

Short Communication: Minimal Quantile Functions Subject to Stochastic Dominance Constraints*

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Abstract. We consider a problem of finding a second-order stochastic dominance (SSD)–minimal quantile function subject to the mixture of first-order stochastic dominance (FSD) and SSD constraints. The SSD-minimal solution is explicitly worked out and has a close relation to the Skorokhod problem. This result is then applied to explicitly solve a risk minimizing problem in financial economics.

Key words. SSD-minimal, stochastic dominance, Skorokhod lemma, complete market, risk minimizing

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1. Introduction. Stochastic dominance (first/second-order) plays an important role in statistics, financial economics, insurance, etc. In this paper we consider a problem of finding a second-order stochastic dominance (SSD)–minimal quantile function subject to the mixture of first-order stochastic dominance (FSD) and SSD constraints. More precisely, given two quantile functions Q_1 and Q_2 , the problem is to find a quantile function Q^* such that

- (i) Q^* first-order stochastically dominates Q_1 and second-order stochastically dominates Q_2 ;
- (ii) Q^* is the SSD-minimal one, in the sense that if a quantile function Q also satisfies condition (i), then Q second-order stochastically dominates Q^* .

Such an SSD-minimal solution has applications to determine the lower optimal stopping value of a sequence of random variables whose joint distribution is not known (only the marginal distributions are known); see Müller and Rüschendorf [11]. For the special case when the benchmark distribution functions have at most countable crossing points, Müller and Rüschendorf [11] provided an explicit construction of the SSD-minimal solution. In the iterated application of the SSD-minimal solution of Müller and Rüschendorf [11, Proposition 2.2 and Theorem 2.1],

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it is, however, not easy to verify that F_i and G_{i+1}^* have at most countable crossing points since G_{i+1}^* itself is not a given distribution but a part of the solution of the problem. To facilitate the applications of SSD-minimal quantile functions, we need to find an explicit construction for the general case. Such an observation motivates us to discuss the SSD-minimal problem for the general case.

We will study the general case and show that the problem is highly related to the Skorokhod problem. The main contribution of this paper is to show that such an SSD-minimal quantile function always exists in the general case and to provide an explicit expression for the unique solution.

Apart from the problem of the lower optimal stopping value of random variables in Müller and Rüschemdorf [11], the SSD-minimal quantile function can also be applied to explicitly solve a risk minimizing problem in financial economics, which reduces to an expenditure minimizing problem (EMP).

The classical EMP is to find an optimal payoff which has a given probability distribution function and minimizes the price of its replication. This problem can go back at least to Dybvig [4] and is further investigated by Schied [13], Carlier and Dana [3], and Jin and Zhou [8]. The related result plays an important role in the so called “quantile formulation” of the problem to find the optimal payoffs for preferences described by nonexpected utilities such as rank-dependent utilities, rank linear utilities, and cumulative prospect theory; see, e.g., Carlier and Dana [3], Jin and Zhou [8], He and Zhou [7], Xia and Zhou [15], and Xu [17].

In a nonatomic probability space and a complete market, the EMP with an FSD constraint is essentially equivalent to the problem with the constraint that the payoff has a given distribution. Moreover, the problem with an SSD constraint was investigated by Föllmer and Schied [6] and has the same solution as the one with an FSD constraint; see Föllmer and Schied [6, Theorem 3.44]. The risk minimizing problem in this paper is related to the EMP with a mixture of an FSD constraint and an SSD constraint. We will investigate this problem and provide an explicit optimal solution.

The rest of the paper is organized as follows. Section 2 formulates a problem of finding an SSD-minimal quantile function subject to the mixture of FSD and SSD constraints and presents our main result. Section 3 provides an application of the main result to a risk minimizing problem in financial economics. The main result is proved in section 4.

2. Problem formulation and main result. We will use quantile functions,¹ instead of probability distribution functions, to formulate the problem. Let \mathcal{Q} denote the set of (upper) quantile functions of all probability distribution functions; then

$$\mathcal{Q} = \{Q : (0, 1) \rightarrow \mathbb{R} \mid Q \text{ is increasing and right-continuous}\}.$$
²

For more details about quantile functions, see Föllmer and Schied [6, Appendix A.3].

Let $Q, Q_0 \in \mathcal{Q}$. We say that Q first-order stochastically dominates Q_0 and write it $Q \succeq_{(1)} Q_0$ if

$$Q(t) \geq Q_0(t) \quad \text{for all } t \in (0, 1).$$

¹Quantile functions are the general inverse of probability distribution functions.

