x := \varphi(d) \in \mathbf{R}^{E(G^\nabla)}$ by

$$x_{ai} := 1 - 2d_i \ (i \in V_n), \quad x_{ij} := 1 - 2d_i - 2d_j \ (ij \in E). \tag{6.1}$$

Then $\varphi$ is a bijection between $\mathbf{R}^{V_n}$ and $\mathbf{R}^{E(G^\nabla)}$. For $S \subseteq V_n$, the $(\pm 1)$-incidence vector of the cut $\delta(S)$ (in $G^\nabla$) lies in $\mathcal{L}_G$ if and only if $S$ is a stable set in $G$. This shows the following well-known fact (cf., e.g., [25]):

$$\varphi(\mathrm{STAB}(G)) = \mathrm{CUT}(G^\nabla) \cap \mathcal{L}_G. \tag{6.2}$$

As $\varphi(\mathrm{STAB}(G))$ is a face of $\mathrm{CUT}(G^\nabla)$, every valid inequality for $\mathrm{CUT}(G^\nabla)$ gives rise to a valid inequality for $\mathrm{STAB}(G)$. For instance, if $C$ is an odd circuit in $G$, the circuit inequality $\sum_{ij \in E(C)} x_{ij} \geq 2 - |C|$ for $\mathrm{CUT}(G^\nabla)$ gives rise to the odd hole inequality $\sum_{i \in V(C)} d_i \leq \frac{|C|-1}{2}$ for $\mathrm{STAB}(G)$ (as $\sum_{ij \in E(C)} x_{ij} = |C| - 4 \sum_{i \in V(C)} d_i$); one can verify that the (switching of the) bicycle odd wheel inequality

$$-x_{au} + \sum_{i \in V(C)} (-x_{ai} + x_{ui}) + \sum_{ij \in E(C)} x_{ij} \geq 1 - |C|$$

for $\mathrm{CUT}(G^\nabla)$ gives rise to the odd wheel inequality $\sum_{i \in V(C)} d_i + \frac{|C|-1}{2} d_u \leq \frac{|C|-1}{2}$ for $\mathrm{STAB}(G)$, and that the gap inequality for $(b_a, b_1, \ldots, b_n) = (-(n-3), 1, \ldots, 1) \in \mathbf{Z}^{n+1}$ for $\mathrm{CUT}(G^\nabla)$ gives rise to the clique inequality $\sum_{i=1}^n d_i \leq 1$. It is shown in [20] that the correspondence (6.2) extends at the level of the basic linear and semidefinite relaxations; namely,

$$\varphi(\mathrm{ODD}(G)) = \mathrm{MET}(G^\nabla) \cap \mathcal{L}_G \quad \text{and} \quad \varphi(\mathrm{TH}(G)) = \mathcal{E}(G^\nabla) \cap \mathcal{L}_G, \tag{6.3}$$

where $\mathcal{E}(G^\nabla)$ is the projection of $\mathcal{E}_{n+1}$ on $\mathbf{R}^{E(G^\nabla)}$ and TH$(G)$ is the *theta body* defined as the set of vectors $x \in \mathbf{R}^{V_n}$ for which $(1, x) = Xe_0$ for some positive semidefinite matrix $X = (x_{ij})_{i,j=0}^n$ satisfying $x_{0i} = x_{ii}$ $(i = 1, \ldots, n)$ and $x_{ij} = 0$ $(ij \in E)$. It follows from the above that

$$\varphi\left(\mathrm{STAB}(G)\right) \subseteq \mathrm{MET}(G^\nabla) \cap \mathcal{L}_G = \varphi\left(N(\mathrm{FRAC}(G))\right).$$

We now examine how the correspondence between the relaxations $\nu(\mathrm{MET}(G^\nabla))$ and $\nu(\mathrm{FRAC}(G))$ carries out for $\nu = N, N_+, N', N'_+$ and their iterates.

PROPOSITION 6.1. *Let $k \geq 0$ be an integer. Then*

$$\varphi\left(\mathrm{STAB}(G)\right) \subseteq N^k(\mathrm{MET}(G^\nabla)) \cap \mathcal{L}_G \subseteq \varphi\left(N^{k+1}(\mathrm{FRAC}(G))\right),$$

*and, for $\nu = N_+, N', N'_+$,*

$$\varphi\left(\mathrm{STAB}(G)\right) \subseteq \nu^k(\mathrm{MET}(G^\nabla)) \cap \mathcal{L}_G \subseteq \varphi\left(\nu^k(\mathrm{FRAC}(G))\right).$$

*Proof.* The left inclusions follow from (6.2). We show that $N^k(\mathrm{MET}(G^\nabla)) \cap \mathcal{L}_G$ is contained in $\varphi(N^{k+1}(\mathrm{FRAC}(G)))$ by induction on $k \geq 0$. The inclusion holds for $k = 0$. Let $k \geq 1$ and suppose that the inclusion holds for $k-1$. Let $x \in N^k(\mathrm{MET}(G^\nabla)) \cap \mathcal{L}_G$; then $(1, x) = Ye_0$ for some $Y \in M(N^{k-1}(\mathrm{MET}(G^\nabla)))$. Let $Z$ denote the matrix indexed by $\{0\} \cup V_n$ defined by

$$\text{(6.4)} \qquad \begin{array}{l} Z_{0,0} := 1,\ Z_{0,i} = Z_{i,i} := \frac{1}{2}(1 - Y_{0,ai}) \quad (i \in V_n), \\ Z_{i,j} := \frac{1}{4}(1 + Y_{ai,aj} - Y_{0,ai} - Y_{0,aj}) \quad (i, j \in V_n). \end{array}$$

Then $\varphi^{-1}(x) = (Z_{0,i})_{i \in V_n}$. Therefore the result will follow if we can show that the matrix $Z$ belongs to $M(N^k(\mathrm{FRAC}(G)))$, i.e., that $Z(e_k)$, $Z(e_0 - e_k)$ belong to $N^k(\widetilde{\mathrm{FRAC}(G)})$. By assumption, $Y(e_0 \pm e_f) \in N^{k-1}(\widetilde{\mathrm{MET}(G^\nabla)})$ for all $f \in E(G^\nabla)$. As $Ye_0 \in \widetilde{\mathcal{L}_G}$ and $Ye_0 = \frac{1}{2}(Y(e_0 + e_f) + Y(e_0 - e_f))$, we deduce that $Y(e_0 \pm e_f) \in \widetilde{\mathcal{L}_G}$, and thus $Ye_f \in \widetilde{\mathcal{L}_G}$ for all $f \in E(G^\nabla)$, which can be rewritten as

$$\text{(6.5)}\ 1 + Y_{0,ij} - Y_{0,ai} - Y_{0,aj} = 0,\ Y_{0,f} + Y_{ij,f} - Y_{ai,f} - Y_{aj,f} = 0 \ \text{ for } f \in E(G^\nabla).$$

