

On the initial-value problem in the Lifshitz-Slyozov-Wagner theory of Ostwald ripening

Barbara Niethammer
Inst. für Angew. Math.
Universität Bonn
Wegelerstr. 6
53115 Bonn, Germany

Robert L. Pego
Dept. of Mathematics &
Inst. Phys. Sci. Tech.
University of Maryland
College Park, MD 20742 USA

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Abstract

The LSW theory of Ostwald ripening concerns the time evolution of the size distribution of a dilute system of particles that evolve by diffusional mass transfer with a common mean field. We prove global existence, uniqueness and continuous dependence on initial data for measure-valued solutions with compact support in particle size. These results are established with respect to a natural topology on the space of size distributions, one given by the Wasserstein metric which measures the smallest maximum volume change required to rearrange one distribution into another.

1 Introduction

The classical theory of Ostwald ripening, formulated by Lifshitz and Slyozov [3] and Wagner [5] concerns the evolution of the size distribution of a large number of small particles of one phase embedded in a matrix of another phase. Particles are assumed to be widely separated spheres that evolve by diffusional mass transfer with a common mean field. In the late stages of the phase transformation, diffusion is quasi-steady and the particle growth rate is determined by the mass flux at the particle boundary. The mass flux is proportional to the gradient of a potential that is harmonic, is proportional to curvature on the particle boundaries, and is close to constant in the mean field between particles.

In appropriate units, it is found that any particle radius $R(t)$ evolves according to

$$\frac{dR}{dt} = V(R, R_c(t)) := \frac{a}{R^2} \left(\frac{R}{R_c(t)} - 1 \right), \quad (1)$$

where a is a constant and the critical radius $R_c(t)$ is the same for all particles. The value of $R_c(t)$ is determined from conservation of mass. If mass changes in the diffusion field can be neglected, the particle volume is conserved and one finds that the critical radius

equals the average radius of currently existing particles. Particles with radius larger than $R_c(t)$ are growing, and particles with smaller radius shrink and can disappear in finite time.

Classically, the size distribution of particles is described by a particle radius distribution $n(t, R)$. This is a normalized number density that we may scale so that $\int_0^R n(t, r) dr$ is the number of (currently existing) particles with radius less than R , divided by the number N of initially existing particles. The number of particles with size between $R_1(t)$ and $R_2(t)$ for any two solutions of (1) is conserved, so $n(t, R)$ should satisfy the conservation law

$$\partial_t n + \partial_R(Vn) = 0, \quad (2)$$

where the critical radius is given by

$$R_c(t) = \int_0^\infty Rn(t, R) dR \Big/ \int_0^\infty n(t, R) dR. \quad (3)$$

The initial number density $n_0(R) = n(0, R)$ satisfies $\int_0^\infty n_0(R) dR = 1$ in this normalization.

Our aim in this paper is to develop a satisfactory theory of well-posedness for the initial value problem for the particle size distribution. From the physical point of view, it is reasonable to suppose that a positive fraction of the particles can have the same radius, in which case the size distribution contains one or more Dirac deltas. Mathematically, the ideal is to allow the initial data $n_0(R) dR$ to be an arbitrary probability measure such that the total volume $\int_0^\infty \frac{4}{3}\pi R^3 n_0(R) dR$ is finite.

It will be convenient to work with particle volume v instead of radius R , and to work with a cumulative number distribution function φ instead of the number density n . We say that

$$\varphi \text{ is the fraction of (initially existing) particles with volume } \geq v. \quad (4)$$

As a function of volume v at time t , $\varphi(t, v)$ is a monotonically decreasing function which is left continuous at jumps with $\varphi(t, 0) = 1$, and $\int_0^\infty \varphi(t, v) dv$ (the total volume) is independent of time. The particle volume distribution, defined by $f(t, v) dv = -d\varphi(t, v)$ for each fixed t , is formally related to n via $f(t, v) dv = n(t, R) dR$.

We normalize the time scale by the factor $4\pi a$ and let $\theta(t) = (4\pi R_c(t)^3/3)^{-1/3}$, so that the volume $v(t)$ of any existing particle should satisfy

$$\frac{dv}{dt} = \Lambda(v, \theta(t)) := v^{1/3}\theta(t) - 1. \quad (5)$$

If $v(t)$ is a positive solution of (5) on some time interval, then $\varphi(t, v(t))$ should remain constant. This means $\varphi(t, v)$ should be a solution of the hyperbolic equation

$$\partial_t \varphi + \Lambda(v, \theta(t))\partial_v \varphi = 0, \quad (6)$$

whose characteristics satisfy (5). The value of $\theta(t)$ is obtained from φ in terms of Riemann-Stieltjes integrals by

$$\theta(t) = \int_{0^+}^\infty d\varphi(t, v) \Big/ \int_0^\infty v^{1/3} d\varphi(t, v). \quad (7)$$

The numerator is -1 times the quantity $\varphi_0(t) := \lim_{v \rightarrow 0} \varphi(t, v)$, which is the fraction of initially existing particles that still exist at time t .

It turns out to be still better to regard the volume v as a function of the fraction φ , $0 \leq \varphi \leq 1$. We take the map $\varphi \mapsto v(t, \varphi)$ to be right continuous and decreasing with $v(t, 1) = 0$. Mathematically, given $\varphi(t, v)$ we obtain $v(t, \varphi)$ via the prescription

$$v(t, x) = \sup\{y \mid \varphi(t, y) > x\} \quad \text{for } 0 \leq x < 1 = \max \varphi. \quad (8)$$

This is most easily understood when the size distribution corresponds to a finite number of particles. If we list the particle volumes in decreasing order, $v_0(t) \geq \dots \geq v_{N-1}(t)$, then $v(t, \varphi) = v_j$ for $\varphi \in [j/N, (j+1)/N)$. We shall call $\varphi \mapsto v(t, \varphi)$ a *volume ordering* for the system at time t .

For technical simplicity we shall assume that the particle volumes in the system are bounded. This seems reasonable physically, and corresponds to assuming that the particle volume distribution has compact support in v . We then introduce function spaces as follows. Let $rcd([0, 1])$ be the set of functions $v: [0, 1] \rightarrow \mathbb{R}$ that are right continuous, decreasing, and satisfy $v(1) = 0$. (To be precise, we say v is *decreasing* if $v(x_1) \leq v(x_2)$ whenever $x_1 \geq x_2$, and similarly for *increasing*. A decreasing function need not be strictly decreasing.) The set $rcd([0, 1])$ is contained in the space $bdd([0, 1])$ of real-valued bounded functions on $[0, 1]$, equipped with the sup norm $\|v\| = \sup_{\varphi} |v(\varphi)|$. $rcd([0, 1])$ is a complete metric space in the induced topology.

If X is a Banach space and $I \subset \mathbb{R}$ is an interval, then $C(I, X)$ is the space of continuous X -valued functions on I , and $L_{loc}^{\infty}(I)$ is the space of equivalence classes of measurable functions locally bounded on I , where two functions are considered equivalent if they agree almost everywhere.

Our main results are the following.

Theorem 1.1 (*Global existence and uniqueness*) *Let $v_0 \in rcd([0, 1])$. Then there exist unique functions $\theta \in L_{loc}^{\infty}(0, \infty)$ and $v \in C([0, \infty), rcd([0, 1]))$, such that*

$$\int_0^1 v(t, \varphi) d\varphi = \int_0^1 v_0(\varphi) d\varphi$$

for all $t \geq 0$, and

$$v(t, \varphi) = v_0(\varphi) + \int_0^t (v(s, \varphi)^{1/3} \theta(s) - 1) ds$$

for all (t, φ) such that $v(t, \varphi) > 0$.

Theorem 1.2 (*Continuous dependence on initial data*) *Given positive constants T and C_0 , there exists a positive constant C such that, if (v_1, θ_1) and (v_2, θ_2) are two solutions with the properties stated for (v, θ) in Theorem 1.1, and if $\max(v_1(0, 0), v_2(0, 0)) \leq C_0$, then*

$$\sup_{0 \leq t \leq T} \|v_1(t, \cdot) - v_2(t, \cdot)\| \leq C \|v_1(0, \cdot) - v_2(0, \cdot)\|.$$

Consequently, the map $v_0 \mapsto v$ is locally Lipschitz from $rcd([0, 1])$ into $C([0, T], rcd([0, 1]))$.

