TROPICAL LINEAR REGRESSION AND MEAN PAYOFF GAMES: OR, HOW TO MEASURE THE DISTANCE TO EQUILIBRIA

MARIANNE AKIAN, STÉPHANE GAUBERT, YANG QI, AND OMAR SAADI

ABSTRACT. We study a tropical linear regression problem consisting in finding the best approximation of a set of points by a tropical hyperplane. We establish a strong duality theorem, showing that the value of this problem coincides with the maximal radius of a Hilbert's ball included in a tropical polyhedron. We also show that this regression problem is polynomial-time equivalent to mean payoff games. We illustrate our results by solving an inverse problem from auction theory. In this setting, a tropical hyperplane represents the set of equilibrium prices. Tropical linear regression allows us to quantify the distance of a market to the set of equilibria, and infer secret preferences of a decision maker.

1. INTRODUCTION

1.1. **The tropical linear regression problem.** A *tropical hyperplane* in the n-dimensional tropical vector space $(\mathbb{R} \cup \{-\infty\})^n$ is a set of vectors of the form

$$(1) \qquad \mathcal{H}_a = \{x \in (\mathbb{R} \cup \{-\infty\})^n, \qquad \max_{1 \leqslant i \leqslant n} a_i + x_i \text{ is achieved at least twice}\} \ .$$

Such a hyperplane is parametrized by the vector $a = (a_1, \dots, a_n) \in (\mathbb{R} \cup \{-\infty\})^n$, which is required to be non-identically $-\infty$.

Tropical hyperplanes are among the most basic objects in tropical geometry. They are images by the valuation of hyperplanes over non-archimedean fields, and so, they are the simplest examples of tropical linear spaces [SS04, FR15] and tropical hypersurfaces [EKL06]. Tropical hyperplanes arise in tropical convexity [CGQ04, DS04], since closed tropical convex sets can be described as intersections of tropical half-spaces. A further motivation arises from the study of pricing problems: tropical hypersurfaces have been used in [BK19] to represent the influence of prices on the decision of agents buying bundles of elementary products. The "unit demand" case (bundles of cardinality one) is modelled by tropical hyperplanes.

In this paper, we address the following tropical analogue of the linear regression problem. Given a finite set of points $\mathcal{V} \subset (\mathbb{R} \cup \{-\infty\})^n$, we look for the best approximation of these points by a tropical hyperplane. Of course, the notion of "best approximation" depends on the metric. A canonical choice in tropical geometry is the (additive version of) Hilbert's projective metric. Its restriction to $\mathbb{R}^n$ is induced by the so called *Hilbert's seminorm* or *Hopf oscillation*

$$\|x\|_H := \max_{i \in [n]} x_i - \min_{i \in [n]} x_i \ .$$

It is a projective metric, in the sense that the distance between two points is zero if and only if these two points differ by an additive constant. Hence, we formulate the *tropical linear regression*

2020 *Mathematics Subject Classification.* 14T90, 91A25 , 91B26.

Key words and phrases. Tropical geometry, best approximation, tropical linear spaces, regression, inner radius, mean payoff games, equilibria, auction.

O. Saadi acknowledges the support of the Hassan II Academy of Science and Technology. We acknowledge the support of Fondation Mathématique Jacques Hadamard and EDF through the Gaspard Monge Program for Operations Research and their Interactions with Data Sciences (PGMO).

problem as the following optimization problem:

$$\operatorname*{Min}_{a} \max_{v \in \mathcal{V}} \min_{x \in \mathcal{H}_a \cap \mathbb{R}^n} \|v - x\|_H \tag{2}$$

where the minimum is taken over the space of parameters of tropical hyperplanes. For simplicity, we assume for the moment that the vectors $v \in \mathcal{V}$ have finite entries, this assumption will be relaxed in the body of the article.

Equation (2) is a non-convex optimization problem, which is of a disjunctive nature since a tropical hyperplane is a union of convex cones.

The tropical linear regression problem (2) is not only of theoretical interest. We shall see that it allows one to quantify the "distance to equilibrium" of a market model, and to infer hidden preferences of a decision maker.

1.2. **Results.** We show that tropical linear regression is tractable, theoretically, and to some extent, computationally. Our main result is a strong duality theorem, Theorem 20, showing that the infimum of the distance of the set of points $\mathcal{V}$ to a tropical hyperplane coincides with the supremum of the radii of Hilbert's balls included in the tropical convex cone generated by the elements of $\mathcal{V}$. This provides optimality certificates which can be interpreted geometrically as collections of n "witness" points among the elements of $\mathcal{V}$. Our approach also entails that tropical linear regression is polynomial-time equivalent to solving mean payoff games. The latter games, originally studied in [EM79, GKK88], are among the problems in the complexity class NP $\cap$ co-NP [ZP96] for which no polynomial time algorithm is known. However, several effective methods are available [GKK88, ZP96, BV07, DG06]. In particular, policy iteration allows to solve large scale instances [Cha09], even if it is generally super-polynomial [Fri09]. Thus, the present results lead to a practical solution of the tropical linear regression problem.