Using the induction assumption, we obtain that $\varphi^{-1}(Y(e_0 \pm e_{ak}))$ $(k \in V_n)$ belongs to $N^k(\widetilde{\mathrm{FRAC}(G)})$. (We have extended the bijection $\varphi$ as a bijection between the homogenized spaces $\mathbf{R}^{V_n \cup \{0\}}$ and $\mathbf{R}^{E(G^\nabla) \cup \{0\}}$ in the obvious way; namely, $(x_0, x) = \varphi(d_0, d)$ if $x_0 = d_0$, $x_{ai} = d_0 - 2d_i$, and $x_{ij} = d_0 - 2d_i - 2d_j$.) In order to conclude, it suffices now to observe that $Ze_k = \varphi^{-1}(\frac{1}{2}Y(e_0 - e_{ak}))$ and $Z(e_0 - e_k) = \varphi^{-1}(\frac{1}{2}Y(e_0 + e_{ak}))$ for $k \in V_n$; this is an easy verification using the relation (6.5).

We now show the result for the $N'$ operator. In view of the above, it suffices to show the following result: If $Y \in M'((N')^{k-1}(\mathrm{MET}(G^\nabla)))$ satisfies $Ye_0 \in \widetilde{\mathcal{L}_G}$ and if $Z$ is the associated matrix defined by (6.5), then $Z \in M'((N')^{k-1}(\mathrm{FRAC}(G)))$; that is, $Ze_k$, $Z(e_0 - e_h - e_k)$ belong to $(N')^{k-1}(\widetilde{\mathrm{FRAC}(G)})$ for all $k \in V_n$, all $hk \in E(G)$, respectively. By assumption, the vectors $Y(e_0 \pm e_f)$ $(f \in E(G^\nabla))$ and $Y(e_0 \pm e_{ai} \pm e_{aj} \pm e_{ij})$ (with an even number of minus signs) $(ij \in E(G))$ belong to $(N')^{k-1}(\mathrm{MET}(G^\nabla))$; as $Ye_0 \in \widetilde{\mathcal{L}_G}$, their images under $\varphi^{-1}$ belong to $(N')^{k-1}(\mathrm{FRAC}(G))$ (by the induction assumption) and (6.5) holds. To conclude the proof it suffices to verify (using (6.5)) that $Z(e_0 - e_h - e_k) = \varphi^{-1}\left(\frac{1}{4}Y(e_0 + e_{ah} + e_{ak} + e_{hk})\right)$ for $hk \in E(G)$.

The result for the $N_+$ and $N'_+$ operators follows, using the fact that $Y \succeq 0 \Longrightarrow Z \succeq 0$, which holds because $b^T Z b = c^T Y c$, where $b \in \mathbf{R}^{n+1}$ and $c := (-(b_0 + \sum_{i=0}^n b_i), b_1, \ldots, b_n)$. $\square$

It is shown in [22] that the smallest integer $k$ for which $N^k(\mathrm{FRAC}(G)) = \mathrm{STAB}(G)$ is less than or equal to $n - \alpha(G) - 1$ if $G$ has at least one edge. On the other hand, by (4.7), $\eta_N(G^\nabla) \leq n + 1 - \alpha(G^\nabla) - 3 = n - \alpha(G) - 2$ if $\alpha(G) \leq n - 2$. The similarity between the two bounds reflects the fact that $\mathrm{STAB}(G)$ arises as a face of $\mathrm{CUT}(G^\nabla)$. In fact the two upper bounds match, as the discrepancy of 1 can be explained by the fact that in the case of the cut polytope we start with a stronger relaxation than in the case of the stable set polytope; indeed, in view of (6.3), we "win" one iteration at the beginning step.

The inclusion $N^k(\mathrm{MET}(G^\nabla)) \cap \mathcal{L}_G \subseteq \varphi\left(N^{k+1}(\mathrm{FRAC}(G))\right)$ holds at equality for $k = 0$ for all graphs and is strict for $k \geq 1$ for certain graphs. Indeed, for $k \geq 1$,

$$\mathrm{STAB}(K_{k+4}) = \varphi^{-1}\left(N^k(\mathrm{MET}(K_{k+4}^\nabla)) \cap \mathcal{L}_{K_{k+4}}\right) \subset N^{k+1}(\mathrm{FRAC}(K_{k+4})).$$

To see it, note that the clique inequality $\sum_{i=1}^{k+4} d_i \leq 1$ is not valid for $N^{k+1}(\mathrm{FRAC}(K_{k+4}))$, while it is valid for $\varphi^{-1}\left(N^k(\mathrm{MET}(K_{k+4}^\nabla)) \cap \mathcal{L}_{K_{k+4}}\right)$. The latter holds because the clique inequality $\sum_{i=1}^{k+4} d_i \leq 1$ arises from the gap inequality for $(-(k+1), 1, \ldots, 1) \in \mathbf{Z}^{k+5}$ (assigning $-(k+1)$ to the apex node), which is valid for $N^k(\mathrm{MET}(K_{k+5}))$ when $k \geq 2$ by Proposition 5.2; in the case $k = 1$, while not valid for $N(\mathrm{MET}(K_6))$, the gap inequality for $(-2, 1, 1, 1, 1, 1)$ is valid for $N(\mathrm{MET}(K_5^\nabla)) \cap \mathcal{L}_{K_5}$ (cf. Lemma 7.5).

We know that clique and odd antihole inequalities are valid for $N_+(\mathrm{MET}(G^\nabla)) \cap \mathcal{L}_G$ (as they are valid for $N_+(\mathrm{FRAC}(G))$). It would be interesting to find for them some "parent" inequality for $\mathrm{CUT}(G^\nabla)$ which would be valid for $N_+(\mathrm{MET}(G^\nabla))$.