²Throughout the paper “increasing” means “nondecreasing,” and “decreasing” means “nonincreasing.”

We say that Q second-order stochastic dominates Q_0 and write it $Q \succeq_{(2)} Q_0$ if

$$\int_0^t Q(s) ds \geq \int_0^t Q_0(s) ds \quad \text{for all } t \in (0, 1).$$

Clearly, $Q \succeq_{(1)} Q_0$ implies $Q \succeq_{(2)} Q_0$. Given a quantile function $Q_0 \in \mathcal{Q}$, let

$$\mathcal{Q}_i(Q_0) \triangleq \{Q \in \mathcal{Q} \mid Q \succeq_{(i)} Q_0\}, \quad i \in \{1, 2\}.$$

Both $\mathcal{Q}_1(Q_0)$ and $\mathcal{Q}_2(Q_0)$ are obviously convex sets.

Consider two benchmark quantile functions $Q_1, Q_2 \in \mathcal{Q}$ that satisfy the following assumption.

Assumption 2.1. $\int_0^1 Q_i^-(s) ds < \infty, i \in \{1, 2\}$.

Definition 2.2. A quantile function $Q^* \in \mathcal{Q}_1(Q_1) \cap \mathcal{Q}_2(Q_2)$ is called SSD-minimal in $\mathcal{Q}_1(Q_1) \cap \mathcal{Q}_2(Q_2)$ if $Q \in \mathcal{Q}_2(Q^*)$ for every $Q \in \mathcal{Q}_1(Q_1) \cap \mathcal{Q}_2(Q_2)$.

One can easily show that there is at most one SSD-minimal quantile function in $\mathcal{Q}_1(Q_1) \cap \mathcal{Q}_2(Q_2)$. Müller and Rüschendorf [11] gave an explicit construction of the SSD-minimal quantile function when the two benchmark distributions have at most a countable discrete set of crossing points. We will show that, in the general case, the SSD-minimal quantile function Q^* always exists and can be characterized by the solution to the following ordinary differential equation (ODE)³ with respect to ϕ :

$$(2.1) \quad \begin{cases} \min\{\phi', \phi - f\} = 0 & \text{almost everywhere (a.e.) in } (0, 1), \\ \phi(0) = 0, \end{cases}$$

where the function f is defined as

$$f(t) \triangleq \int_0^t (Q_2(r) - Q_1(r)) dr \quad \text{for all } t \in [0, 1].$$

Obviously, Assumption 2.1 guarantees that f is finite and continuous on $[0, 1]$.

The next theorem characterizes the solution of ODE (2.1), which, as we will see, is closely related to the Skorokhod problem.

Theorem 2.3. Under Assumption 2.1, the unique solution to ODE (2.1) is given by

$$(2.2) \quad \phi(t) = \max_{0 \leq s \leq t} f(s) \quad \text{for all } t \in [0, 1].$$

Proof. ODE (2.1) is essentially the well-known Skorokhod problem; see, e.g., Revuz and Yor [12, Lemma VI.2.1]. Actually, let $z(t) = \phi(t) - f(t)$; then (2.1) is obviously equivalent to

- (i) $z \geq 0$,
- (ii) $\phi(0) = 0$, ϕ is increasing, and
- (iii) ϕ is flat off $\{t \in (0, 1) \mid z(t) = 0\}$, i.e., $\int_0^1 \mathbf{1}_{z(t)>0} d\phi(t) = 0$.⁴

³A solution to (2.1) is an absolutely continuous function $\phi : [0, 1] \rightarrow \mathbb{R}$ that satisfies (2.1).

⁴Hereafter, $\mathbf{1}_A$ is the indicator function for a statement A , whose value is 1 if the statement A is true and 0 otherwise.

Therefore, by the Skorokhod lemma, the unique solution to ODE (2.1) is (2.2). \blacksquare

The next theorem characterizes the SSD-minimal quantile function Q^* , whose proof is deferred to section 4.

Theorem 2.4. *Suppose Assumption 2.1 holds. Let ϕ be given as in Theorem 2.3, and let*

$$(2.3) \quad Q^*(t) \triangleq Q_1(t)\mathbf{1}_{\phi(t)>f(t)} + (Q_1(t) \vee Q_2(t))\mathbf{1}_{\phi(t)=f(t)} \quad \text{for all } t \in (0, 1).$$

Then Q^* is SSD-minimal in $\mathcal{Q}_1(Q_1) \cap \mathcal{Q}_2(Q_2)$.