We subsequently study variants of the tropical linear regression problem, involving in particular the *signed* notion of tropical hyperplane, obtained by requiring the maximum in (1) to be achieved by two indices i, j belonging to prescribed disjoint subsets I, J of $[n]$. We also establish a strong duality theorem in this setting, and provide reductions to mean payoff games for these variants.

We finally illustrate tropical linear regression by an application to an auction model. We consider a market governed by an invitation to tender procedure. We suppose that a decision maker selects repeatedly bids made by firms, based on the bid prices, which are ultimately made public (after the decision is taken), and also on other criteria (assessments of the technical quality of each firm or of environmental impact) or influence factors (like bribes). This is a variant of the classical "first-price sealed-bid auction" [Kri02], with a bias induced by the secret preference. Here, we define the market to be at equilibria if for each invitation, there are at least two best offers. Hence, in the simplest model (unit demand), the set of equilibria prices can be represented by a tropical hyperplane. We distinguish two versions of this problem, one in which only the prices are public, and the other, in which the identities of the winners of the successive invitations are also known. In both cases, we show that solving a tropical linear regression problem allows an observer to quantify the distance of such a market to equilibrium, and also to infer secret preference factors. This solves, in the special case of unit-demand, an inverse problem, consisting in identifying the agent preferences and utilities in auction models, like the one of [BK19]. This might be of interest to a regulation authority wishing to quantify anomalies, or to a bidder, who, seeing the history of the market, would wish to determine how much he should have bidded to win a given invitation or to get the best price for an invitation that he won, thus avoiding the "winner's curse".

1.3. **Related work, and discussion.** Several "best approximation" problems have been studied in tropical geometry. The simplest one consists in finding the nearest point in a (closed) tropical module, in the sense of Hilbert's metric. The solution is given by the tropical projection [CGQ04], see also [AGNS11]. The best approximation in the space of ultrametrics, which is a fundamental example of tropical module in view of its application to phylogenetics, has been thoroughly studied [CF00, LSTY17, Ber20]. Another important special case is the best approximation of a point by a tropical linear space [Ard04, JSY07]. In contrast with the regression problem studied here, these problems concern the approximation of a *single* point.

It is a general principle that regression (best approximation) is somehow dual to separation. Hence, tropical linear regression should be compared with the tropical support vector machines (SVM) introduced in [GJ08], and further studied in [TWY20]. Whereas the input of the tropical SVM problem (a configuration of points in dimension $n-1$ partitioned in n color classes) is the same as the one of the version "with types" of the tropical linear regression problem, we explain in Remark 9 why both problems differ in essential ways.

A different problem of tropical regression consists in finding a vector x minimizing the sup-norm $\|y-Ax\|_\infty$ where y is a vector of observations, and A is a known matrix acting tropically on x. This can be solved in (strongly) polynomial time, again by a tropical projection [But10]. See also [CF00] for a general version of this result. Tropical linear regression problems of this nature have been studied in the context of learning [MCT21]. The sparse version is of practical interest; it arose in the approximation of solutions of Hamilton-Jacobi PDE, where it was shown to be equivalent to a non-metric infinite dimensional facility location problem [GMQ11]. The finite dimension version which is NP-hard is studied in [TM19].

A different tropical regression problem, with a L_1-type error term (instead of sup-norm here), has been solved in [YZZ19, Theorem 4], in the special case of a configuration of n points in dimension $n-1$. The value is given by a *tropical volume* [DGJ17], instead of an inner radius.

Tropical geometry has been applied to economics in [BK19], see also [TY19], and [DKM01] for early results in this direction. Our modelling of agent's responses to prices is inspired by [BK19]. Auction models taking into account bribery have been studied in particular in [CLMV05, BP07, Rac13].

We build on the results of [AGG12], showing the equivalence between tropical linear programming and mean payoff games. Further reductions and equivalences, concerning in particular the problem of the emptiness of tropical linear prevarieties, were given in [GP13]. The relation between the mean payoff of a game and the inner radius of a Shapley operator was first observed in [Sko18, AGKS18], where it was applied to define a condition number and derive complexity results for games. The inner radius of tropical polyhedra defined by n generators in dimension $n-1$ was initially characterized in [Ser07], as a tropical eigenvalue.