**7. Proofs of Propositions 5.2–5.4.** We study here in detail the validity of the gap inequalities for $c_n := (1, \ldots, 1) \in \mathbf{Z}^n$ ($n \geq 7$ odd) and for $b_n := (n-4, 1, \ldots, 1) \in \mathbf{Z}^n$ ($n \geq 6$) for some relaxations $\nu^k(K_n)$. Set

$$C_n := \min\left(\sum_{1 \leq i < j \leq n} x_{ij} \mid x \in \nu^k(K_n)\right), \tag{7.1}$$

$$B_n := \min\left((n-4)\sum_{i=2}^n x_{1i} + \sum_{2 \leq i < j \leq n} x_{ij} \mid x \in \nu^k(K_n)\right). \tag{7.2}$$

Given some scalars $a, c \in \mathbf{R}$, the vector $x(a,c) \in \mathbf{R}^{E_n}$ is defined by

$$x(a,c)_{1i} := a \quad \text{for } i = 2, \ldots, n, \qquad x(a,c)_{ij} := c \quad \text{for } 2 \leq i < j \leq n; \tag{7.3}$$

it is said to have *pattern* $(a,c)$.

A first basic observation is that the minimum in the program (7.1) (resp., (7.2)) is attained at a point of $\nu^k(K_n)$ having some pattern $(a,a)$ (resp., $(a,c)$). Indeed, let $x \in \nu^k(K_n)$ be an optimum solution to program (7.1) and set $x^* := \frac{1}{n!}\sum_\sigma x^\sigma$, where the sum is taken over all permutations $\sigma$ of $[1,n]$; then $x^* \in \nu^k(K_n)$ is still optimum for (7.1) and has pattern $(a,a)$ for some $a \in \mathbf{R}$. The reasoning is similar in the case of program (7.2), except $x^* := \frac{1}{(n-1)!}\sum_\sigma x^\sigma$, where the sum is now taken over all permutations of $[1,n]$ fixing 1; then $x^*$ has pattern $(a,c)$ for some $a, c \in \mathbf{R}$.

For the proofs of Propositions 5.2 and 5.3 we need to determine the conditions on $a, c$ which will permit us to express membership of the vector $x(a,c)$ in $N(K_n)$ and $N'(K_n)$. The study of validity for $N^2(K_n)$ will involve checking the membership in $N(K_n)$ of a more complicated vector $x(a,b,c,d) := x$, defined as follows:

$$(7.4)\ x_{12} := a,\ x_{1i} := b,\ x_{2i} := c \ \text{ for } i = 3, \dots, n, \quad x_{ij} := d \ \text{ for } 3 \le i < j \le n;$$

$(a,b,c,d)$ is again called the *pattern* of the vector $x(a,b,c,d)$. Note that $x(a,b,c,d) = x(a,c)$ if $a = b$ and $c = d$.

The rest of this section is organized as follows. In section 7.1 we determine the conditions on $a,b,c,d$ expressing membership in $N(K_n)$ for the vector $x(a,b,c,d)$ or membership in $N'(K_n)$ for the vector $x(a,c)$. These results are then applied in sections 7.2–7.3 to proving Propositions 5.3–5.4.

**7.1. Vectors with pattern $(a,b,c,d)$.** We begin by determining the conditions on $a,b,c,d$ expressing membership in $N(K_n)$ for a vector with pattern $(a,b,c,d)$. By definition, $x := x(a,b,c,d) \in N(K_n)$ if and only if $(1,x) = Ye_0$ for some matrix $Y \in M(\mathrm{MET}(K_n))$. In fact, such a matrix $Y$ can be assumed to satisfy certain symmetries. Indeed, set $Y^* := \frac{1}{(n-2)!}\sum_\sigma Y^\sigma$, where the sum is taken over all permutations $\sigma$ of $[1,n]$, fixing 1 and 2 (recall the definition of $Y^\sigma$ from (3.5)). Then $Y^* \in M(\mathrm{MET}(K_n))$ and $Y^*e_0 = (1,x)$. Moreover, the matrix $Y^*$ has the property that the value of its $(ij,hk)$th entry depends only on whether the pairs $ij$ and $hk$ meet and whether they contain any of the points 1 and 2. Namely, if the pairs $ij$ and $hk$ meet, then the value of $Y_{ij,hk}$ is determined by relation (3.3) and is thus one of $a,b,c,d$; otherwise,

$$(7.5)\quad \begin{aligned} &Y_{12,ij} = x \quad \text{for } 3 \le i < j \le n,\\ &Y_{1i,2j} = z \quad \text{for } 3 \le i \ne j \le n,\\ &Y_{1i,hk} = y,\ Y_{2i,hk} = u \quad \text{for } 3 \le i \le n,\ 3 \le i < j \le k,\ h,k \ne i,\\ &Y_{ij,hk} = v \quad \text{for } 3 \le i < j \le n,\ 3 \le h < k \le n,\ \{i,j\} \cap \{h,k\} = \emptyset \end{aligned}$$

for some scalars $x,y,z,u,v$; $(a,b,c,d,x,y,z,u,v)$ is then called the *pattern* of $Y$.

Let $\mathcal{Y}_n$ denote the set of matrices $Y \in \mathcal{S}^1_{1+d_n}$ having some pattern $(a,b,c,d,x,y,z,u,v)$ as defined above. A matrix $Y \in \mathcal{Y}_6$ is shown in Figure 7.1. When $a = b$ and $c = d$ (i.e., when $x = x(a,c)$), the matrix $Y$ can be assumed to satisfy the additional symmetry $x = y = z$ and $u = v$, and $(a,c,x,u)$ is then called the *simplified pattern* of $Y$. (Such a matrix is pictured in Figure A.1.)

We first work out the conditions on $a, \dots, v$ for membership of $Y \in \mathcal{Y}_n$ in $M(\mathrm{MET}(K_n))$, and then deduce the conditions on $a,b,c,d$ for membership of $x(a,b,c,d)$ in $N(K_n)$.