Remark 2.5. We know that $Q \in \mathcal{Q}_1(Q_1) \cap \mathcal{Q}_2(Q_2)$ if and only if

$$(2.4) \quad \min \left\{ Q(t) - Q_1(t), \int_0^t Q(r) dr - \int_0^t Q_2(r) dr \right\} \geq 0 \quad \text{for all } t \in (0, 1).$$

Let Q^* be given as in Theorem 2.4. Then by (2.1) we get

$$\min \left\{ Q^*(t) - Q_1(t), \int_0^t Q^*(r) dr - \int_0^t Q_2(r) dr \right\} = 0 \quad \text{for all } t \in (0, 1).$$

Therefore, the SSD-minimal quantile function Q^* makes the constraint in (2.4) binding for all $t \in (0, 1)$.

Remark 2.6. In the previous discussion, the problem has been investigated for the mixture of only one FSD constraint and only one SSD constraint. Actually, the multiple FSD (resp., SSD) constraints can be reduced to a single FSD (resp., SSD) constraint; see, e.g., Wang and Xia [14, Remarks 3.5] for details.

3. Risk minimizing problem. In this section we apply our main result to a risk minimizing problem in financial economics. Consider a nonatomic complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let L^0 denote all \mathcal{F} -measurable random variables and L^1 all integrable $X \in L^0$. For every $X \in L^0$, its (upper) quantile function is defined by

$$Q_X(t) = \inf \{x \in \mathbb{R} : \mathbb{P}(X \leq x) > t\}, \quad t \in (0, 1).$$

Given two benchmark payoffs $X_1 \in L^0$ and $X_2 \in L^1$, consider the acceptance sets

$$\mathcal{A}_i = \{X \in L^0 \mid X \succeq_{(i)} X_i\}, \quad i \in \{1, 2\},$$

where $X \succeq_{(i)} X_i$ is synonymous with $Q_X \succeq_{(i)} Q_{X_i}$. Let the risk measures \mathcal{R}_i , $i \in \{1, 2\}$, be derived from \mathcal{A}_i as follows:

$$\mathcal{R}_i(X) \triangleq \inf \{m \in \mathbb{R} \mid X + m \in \mathcal{A}_i\} \quad \text{with } \inf \emptyset = \infty, \quad X \in L^0.$$

Then \mathcal{R}_1 is called the *loss value at risk* and \mathcal{R}_2 the *benchmark adjusted expected shortfall*; see Bignozzi, Burzoni, and Munari [1] and Burzoni, Munari, and Wang [2]. Moreover, we have the representations

$$\mathcal{R}_1(X) = \sup_{t \in (0,1)} \{\text{VaR}_t(X) - \text{VaR}_t(X_1)\}$$

and

$$\mathcal{R}_2(X) = \sup_{t \in (0,1)} \{\text{ES}_t(X) - \text{ES}_t(X_2)\},$$

where $\text{VaR}_t(X)$ and $\text{ES}_t(X)$ are the *value at risk* and *expected shortfall* of X at the level $t \in (0, 1)$, defined as

$$\text{VaR}_t(X) \triangleq -Q_X(t) \quad \text{and} \quad \text{ES}_t(X) \triangleq -\frac{1}{t} \int_0^t Q_X(s) \, ds.$$

For a general relation between acceptance sets and monetary risk measures, see, e.g., Föllmer and Schied [6].

It is natural to consider an acceptance set \mathcal{A} given by

$$\mathcal{A} \triangleq \{X \in L^0 \mid X \succeq_{(i)} X_i \text{ for } i \in \{1, 2\}\} = \mathcal{A}_1 \cap \mathcal{A}_2.$$

Let the risk measure \mathcal{R} be derived from \mathcal{A} ; that is,

$$(3.1) \quad \mathcal{R}(X) \triangleq \inf \{m \in \mathbb{R} \mid X + m \in \mathcal{A}_1 \cap \mathcal{A}_2\} \text{ with } \inf \emptyset = \infty, \quad X \in L^0.$$

Clearly, \mathcal{R} is monotone and translation invariant, $\mathcal{R} = \mathcal{R}_1 \vee \mathcal{R}_2$, and $X \in \mathcal{A}_1 \cap \mathcal{A}_2$ iff $\mathcal{R}(X) \leq 0$. We proceed with the following technical assumption.