Open problems related to the present work are discussed in the concluding section.

1.4. **Organization.** In Section 2, we recall the needed results concerning tropical algebra, mean payoff games, and non-linear Perron-Frobenius theory. In Section 3, we show that computing the inner radius of a tropical polyhedron given by generators is equivalent to solving a mean payoff game. Section 4 contains our main results, including Theorem 20, the strong duality theorem for tropical linear regression. Several variants of the tropical linear regression problem are dealt with in Section 5. In Section 6, we explain how to solve tropical linear regression problems in practice, using mean payoff games algorithms. In Section 7, we give an application to an auction problem. The appendix provides sufficient conditions for the existence of finite eigenvectors of a class of Shapley operators. These conditions are helpful when dealing with regression problems for configurations of points with $-\infty$ coordinates.

2. Preliminaries

2.1. **Tropical cones.** The max-plus semifield $\mathbb{R}_{\max}$ is the set of real numbers, completed by $-\infty$ and equipped with the addition $(a,b) \mapsto \max(a,b)$ and the multiplication $(a,b) \mapsto a \odot b := a+b$. The name "tropical" will be used in the sequel as a synonym of "max-plus". We shall occasionally use variants of this semifield. These include *the min-plus semifield* $\mathbb{R}_{\min}$, which is the set $\mathbb{R}\cup\{+\infty\}$, equipped with the addition $(a,b)\mapsto\min(a,b)$ and the multiplication $(a,b)\mapsto a\odot b:=a+b$. This semifield is isomorphic to $\mathbb{R}_{\max}$. These also include the subsemifield $\mathbb{Z}_{\max}\subset\mathbb{R}_{\max}$, with ground set $\mathbb{Z}\cup\{-\infty\}$. We refer the reader to [BCOQ92, But10, MS15] for background on tropical algebra.

For any integer n, we set $[n]:=\{1,\dots,n\}$. For all $x,y\in(\mathbb{R}_{\max})^n$, $A\in(\mathbb{R}_{\max})^{n\times m}$, and $\lambda\in\mathbb{R}_{\max}$, $\lambda+x\in(\mathbb{R}_{\max})^n$ denotes the vector with entries $\lambda+x_i$, for $i\in[n]$, $\lambda+A\in(\mathbb{R}_{\max})^{n\times m}$ denotes the matrix with entries $\lambda+A_{ij}$, for $i\in[n], j\in[m]$, $x\vee y=\sup(x,y)$ denotes the vector with entries $\max(x_i,y_i)$, for $i\in[n]$, and $x\wedge y=\inf(x,y)$ denotes the vector with entries $\min(x_i,y_i)$, for $i\in[n]$. The set $(\mathbb{R}_{\max})^n$ equipped with the addition $(x,y)\mapsto x\vee y$ and the action $(\lambda,x)\mapsto\lambda+x$ of $\mathbb{R}_{\max}$ is a tropical module, i.e. a module over the semifield $\mathbb{R}_{\max}$.

A subset $\mathcal{C}$ of $(\mathbb{R}_{\max})^n$ is a *tropical (convex) cone* or equivalently a *tropical submodule* of $(\mathbb{R}_{\max})^n$ if it satisfies $x,y\in\mathcal{C}$ and $\lambda\in\mathbb{R}_{\max}$ implies $\lambda+x\in\mathcal{C}$ and $x\vee y\in\mathcal{C}$. We endow $(\mathbb{R}_{\max})^n$ with the topology defined by the metric $\delta(x,y)=\max_{i\in[n]}|e^{x_i}-e^{y_i}|$. It induces the usual topology in $\mathbb{R}^n$. For any given subset $\mathcal{V}$ of $(\mathbb{R}_{\max})^n$, we denote by $\mathrm{Sp}(\mathcal{V})$ the tropical submodule of $(\mathbb{R}_{\max})^n$ generated by $\mathcal{V}$, that is the minimal tropical submodule of $(\mathbb{R}_{\max})^n$ containing $\mathcal{V}$. A tropical polyhedral cone $\mathcal{C}$ is a tropical cone which is finitely generated, that is such that there exists a finite subset $\mathcal{V}$ such that $\mathcal{C}=\mathrm{Sp}(\mathcal{V})$. For any given matrix V, we also denote by $\mathrm{Col}(V)$ the column space of V, that is the tropical polyhedral cone generated by the columns of V, and we denote by $\mathrm{Row}(V)$ the row space of V, that is the tropical polyhedral cone generated by the rows of V.