Our strategy to prove these results at the same time justifies a method of numerical approximation for the problem that has a direct physical interpretation. We first consider solutions that are piecewise constant, taking a finite number of values $v_0(t) > \dots > v_{N-1}(t)$, as is the case for a finite number of particles. We show that these solutions are determined on a succession of time intervals by solving finite systems of coupled ordinary differential equations with a number of components that decreases as the smallest particles vanish. Once we prove the continuity estimate in Theorem 1.2 (at first for initial data near to each other), uniqueness is immediate and existence for general initial data in $rcd([0, 1])$ follows by an approximation argument.

The solutions constructed in Theorem 1.1 correspond to measure-valued weak solutions of the evolution equation

$$\partial_t f + \partial_v(\Lambda(v, \theta(t))f) = 0 \tag{9}$$

for the particle volume distribution. This means that at each time t , the formal expression $f(t, v) dv$ corresponds to a probability measure ν_t having compact support in $[0, \infty)$, the set of volumes. The notion of distance used in Theorem 1.2 has an interpretation as a natural metric on the space \mathcal{P}_0 of such probability measures. This metric measures the smallest ‘maximum volume change’ required to rearrange one volume distribution into another. Mathematically it is the L^∞ Wasserstein metric [2, 4], which we denote by d_∞ . In section 3 we shall establish the relationship between $v(t, \varphi)$ and ν_t , and deduce the following result as a corollary of Theorems 1.1 and 1.2.

Theorem 1.3 *Let \mathcal{P}_0 denote the set of probability measures on $[0, \infty)$ of compact support, with topology given by d_∞ , the L^∞ Wasserstein metric. Given $\nu_0 \in \mathcal{P}_0$, there exists a unique $\theta \in L_{\text{loc}}^\infty(0, \infty)$ and a unique map $t \mapsto \nu_t$ that is locally Lipschitz from $[0, \infty)$ into \mathcal{P}_0 , such that (θ, ν) is a volume-conserving weak solution of (9), in the sense that*

$$\int_0^\infty v d\nu_t(v) = \int_0^\infty v d\nu_0(v)$$

for all $t \geq 0$, and

$$\int_0^\infty \int_0^\infty \partial_t \zeta(t, v) + \Lambda(v, \theta(t)) \partial_v \zeta(t, v) d\nu_t(v) dt = 0$$

for all smooth $\zeta: (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ with compact support.

Furthermore, given any $T > 0$, $C_0 > 0$, there exists $C > 0$ such that, if two such weak solutions $(\theta_1, \nu^{(1)})$, $(\theta_2, \nu^{(2)})$ are given, such that the supports of $\nu_0^{(1)}$ and $\nu_0^{(2)}$ are contained in $[0, C_0]$, then

$$\sup_{0 \leq t \leq T} d_\infty(\nu_t^{(1)}, \nu_t^{(2)}) \leq C d_\infty(\nu_0^{(1)}, \nu_0^{(2)}).$$

It is arguably natural from the physical point of view to measure distance between volume distributions by using the Wasserstein distance as is done here. A physically

reasonable notion of distance should reflect in a plausible way the effect of small perturbations of the system on size distributions. In late-stage Ostwald ripening one imagines that the nucleation or destruction of large particles is unlikely. Thus the topology should not make it ‘easy’ to change the number of large particles. It is plausible, rather, that small perturbations to the system would involve small changes to particle volumes. These notions are captured here by the use of the sup norm distance between volume orderings, and this is equivalent to using the L^∞ Wasserstein metric to compare volume distributions.

In section 4 we briefly treat a related, but simpler, case that arises in LSW theory, in which mass variations in the diffusion field are not neglected. In this case it is not total particle volume that is conserved in time, but rather a quantity of the form

$$a\theta(t) + \int_0^1 v(t, \varphi) d\varphi,$$

where $a > 0$ is constant. The evolution of particle volumes is still given by (5), but θ is now determined directly from the conserved quantity.

2 A priori estimates and well-posedness

In order to prove the a priori estimate stated in Theorem 1.2, we need a pair of lemmas that yield strengthened variants of Gronwall’s inequality.

Lemma 2.1 *Suppose $G: [0, T] \rightarrow \mathbb{R}$ is increasing with $G(0) = 0$, $K \geq 0$ is a constant and $f: [0, T] \rightarrow \mathbb{R}$ is continuous and satisfies*

$$0 \leq f(t) \leq K + \int_{0^+}^t f(s) dG(s), \quad 0 \leq t \leq T.$$

Then $f(t) \leq Ke^{G(t)}$ for $0 \leq t \leq T$.

Proof: Let

$$U(t) = K + \int_{0^+}^t f(s) dG(s),$$

then $U(0) = K$ and U is increasing. To prove the lemma it suffices to show that $e^{-G}U \leq K$. Let $\{t_j\}_{j=0}^n$ be a partition of $[0, T]$ and define

$$\Delta t = \sup_{1 \leq j \leq n} (t_j - t_{j-1}), \quad \epsilon(\Delta t) = \sup_{|t-s| \leq \Delta t} |f(t) - f(s)|.$$

Put $U_j = U(t_j)$, $G_j = G(t_j)$. Then

$$\begin{aligned} e^{-G_{j+1}}U_{j+1} - e^{-G_j}U_j &= e^{-G_{j+1}}(U_{j+1} - U_j) - U_j(e^{-G_j} - e^{-G_{j+1}}) \\ &= e^{-G_{j+1}} \left(\int_{t_j}^{t_{j+1}} f(s) dG(s) - U_j (e^{G_{j+1}-G_j} - 1) \right) \\ &\leq e^{-G_{j+1}} (f(t_j) + \epsilon(\Delta t) - U_j) (G_{j+1} - G_j) \\ &\leq \epsilon(\Delta t)(G_{j+1} - G_j), \end{aligned}$$

where we used that $e^x - 1 \geq x$ for all x and $f(t_j) \leq U_j$. Summing, we find that $e^{-G_j} U_j \leq K + \epsilon(\Delta t) G_j$ for all j . Since the partition is arbitrary, $\epsilon(\Delta t)$ can be made arbitrarily small and the result follows.

Lemma 2.2 *Suppose $G: [0, T] \rightarrow \mathbb{R}$ is increasing, and $f: [0, T] \rightarrow \mathbb{R}$ is continuous and nonnegative and increasing. Then as long as $0 \leq t + f(t) \leq T$ we have*

$$\int_0^t (G(s + f(s)) - G(s)) ds \leq \int_0^{f(0)} (G(f(0)) - G(s)) ds + \int_0^t f(s) d\tilde{G}(s)$$

where $\tilde{G}(s) = G(s + f(s))$.

Proof: Let Q denote the quantity on the right hand side of the desired inequality. Observe that since G is increasing, we have that

$$Q + \int_t^{t+f(t)} (G(s + f(s)) - G(s)) ds \geq Q + \int_t^{t+f(t)} \tilde{G}(s) ds - G(t + f(t))f(t).$$

Since $G(t + f(t)) = \tilde{G}(t)$, after integrating by parts in the Riemann-Stieltjes integral and cancelling boundary terms we find that the last right hand side equals

$$\begin{aligned} & - \int_0^{f(0)} G(s) ds - \int_0^t \tilde{G}(s) df(s) + \int_t^{t+f(t)} \tilde{G}(s) ds \\ & = - \int_0^{f(0)} G(s) ds - \int_0^t \tilde{G}(s) d(s + f(s)) + \int_0^{t+f(t)} \tilde{G}(s) ds \\ & = \int_0^{t+f(t)} (G(s + f(s)) - G(s)) ds. \end{aligned}$$

Cancelling the part of the integral from t to $t + f(t)$ finishes the proof.

Next we establish some basic properties of solutions of the initial value problem as described in Theorem 1.2. Fixing $T > 0$, we shall consider $t \in [0, T]$. Let $\theta \in L^\infty(0, T)$ be positive and let $v \in C([0, T], \text{red}([0, 1]))$ be such that

$$\int_0^1 v(t, \varphi) d\varphi = \int_0^1 v(0, \varphi) d\varphi \tag{10}$$

for all t and

$$v(t, \varphi) = v(0, \varphi) + \int_0^t (v(s, \varphi)^{1/3} \theta(s) - 1) ds \tag{11}$$

whenever $v(t, \varphi) > 0$. By scaling, we may assume $\int_0^1 v(t, \varphi) d\varphi = 1$ for all $t \in [0, T]$.