Assumption 3.1. $\int_0^1 |Q_{X_2}(s)| \, ds < \infty$ and $\int_0^1 |Q_{X_i}(s)| Q_\rho(1-s) \, ds < \infty$ for $i \in \{1, 2\}$.

Consider an arbitrage-free market. Assume the market is complete and has a unique stochastic discount factor (SDF)⁵ $\rho \in L^0$ satisfying $\mathbb{P}(\rho > 0) = 1$ and $\mathbb{E}[\rho] < \infty$. Given a budget level $x > 0$, the risk minimizing problem for risk measure \mathcal{R} is defined as

$$(3.2) \quad \underset{X \in \mathcal{X}_\rho}{\text{minimize}} \mathcal{R}(X) \quad \text{subject to} \quad \mathbb{E}[\rho X] \leq x,$$

where

$$\mathcal{X}_\rho \triangleq \{X \in L^0 \mid \mathbb{E}[\rho X^+] < \infty\}.$$

Thanks to the monotonicity of the risk measure \mathcal{R} , solving problem (3.2) can reduce to solving the following EMP:

$$(3.3) \quad \underset{X \in \mathcal{X}_\rho}{\text{minimize}} \mathbb{E}[\rho X] \quad \text{subject to} \quad \mathcal{R}(X) \leq 0.$$

Lemma 3.2 below states quantitatively the relationship between the optimal solution to (3.2) and that to (3.3).

Lemma 3.2. *Suppose Assumption 3.1 holds. Then the optimal values of problems (3.2) and (3.3) are finite. If X^* is an optimal solution to problem (3.2), then $X^* + \mathcal{R}(X^*)$ is an optimal solution to problem (3.3). On the other hand, if X^* is an optimal solution to problem (3.3), then $X^* + \frac{x - \mathbb{E}[\rho X^*]}{\mathbb{E}[\rho]}$ is an optimal solution to problem (3.2).*

⁵Also sometimes termed “pricing kernel” or “state price density” in the financial economics literature.

Proof. Without loss of generality, we can assume that X_1 and X_2 are anticomonotonic with ρ because \mathcal{R} is law-invariant. In this case, Assumption 3.1 reads $\mathbb{E}[|X_2|] < \infty$ and $\mathbb{E}[\rho|X_i|] < \infty, i \in \{1, 2\}$. Let v_1 and v_2 denote, respectively, the optimal values of problems (3.2) and (3.3). Let c be any real constant. Then

$$\mathcal{R}_1(|X_1| + c) = \mathcal{R}_1(|X_1|) - c \leq \mathcal{R}_1(X_1) - c = -c < \infty.$$

By the well-known fact that $\text{ES}_t(X_2) \geq -\mathbb{E}[X_2]$,

$$\mathcal{R}_2(|X_1| + c) \leq \mathcal{R}_2(c) = \sup_{t \in (0,1)} \{-c - \text{ES}_t(X_2)\} \leq -c + \mathbb{E}[X_2] < \infty.$$

Since $\mathbb{E}[\rho] > 0$, for sufficiently small c , we have that $\mathbb{E}[\rho(|X_1| + c)] \leq x$ and $|X_1| + c$ is a feasible solution to problem (3.2), which leads to $v_1 \leq \mathcal{R}(|X_1| + c) < \infty$. Similarly, for sufficiently large c , we have that $\mathcal{R}(|X_1| + c) \leq 0$ and $|X_1| + c$ is a feasible solution to problem (3.3), which yields $v_2 \leq \mathbb{E}[\rho(|X_1| + c)] < \infty$.

On the other hand, using that X_1 is anticomonotonic with ρ and the Hardy–Littlewood inequality, we have, for any $X \in \mathcal{X}_\rho$,

$$\begin{aligned} \mathcal{R}_1(X)\mathbb{E}[\rho] &= \sup_{t \in (0,1)} (Q_{X_1}(t) - Q_X(t)) \int_0^1 Q_\rho(1-s) \, ds \\ (3.4) \quad &\geq \int_0^1 (Q_{X_1}(s) - Q_X(s))Q_\rho(1-s) \, ds \geq \mathbb{E}[\rho X_1] - \mathbb{E}[\rho X]. \end{aligned}$$

For any feasible solution X to problem (3.2), it follows that

$$\mathcal{R}(X) \geq \mathcal{R}_1(X) \geq \frac{\mathbb{E}[\rho X_1] - \mathbb{E}[\rho X]}{\mathbb{E}[\rho]} \geq \frac{\mathbb{E}[\rho X_1] - x}{\mathbb{E}[\rho]}.$$

This together with $v_1 < \infty$ implies that v_1 is finite. For any feasible solution X to problem (3.3), since $\mathcal{R}_1(X) \leq \mathcal{R}(X) \leq 0$, we see from (3.4) that

$$\mathbb{E}[\rho X] \geq \mathbb{E}[\rho X_1].$$

This together with $v_2 < \infty$ implies that v_2 is finite.