LEMMA 7.1. *Let $Y \in \mathcal{Y}_n$ with pattern $(a,b,c,d,x,y,z,u,v)$ and $n \ge 6$. Then $Y$*

$$
\begin{array}{r@{\;}c}
 & \begin{array}{cccccccccccccccc} 0 & 12 & 13 & 14 & 15 & 16 & 23 & 24 & 25 & 26 & 34 & 35 & 36 & 45 & 46 & 56 \end{array} \\
\begin{array}{r} 0\\ 12\\ 13\\ 14\\ 15\\ 16\\ 23\\ 24\\ 25\\ 26\\ 34\\ 35\\ 36\\ 45\\ 46\\ 56 \end{array} &
\begin{pmatrix}
1 & a & b & b & b & b & c & c & c & c & d & d & d & d & d & d \\
a & 1 & c & c & c & c & b & b & b & b & x & x & x & x & x & x \\
b & c & 1 & d & d & d & a & z & z & z & b & b & b & y & y & y \\
b & c & d & 1 & d & d & z & a & z & z & b & y & y & b & b & y \\
b & c & d & d & 1 & d & z & z & a & z & y & b & y & b & y & b \\
b & c & d & d & d & 1 & z & z & z & a & y & y & b & y & b & b \\
c & b & a & z & z & z & 1 & d & d & d & c & c & c & u & u & u \\
c & b & z & a & z & z & d & 1 & d & d & c & u & u & c & c & u \\
c & b & z & z & a & z & d & d & 1 & d & u & c & u & c & u & c \\
c & b & z & z & z & a & d & d & d & 1 & u & u & c & u & c & c \\
d & x & b & b & y & y & c & c & u & u & 1 & d & d & d & d & v \\
d & x & b & b & b & y & c & u & c & u & d & 1 & d & d & v & d \\
d & x & b & y & y & b & c & u & u & c & d & d & 1 & v & d & d \\
d & x & y & b & b & y & u & c & c & u & d & d & v & 1 & d & d \\
d & x & y & b & y & b & u & c & u & c & d & v & d & d & 1 & d \\
d & x & y & y & b & b & u & u & c & c & v & d & d & d & d & 1
\end{pmatrix}
\end{array}
$$

FIG. 7.1. *A matrix $Y \in \mathcal{Y}_6$ with pattern $(a, b, c, d, x, y, z, u, v)$.*

*belongs to $M(\mathrm{MET}(K_n))$ if and only if $a, \ldots, v$ satisfy the linear inequalities*

$$
\begin{gathered}
a + 2b + 2c + d + x \geq -1, \quad a - 2b - 2c + d + x \geq -1, \\
-a + 2b - 2c + d - x \geq -1, \quad -a - 2b + 2c + d - x \geq -1, \\
a - d - x \geq -1, \ -a - d + x \geq -1, \quad a + 3d + 3x \geq -1, \quad -a + 3d - 3x \geq -1, \\
a + 2b + 2c + d + z \geq -1, \quad -a + 2b - 2c + d - z \geq -1, \quad a - d - z \geq -1, \\
-a - d + z \geq -1, \quad a - 2b - 2c + d + z \geq -1, \quad -a - 2b + 2c + d - z \geq -1, \\
3b + 3d + y \geq -1, \quad -3b + 3d - y \geq -1, \quad -b - d + y \geq -1, \quad b - d - y \geq -1, \\
b + 2c + d + y + 2z \geq -1, \quad b - 2c + d + y - 2z \geq -1, \quad b - d - y \geq -1, \\
-b + 2c + d - y - 2z \geq -1, \quad -b - 2c + d - y + 2z \geq -1, \quad -b - d + y \geq -1, \\
b + 3d + 3y \geq -1, \quad -b + 3d - 3y \geq -1, \quad 3c + 3d + u \geq -1, \quad -3c + 3d - u \geq -1, \\
2b + c + d + 2z + u \geq -1, \quad -2b + c + d - 2z + u \geq -1, \quad c - d - u \geq -1, \\
2b - c + d - 2z - u \geq -1, \quad -2b - c + d + 2z - u \geq -1, \quad -c - d + u \geq -1, \\
c + 3d + 3u \geq -1, \quad -c + 3d - 3u \geq -1, \quad 6d + v \geq -1, \quad -2d + v \geq -1, \quad v \leq 1.
\end{gathered} \tag{7.6}
$$

*Proof.* By definition, $Y \in M(\mathrm{MET}(K_n))$ if and only if, for all $ij \in E_6$, $y := Y(e_0 \pm e_{ij})$ satisfies all triangle inequalities. By symmetry, it suffices to consider the cases when $ij = 12$, 13, 23, or 34. Let $ij = 12$. Due to symmetry and to the fact that $y_{12} = \pm y_0$, it suffices to consider the triangle inequalities based on the triples 134 and 345. The triangle inequalities based on triple 134 can be reformulated as

$$
\begin{gathered}
a + 2b + 2c + d + x \geq -1, \quad a - 2b - 2c + d + x \geq -1, \quad a - d - x \geq -1, \\
-a + 2b - 2c + d - x \geq -1, \quad -a - 2b + 2c + d - x \geq -1, \quad -a - d - x \geq -1,
\end{gathered}
$$

and those based on triple 345 give

$$
a + 3d + 3x \geq -1, \qquad -a + 3d - 3x \geq -1.
$$

Next let $ij$ be one of $13, 23, 34$. Due to symmetry and to the fact that $y_{ij} = \pm y_0$, it suffices to consider the triangle inequalities based on the triples 124, 145, 245, and

456. When $ij = 13$, we find from (7.6) the relations $a + 2b + 2c + d + z \geq -1$ until $-b + 3d - 3y \geq -1$. When $ij = 23$, we find the relations $3c + 3d + u \geq -1$ until $-c + 3d - 3u \geq -1$. When $ij = 34$, we find the relations $6d + v \geq -1, -2d + v \geq -1$, $v \leq 1$. □

COROLLARY 7.2. *The vector $x(a, b, c, d)$ belongs to $N(K_n)$ ($n \geq 6$) if and only if*

$$\begin{aligned} d \leq 1, \quad \pm 2b + d \geq -1, \quad \pm 2c + d \geq -1, \quad \pm 2b + 3d \geq -1, \quad \pm 2c + 3d \geq -1, \\ \pm a \pm b \pm c \geq -1, \quad \pm a \pm 3b \pm 3c + 3d \geq -2, \\ \pm 3a \pm 5b \pm 9c + 6d \geq -5, \quad \pm 3a \pm 9b \pm 5c + 6d \geq -5, \end{aligned} \tag{7.7}$$

*where in lines 2 and 3 of the above system there is an even number of minus signs (e.g., $a + b + c \geq -1$, $-a - b + c \geq -1$, etc.). The vector $x(a, c)$ belongs to $N(K_n)$ ($n \geq 6$) if and only if $a, c$ satisfy*

$$\pm 2a + c \geq -1, \quad \pm 2a + 3c \geq -1, \quad \pm 12a + 11c \geq -5, \quad -\tfrac{1}{5} \leq c \leq 1. \tag{7.8}$$

*Proof.* We saw above that $x = x(a, b, c, d) \in N(K_n)$ if and only if $(1, x) = Ye_0$ for some matrix $Y \in M(\mathrm{MET}(K_n))$ having pattern $(a, b, c, d, x, y, z, u, v)$ for some $x, y, z, u, v$. Using the computer code **cdd+** of Fukuda [11] for polyhedral computations, we have verified that the projection on the subspace indexed by the variables $a, b, c, d$ of the polytope defined by linear system (7.6) is described by linear system (7.7). One can then verify that for $a = b$ and $c = d$, system (7.7) is equivalent to (7.8). □

We now characterize membership in $N'(K_n)$ for a vector with pattern $(a, c)$.