A tropical polyhedral cone can also be defined externally by a system of tropical linear inequalities of the form

$$\max_{j\in[n]}(A_{ij}+x_j)\leqslant\max_{j\in[n]}(B_{ij}+x_j),\qquad i\in[m]\ , \tag{3}$$

where A_{ij},B_{ij} belong to $\mathbb{R}_{\max}$, see [GK11]. Then, A and B will be thought of as $m\times n$ matrices with entries in $\mathbb{R}_{\max}$.

Let $A\in(\mathbb{R}_{\max})^{m\times n}$ and $x\in(\mathbb{R}_{\max})^n$. We denote by Ax the vector in $(\mathbb{R}_{\max})^m$ with entries $(Ax)_i=\max_{j\in[n]}(A_{ij}+x_j)$, for $i\in[m]$. To a matrix $A\in(\mathbb{R}_{\max})^{m\times n}$, we associate the operator $A^\sharp:(\mathbb{R}_{\min})^m\to(\mathbb{R}_{\min})^n$, given by:

$$\forall y\in(\mathbb{R}_{\min})^m,\forall j\in[n],(A^\sharp y)_j=\min_{i\in[m]}(-A_{ij}+y_i)\ ,$$

with the convention $-\infty+\infty=+\infty$. The operator $A^\sharp$ is called the *adjoint* of A and we can easily check that it satisfies the following property:

$$\forall x\in(\mathbb{R}_{\max})^n,\forall y\in(\mathbb{R}_{\min})^m, Ax\leqslant y\Leftrightarrow x\leqslant A^\sharp y\ .$$

We define the identity matrix $\mathrm{I}\in(\mathbb{R}_{\max})^{n\times n}$ by $\forall i\in[n],\mathrm{I}_{ii}=0$, and $\forall i,j\in[n],i\neq j,\mathrm{I}_{ij}=-\infty$.

A scalar μ is a *tropical eigenvalue* of a matrix $M\in(\mathbb{R}_{\max})^{n\times n}$ if there exists a vector $u\in(\mathbb{R}_{\max})^n$, not identically $-\infty$, such that $Mu=\mu+u$ in the tropical sense. The eigenvalue is known to be unique when the digraph of M is strongly connected, then it coincides with the maximum weight-to-length ratio of the circuits of the digraph of M. We denote it by $\lambda(M)$. See [BCOQ92, But10] for more information.

2.2. **Mean payoff games.** We consider zero-sum deterministic games, with perfect information, defined as follows. There are two players, “Max” and “Min” (the maximizer and the minimizer), who will move a token on a weighted digraph. We assume this digraph is finite and bipartite: the node set is the disjoint union of two non-empty sets $S^{\max}$ and $S^{\min}$, and the arc set $\mathcal{A}$ is included in $(S^{\max}\times S^{\min})\cup(S^{\min}\times S^{\max})$. The set of states of the game is the set of nodes of the digraph. We associate a real weight w_{rs} to each arc (r,s).

The two players alternate their actions. When the token is in node $i\in S^{\min}$, Player Min must choose an arc (i,j) in the digraph, meaning he moves the token to node j, and pays w_{ij} to player

Max. When Player Min has no possible action, that is, when there are no arcs of the form (i,j) in the digraph, the game terminates, and Player Max receives $+\infty$. Similarly, when the token is in node $j \in S^{\max}$, Player Max must choose an arc (j,i) in the digraph. Then he moves the token from node j to node i, and receives w_{ji} from Player Min. When Player Max has no possible action, that is there are no arcs of the form (j,i) in the digraph, the game terminates, and Player Max receives $-\infty$.

We measure the time in turns, i.e., a time step consists of two half-turns (a move made by Player Min followed by a move made by Player Max). We consider the following game in horizon k: starting from an initial state $\bar{\imath} \in S^{\min}$. the two players make k moves each, unless the game terminates before. So, if the game does not terminate before time k, the *history* of the game is described by a sequence of nodes $\bar{\imath} = i_0, j_1, i_1, \dots, j_k, i_k$, belonging alternatively to $S^{\min}$ and $S^{\max}$, and the total payment received by Player Max is given by

$$R^k_{\bar{\imath}} = w_{i_0 j_1} + w_{j_1 i_1} + w_{i_1 j_2} + \cdots + w_{j_k i_k} \enspace .$$

If the game terminates before time k, we set $R^k_{\bar{\imath}} = \pm\infty$ depending on the player who had no available action. The following assumption requires Player Min to have at least one available action in every state:

Assumption 1. For all $i \in S^{\min}$, there exists $j \in S^{\max}$ such that (i,j) is an arc of the digraph of the zero-sum deterministic game.