Now suppose X^* is an optimal solution to problem (3.2). Then $d = \mathcal{R}(X^*)$ is finite. Let $Y = X^* + d$. Then

$$\mathcal{R}(Y) = \mathcal{R}(X^*) - d = 0.$$

Suppose on the contrary that Y is not optimal to (3.3); then there exists a $Z \in \mathcal{X}_\rho$ such that $\mathbb{E}[\rho Z] < \mathbb{E}[\rho Y]$ and $\mathcal{R}(Z) \leq 0$. In this case, as $\mathbb{E}[\rho] < \infty$,

$$\mathbb{E}[\rho(Z - d)] < \mathbb{E}[\rho(Y - d)] = \mathbb{E}[\rho X^*] \leq x.$$

Hence, there exists a constant $\delta > 0$ such that $\mathbb{E}[\rho(Z - d + \delta)] \leq x$. Obviously, $Z - d + \delta \in \mathcal{X}_\rho$. But

$$\mathcal{R}(Z - d + \delta) = \mathcal{R}(Z) + d - \delta < d = \mathcal{R}(X^*),$$

which contradicts the optimality of X^* to (3.2). Hence, $X^* + d$ is an optimal solution to (3.3).

Suppose X^* is an optimal solution to problem (3.3). Then $d = \frac{x - \mathbb{E}[\rho X^*]}{\mathbb{E}[\rho]}$ is finite. Let $Y = X^* + d$. Clearly

$$\mathbb{E}[\rho Y] = \mathbb{E}[\rho X^*] + \mathbb{E}[\rho d] = x.$$

Suppose on the contrary that Y is not optimal to (3.2); then there exists a $Z \in \mathcal{X}_\rho$ such that $\mathbb{E}[\rho Z] \leq x$ and $\mathcal{R}(Z) < \mathcal{R}(Y)$. In this case,

$$\mathcal{R}(Z - d) < \mathcal{R}(Y - d) = \mathcal{R}(X^*) \leq 0.$$

Hence, there exists a small constant $\delta > 0$ such that

$$\mathcal{R}(Z - d - \delta) = \mathcal{R}(Z - d) + \delta < 0.$$

Obviously, $Z - d + \delta \in \mathcal{X}_\rho$. But

$$\mathbb{E}[\rho(Z - d - \delta)] = \mathbb{E}[\rho Z] - d\mathbb{E}[\rho] - \delta\mathbb{E}[\rho] \leq x - (x - \mathbb{E}[\rho X^*]) - \delta\mathbb{E}[\rho] < \mathbb{E}[\rho X^*],$$

which clearly contradicts the optimality of X^* to (3.3). Therefore, $X^* + d$ is an optimal solution to (3.2). ■

By Lemma 3.2, solving problem (3.2) is essentially equivalent to solving problem (3.3), so we focus on the latter from now on.

As \mathcal{R} is law-invariant and monotone, using the Hardy–Littlewood inequality, we know that any solution X^* to problem (3.3) satisfies $X^* = Q_{X^*}(1 - \xi)$, and the minimum is

$$\mathbb{E}[\rho X^*] = \int_0^1 Q_{X^*}(s)Q_\rho(1 - s) ds,$$

where $\xi \in \Xi$ and⁶

$$\Xi \triangleq \{\xi \mid \xi \text{ is uniformly distributed on } (0, 1) \text{ and } \rho = Q_\rho(\xi) \text{ a.s.}\};$$

see, e.g., Dybvig [4], Schied [13], Carlier and Dana [3], Jin and Zhou [8], Xu [16], and Liebrich and Munari [9]. Recalling that $\mathcal{R}(X) \leq 0$ if and only if $X \in \mathcal{A}_1 \cap \mathcal{A}_2$, we know that problem (3.3) reduces to the following problem:

$$(3.5) \quad \underset{X \in \mathcal{X}_\rho}{\text{minimize}} \int_0^1 Q_X(s)Q_\rho(1 - s) ds \quad \text{subject to } X \in \mathcal{A}_1 \cap \mathcal{A}_2.$$