From (11) it follows that $t \mapsto v(t, \varphi)$ is Lipschitz and satisfies

$$\frac{\partial v}{\partial t} = v^{1/3} \theta(t) - 1 \tag{12}$$

for almost every t in any interval where $v > 0$. Since $v^{1/3}\theta - 1 \leq -\frac{1}{2}$ for $v < \varepsilon_0$ where

$$\varepsilon_0^{-1} = 8 \operatorname{ess\,sup}_{0 \leq t \leq T} \theta(t),$$

it follows easily that if $v(t_0, \varphi) = 0$ then $v(t, \varphi) = 0$ for all $t \geq t_0$.

We define $\bar{v}(t) = v(t, 0) = \max_{\varphi} v(t, \varphi)$ and with the notation $a \wedge b = \min(a, b)$ we define

$$\begin{aligned} \bar{t}(\varphi) &= \inf\{t \in [0, T] \mid v(t, \varphi) = 0\} \wedge T, \\ \bar{\varphi}(t) &= \sup\{\varphi \in [0, 1] \mid v(t, \varphi) > 0\}. \end{aligned}$$

The functions \bar{t} and $\bar{\varphi}$ are decreasing functions, and $\bar{\varphi}(t) > 0$ for all t , since $v(t, \cdot)$ can never vanish identically by volume conservation. We call $\bar{t}(\varphi)$ the *vanishing time* for $v(t, \varphi)$ at φ if $\bar{t}(\varphi) < T$ (but note that $\bar{t}(\varphi) = T$ if $v(T, \varphi) > 0$).

Lemma 2.3 *For almost every $t \in [0, T]$ we have*

$$0 < \theta(t) = \frac{\bar{\varphi}(t)}{\int_0^1 v(t, \varphi)^{1/3} d\varphi} \leq \bar{v}(t)^{2/3} \leq (e^t \bar{v}(0))^{2/3}.$$

Proof: Evaluate (11) at $\min(t, \bar{t}(\varphi))$ and integrate over $\varphi \in [0, 1]$. Changing the order of integration and using the fact that $\bar{v}(\bar{t}(\varphi), \varphi) = 0$ if $\bar{t}(\varphi) < t$, we obtain

$$\begin{aligned} 0 &= \int_0^{\bar{\varphi}(t)} v(t, \varphi) d\varphi - \int_0^1 v(0, \varphi) d\varphi \\ &= \int_0^1 \int_0^{\min(t, \bar{t}(\varphi))} (v(s, \varphi)^{1/3} \theta(s) - 1) ds d\varphi \\ &= \int_0^t \int_0^{\bar{\varphi}(s)} (v(s, \varphi)^{1/3} \theta(s) - 1) d\varphi ds. \end{aligned}$$

Since t is arbitrary the formula for $\theta(t)$ follows. To get the inequalities, we use that $\bar{\varphi}(t) \leq 1$ and $\int_0^1 v^{1/3} d\varphi \geq \bar{v}(t)^{-2/3} \int_0^1 v d\varphi$. Then since $d\bar{v}/dt \leq \bar{v}^{1/3} \theta \leq \bar{v}$ we find

$$\bar{v}(t) \leq e^t \bar{v}(0). \tag{13}$$

Lemma 2.4 *Whenever $v(t_1, \varphi) < \varepsilon_0$, we have $\partial v / \partial t < -\frac{1}{2}$ for almost every $t \in [t_1, \bar{t}(\varphi)]$ and*

$$\frac{1}{2}(\bar{t}(\varphi) - t) \leq v(t, \varphi) < \varepsilon_0 - \frac{1}{2}(t - t_1)$$

for all $t \in [t_1, \bar{t}(\varphi)]$.

Proof: $v < \varepsilon_0$ implies $v^{1/3}\theta - 1 < -\frac{1}{2}$ almost everywhere, and the results follow easily.

Corollary 2.5 *There is a constant $C = C(T, C_0)$ such that*

$$\int_0^{\bar{t}(\varphi)} v(t, \varphi)^{-2/3} dt \leq C$$

for all $\varphi \in [0, 1]$. Furthermore, the function β given by

$$\beta(t) = \int_0^{\bar{\varphi}(t)} v(t, \varphi)^{-2/3} d\varphi$$

is finite for a.e. $t \in [0, T]$ and $\int_0^T \beta(t) dt \leq C$.

Proof: The first assertion follows directly from the estimates of the preceding lemma. The second follows from Fubini's theorem.

Our plan now is to first prove a restricted version of Theorem 1.2, for two solutions that are initially close together. This restricted result will suffice to establish the existence and uniqueness theorem, after which the results of Theorem 1.2 without restriction can be proved.

Proposition 2.6 *Given $T > 0$, $C_0 > 0$, there exist $C > 0$ and $\delta > 0$ such that the bound asserted in Theorem 1.2 holds under the additional assumption that*

$$\|v_1(0, \cdot) - v_2(0, \cdot)\| \leq \delta.$$

To start the proof of this restricted version of Theorem 1.2, we suppose that $T, C_0 > 0$ are given and put

$$\varepsilon_1 = (8e^T C_0)^{-1}.$$

We suppose that (θ_1, v_1) and $(\theta_2, v_2) \in L^\infty(0, T) \times C([0, T], rcd([0, 1]))$ are two solutions of (10) and (11) such that $\max(v_1(0, 0), v_2(0, 0)) \leq C_0$. We define

$$M(t) = \sup_{0 \leq s \leq t} \|v_1(s, \cdot) - v_2(s, \cdot)\|$$

and assume that $M(0) < \varepsilon_1$.

Lemma 2.7 *There is a constant $C_1 = C_1(T, C_0)$ such that for $0 \leq t \leq T$ we have*

$$M(t) \leq C_1 \left(M(0) + \int_0^t |\theta_1(s) - \theta_2(s)| ds \right).$$

Proof: Fix $\varphi \in [0, 1]$. We suppose that $\bar{t}_1(\varphi) \geq \bar{t}_2(\varphi)$ without loss of generality. For $t \in [0, \bar{t}_2(\varphi)]$ we may write

$$\begin{aligned} v_1(t, \varphi) - v_2(t, \varphi) &= v_1(0, \varphi) - v_2(0, \varphi) + \int_0^t v_2(s, \varphi)^{1/3} (\theta_1(s) - \theta_2(s)) ds \\ &\quad + \int_0^t \theta_1(s) (v_1(s, \varphi)^{1/3} - v_2(s, \varphi)^{1/3}) ds. \end{aligned}$$

Using the bounds above for θ_1 and v_2 , and the fact that $|a - b| \leq |a^3 - b^3|/a^2$ whenever $a, b > 0$, with $C_* = (e^T C_0)^{1/3}$ we obtain the estimate

$$\begin{aligned} |v_1(t, \varphi) - v_2(t, \varphi)| &\leq |v_1(0, \varphi) - v_2(0, \varphi)| + C_* \int_0^t |\theta_1(s) - \theta_2(s)| ds \\ &\quad + C_*^2 \int_0^t v_1(s, \varphi)^{-2/3} |v_1(s, \varphi) - v_2(s, \varphi)| ds. \end{aligned} \quad (14)$$

For $t \in [\bar{t}_2(\varphi), \bar{t}_1(\varphi)]$, we have $v_2(t, \varphi) = 0$ and may write

$$v_1(t, \varphi) \leq v_1(\bar{t}_2(\varphi), \varphi) + C_*^2 \int_{\bar{t}_2(\varphi)}^t v_1(s, \varphi)^{1/3} ds.$$

Using (14) with $t = \bar{t}_2(\varphi)$ to estimate $v_1(\bar{t}_2(\varphi), \varphi)$, we find that (14) is valid for all $t \in [0, \bar{t}_1(\varphi)]$. Gronwall's inequality then yields that

$$\begin{aligned} \exp\left(-C_*^2 \int_0^t v_1(s, \varphi)^{-2/3} ds\right) |v_1(t, \varphi) - v_2(t, \varphi)| \\ \leq |v_1(0, \varphi) - v_2(0, \varphi)| + C_* \int_0^t |\theta_1(s) - \theta_2(s)| ds. \end{aligned}$$

Using Corollary 2.5 completes the proof.