In this way, we always have $R^k_{\bar{\imath}} \in \mathbb{R} \cup \{-\infty\}$. We shall also consider the dual assumption.

Assumption 2. For all $j \in S^{\max}$, there exists $i \in S^{\min}$ such that (j,i) is an arc of the digraph of the zero-sum deterministic game.

In most works on mean payoff games, both assumptions are required to hold, which entails in particular that $R^k_{\bar{\imath}}$ is finite. Here, we shall occasionally relax Assumption 2, but always require Assumption 1, so that $R^k_{\bar{\imath}} \in \mathbb{R} \cup \{-\infty\}$. This leads to an unpleasant symmetry breaking. However, we shall see that this generality will be sometimes needed to handle the application to tropical linear regression. Indeed, from a tropical perspective, $-\infty$ is the zero element, hence a meaningful value.

A strategy of a player is a map which associates to the history of the game an action of this player. Assuming that Player Min plays according to strategy σ, and that Player Max plays according to strategy τ, we shall write $R^k_{\bar{\imath}} = R^k_{\bar{\imath}}(\sigma,\tau)$ to indicate the dependence on these strategies. It follows from standard dynamic programming arguments that the game in horizon k starting from node $\bar{\imath}$ has a value $v^k_{\bar{\imath}}$ and that Players Min and Max have optimal strategies σ^* and τ^*, respectively, see e.g. [MSZ15, Th. IV.3.2]. This means that the payment function has the following saddle point property:

$$R^k_{\bar{\imath}}(\sigma,\tau^*) \leqslant v^k_{\bar{\imath}} = R^k_{\bar{\imath}}(\sigma^*,\tau^*) \leqslant R^k_{\bar{\imath}}(\sigma^*,\tau)$$

for all strategies σ, τ. Moreover, the *value vector* $v^k := (v^k_i)_{i \in S^{\min}}$ is determined by the following dynamic programming equation

$$v^k = T(v^{k-1}), \qquad v^0 = 0$$

where $T : (\mathbb{R} \cup \{-\infty\})^n \to (\mathbb{R} \cup \{-\infty\})^n$ is the *Shapley operator*, defined, for $i \in S^{\min}$, by

$$T_i(x) = \min_{j,\, (i,j) \in \mathcal{A}} \big(w_{ij} + \max_{l,\, (j,l) \in \mathcal{A}} (w_{jl} + x_l)\big) \enspace . \tag{4}$$

Owing to Assumption 1, the above minimum is never taken over an empty set, whereas the above maximum is never taken over an empty set when Assumption 2 is made. By convention, the maximum of an empty set is $-\infty$. When both assumptions hold, T sends $\mathbb{R}^n$ to $\mathbb{R}^n$.

We are interested in the limit

$$\chi(T) := \lim_{k \to \infty} T^k(0)/k = \lim_{k \to \infty} v^k/k \enspace .$$

Thus, $\chi_i(T)$ yields the limit of the mean payoff per time unit, for the game starting from the initial state i, when the horizon tends to infinity. It follows from [Koh80] that the limit does exist, and that $\chi(T) \in \mathbb{R}^n$, when Assumption 1 and Assumption 2 hold (and more generally, when T is a piecewise affine self-map of $\mathbb{R}^n$ that is non-expansive in some norm, see [AGG12] and Section 2.3 below for details). Alternatively, under the same assumptions, $\chi_i(T)$ can be characterized as the value of an infinite mean-payoff game, in which player Max wishes to maximize the liminf of the average payment received per time unit, whereas player Min wishes to minimize the liminf of the same quantity — this is the approach originally described by Ehrenfeucht and Mycielski [EM79]. It follows from a general result of Mertens and Neyman, on the existence of the so called *uniform value* [MN81], that this approach leads to the same notion of mean payoff. Hence, we shall refer to $\chi_i(T)$ as the (asymptotic) mean payoff starting from node i.

More generally, the limit $\chi(T) \in (\mathbb{R} \cup \{-\infty\})^n$ does exist as soon as Assumption 1 is satisfied. To see this, observe that we can always construct in polynomial time an equivalent game satisfying also Assumption 2. Indeed, let us delete any node i of $S^{\min}$ in which Player Min has at least one action $(i, j) \in \mathcal{A}$ such that Player Max has no available action in state j. After at most $|S^{\min}|$ of such deletions, we arrive at a new game, played on a bipartite subdigraph of the original graph, induced by a subset of nodes belonging to Player Min $S'^{\max} \subset S^{\min}$. Note that $S'^{\max}$ may be empty. It is immediate that this subdigraph satisfies both Assumption 1 and 2. So, for $i \in S^{\min}$, the existence of $\lim_k v_i^k/k$ follows from the result already established, whereas for $i \in S^{\min} \setminus S'^{\max}$, we have $v_i^k = -\infty$ for k large enough, implying $\lim_k v_i^k/k = -\infty$.