Its quantile formulation is then given by

$$(3.6) \quad \underset{Q \in \mathcal{Q}_\rho}{\text{minimize}} \int_0^1 Q(s)Q_\rho(1 - s) ds \quad \text{subject to } Q \in \mathcal{Q}_1(Q_{X_1}) \cap \mathcal{Q}_2(Q_{X_2}),$$

⁶For the existence of such a ξ , see, e.g., Föllmer and Schied [6, Lemma A.28]. Moreover, let ξ be a random variable uniformly distributed on $(0, 1)$. Then by Xu [16, Theorem 5], $\xi \in \Xi$ if and only if (ξ, ρ) is comonotonic.

where

$$\mathcal{Q}_\rho \triangleq \left\{ Q \in \mathcal{Q} \mid \int_0^1 Q^+(s)Q_\rho(1-s) ds < \infty \right\}.$$

By Föllmer and Schied [6, Theorem 2.57 and Lemma 3.45], we have, for any $\bar{Q} \in \mathcal{Q}$,

$$(3.7) \quad Q \in \mathcal{Q}_2(\bar{Q}) \implies \int_0^1 Q(s)Q_\rho(1-s) ds \geq \int_0^1 \bar{Q}(s)Q_\rho(1-s) ds.$$

Under Assumption 3.1,⁷ let Q^* be the SSD-minimal quantile function in $\mathcal{Q}_1(Q_{X_1}) \cap \mathcal{Q}_2(Q_{X_2})$, which is given by Theorem 2.4. Then Assumption 3.1 and (3.7) imply that Q^* solves problem (3.6). Therefore, problem (3.3) is completely solved.

Remark 3.3. Problems (3.2)–(3.3) and their variants for convex, coherent, distortion, or SSD-consistent risk measures have been investigated in Schied [13], He and Zhou [7], Föllmer and Schied [6], Mao and Wang [10], Embrechts, Schied, and Wang [5], and Burzoni, Munari, and Wang [2]. Risk measure \mathcal{R} in this paper lies in none of the aforementioned classes of risk measures.

Remark 3.4. The classical EMP is as follows: given a quantile function $Q \in \mathcal{Q}$,

$$(3.8) \quad \underset{X \in L^0}{\text{minimize}} \quad \mathbb{E}[\rho X] \quad \text{subject to } X \sim Q.$$

Using the Hardy–Littlewood inequality, it turns out that a solution to problem (3.8) is given by $X = Q(1 - \xi)$, and the minimum is $\int_0^1 Q(s)Q_\rho(1-s) ds$, where $\xi \in \Xi$. The EMP subject to an FSD constraint or an SSD constraint is as follows:

$$(3.9) \quad \underset{X \in L^0}{\text{minimize}} \quad \mathbb{E}[\rho X] \quad \text{subject to } Q_X \in \mathcal{Q}_i(Q_0).$$

For the case of an FSD constraint, i.e., $i = 1$, problem (3.9) is essentially equivalent to problem (3.8) (with $Q = Q_0$ there), and its solution is thus given by $X = Q_0(1 - \xi)$, and the minimum is

$$x_0 = \int_0^1 Q_0(s)Q_\rho(1-s) ds,$$

where $\xi \in \Xi$. For the case of an SSD constraint, i.e., $i = 2$, problem (3.9) has the same minimum x_0 as in the case of an FSD constraint; see, e.g., Föllmer and Schied [6, Theorem 3.44]. Moreover, a solution of problem (3.9) with $i = 2$ is also given by $X = Q_0(1 - \xi)$, where $\xi \in \Xi$.

4. Proof of Theorem 2.4. To prove Theorem 2.4, it suffices to prove the following assertions.

1. Q^* is right-continuous.
2. Q^* is increasing, which together with the first assertion implies that Q^* is a quantile function in \mathcal{Q} .

⁷Obviously, $\int_0^1 |Q_{X_i}(s)|Q_\rho(1-s) ds < \infty$ implies that $\int_0^1 Q_{X_i}^-(s) ds < \infty$, so $Q_i = Q_{X_i}$, $i \in \{1, 2\}$, fulfills Assumption 2.1.

- 3. $Q^* \in \mathcal{Q}_1(Q_1) \cap \mathcal{Q}_2(Q_2)$.
- 4. For any $Q \in \mathcal{Q}_1(Q_1) \cap \mathcal{Q}_2(Q_2)$, we have $Q \in \mathcal{Q}_2(Q^*)$.