LEMMA 7.3. *Let $Y \in \mathcal{Y}_n$ with pattern $(a, c, x, u)$ and $n \geq 6$. Then $Y \in M'(\mathrm{MET}(K_n))$ if and only if $a$, $c$, $x$, $u$ satisfy the linear inequalities*

$$\begin{aligned} -2c + u \geq -1, \quad 2c - 3u \geq -1, \quad 10c + 5u \geq -1, \quad 2a - 2x - u \geq -1, \\ -2a + 2x - u \geq -1, \quad 4a + 6c + 4x + u \geq -1, \quad -4a + 6c - 4x + u \geq -1, \\ 2a + 4c + 6x + 3u \geq -1, \quad -2a + 4c - 6x + 3u \geq -1, \end{aligned} \tag{7.9}$$

*as well as $6c + 9u \geq -1$ when $n \geq 7$.*

*Proof.* By definition, $Y \in M'(\mathrm{MET}(K_n))$ if and only if, for all $1 \leq i < j < k \leq n$, the vector $Y(e_0 \pm e_{ij} \pm e_{ik} \pm e_{jk})$ (with 0 or 2 minus signs) satisfies all triangle inequalities. By symmetry, it suffices to consider the two cases when $ijk = 123$ or 234. Consider first the case when $ijk = 123$. Due to symmetry, it suffices to consider the triangle inequalities for the vectors $x := Y(e_0 + e_{12} + e_{13} + e_{23})$, $y := Y(e_0 + e_{12} - e_{13} - e_{23})$, and $z := Y(e_0 - e_{12} - e_{13} + e_{23})$, based on the triples 145 and 456 (we also use the fact that $x_{12} = x_{13} = x_{23} = x_0$, $y_{13} = y_{23} = -y_{12} = -y_0$, and $z_{12} = z_{13} = -z_{23} = -z_0$). The triangle inequalities for $x$ based on triple 145 are equivalent to

$$4a + 6c + 4x + u \geq -1, \quad 2a - 2x - u \geq -1, \quad -2c + u \geq -1, \tag{a}$$

and those based on triple 456 give the new relation

$$2a + 4c + 6x + 3u \geq -1. \tag{b}$$

The triangle inequalities for $y$ based on triples 145 and 456 yield, respectively,

$$-2a + 2x - u \geq -1, \qquad 2c - 3u \geq -1. \tag{c}$$

The triangle inequalities for $z$ based on triples 145 and 456 give, respectively, the relations:

$$\text{(d)} \qquad -4a + 6c - 4x + u \geq -1, \qquad -2a + 4c - 6x + 3u \geq -1.$$

Consider now the case when $ijk = 234$. Due to symmetry, it suffices to look at the triangle inequalities for the vectors $x := Y(e_0 + e_{23} + e_{24} + e_{34})$ and $y := Y(e_0 - e_{23} - e_{24} + e_{34})$, based on the triples 125, 156, 256, and 567 (the last occurring only for $n \geq 7$). The triangle inequalities for $x$ based on triples 125 and 156 give no new condition; those for triple 256 give the condition

$$\text{(e)} \qquad 10c + 5u \geq -1,$$

and, when $n \geq 7$, those for triple 567 yield

$$\text{(f)} \qquad 6c + 9u \geq -1.$$

No new condition is obtained when looking at the triangle inequalities for $y$. The inequalities from (a)–(f) are those from (7.9). $\square$

COROLLARY 7.4. *For $n = 6$, $x(a, c) \in N'(K_n)$ if and only if*

$$\pm 2a + c \geq -1, \quad \pm 5a + 5c \geq -2, \quad -\frac{1}{5} \leq c \leq 1, \tag{7.10}$$

*and, for $n \geq 7$, $x(a, c) \in N'(K_n)$ if and only if $a, c$ satisfy* (7.10) *together with the inequalities $\pm 18a + 15c \geq -7$.*

*Proof.* We have verified (using the computer program **cdd+** [11]) that the projection on the subspace indexed by the variables $a$ and $c$ of the polytope defined by the linear system (7.9) (resp., (7.9) together with $6c + 5u \geq -1$) is described by the linear system (7.10) (resp., (7.10) together with $\pm 18a + 15c \geq -7$). $\square$

We will also need to check whether a matrix $Y \in \mathcal{Y}_n$ is sdp. For concrete examples this can be checked using a computer. However, for a matrix $Y$ with simplified pattern $(a, c, x, u)$ one can explicitly describe the conditions on $a, c, x, u$ ensuring $Y \succeq 0$. Indeed, the positive semidefiniteness of $Y$ can be reformulated as the positive semidefiniteness of some smaller matrix $Z$ whose eigenvalues can be computed because $Z$ belongs to an association scheme. Details will be given in the appendix.

**7.2. Proof of Proposition 5.2.** We show here the (non)validity of the gap inequality for $b_n = (n - 4, 1, \ldots, 1) \in \mathbf{Z}^n$ for the relaxations $\nu(K_n)$ ($\nu = N, \ldots, N'_+$). Validity over $\nu(K_n)$ means that $B_n \geq \rho_n := -\frac{1}{2}(n^2 - 7n + 14)$, where $B_n$ is defined in (7.2) (with $k = 1$); note that $\rho_6 = -4$, $\rho_7 = -7$, $\rho_8 = -11$. As the program (7.2) admits an optimum solution $x$ having some pattern $(a, c)$ we can, using the results from the preceding subsection, reformulate (7.2) as a program in the variables $a$ and $c$. In particular, for $\nu = N'$ and $n = 6$, (7.2) can be reformulated as

$$\min(10a + 10c \mid \ a, c \text{ satisfy } (7.10)),$$

and, for $\nu = N'$ and $n = 7$, (7.2) is reformulated as

$$\min(18a + 15c \mid \ a, c \text{ satisfy } (7.10) \text{ and } \pm 18a + 15c \geq -7).$$

Hence we deduce that the gap inequality for $b_n$ is valid for $N'(K_n)$ when $n = 6, 7$.