A (stationary) *policy* of Player Min is a map $\sigma : S^{\min} \to S^{\max}$ such that $(i, \sigma(i)) \in \mathcal{A}$ for all $i \in S^{\min}$. Such a policy determines a one-player game, in which Player Min always selects moves $i \to \sigma(i)$. This one-player game corresponds to the Shapley operator T^σ, defined by

$$T_i^\sigma(x) = w_{i\sigma(i)} + \max_{l,\,(\sigma(i),l) \in \mathcal{A}} (w_{\sigma(i)l} + x_l) \enspace .$$

Similarly, a policy of Player Max is a map $\tau : S^{\max} \to S^{\min}$ such that $(j, \tau(j)) \in \mathcal{A}$ for all $j \in S^{\max}$. It determines a one-player game, with Shapley operator ${}^\tau T$ defined by

$${}^\tau T_i(x) = \min_{j,\,(i,j) \in \mathcal{A}} (w_{ij} + w_{j\tau(j)} + x_{\tau(j)}) \enspace .$$

A result of Liggett and Lippman [LL69] entails that each player has optimal strategies in a mean payoff game, which are obtained by applying a stationary policy. This entails in particular that

$$\chi(T) = \min_\sigma \chi(T^\sigma) = \max_\tau \chi({}^\tau T) \enspace ,$$

under Assumptions 1 and 2. The mean payoff $\chi_i(T)$ is known to coincide with the weight-to-length ratio of a circuit of the bipartite digraph of the game, the "length" being measured as the number of full turns, i.e., as the number of Min nodes of the circuit (one half of the ordinary length). In particular, if the payments w_{rs} are integers, the mean payoff is a rational number p/q, where p, q are integers, and q is a positive integer bounded by the maximal length of a circuit of the bipartite digraph of the game (measuring the length as the number of nodes of Min that are visited) and $|p/q|$ is bounded by $2|\max_{rs} w_{rs}|$.

Now we formalize the following problem.

Problem 1 (Mean payoff games)**.** *Input: A finite bipartite directed graph with integer weights, satisfying Assumptions 1 and 2, together with an initial node $\imath$. Question: Is the mean payoff $\chi_\imath(T)$ starting from node $\imath$ nonnegative?*

As discussed in the introduction, Problem 1 is a fundamental problem in algorithmic game theory [GKK88]. It belongs to the class NP $\cap$ coNP [ZP96], no polynomial time algorithm is known.

It will be useful to keep in mind several equivalent versions of this problem.

As a first variant, one may ask whether $\chi_\imath(T)$ is positive, instead of non-negative. This variant is equivalent to the negated version of Problem 1: considering $\tilde{T}(x) = -T(-x)$, i.e., the Shapley operator of the game in which all payments are negated, we have that $\chi(\tilde{T}) = -\chi(T)$, and so, $\chi_\imath(T) > 0$ iff $\chi_\imath(\tilde{T}) \leqslant 0$.

As observed above, the variant of mean payoff games in which Assumption 2 is relaxed reduces to the variant in which this assumption holds by a preprocessing, so there is no restriction on requiring Assumption 2 in Problem 1.

Another variant consists in computing $\chi_\imath(T)$, instead of deciding whether $\chi_\imath(T)$ is nonnegative. This problem of computation polytime Turing-reduces to Problem 1 by binary search. Indeed, given a rational number $\alpha = p/q$, we can consider the modified game with integer weights $w^\alpha_{rs} = 2q(w_{rs} - \alpha/2)$, which corresponds to replacing the Shapley operator T by $T^\alpha := 2q(-\alpha + T)$. Thus $\chi_\imath(T^\alpha) \geqslant 0$ iff $\chi_\imath(T) \geqslant \alpha$. Then, since the mean payoff $\chi_\imath(T)$ is a rational number whose absolute value is bounded by $2\max_{rs}|w_{rs}|$ and whose denominator is bounded by $|S^{\min}|$, we can compute $\chi_\imath(T)$ by a dichotomy argument, calling at each step an oracle solving Problem 1 for a modified game with weights w^α_{rs}.

2.3. **Perron-Frobenius tools.** We now recall some tools from Perron-Frobenius theory, in relation with mean payoff games. We refer the reader to [AGG12] for more information.