Recall that Assumption 2.1 guarantees that f is continuous on $[0, 1]$.

Let us start with the first assertion that Q^* is right-continuous. For any $t \in (0, 1)$, since $\phi \geq f$, there are two possible cases.

- If $\phi(t) > f(t)$, then by the continuity of ϕ and f , we have $\phi(t + \Delta) > f(t + \Delta)$ for all sufficiently small $\Delta > 0$. Therefore, by definition and the right-continuity of quantiles,

$$\lim_{\Delta \rightarrow 0^+} Q^*(t + \Delta) = \lim_{\Delta \rightarrow 0^+} Q_1(t + \Delta) = Q_1(t) = Q^*(t).$$

- If $\phi(t) = f(t)$, then by definition and the right-continuity of quantiles,

$$\lim_{\Delta \rightarrow 0^+} Q^*(t + \Delta) \leq \lim_{\Delta \rightarrow 0^+} Q_1(t + \Delta) \vee Q_2(t + \Delta) = Q_1(t) \vee Q_2(t) = Q^*(t).$$

By the monotonicity of quantiles, we also have

$$\lim_{\Delta \rightarrow 0^+} Q^*(t + \Delta) \geq Q^*(t).$$

The above two inequalities show that Q^* is right-continuous.

To prove the second and fourth assertions, we first show the following result:

$$(4.1) \quad \begin{cases} \phi' = g & \text{a.e. in } (0, 1), \\ \phi(0) = 0, \end{cases}$$

where

$$g(t) \triangleq (Q_2(t) - Q_1(t))^+ \mathbf{1}_{\phi(t)=f(t)}.$$

In fact, suppose the ODE in (2.1) holds at $t \in (0, 1)$; since $\phi \geq f$, there are two possible cases.

- If $\phi(t) > f(t)$, then by continuity, we have $\phi(s) > f(s)$ when s is sufficiently close to t . It then follows from (2.1) that ϕ is a constant near t , so $\phi'(t) = 0 = g(t)$.
- If $\phi(t) = f(t)$, then by Theorem 2.3, for any $0 < \Delta < 1 - t$,

$$\phi(t + \Delta) - \phi(t) = \max_{s \leq t + \Delta} f(s) - f(t) \geq f(t + \Delta) - f(t),$$

so

$$\phi'(t) = \liminf_{\Delta \rightarrow 0^+} \frac{\phi(t + \Delta) - \phi(t)}{\Delta} \geq \liminf_{\Delta \rightarrow 0^+} \frac{f(t + \Delta) - f(t)}{\Delta} = Q_2(t) - Q_1(t),$$

by virtue of the right-continuity of quantiles. Also, trivially $\phi' \geq 0$; hence

$$\phi'(t) \geq \max\{Q_2(t) - Q_1(t), 0\} = g(t).$$

On the other hand,

$$\begin{aligned} \phi(t + \Delta) - \phi(t) &= \max_{t \leq s \leq t + \Delta} f(s) - f(t) \\ &= \max_{t \leq s \leq t + \Delta} \int_t^s (Q_2(r) - Q_1(r)) \, dr \\ &\leq \int_t^{t + \Delta} (Q_2(r) - Q_1(r))^+ \, dr, \end{aligned}$$

so

$$\begin{aligned} \phi'(t) &= \limsup_{\Delta \rightarrow 0^+} \frac{\phi(t + \Delta) - \phi(t)}{\Delta} \\ &\leq \limsup_{\Delta \rightarrow 0^+} \frac{1}{\Delta} \int_t^{t+\Delta} (Q_2(r) - Q_1(r))^+ dr \\ &= (Q_2(t) - Q_1(t))^+ = g(t), \end{aligned}$$

thanks to the right-continuity of quantiles. Therefore, $\phi'(t) = g(t)$.

Now we have established (4.1).

By virtue of (4.1), we now prove the second assertion that Q^* is increasing. For any $0 \leq s < t < 1$, as $\phi \geq f$, there are three possible cases.