Lemma 2.8 *Suppose $M(t) \leq \varepsilon_1$ for $0 \leq t \leq \tau$. Then*

$$|\bar{\varphi}_1(t) - \bar{\varphi}_2(t)| \leq \bar{\varphi}_1(t) - \bar{\varphi}_1(t + 2M(t)) + \bar{\varphi}_2(t) - \bar{\varphi}_2(t + 2M(t)).$$

as long as $t + 2M(t) \leq \tau$.

Proof: Fixing t , by relabeling we can assume $\bar{\varphi}_1(t) \leq \bar{\varphi}_2(t)$. For $\varphi \in [\bar{\varphi}_1(t), \bar{\varphi}_2(t)]$, $s \in [t, \tau]$ we have $v_1(s, \varphi) = 0$ and $v_2(s, \varphi) \leq M(s) \leq \varepsilon_1$ by assumption. By Lemma 2.4, for $s \leq \bar{t}_2(\varphi)$ we have $\partial v_2 / \partial t \leq -\frac{1}{2}$ and therefore $\bar{t}_2(\varphi) \leq \min(t + 2M(t), T)$. Hence $\bar{\varphi}_2(t + 2M(t)) \leq \bar{\varphi}_1(t)$, and the result follows.

Lemma 2.9 *There is a constant $C_2 = C_2(T, C_0)$ and an increasing function $H: [0, T] \rightarrow \mathbb{R}$ depending on v_1 and v_2 , satisfying $H(0) = 0$ and $H(T) \leq C_2$, such that if $M(t) \leq \varepsilon_1$ for $0 \leq t \leq \tau$, then*

$$\int_0^t |\theta_1(s) - \theta_2(s)| ds \leq C_2 M(0) + \int_{0+}^t M(s) dH(s)$$

as long as $t + 2M(t) \leq \tau$.

Proof: Using that $\int v_j^{1/3} d\varphi \geq \bar{v}_j^{-2/3} \geq C_*^{-2}$, from the formula for $\theta(t)$ we obtain that

$$|\theta_1(t) - \theta_2(t)| \leq C_*^2 |\bar{\varphi}_1(t) - \bar{\varphi}_2(t)| + C_*^4 \int_0^1 |v_1^{1/3} - v_2^{1/3}| d\varphi.$$

Let $\varphi_+(t) = \max(\bar{\varphi}_1(t), \bar{\varphi}_2(t))$, then for $\varphi < \varphi_+$ we have

$$|v_1^{1/3} - v_2^{1/3}| \leq \frac{|v_1 - v_2|}{v_1^{2/3} + v_2^{2/3}}.$$

Note that from Corollary 2.5, it follows that with $t_+(\varphi) = \max(\bar{t}_1(\varphi), \bar{t}_2(\varphi))$ we have

$$\int_0^{t_+(\varphi)} \frac{1}{v_1^{2/3} + v_2^{2/3}} dt \leq C(T, C_0).$$

By Fubini's theorem it follows that the function defined by

$$h_0(t) = \int_0^{\varphi_+(t)} \frac{1}{v_1^{2/3} + v_2^{2/3}} d\varphi$$

is finite for a.e. t and is integrable with $\int_0^T h_0(t) dt \leq C(T, C_0)$. Then we have

$$\int_0^1 |v_1^{1/3} - v_2^{1/3}| d\varphi \leq M(t) h_0(t) \tag{15}$$

for a.e. $t \in [0, T]$.

Next, for $j = 1$ and 2 we invoke Lemma 2.2 with $G(t) = -\bar{\varphi}_j(t)$, $f(t) = 2M(t)$, and conclude that as long as $t + 2M(t) \leq \tau$, then

$$\int_0^t \bar{\varphi}_j(s) - \bar{\varphi}_j(s + 2M(s)) ds \leq 2M(0) + \int_{0+}^t 2M(s) dH_j(s)$$

where $H_j(t) = -\bar{\varphi}_j(t + 2M(t)) + \bar{\varphi}_j(2M(0))$. Evidently H_j satisfies $H_j(t) \leq 1$ for all t .

Putting these estimates together with the result of Lemma 2.8, we find that

$$\int_0^t |\theta_1(s) - \theta_2(s)| ds \leq 4C_*^2 M(0) + \int_{0+}^t M(s) dH(s)$$

where

$$H(t) = 2C_*^2(H_1(t) + H_2(t)) + C_*^4 \int_0^t h_0(s) ds.$$

The desired result follows.

The proof of Proposition 2.6 uses a continuation argument based on the estimates above together with the estimate

$$M(\tau) - M(t) \leq 2C_*^3(\tau - t) \quad (16)$$

whenever $0 \leq t \leq \tau \leq T$, which follows from $|\partial v / \partial t| \leq C_*^3$. Since M is increasing, we can find $\tilde{T} \leq T$ such that $\tilde{T} + 2M(\tilde{T}) = T$. With $\tau = t + 2M(t)$, inequality (16) yields

$$M(t + 2M(t)) \leq M(t)(1 + 4C_*^3) \quad (17)$$

whenever $t \leq \tilde{T}$. Now let

$$\Omega = \{t \in [0, \tilde{T}] \mid M(t + 2M(t)) \leq \varepsilon_1\}.$$

If $M(0) \leq \delta_0 := \varepsilon_1 / (1 + 4C_*^3)$, then $0 \in \Omega$ so Ω is nonempty, and clearly Ω is closed. We claim Ω is open in $[0, \tilde{T}]$ if $M(0)$ is sufficiently small.

Given any $t_1 \in \Omega$ we can apply Lemmas 2.7 and 2.9 to deduce that

$$M(t) \leq C_1(1 + C_2)M(0) + C_1 \int_{0^+}^t M(s) dH(s) \quad (18)$$

for $0 \leq t \leq t_1$. Then Lemma 2.1 implies

$$M(t) \leq C_3 M(0) \quad (19)$$

for $0 \leq t \leq t_1$, where $C_3(T, C_0) = \exp(C_1 C_2) C_1 (1 + C_2)$. Using (17) we infer that $M(t_1 + 2M(t_1)) \leq C_4 M(0)$ with $C_4 = C_3(1 + 4C_*^3)$. Provided we assume

$$M(0) \leq \delta_1 := \frac{1}{2} \frac{\varepsilon_1}{C_4},$$

it follows that $M(t_1 + 2M(t_1)) < \varepsilon_1$, and since M is continuous, Ω is open in $[0, \tilde{T}]$.

Consequently we have $\tilde{T} \in \Omega$. Putting $t_1 = \tilde{T}$, this means we have $M(T) \leq \varepsilon_1$ and $M(T) \leq C_4 M(0)$ if $M(0) \leq \delta_1$. This finishes the proof of Proposition 2.6.

Proof of Theorem 1.1: Uniqueness follows immediately from Proposition 2.6. To prove existence for arbitrary $v_0 \in rcd([0, 1])$, by Proposition 2.6 and Lemma 2.3 it evidently suffices to prove global existence for v_0 in a dense set of $rcd([0, 1])$. Solutions in general are constructed by passing to the limit in $C([0, T], rcd([0, 1]))$ for every $T > 0$.

Lemma 2.10 *The set of functions in $rcd([0, 1])$ that take a finite number of values is dense in $rcd([0, 1])$.*

Proof: Let $v_0 \in rcd([0, 1])$ and let $\varepsilon > 0$. Let $y_j = \frac{1}{2}\varepsilon j$ for $j = 0, 1, \dots$, and let

$$v_\varepsilon(\varphi) = \min\{y_j \mid y_j \geq v_0(\varphi)\}$$

for $\varphi \in [0, 1]$. It is easy to see that v_ε has a finite number of values, that $v_\varepsilon \in rcd([0, 1])$, and $\|v_\varepsilon - v_0\| < \varepsilon$. This proves the lemma.