We now show nonvalidity for $N'(K_n)$ ($n \geq 8$) and $N_+(K_n)$ ($n \geq 6$). We first observe that it suffices to consider the two bottom cases: $n = 8$ for $N'$ and $n = 6$

for $N_+$. Indeed, the gap inequality for $b_n = (n-4, 1, \ldots, 1) \in \mathbf{Z}^n$ coincides with the inequality obtained from the gap inequality for $b_{n+1} = (n-3, 1, \ldots, 1) \in \mathbf{Z}^{n+1}$ by anticollapsing the nodes 1 and $n+1$. Therefore, if $x \in \nu(K_n)$ violates the gap inequality for $b_n$, then by taking successive $(-1)$-extensions of $x$ we construct a point $y \in \nu(K_m)$ violating the gap inequality for $b_m$ for any $m \geq n+1$.

Let $a := -\frac{2}{3}$ and $c := \frac{1}{3}$. Then $x(a,c) \in N'(K_n)$ for any $n \geq 7$, and $(n-4)(n-1)a + \binom{n-1}{2}c < \rho_n$ for any $n \geq 8$. This shows that the gap inequality for $b_n$ is not valid for $N'(K_n)$ for $n \geq 8$.

Let $a := -\frac{5}{12}$, $c := \frac{1}{120}$, $x := \frac{11}{45}$, $u := -\frac{29}{90}$. Then $x(a,c) \in N_+(K_6)$. Indeed, the matrix $Y \in \mathcal{Y}_6$ with simplified pattern $(a,c,x,u)$ belongs to $M_+(\mathrm{MET}(K_6))$; that is, $a,c,x,u$ satisfy (7.6) and (A.2). (Note that $\lambda_0(X) = 0$ in (A.2).) As $10a + 10c = -\frac{49}{12} < -4$, $x(a,c)$ violates the gap inequality for $b_6$. We have found those values of $a,c,x,u$ with the help of the software package SDPPACK [1]. Using SDPPACK, we have solved the semidefinite programming problem

$$\min(10a + 10c \mid Y \in M_+(\mathrm{MET}(K_6)) \text{ having some pattern } (a,c,x,u))$$

and found that the optimum is attained at the above values of $a,c,x,u$. (This is a problem in dimension $1 + \binom{6}{2} = 16$ with $\binom{16}{2} - 4 + 14 + 16 = 146$ linear (in)equalities; indeed, one can replace the $2\binom{6}{2} \times 4\binom{6}{3} = 2400$ triangle inequalities expressing $Y \in M(\mathrm{MET}(K_6))$ by the 14 linear inequalities from (7.6).)

Note that $\min(10a + 10c \mid x(a,c) \in N(K_6)) = -\frac{30}{7}$, attained at $a = -\frac{2}{7}$, $c = -\frac{1}{7}$. This again shows that the gap inequality for $b_6$ is not valid for $N(K_6)$ or, moreover, for the strict inclusion $N_+(K_6) \subset N(K_6)$. The following result has been referred to earlier in the paper.

LEMMA 7.5. *Although it is not valid for* $N(\mathrm{MET}(K_6))$*, the gap inequality for* $(-2,1,1,1,1,1)$ is *valid for* $N(\mathrm{MET}(K_5^\nabla)) \cap \mathcal{L}_{K_5}$ *(assigning* $-2$ *to the apex node).*

*Proof.* Indeed, $x(a,c)$ belongs to $\mathcal{L}_{K_5}$ if and only if $c = 2a - 1$. Then $x(a,c) \in N(\mathrm{MET}(K_6))$ implies that $-12a + 11c \geq -5$ and thus $10a \geq 6$; that is, $-10a + 10c \geq -4$. ∎

We now show that the gap inequality for $b_n$ is valid for $N^{n-5}(K_n)$ for $n \geq 7$. Again it suffices to show the result for the bottom case $n = 7$, as the general result follows using induction. (Indeed, consider $b_{n+1} = (n-3, 1, \ldots, 1) \in \mathbf{Z}^{n+1}$. Anticollapsing of nodes 1 and $n+1$ yields the gap inequality for $b_n$, which is valid for $N^{n-5}(K_n)$ by the induction assumption, while collapsing of these two nodes yields the gap inequality for $(n-2, 1, \ldots, 1) \in \mathbf{Z}^n$, which is valid for $\mathrm{MET}(K_n)$ (as it is a sum of triangle inequalities). Therefore we deduce, using Proposition 4.10, that the gap inequality for $b_{n+1}$ is valid for $N^{n-4}(K_{n+1})$.) Our task is now to show that

$$\min(18a + 15c \mid x(a,c) \in N^2(K_7)) \geq -7.$$

For this we need to characterize when $x(a,c) \in N^2(K_7)$. By definition, $x(a,c) \in N^2(K_7)$ if and only if $(1, x(a,c)) = Ye_0$ for some matrix $Y \in M(N(K_7))$ with simplified pattern $(a,c,x,u)$ for some $x,u$. Due to symmetry, $Y \in M(N(K_7))$ if and only if $Y(e_0 \pm e_{12})$, $Y(e_0 \pm e_{23}) \in N(K_7)$. Note that the vector $Y(e_0 + e_{12})$ is the 1-extension of a vector in $\mathbf{R}^{E_6}$ with pattern $(\frac{a+c}{1+a}, \frac{c+x}{1+a})$; the vector $Y(e_0 - e_{12})$ is the $(-1)$-extension of $x(\frac{a-c}{1-a}, \frac{c-x}{1-a}) \in \mathbf{R}^{E_6}$; the vector $Y(e_0 + e_{23})$ is the 1-extension of $x(\frac{2a}{1+c}, \frac{a+x}{1+c}, \frac{2c}{1+c}, \frac{c+u}{1+c}) \in \mathbf{R}^{E_6}$; the vector $Y(e_0 - e_{23})$ is the $(-1)$-extension of $x(0, 0, \frac{a-x}{1-c}, \frac{c-u}{1-c}) \in \mathbf{R}^{E_6}$. Using Corollary 7.2, we find that $x(\frac{a+c}{1+a}, \frac{c+x}{1+a})$, $x(\frac{a-c}{1-a}, \frac{c-x}{1-a})$

belong to $N(K_6)$ if and only if

$$\begin{gathered}a-c-x\geq -1,\quad -a-c+x\geq -1,\quad 3a+3c+x\geq -1,\quad -3a+3c-x\geq -1,\\ 17a+23c+11x\geq -5,\quad -17a+23c-11x\geq -5,\quad 7a-c-11x\geq -5,\\ -7a-c+11x\geq -5,\quad 3a+5c+3x\geq -1,\quad -3a+5c-3x\geq -1,\\ a+c-3x\geq -1,\quad -a+c+3x\geq -1,\quad a+5c+5x\geq -1,\quad -a+5c-5x\geq -1.\end{gathered}\tag{7.11}$$