- If $\phi(t) = f(t)$, then by definition and the monotonicity of quantiles,

$$Q^*(t) = Q_1(t) \vee Q_2(t) \geq Q_1(s) \vee Q_2(s) \geq Q^*(s).$$

- If $\phi(t) > f(t)$ and $\phi(s) > f(s)$, then by definition and the monotonicity of quantiles,

$$Q^*(t) = Q_1(t) \geq Q_1(s) = Q^*(s).$$

- If $\phi(t) > f(t)$ and $\phi(s) = f(s)$, then it follows from (4.1) that

$$\begin{aligned} \int_s^t Q^*(r) dr &= \int_s^t (Q_1(r) + g(r)) dr = \phi(t) - \phi(s) + \int_s^t Q_1(r) dr \\ &> f(t) - f(s) + \int_s^t Q_1(r) dr = \int_s^t Q_2(r) dr. \end{aligned}$$

So there exists some $r \in (s, t)$ such that $Q^*(r) > Q_2(r)$. It follows from the definition (2.3) that $Q_1(r) > Q_2(r)$, so $Q_1(r) = Q_1(r) \vee Q_2(r)$. Consequently,

$$Q^*(t) = Q_1(t) \geq Q_1(r) = Q_1(r) \vee Q_2(r) \geq Q_1(s) \vee Q_2(s) \geq Q^*(s)$$

by the monotonicity of quantiles.

We have now proved the second assertion.

Thanks to (4.1) again, we can now show the third assertion that $Q^* \in \mathcal{L}_1(Q_1) \cap \mathcal{L}_2(Q_2)$. By (2.1) and (4.1), we see that for a.e. $t \in (0, 1)$,

$$\min \left\{ Q^*(t) - Q_1(t), \int_0^t Q^*(r) dr - \int_0^t Q_2(r) dr \right\} = \min \left\{ g(t), \int_0^t g(r) dr - f(t) \right\} = 0.$$

By the right-continuity of quantiles and integrals, the above holds for all $t \in (0, 1)$. Hence, $Q^* \in \mathcal{L}_1(Q_1) \cap \mathcal{L}_2(Q_2)$.

We now show the fourth assertion that $Q \in \mathcal{L}_2(Q^*)$ for any $Q \in \mathcal{L}_1(Q_1) \cap \mathcal{L}_2(Q_2)$. In fact, if $Q \in \mathcal{L}_1(Q_1) \cap \mathcal{L}_2(Q_2)$, then

$$(4.2) \quad \min \left\{ Q(t) - Q_1(t), \int_0^t Q(r) dr - \int_0^t Q_2(r) dr \right\} \geq 0 \quad \text{for all } t \in (0, 1).$$

The following comparison principle shows that $\int_0^t Q(r) dr \geq \int_0^t Q^*(r) dr$ for all $t \in (0, 1)$. Hence, $Q \in \mathcal{L}_2(Q^*)$, which completes the proof of Theorem 2.4.

Lemma 4.1 (comparison principle). Assume φ_1 and φ_2 are absolutely continuous functions that satisfy the following variational inequalities on $[0, 1)$, respectively:

$$\min\{\varphi_1' - f_1, \varphi_1 - g_1\} \geq 0 \text{ a.e.}, \quad \min\{\varphi_2' - f_2, \varphi_2 - g_2\} \leq 0 \text{ a.e.}$$

If $f_1 \geq f_2$, $g_1 \geq g_2$ a.e. on $[0, 1)$, and $\varphi_1(0) \geq \varphi_2(0)$, then $\varphi_1 \geq \varphi_2$ on $[0, 1)$.

Proof. Suppose, on the contrary, that there exists a $t \in (0, 1)$ such that $\varphi_1(t) < \varphi_2(t)$. Let

$$s = \sup\{0 \leq r < t \mid \varphi_1(r) \geq \varphi_2(r)\}.$$

Then by the continuity of φ_1 and φ_2 and $\varphi_1(0) \geq \varphi_2(0)$, we get $0 \leq s < t$, $\varphi_1(s) \geq \varphi_2(s)$, and $\varphi_1(r) < \varphi_2(r)$ for all $r \in (s, t]$. This together with the first variational inequality and the fact $g_1 \geq g_2$ gives

$$\varphi_2 > \varphi_1 \geq g_1 \geq g_2 \quad \text{a.e. in } (s, t].$$

Then, by the two variational inequalities and the fact that $f_1 \geq f_2$, we get

$$\varphi_2' \leq f_2 \leq f_1 \leq \varphi_1' \quad \text{a.e. in } (s, t].$$

It follows that

$$\varphi_2(t) - \varphi_2(s) = \int_s^t \varphi_2'(r) \, dr \leq \int_s^t \varphi_1'(r) \, dr = \varphi_1(t) - \varphi_1(s),$$

contradicting the fact that $\varphi_1(t) < \varphi_2(t)$ and $\varphi_1(s) \geq \varphi_2(s)$. ■

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