Suppose, then, that $v_0 \in rcd([0, 1])$ takes a finite number of values $y_0 > \dots > y_N = 0$. Then with $\varphi_j = \inf\{\varphi \mid v_0(\varphi) = y_j\}$, we have $0 = \varphi_0 < \dots < \varphi_N \leq 1$ and $v_0(\varphi) = y_j$ for $\varphi \in [\varphi_j, \varphi_{j+1})$, $j = 0, \dots, N - 1$. We start to construct a solution by solving the system of ordinary differential equations

$$w'_j(t) = w_j(t)^{1/3}\Theta(t) - 1, \quad j = 0, \dots, N - 1, \quad (20)$$

with

$$\Theta(t) = \varphi_N \int \sum_{j=0}^{N-1} w_j(t)^{1/3} (\varphi_{j+1} - \varphi_j) \quad (21)$$

and $w_j(0) = y_j$, on a maximal interval $[0, t_N)$ in which $\min w_j(t) > 0$. The solution is smooth and $w_j(t) > w_{j+1}(t)$ by backwards uniqueness for the equation $w' = w^{1/3}\Theta - 1$. The quantity

$$\sum_{j=0}^{N-1} w_j(t) (\varphi_{j+1} - \varphi_j)$$

is conserved in time. Without loss of generality we can assume this quantity is 1.

We can estimate $\Theta(t) \leq w_0(t)^{2/3}$ so $w'_0 \leq w_0$ and hence $w_0(t) \leq e^t y_0$. If $t_N < \infty$, then, it follows that the smallest component vanishes, i.e., $w_{N-1}(t_N^-) = \lim_{t \nearrow t_N} w_{N-1}(t) = 0$.

For $t \in [0, t_N)$ we define $v(t, \varphi) = w_j(t)$ for $\varphi \in [\varphi_j, \varphi_{j+1})$, $j = 0, \dots, N - 1$, and let $\theta = \Theta$. This yields a solution of equations (11) and (10) for $t \in [0, t_N)$. As $t \rightarrow t_N$ from below, the limits $v(t_N^-, \varphi)$ and $\theta(t_N^-)$ exist. The solution can then be re-initialized at time t_N with one less component (N replaced by $N - 1$). After some finite number of such steps the solution must exist globally.

Thus, for $v_0 \in rcd([0, 1])$ with a finite number of values, a global solution exists. Theorem 1.1 follows.

Proof of Theorem 1.2: The additional restriction imposed in Proposition 2.6 can be removed now by considering convex combinations of initial data. Given T, C_0, v_1 , and v_2 as stated, let $C > 0, \delta > 0$ be as given by Proposition 2.6, and let $M_0 = \|v_1(0, \cdot) - v_2(0, \cdot)\|$. Fix an integer $n > M_0/\delta$, and for $j = 0, 1, \dots, n$ let

$$x_j(\varphi) = \left(1 - \frac{j}{n}\right) v_1(0, \varphi) + \left(\frac{j}{n}\right) v_2(0, \varphi)$$

for $\varphi \in [0, 1]$. Then $x_j \in rcd([0, 1])$, $x_j(0) \leq C_0$ for all j and $\|x_{j+1} - x_j\| = M_0/n < \delta$. By the existence theorem there exist corresponding solutions $v = \tilde{v}_j$ to (10)–(11) with $\tilde{v}_j(0, \cdot) = x_j$, and by Proposition 2.6 we have

$$\sup_{0 \leq t \leq T} \|\tilde{v}_{j+1}(t, \cdot) - \tilde{v}_j(t, \cdot)\| \leq C \|x_{j+1} - x_j\| = CM_0/n.$$

Since $v_1 - v_2 = \sum_{j=0}^{n-1} (\tilde{v}_{j+1} - \tilde{v}_j)$, using the triangle inequality we find that

$$\sup_{0 \leq t \leq T} \|v_1(t, \cdot) - v_2(t, \cdot)\| \leq CM_0,$$

as desired.

3 Measure-valued solutions

Our aim here is to describe a precise correspondence between the solutions $v(t, \varphi)$ of Theorem 1.1 and measure-valued weak solutions ν_t of (9), and to show that the metric $\|v_1 - v_2\|$ on $rcd([0, 1])$ corresponds to the L^∞ Wasserstein metric on the space \mathcal{P}_0 of (Borel) probability measures on $[0, \infty)$ with compact support. Theorem 1.3 then follows as a corollary of Theorems 1.1 and 1.2.

We begin with a technical lemma on generalized inverses of increasing functions.

Lemma 3.1 *Suppose $b > 0$ and $w: [0, b] \rightarrow \mathbb{R}$ is a left continuous increasing function with $w(0) = 0$. Let $b^\dagger = w(b)$ and define $w^\dagger: [0, b^\dagger] \rightarrow \mathbb{R}$ by*

$$w^\dagger(y) = \begin{cases} \sup\{x \mid w(x) < y\}, & 0 < y \leq b^\dagger, \\ 0, & y = 0. \end{cases}$$

Then w^\dagger is left continuous and increasing, and moreover,

$$w^{\dagger\dagger} = w.$$

Proof: Clearly w^\dagger is increasing. Given $y \in (0, b^\dagger]$ and $\varepsilon > 0$, put $\bar{x} = w^\dagger(y)$ and $2\delta = y - w(\bar{x} - \varepsilon)$. Then $\delta > 0$ and $w(\bar{x} - \varepsilon) < y - \delta$, hence $\bar{x} - \varepsilon < w^\dagger(y - \delta) \leq \bar{x}$. It follows w^\dagger is left continuous.

To show $w^{\dagger\dagger} = w$, it suffices to show that for $0 < x < b$,

$$w(x - \varepsilon) \leq w^{\dagger\dagger}(x) \leq w(x + \varepsilon)$$

for all sufficiently small $\varepsilon > 0$. Let $\bar{y} = w^{\dagger\dagger}(x) = \sup\{y \mid w^\dagger(y) < x\}$. Then for all $\varepsilon_0 > 0$, $w^\dagger(\bar{y} + \varepsilon_0) \geq x$, hence for any small $\varepsilon > 0$ we have $w(x - \varepsilon) < \bar{y} + \varepsilon_0$, therefore $w(x - \varepsilon) \leq \bar{y}$.

For the reverse inequality there are two cases: If $\bar{x} = w^\dagger(\bar{y}) < x$ then for small $\varepsilon > 0$ we have

$$\bar{y} \leq w(\bar{x} + \varepsilon) \leq w(x + \varepsilon). \quad (22)$$

Otherwise $\bar{x} \geq x$, and since w is left continuous, $\bar{x} = x$. In this case, (22) again holds. This finishes the proof.

If w is continuous and strictly increasing, then w^\dagger is the inverse function of w .

Given a probability measure ν with compact support $[0, \bar{v}] \subset [0, \infty)$, we associate the distribution function $F_\nu: [0, \infty) \rightarrow [0, 1]$ given by

$$F_\nu(x) = \begin{cases} \nu([0, x]) & x > 0 \\ 0, & x = 0. \end{cases} \quad (23)$$

F_ν is left continuous and increasing, and F_ν determines ν (that is, the values of F_ν determine the values of ν on all Borel sets). We associate a decreasing function $v = \hat{v}(\nu)$ to ν via $v(x) = F_\nu^\dagger(1-x)$ for $x \in [0, 1]$. (Here, F_ν^\dagger is the generalized inverse of the restriction of F_ν to $[0, \bar{v} + 1]$.) That is,

$$v(x) = \begin{cases} \sup\{y \mid F_\nu(y) < 1-x\}, & 0 \leq x < 1, \\ 0, & x = 1. \end{cases} \quad (24)$$

With the notation $Rv(x) = v(1-x)$ we have $\hat{v}(\nu) = R(F_\nu^\dagger)$. The first part of Lemma 3.1 implies $\hat{v}(\nu) \in rcd([0, 1])$, thus the map $\hat{v}: \mathcal{P}_0 \rightarrow rcd([0, 1])$. (Recall \mathcal{P}_0 is the set of probability measures on $[0, \infty)$ with compact support.)