Moreover, $x(\frac{2a}{1+c},\frac{a+x}{1+c},\frac{2c}{1+c},\frac{c+u}{1+c})\in N(K_6)$ if and only if

$$\begin{gathered}-\tfrac{1}{3}\leq u\leq 1,\quad 2a+2c+2x+u\geq -1,\quad -2a+2c-2x+u\geq -1,\\ 2a+4c+2x+3u\geq -1,\quad -2a+4c-2x+3u\geq -1,\quad 6c+u\geq -1,\\ -2c+u\geq -1,\quad 8c+3u\geq -1,\quad 3a+3c+x\geq -1,\quad -3a+3c-x\geq -1,\\ a-c-x\geq -1,\quad -a-c+x\geq -1,\quad 5a+11c+3x+3u\geq -2,\\ -5a+11c-3x+3u\geq -2,\quad -a-c-3x+3u\geq -2,\quad a-c+3x+3u\geq -2,\\ 11a+29c+5x+6u\geq -5,\quad -11a+29c-5x+6u\geq -5,\quad a-7c-5x+6u\geq -5,\\ -a-7c+5x+6u\geq -5,\quad 15a+21c+9x+6u\geq -5,\quad -15a+21c-9x+6u\geq -5,\\ -3a+c-9x+6u\geq -1,\quad 3a+c+9x+6u\geq -5.\end{gathered}\tag{7.12}$$

Finally, after noting that $x(0,0,x,u)\in N(K_6)$ if and only if $-\frac{1}{3}\leq u\leq 1$, $-1\leq x\leq 1$, $\pm 2x+u\geq -1$, $\pm 2x+3u\geq -1$, we find that $x(0,0,\frac{a-x}{1-c},\frac{c-u}{1-c})\in N(K_6)$ if and only if

$$\begin{gathered}-a-c+x\geq -1,\quad a-c-x\geq -1,\quad -2c+u\geq -1,\quad 2c-3u\geq -1,\\ 2a-2x-u\geq -1,\quad -2a+2x-u\geq -1,\quad 2a+2c-2x-3u\geq -1,\\ -2a+2c+2x-3u\geq -1.\end{gathered}\tag{7.13}$$

Using a computer, we verified that the minimum value of $18a+15c$ subject to $a,c,x,u$ satisfying the linear system (7.11), (7.12), and (7.13) is equal to $-7$ (attained at $a=-\frac{1}{3}$, $c=-\frac{1}{15}$, $x=\frac{1}{5}$, $u=-\frac{1}{15}$). This shows that the gap inequality for $b_7$ is valid for $N^2(K_7)$.

**7.3. Proof of Propositions 5.3 and 5.4.** We begin by showing that the gap inequality for $c_n=(1,\dots,1)\in\mathbf{Z}^n$ is not valid for $N'_+(K_n)$ for $n\geq 7$ odd. First let $n=7$ and set $a=c:=-\frac{11}{70}$ and $x=u:=\frac{4}{35}$. Then the matrix $Y\in\mathcal{Y}_7$ with pattern $(a,c,x,u)$ belongs to $M'_+(K_7)$, because $a,c,x,u$ satisfy (7.9) and (A.1) (for $n=7$). Hence $x(a,a)$ belongs to $N'_+(K_7)$ and violates the gap inequality for $c_7$ as $21a<-3$. We extend the result for any odd $n\geq 7$ by induction. Suppose $x\in N'_+(K_n)$ violates the gap inequality for $c_n$ for some odd $n\geq 7$. For $\epsilon=\pm 1$, the $\epsilon$-extension $x^\epsilon$ of $x$ belongs to $N'_+(K_{n+1})$, and thus $\hat{x}:=\frac{1}{2}(x^1+x^{-1})\in N'_+(K_{n+1})$ with $\hat{x}_{i,n+1}=0$ $(1\leq i\leq n)$ and $\hat{x}_{ij}=x_{ij}$ $(ij\in E_n)$. Consider now the $(-1)$-extension $y$ of $\hat{x}$ defined by $y_{n+1,n+2}=-1$. Then $y\in N'_+(K_{n+2})$ and violates the gap inequality for $c_{n+2}$. This proves the first part of Proposition 5.3 and the strict inclusion $\mathrm{CUT}(K_n)\subset N'_+(K_n)$ $(n\geq 7)$.

We now show that the gap inequality for $c_n$ is not valid for $N^2(K_n)$ for odd $n\geq 7$. As observed above, it suffices to consider the case $n=7$. We show that

$$\min(21a\mid x(a,a)\in N^2(K_7))<-3.$$

Using the results from the preceding subsection, we find that $x(a,a)\in N^2(K_7)$ if and only if there exists $x\in\mathbf{R}$ satisfying $x(\frac{2a}{1+a},\frac{a+x}{1+a}),\ x(0,\frac{a-x}{1-a})\in N(K_6)$, which in turn is equivalent to the following linear system:

$$\begin{gathered}-\tfrac{1}{3}\leq x\leq 1,\quad -2a+x\geq -1,\quad 6a+x\geq -1,\\ 40a+11x\geq -5,\quad -8a+11x\geq -5,\\ 8a+3x\geq -1,\quad 2a-3x\geq -1,\\ 6a+5x\geq -1,\quad 4a-5x\geq -1.\end{gathered}\tag{7.14}$$

One can verify that the minimum value of $a$ for which (7.14) holds is $-\frac{9}{61}$ (attained at $a = -\frac{9}{61}$, $x = \frac{5}{61}$), and thus

$$x(a, a) \in N^2(K_7) \Longleftrightarrow -\frac{9}{61} \cdot 21 \le a \le 1.$$

As $-\frac{9}{61} \cdot 21 < -3$, we deduce that the gap inequality for $c_7$ is not valid for $N^2(K_7)$.