We denote by $\bot$ the vector of $(\mathbb{R}_{\max})^n$ identically equal to $-\infty$. We consider the *Hilbert's projective metric*, defined for vectors $x, y \in (\mathbb{R}_{\max})^n$ where at least one of them is not equal to $\bot$, by

$$d(x,y) = \inf\{\lambda - \mu \mid \lambda, \mu \in \mathbb{R},\ \mu + y_i \leqslant x_i \leqslant \lambda + y_i\ \forall i \in [n]\} \in \mathbb{R}_{\geqslant 0} \cup \{+\infty\} \ .$$

In addition, we set $d(\bot, \bot) := 0$.

The *support* of a vector $x \in (\mathbb{R}_{\max})^n$ is defined by $\operatorname{supp} x := \{i \in [n] \mid x_i \neq -\infty\}$. Each subset $I \subset [n]$ yields a *part* P_I of $(\mathbb{R}_{\max})^n$, consisting of vectors with support I.

Observe that $d(x, y)$ is finite if and only if x and y belong to the same part P_I. Moreover, if $I \neq \emptyset$,

$$d(x,y) = \max_{i \in I}(x_i - y_i) - \min_{i \in I}(x_i - y_i) \ .$$

We denote by $\mathbb{P}(\mathbb{R}_{\max})^n$ the *tropical projective space*, i.e., the quotient of the set of non-identically $-\infty$ vectors of $(\mathbb{R}_{\max})^n$ by the equivalence relation $\sim$ which identifies tropically proportional vectors. We shall abuse notation and denote by the same symbol a vector and its equivalence class. Similarly, we shall think of a part P_I with $I \neq \emptyset$ as a subset of the tropical projective space.

Observe that $d(x, y)$ vanishes if and only x and y represent the same point of the tropical projective space, so that d yields a well defined metric on each part of the tropical projective space. We denote by $B(a, r)$ the closed ball centered at $a \in \mathbb{R}^n$ with radius r under Hilbert's projective metric.

It will be convenient to consider an abstract version of the concrete Shapley operators used so far. We call (abstract) *Shapley operator* a map $T : (\mathbb{R}_{\max})^n \to (\mathbb{R}_{\max})^n$ that is order preserving, continuous, and such that $T(\alpha + x) = \alpha + T(x)$ for all $\alpha \in \mathbb{R}_{\max}$ and $x \in (\mathbb{R}_{\max})^n$. Observe that the operator T defined by (4), with $S^{\min} = [n]$, is a special case of abstract Shapley operator, as soon as Assumption 1 holds. We shall often consider situations in which an abstract Shapley operator restricts to a map $\mathbb{R}^n \to \mathbb{R}^n$, we will still use the term *Shapley operator* for the restricted map.

We are interested in the non-linear spectral problem for T, consisting in finding a vector $u \in (\mathbb{R}_{\max})^n$, non-identically $-\infty$, and a scalar $\lambda \in \mathbb{R}_{\max}$ such that $T(u) = \lambda + u$. The *spectral radius* of T is defined as

$$\rho(T) = \sup\{\lambda \in \mathbb{R} \cup \{-\infty\} \mid \exists u \in (\mathbb{R} \cup \{-\infty\})^n, u \neq \bot,\ T(u) = \lambda + u\} \ . \tag{5}$$

Variants of this spectral radius are given by the Collatz-Wielandt number cw,

$$\operatorname{cw}(T) = \inf\{\lambda \in \mathbb{R} \mid \exists u \in \mathbb{R}^n,\ T(u) \leqslant \lambda + u\} \ . \tag{6}$$

and by the dual Collatz-Wielandt number

$$\operatorname{cw}'(T) = \sup\{\lambda \in \mathbb{R} \cup \{-\infty\} \mid \exists u \in (\mathbb{R} \cup \{-\infty\})^n, u \neq \bot,\ T(u) \geqslant \lambda + u\} \ . \tag{7}$$

For all $x \in (\mathbb{R}_{\max})^n$, we define $\operatorname{top} x := \max_{i \in [n]} x_i$. We shall also consider

$$\overline{\chi}(T) := \lim_k \operatorname{top}(T^k(0))/k = \inf_{k \geqslant 1} \operatorname{top}(T^k(0))/k \ .$$

The existence of the limit and the fact it coincides with the infimum follow from the subadditivity property $\operatorname{top}(T^{k+l}(0)) \leqslant \operatorname{top}(T^k(0)) + \operatorname{top}(T^l(0))$. Of course, when the limit $\chi(T) = \lim_k T^k(0)/k$ exists, we have $\overline{\chi}(T) = \operatorname{top} \chi(T) = \max_{i \in [n]} \chi_i(T)$. Then, $\overline{\chi}(T)$ may be interpreted as the value of a modified mean payoff game, in which Player Max chooses first the initial state $i \in [n]$, and then, the games starts from this state as described in Section 2.2. Thus, in the sequel, we shall refer to $\overline{\chi}(T)$ as the *upper mean payoff* associated to the operator T.