The inverse map to \hat{v} is given as follows. If $v \in rcd([0, 1])$ we let $F = (Rv)^\dagger$ on $[0, v(0)]$ and put $F(x) = 1$ for $x > v(0)$. Then F is increasing and left continuous, and determines a (Borel) probability measure ν for which $F = F_\nu$. For later use we note that for any continuous $f: (0, \infty) \rightarrow \mathbb{R}$ with compact support, we have

$$\int_0^1 f(v(x)) dx = \int_0^1 f(F^\dagger(x)) dx = \int_0^\infty f(y) dF(y) = \int_0^\infty f(y) d\nu(y). \quad (25)$$

This follows from [1, 2.5.18(3)], for example. The identity function $y \mapsto y$ can be approximated uniformly on compact sets in $[0, \infty)$ by such functions f , hence

$$\int_0^1 v(x) dx = \int_0^\infty y d\nu(y). \quad (26)$$

We let $\hat{\nu}(v) = \nu$, so $\hat{\nu}: rcd([0, 1]) \rightarrow \mathcal{P}_0$. Lemma 3.1 implies that we have

Lemma 3.2 *\hat{v} and $\hat{\nu}$ are inverse maps: $\hat{\nu}(\hat{v}(v)) = v$ for all $v \in rcd([0, 1])$, and $\hat{v}(\hat{\nu}(\nu)) = \nu$ for all $\nu \in \mathcal{P}_0$.*

We now recall from [2] that the L^p Wasserstein metric can be defined on \mathcal{P}_0 as follows. Given ν_1 and ν_2 in \mathcal{P}_0 , let $D(\nu_1, \nu_2)$ be the set of probability measures μ on $[0, \infty) \times [0, \infty)$ with marginal distributions ν_1 and ν_2 , that is, for all continuous $\zeta: [0, \infty) \rightarrow \mathbb{R}$,

$$\int_0^\infty \int_0^\infty \zeta(x) d\mu(x, y) = \int_0^\infty \zeta(x) d\nu_1(x)$$

and

$$\int_0^\infty \int_0^\infty \zeta(y) d\mu(x, y) = \int_0^\infty \zeta(y) d\nu_2(y).$$

If $1 \leq p < \infty$ then the L^p Wasserstein metric is defined by

$$d_p(\nu_1, \nu_2) = \left(\inf_{\mu \in D(\nu_1, \nu_2)} \int |x - y|^p d\mu(x, y) \right)^{1/p}$$

The L^∞ Wasserstein metric is defined by

$$d_\infty(\nu_1, \nu_2) = \inf_{\mu \in D(\nu_1, \nu_2)} \mu\text{-ess sup } |x - y|.$$

The measures μ represent ways to ‘rearrange mass’ from one distribution into the other, and the L^p Wasserstein metrics measure the least costly way to do this according to the notion of cost indicated.

Lemma 3.3 *Given $\nu_1, \nu_2 \in \mathcal{P}_0$, let $v_1 = \hat{v}(\nu_1)$, $v_2 = \hat{v}(\nu_2)$. Then for $1 \leq p < \infty$ we have*

$$d_p(\nu_1, \nu_2) = \left(\int_0^1 |v_1(\varphi) - v_2(\varphi)|^p d\varphi \right)^{1/p},$$

and

$$d_\infty(\nu_1, \nu_2) = \|v_1 - v_2\|.$$

Proof: The assertion for $1 \leq p < \infty$ follows from [4], see pp 28–30 and Corollary 7.3.6, which yields that

$$d_p(\nu_1, \nu_2) = \left(\int_0^1 |F_{\nu_1}^\dagger(\varphi) - F_{\nu_2}^\dagger(\varphi)|^p d\varphi \right)^{1/p}.$$

Then Proposition 3 of [2] asserts that $\lim_{p \rightarrow \infty} d_p(\nu_1, \nu_2) = d_\infty(\nu_1, \nu_2)$. Since v_1 and v_2 are right continuous, it follows

$$\begin{aligned} d_\infty(\nu_1, \nu_2) &= \lim_{p \rightarrow \infty} \left(\int_0^1 |v_1(\varphi) - v_2(\varphi)|^p d\varphi \right)^{1/p} \\ &= \operatorname{ess\,sup}_{[0,1]} |v_1(\varphi) - v_2(\varphi)| = \sup_{[0,1]} |v_1(\varphi) - v_2(\varphi)|. \end{aligned}$$

Corollary 3.4 *Let \mathcal{P}_0 have the topology induced by d_∞ . Then \mathcal{P}_0 is complete, and the map $\hat{v}: \mathcal{P}_0 \rightarrow \operatorname{rcd}([0, 1])$ is an isometric isomorphism of complete metric spaces.*

The completeness of \mathcal{P}_0 with respect to the metric d_∞ was established in [2].

The correspondence between volume orderings $v \in \operatorname{rcd}([0, 1])$ and volume distributions $\nu \in \mathcal{P}_0$ has been established. Now we seek to show that this correspondence maps solutions to weak solutions and vice-versa.

Proposition 3.5 *Let $\theta \in L_{\text{loc}}^\infty(0, \infty)$, $v \in C([0, \infty), \operatorname{rcd}([0, 1]))$ be a solution of (11). For each $t \geq 0$, let $\nu_t = \hat{v}(v(t, \cdot))$. Then $\nu: [0, \infty) \rightarrow \mathcal{P}_0$ is locally Lipschitz, and ν is a weak solution of (9).*

Proof: From (11) we have that $v: [0, \infty) \rightarrow \operatorname{rcd}([0, 1])$ is locally Lipschitz, therefore $\nu: [0, \infty) \rightarrow \mathcal{P}_0$ is locally Lipschitz by Corollary 3.4.

Let $\zeta: (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ be smooth with compact support. Then for all $\varphi \in [0, 1]$, $t \mapsto \zeta(t, v(t, \varphi))$ is Lipschitz continuous and we have

$$\begin{aligned} 0 &= \int_0^\infty \frac{d}{dt} \zeta(t, v(t, \varphi)) dt \\ &= \int_0^\infty \partial_t \zeta(t, v(t, \varphi)) + \Lambda(v(t, \varphi), \theta(t)) \partial_v \zeta(t, v(t, \varphi)) dt. \end{aligned}$$

Let $F(t, \cdot) = (Rv(t, \cdot))^\dagger$ and let $\varphi(t, \cdot) = 1 - F(t, \cdot)$. We integrate over $\varphi \in [0, 1]$, use Fubini's theorem, and change variables using (25). We obtain

$$\begin{aligned} 0 &= - \int_0^\infty \int_0^\infty (\partial_t \zeta(t, v) + \Lambda(v, \theta(t)) \partial_v \zeta(t, v)) d\varphi(t, v) dt \\ &= \int_0^\infty \int_0^\infty (\partial_t \zeta(t, v) + \Lambda(v, \theta(t)) \partial_v \zeta(t, v)) d\nu_t(v) dt. \end{aligned}$$

Thus (θ, ν) is a weak solution in the sense of Theorem 1.3, as claimed.

Proposition 3.6 *Suppose that $\theta \in L_{\text{loc}}^\infty(0, \infty)$ and $\nu: [0, \infty) \rightarrow \mathcal{P}_0$ is locally Lipschitz, and ν is a weak solution of (9). Let $v(t, \cdot) = \hat{v}(\nu_t)$ for each $t \geq 0$. Then v is a solution of (11).*

Proof: Given θ and ν as described, the map $v: [0, \infty) \rightarrow \text{rcd}([0, 1])$ is locally Lipschitz. We consider test functions ζ of the form

$$\zeta(t, v) = \xi(t)\eta(v) \tag{27}$$

where the functions $\xi, \eta: \mathbb{R} \rightarrow \mathbb{R}$ are smooth with compact support in $(0, \infty)$. Using this form together with the fact that (θ, ν) form a weak solution to (9), and using the change of variables from (25) as previously, we find that

$$0 = \int_0^\infty \int_0^1 \xi'(t)\eta(v(t, \varphi)) + \xi(t)\eta'(v(t, \varphi))\Lambda(v(t, \varphi), \theta(t)) d\varphi dt.$$

Using Fubini's theorem and integrating by parts in time, this gives

$$0 = \int_0^1 \int_0^\infty \xi(t)\tilde{\eta}(v(t, \varphi))(\Lambda(v(t, \varphi), \theta(t)) - \partial_t v(t, \varphi)) dt d\varphi. \tag{28}$$

where $\tilde{\eta} = \eta'$. This formula is justified since for each $\varphi \in [0, 1]$, $v(\cdot, \varphi)$ is Lipschitz, hence differentiable almost everywhere. We note that since v is bounded on compact sets, $\tilde{\eta}$ can be chosen to agree on the range of v with an arbitrary smooth function with compact support in $(0, \infty)$. We do this and drop the tilde. Furthermore, we note that by Lebesgue's dominated convergence theorem, (28) remains valid for any ξ with compact support in $(0, \infty)$ that is the bounded pointwise limit $\xi = \lim_{n \rightarrow \infty} \xi_n$ of a sequence of smooth ξ_n with compact support in $(0, \infty)$, and similarly for $\tilde{\eta}$. For the moment it will suffice to consider $\xi, \tilde{\eta} \in C_c(\mathbb{R}^+)$, the set of continuous functions on $(0, \infty)$ with compact support.