Finally we prove Proposition 5.4. The equality $\mathrm{CUT}(K_n) = N(K_n)$ $(n \le 5)$ follows from Corollary 4.4, and $\mathrm{CUT}(K_6) = N'(K_6) \subset N_+(K_6)$ from Proposition 5.2. We now verify the strict inclusions $N'_+(K_n) \subset N_+(K_n) \subset N(K_n)$ for $n \ge 6$. It suffices to check them for $n = 6$; the first one follows from the above. For the second one note that $x(-\frac{2}{7}, -\frac{1}{7}) \in N(K_6) \setminus N_+(K_6)$. Indeed, if $x \in N_+(K_6)$, then there exist $x, u$ for which the matrix $Y$ with pattern $(-\frac{2}{7}, -\frac{1}{7}, x, u)$ belongs to $M_+(K_6)$. The inequalities $a + 2b + 2c + d + x \ge -1$ and $-a + 3d - 3x \ge -1$ from (7.6) imply that $x = \frac{2}{7}$, and the inequalities $3c + 3d + u \ge -1$ and $2b - c + d - 2z - u \ge -1$ imply that $u = -\frac{1}{7}$ (we have here $a = b, c = d, x = y = z, u = v$). However, the matrix $Y$ is not sdp since the eigenvalue $\lambda_0$ (from (A.2)) is negative.

**Appendix. Positive semidefinite matrices with a simplified pattern.** We will use the following standard result about Schur complements (see, e.g., [15]).

LEMMA A.1. *Let* $X = \begin{pmatrix} A & B^T \\ B^T & C \end{pmatrix}$ *be a symmetric matrix. If $A$ is nonsingular, then*

$$X \succeq 0 \Longleftrightarrow A \succeq 0 \quad \text{and} \quad C - B^T A^{-1} B \succeq 0.$$

*The matrix $C - B^T A^{-1} B \succeq 0$ is known as the* Schur complement *of $A$ in $X$.*

$$\begin{array}{c} \\ 0 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 23 \\ 24 \\ 25 \\ 26 \\ 34 \\ 35 \\ 36 \\ 45 \\ 46 \\ 56 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc} 0 & 12 & 13 & 14 & 15 & 16 & 23 & 24 & 25 & 26 & 34 & 35 & 36 & 45 & 46 & 56 \end{array} \\
\begin{pmatrix}
1 & a & a & a & a & a & c & c & c & c & c & c & c & c & c & c \\
a & 1 & c & c & c & c & a & a & a & a & x & x & x & x & x & x \\
a & c & 1 & c & c & c & a & x & x & x & a & a & a & x & x & x \\
a & c & c & 1 & c & c & x & a & x & x & a & x & x & a & a & x \\
a & c & c & c & 1 & c & x & x & a & x & x & a & x & a & x & a \\
a & c & c & c & c & 1 & x & x & x & a & x & x & a & x & a & a \\
c & a & a & x & x & x & 1 & c & c & c & c & c & c & u & u & u \\
c & a & x & a & x & x & c & 1 & c & c & c & u & u & c & c & u \\
c & a & x & x & a & x & c & c & 1 & c & u & c & u & c & u & c \\
c & a & x & x & x & a & c & c & c & 1 & u & u & c & u & c & c \\
c & x & a & a & x & x & c & c & u & u & 1 & c & c & c & c & u \\
c & x & a & x & a & x & c & u & c & u & c & 1 & c & c & u & c \\
c & x & a & x & x & a & c & u & u & c & c & c & 1 & u & c & c \\
c & x & x & a & a & x & u & c & c & u & c & c & u & 1 & c & c \\
c & x & x & a & x & a & u & c & u & c & c & u & c & c & 1 & c \\
c & x & x & x & a & a & u & u & c & c & u & c & c & c & c & 1
\end{pmatrix}
\end{array}$$

FIG. A.1. *A matrix $Y \in \mathcal{Y}_6$ with simplified pattern $(a, c, x, u)$.*

Let $Y \in \mathcal{Y}_n$ with simplified pattern $(a, c, x, u)$ (i.e., $a = b$, $c = d$, $x = y = z$, $u = v$) and let $Z$ denote the Schur complement in $Y$ of its $(0, 0)$-entry. Suppose first that $a = c$ and $x = u$. Then $Z$ has the property that the value of its $(ij, hk)$th entry depends only on whether the pairs $ij$ and $hk$ meet. Let $A_n$ (resp., $B_n$) denote the symmetric matrix indexed by $E_n$ whose entries are all equal to 0, except entry $(ij, hk)$ equal to 1 if $|\{i,j\} \cap \{h,k\}| = 1$ (resp., $= 0$). Then

$$Z = (1-a^2)I_{d_n} + (a-a^2)A_n + (x-a^2)B_n$$

(where $I_{d_n}$ is the identity matrix of order $d_n$). The matrices $A_n$ and $B_n$ commute (they are the adjacency matrices of the Johnson scheme $J(n,2)$) and thus have a common basis of eigenvectors. From this it follows that a matrix $X = \alpha A_n + \beta B_n + \gamma I_{d_n}$ has three distinct eigenvalues

$$\lambda_0(X) = 2(n-2)\alpha + \binom{n-2}{2}\beta + \gamma, \quad \lambda_1(X) = -2\alpha + \beta + \gamma,$$
$$\lambda_3(X) = (n-4)\alpha - (n-3)\beta + \gamma.$$

Therefore we deduce that $Y \succeq 0$ if and only if

$$\text{(A.1)} \qquad \begin{array}{c} \lambda_0(Z) = 2(n-2)(a-a^2) + \binom{n-2}{2}(x-a^2) + 1 - a^2 \geq 0, \\ \lambda_1(Z) = -2a + x + 1 \geq 0, \quad \lambda_2(Z) = (n-4)a - (n-3)x + 1 \geq 0. \end{array}$$

In the general case, the matrix $Z$ is not of the form $\alpha A_n + \beta B_n + \gamma_{d_n}$. Let $Z_1$ be its principal submatrix indexed by $\{12, \ldots, 1n\}$; its eigenvalues are $1-c$ and $1+(n-2)c-(n-1)a^2$. If $1-c \neq 0$ and $1+(n-2)c-(n-1)a^2 \neq 0$, we can define the Schur complement $X$ of $Z_1$ in $Z$, which turns out to be of the form $\alpha A_{n-1} + \beta B_{n-1} + \gamma I_{d_{n-1}}$, and whose eigenvalues are therefore computable. We mention the result only in the case $n=6$: Assuming that $c \neq 1$, $1+4c-5a^2 \neq 0$, $Y \succeq 0$ if and only if

$$\text{(A.2)} \qquad \begin{array}{c} c \leq 1, \quad 1+4c-5a^2 \geq 0, \\ \lambda_0(X) = 1 + 6c - 10c^2 + 3u - 2\frac{(2a+3x-5ac)^2}{1+4c-5a^2} \geq 0, \\ \lambda_1(X) = 1 - 2c + u \geq 0, \quad \lambda_2(X) = 1 + c - 2u - 3\frac{(a-x)^2}{1-c} \geq 0. \end{array}$$

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