For what follows, we take some care regarding joint measurability in (t, φ) and sets of measure zero. We fix a representative $\tilde{\theta}$ in the equivalence class θ , then drop the tilde. $\partial_t v(t, \varphi)$ need not exist at every point, but equation (28) also holds if $\partial_t v$ is replaced by the upper derivative $\bar{\partial}_t v$ or the lower derivative $\underline{\partial}_t v$, defined by

$$\bar{\partial}_t v(t, \varphi) = \lim_{\varepsilon \rightarrow 0} \sup_{0 < |h| < \varepsilon} \delta^h v(t, \varphi), \quad \underline{\partial}_t v(t, \varphi) = \lim_{\varepsilon \rightarrow 0} \inf_{0 < |h| < \varepsilon} \delta^h v(t, \varphi),$$

where

$$\delta^h v(t, \varphi) = \frac{v(t+h, \varphi) - v(t, \varphi)}{h}.$$

Since v is locally Lipschitz in t uniformly in φ , $\bar{\partial}_t v$ and $\underline{\partial}_t v$ are bounded on compact sets, and $\underline{\partial}_t v \leq \bar{\partial}_t v$.

Lemma 3.7 *As maps from $(0, \infty) \times [0, 1] \rightarrow \mathbb{R}$, $\bar{\partial}_t v$ and $\underline{\partial}_t v$ are Borel measurable. Moreover, $\bar{\partial}_t v = \underline{\partial}_t v$ almost everywhere in $(0, \infty) \times [0, 1]$.*

Proof: Since v is continuous in t uniformly in φ and is right continuous and decreasing in φ , v is lower semicontinuous, hence Borel. Suppose $0 < t_1 < t_2 < \infty$, then for $0 < |h| < t_1$ the map $\delta^h v$ is Borel on $[t_1, t_2] \times [0, 1]$. Let $\{h_j\}$ be a dense sequence in $(-1, 0) \cup (0, 1)$. Since the maximum of two Borel functions is Borel and pointwise limits of sequences of Borel functions are Borel, and pointwise we have

$$\sup_{0 < |h| < \varepsilon} \delta^h v(t, \varphi) = \sup_{|h_j| < \varepsilon} \delta^{h_j} v(t, \varphi) = \lim_{k \rightarrow \infty} \max_{\substack{j \leq k \\ |h_j| < \varepsilon}} \delta^{h_j} v(t, \varphi),$$

by taking ε to zero along a sequence it follows that $\bar{\partial}_t v$ is Borel on $[t_1, t_2] \times [0, 1]$, hence on $(0, \infty) \times [0, 1]$. A similar argument applies for $\underline{\partial}_t v$.

Now we have that the set $Z = \{(t, \varphi) \mid (\bar{\partial}_t v - \underline{\partial}_t v)(t, \varphi) > 0\}$ is a Borel set. We know that for each φ , $v(\cdot, \varphi)$ is differentiable almost everywhere, so $(\bar{\partial}_t v - \underline{\partial}_t v)(t, \varphi) = 0$ for almost every $t > 0$. Fubini's theorem now implies that Z has Lebesgue measure zero in $(0, \infty) \times [0, 1]$.

Returning to (28), we can now apply Fubini's theorem and deduce that for almost every t , $(\bar{\partial}_t v - \underline{\partial}_t v)(t, \varphi) = 0$ for almost every φ , and with

$$J_\eta(t) = \int_0^1 \eta(v(t, \varphi)) (\Lambda(v(t, \varphi), \theta(t)) - \bar{\partial}_t v(t, \varphi)) d\varphi,$$

we have that for any $\eta \in C_c(\mathbb{R}^+)$, $\int_0^\infty \xi(t) J_\eta(t) dt = 0$ for all $\xi \in C_c(\mathbb{R}^+)$. Therefore, given η there is a set $\Omega_\eta \subset (0, \infty)$ of full measure (meaning the complement has measure zero), such that $J_\eta(t) = 0$ for all $t \in \Omega_\eta$.

The set $C_c(\mathbb{R}^+)$ is separable, so if we do this for a dense sequence $\{\eta_n\}$ we find there is a set $\Omega \subset \cap \Omega_{\eta_n}$ of full measure in $(0, \infty)$ such that for $t \in \Omega$, $J_{\eta_n}(t) = 0$ for all n . Since any $\eta \in C_c(\mathbb{R}^+)$ can be approximated uniformly by a subsequence of $\{\eta_n\}$, we infer that:

Lemma 3.8 *There is a set $\Omega \subset (0, \infty)$ of full measure, such that for all $t \in \Omega$, $\partial_t v(t, \varphi)$ exists for almost every $\varphi \in [0, 1]$, and $J_\eta(t) = 0$ for all $\eta \in C_c(\mathbb{R}^+)$.*

Lemma 3.9 *Let $t \in \Omega$, and suppose $v(t, x) = v(t, y)$ where $0 \leq x < y \leq 1$. Then $\partial_t v(t, \varphi)$ exists for all $\varphi \in (x, y)$, and is constant on this interval.*

The proof of this lemma is straightforward, using the facts that $v(t, \cdot)$ is decreasing for every t , and $\partial_t v(t, \varphi)$ exists for almost every φ .

Lemma 3.10 *Let $t \in \Omega$, and let $\bar{\varphi}(t) = \sup\{\varphi \mid v(t, \varphi) > 0\}$. Then*

$$\Lambda(v(t, \varphi), \theta(t)) - \partial_t v(t, \varphi) = 0$$

for almost all $\varphi \in [0, \bar{\varphi}(t)]$.

Proof: We thank B. Kirchheim for the main idea of the following proof. Since t is fixed, we suppress indicating dependence on t , and we let $g(\varphi) = \Lambda(v(t, \varphi), \theta(t)) - \bar{\partial}_t v(t, \varphi)$. We know g is measurable and bounded. $\varphi \mapsto v(\varphi)$ is decreasing, so if $0 \leq y$ is in the range of v , either the pre-image $v^{-1}(y)$ is a singleton or an interval of nonzero width. There can be only a countable set of y of the latter type. Let Δ be the (countable) set of endpoints of such intervals. For $x, y \in [0, 1] \setminus \Delta$, we know that $v(x) = v(y)$ implies $\bar{\partial}_t v(x) = \bar{\partial}_t v(y)$, and so $g(x) = g(y)$.

Given any $\varepsilon > 0$, let $A_\varepsilon = [0, \bar{\varphi}(t)] \cap \{x \in [0, 1] \mid g(x) > \varepsilon\} \setminus \Delta$. Then A_ε is measurable, and we claim the measure of A_ε is zero for any $\varepsilon > 0$. Suppose not, so $|A_\varepsilon| = 2\delta > 0$ for some $\varepsilon > 0$. By Lusin's theorem, there is a compact $K \subset A_\varepsilon$ such that $|K| \geq \delta$ and $v|_K$ is continuous. Then $v(K)$ is compact and is contained in $(0, \infty)$ since v is positive at each point of $[0, \bar{\varphi}(t)]$.

Apply Lemma 3.8 with $\eta(\varphi) = \eta_n(\varphi) = \max\{0, 1 - n \operatorname{dist}(\varphi, v(K))\}$ for $n = 1, 2, \dots$. Then η_n has compact support in $(0, \infty)$ for n sufficiently large and converges boundedly pointwise to the characteristic function $\chi_{v(K)}$. It follows that

$$0 = \int_0^1 \chi_{v(K)}(v(\varphi))g(\varphi) d\varphi.$$

Now if $v(x) \in v(K)$, then $v(x) = v(y)$ for some $y \in K$, and either $g(x) = g(y) > \varepsilon$ or $x \in \Delta$. It follows

$$\int_0^1 \chi_{v(K)}(v(\varphi))g(\varphi) d\varphi \geq \varepsilon|K| > 0,$$

yielding a contradiction. Hence $|A_\varepsilon| = 0$ for any $\varepsilon > 0$. A similar argument applies for $\{x \mid g(x) < -\varepsilon\}$, and we then deduce that $g(\varphi) = 0$ for almost every $\varphi \in [0, \bar{\varphi}(t)]$. This proves the Lemma.

Note that for $t \in \Omega$ and $\varphi \in (\bar{\varphi}(t), 1]$ we have that $v(t, \varphi) = 0$, and $\partial_t v(t, \varphi) = 0$.

Since $(t, \varphi) \mapsto v(t, \varphi)$ is right continuous in φ and locally Lipschitz in t uniformly in φ , the set $Q = \{(t, \varphi) \mid v(t, \varphi) > 0\}$ is open in $(0, \infty) \times [0, 1]$. Define

$$g(t, \varphi) = \Lambda(v(t, \varphi), \theta(t)) - \bar{\partial}_t v(t, \varphi),$$

then g is measurable on $(0, \infty) \times [0, 1]$ and by Lemma 3.10 we have

$$\int_0^\infty \int_0^1 \chi_Q |g| d\varphi dt = 0.$$

By Fubini's theorem, we have $\chi_Q g = 0$ almost everywhere. Hence there exists a set S of full measure in $[0, 1]$, such that if $\varphi \in S$ then $(\chi_Q g)(t, \varphi) = 0$ for almost every t , and $t \mapsto v(t, \varphi)$ is differentiable almost everywhere.

Lemma 3.11 *If $\varphi \in S$ and $v(t, \varphi) > 0$, then $v(s, \varphi) > 0$ for all $s \in [0, t]$ and*

$$v(t, \varphi) = v(0, \varphi) + \int_0^t \Lambda(v(s, \varphi), \theta(s)) ds. \quad (29)$$

Proof: For any $t_1 \in (0, t)$, if $v(s, \varphi) > 0$ for all $s \in [t_1, t]$ then since $s \mapsto v(s, \varphi)$ is differentiable and $g(s, \varphi) = 0$ almost everywhere in $[t_1, t]$, we have

$$v(t, \varphi) = v(t_1, \varphi) + \int_{t_1}^t \Lambda(v(s, \varphi), \theta(s)) ds.$$

We claim that the set $\{t_1 \in [0, t] \mid v(s, \varphi) > 0 \text{ for all } s \in [t_1, t]\}$ has the infimum $t_* = 0$. Note that the set is nonempty by the continuity of $s \mapsto v(s, \varphi)$. Suppose the infimum t_* is positive. Then $v(t_*, \varphi) = 0 < v(s, \varphi)$ for $s \in (t_*, t]$. We know that $\theta(s)$ is bounded for $s \in [0, t]$, so for some sufficiently small $h > 0$ it follows that $\Lambda(v(s, \varphi), \theta(s)) < -1/2$ for $t_* < s < t_* + h$. Then we have

$$0 < v(t_* + h, \varphi) = 0 + \int_{t_*}^{t_*+h} \Lambda(v(s, \varphi), \theta(s)) ds < -h/2,$$

a contradiction. Hence our claim holds: $v(s, \varphi) > 0$ for all $s \in [0, t]$, and the formula asserted in the Lemma follows.

Now we can finish the proof of Proposition 3.6. Suppose $t > 0$, $\varphi \in [0, 1]$ are arbitrary and $v(t, \varphi) > 0$. Since $v(t, \cdot)$ is right continuous and decreasing, there exists a sequence of numbers $\varphi_n \in S$ such that $\varphi_n > \varphi$, $\varphi_n \rightarrow \varphi$ as $n \rightarrow \infty$ and $v(t, \varphi_n) > 0$. Using Lemma 3.11 it follows that $v(s, \varphi) > 0$ for all $s \in [0, t]$ and

$$\begin{aligned} v(t, \varphi) - v(0, \varphi) - \int_0^t \Lambda(v(s, \varphi), \theta(s)) ds \\ = \lim_{n \rightarrow \infty} \left(v(t, \varphi_n) - v(0, \varphi_n) - \int_0^t \Lambda(v(s, \varphi_n), \theta(s)) ds \right) = 0. \end{aligned}$$

This completes the proof of Proposition 3.6.

The results asserted in Theorem 1.3 now follow directly from Theorems 1.1 and 1.2, with the help of Propositions 3.5 and 3.6, Corollary 3.4, and equation (26).

4 A different conserved quantity

In the theory of Ostwald ripening, one also encounters an alternative to the condition that the total particle volume is conserved in time. If mass in the diffusion field is taken into account, one finds that a quantity of the form

$$Q = a\theta(t) + \int_0^1 v(t, \varphi) d\varphi \quad (30)$$

is conserved instead, where $a > 0$ is a constant. $\theta(t)$ need no longer be positive.

In terms of the theory developed in this paper, the constraint (30) is simpler to deal with than the constraint of constant volume. One has the bound

$$\theta(t) \leq Q/a,$$

and when comparing two solutions of (11), one can use the arguments of Lemma 2.7 and replace the use of Lemma 2.9 by the simpler estimate

$$|\theta_1(t) - \theta_2(t)| \leq a^{-1} \|v_1(t, \cdot) - v_2(t, \cdot)\| + a^{-1} |Q_1 - Q_2|. \quad (31)$$

From the standard Gronwall's inequality, one easily deduces the a priori estimate asserted in the following result. The existence and uniqueness proofs go the same as in section 3.

Theorem 4.1 *Let $v_0 \in \text{rcd}([0, 1])$, $Q \in \mathbb{R}$. Then there exists a unique function $v \in C([0, \infty), \text{rcd}([0, 1]))$ such that, with $\theta(t)$ determined by (30), we have*

$$v(t, \varphi) = v_0(\varphi) + \int_0^t (v(s, \varphi)^{1/3} \theta(s) - 1) ds$$

whenever $v(t, \varphi) > 0$.

Given $T > 0$, $C_0 > 0$, there exists a positive constant C such that, given two solutions as above which also satisfy $\max(Q_1, Q_2) \leq C_0$, then

$$\sup_{0 \leq t \leq T} \|v_1(t, \cdot) - v_2(t, \cdot)\| \leq C (\|v_1(0, \cdot) - v_2(0, \cdot)\| + |Q_1 - Q_2|).$$

Using the correspondence $v(t, \cdot) \mapsto \nu_t = \hat{\nu}(v(t, \cdot))$ and its inverse as in section 3, from Propositions 3.5 and 3.6 one may deduce directly the following corollary of Theorem 4.1.

Theorem 4.2 *Given $\nu_0 \in \mathcal{P}_0$, $Q \in \mathbb{R}$, there exists a unique map $t \mapsto \nu_t$ that is locally Lipschitz from $[0, \infty)$ into \mathcal{P}_0 such that, with $\theta(t)$ determined by the relation*

$$Q = a\theta(t) + \int_0^\infty v d\nu_t(v),$$

we have

$$\int_0^\infty \int_0^\infty \partial_t \zeta(t, v) + \Lambda(v, \theta(t)) \partial_v \zeta(t, v) d\nu_t(v) dt = 0$$

for all smooth $\zeta: (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ with compact support.

Given any $T > 0$, $C_0 > 0$, there exists $C > 0$ such that, if two such weak solutions $\nu^{(1)}, \nu^{(2)}$ are given, which satisfy $\max(Q_1, Q_2) \leq C_0$, then

$$\sup_{0 \leq t \leq T} d_\infty(\nu_t^{(1)}, \nu_t^{(2)}) \leq C \left(d_\infty(\nu_0^{(1)}, \nu_0^{(2)}) + |Q_1 - Q_2| \right).$$

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References

- [1] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.
- [2] C. R. Givens and R. M. Shortt, A class of Wasserstein metrics for probability distributions, *Mich. Math. J.* **31** (1984) 231–240.
- [3] I. M. Lifshitz and V. V. Slyozov, The kinetics of precipitation from supersaturated solid solutions, *J. Phys. Chem. Solids* **19** (1961) 35–50.
- [4] S. T. Rachev, *Probability Metrics and the Stability of Stochastic Models*, Wiley, New York, 1991.
- [5] C. Wagner, Theorie der Alterung von Niederschlägen durch Umlösen, *Z. Elektrochem.* **65** (1961) 581–594.