The following result, which follows from [AGG12], provides several spectral characterizations of this upper mean payoff. We say that a map F from $\mathbb{R}^n$ to $(\mathbb{R} \cup \{-\infty\})^n$ is *piecewise affine* if we can cover $\mathbb{R}^n$ by finitely many polyhedra in such a way that each coordinate map F_i is either affine, or identically $-\infty$, on each of these polyhedra.

Theorem 1. *Let $T : (\mathbb{R}_{\max})^n \to (\mathbb{R}_{\max})^n$ be a Shapley operator. Then,*

$$\operatorname{cw}'(T) = \rho(T) = \overline{\chi}(T) = \operatorname{cw}(T) \ , \tag{8}$$

and the suprema in (5) and (7) are always achieved.

Moreover, if the restriction of T to $\mathbb{R}^n$ is piecewise affine, and if $\rho(T) \neq -\infty$, then the infimum in (6) is also achieved.

Before giving the details of the derivation of Theorem 1 from [AGG12], we need to recall a result of Kohlberg. An *invariant half-line* of a Shapley operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is a pair $(u, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$T(u + s\eta) = u + (s+1)\eta, \qquad \forall s \geqslant 0 \ .$$

Recall that a self-map of $\mathbb{R}^n$ is *non-expansive* for a fixed norm $\|\cdot\|$ if $\|T(x) - T(y)\| \leqslant \|x - y\|$. Observe that a Shapley operator that preserves $\mathbb{R}^n$ is automatically non-expansive in the sup-norm (see e.g. [GG04]).

Theorem 2 ([Koh80])**.** *A piecewise affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ that is nonexpansive in some norm admits an invariant half-line.*

If a Shapley operator $T : \mathbb{R}^n \to \mathbb{R}^n$ has an invariant half-line (u, η), it is immediate, using the fact that T is nonexpansive in the sup-norm, that $\chi(T) = \lim_k T^k(0)/k = \lim_k T^k(u)/k = \lim_k (u + k\eta)/k = \eta$. Thus, the invariant half-line determines the mean payoff vector.

Proof of Theorem 1. The equalities in (8) are established in [AGG12, Lemma 2.8], where they are derived from a theorem of Nussbaum concerning continuous, order preserving and positively homogeneous self-maps of the orthant, see [Nus86, Theorem 3.1] and also [GG04, Prop. 1]. If $T(u) = \rho(T) + u$ with $u \neq \bot$, we also have $T(v) = \rho(T) + v$, where $v := u - \operatorname{top} u$ is such that $\operatorname{top} v = 0$. Using the compactness of $\{v \in (\mathbb{R}_{\max})^n \mid \operatorname{top} v = 0\}$ and the continuity of T on this set, we deduce that the supremum in (5) is always achieved. A similar argument shows that the supremum in (7) is also achieved.

Consider now $F(x) = T(x) \vee (\operatorname{cw}(T) + x)$, which sends $\mathbb{R}^n$ to $\mathbb{R}^n$, and which is piecewise affine because the action spaces are finite. It is immediate that $\operatorname{cw}(F) = \operatorname{cw}(T)$. Let us take an invariant half-line (u, η) of F. Then, it follows from $F^k(u) = u + k\eta$, and from the nonexpansiveness of F in the sup-norm that $\bar{\chi}(T) = \lim_{k\to\infty} \operatorname{top} F^k(0)/k = \lim_{k\to\infty} \operatorname{top} F^k(u)/k = \operatorname{top} \eta$. Moreover, $F(u) = u + \eta \leqslant u + \operatorname{cw}(F)$, and so, $T(u) \leqslant u + \operatorname{cw}(T)$. □

Proposition 3. *A piecewise affine Shapley operator $T : \mathbb{R}^n \to \mathbb{R}^n$ admits a finite eigenvector if and only if the mean payoff $\chi_i(T)$ is independent of the choice of the initial state $i \in [n]$.*

Proof. By Theorem 2, T has an invariant half-line (u, η) and $\chi(T) = \eta$. So, if $\chi_i(T) = \lambda$ for all i, we have $T(u) = \lambda + u$, showing that u is a finite eigenvector of T. Conversely, if $T(u) = \lambda + u$ for some $u \in \mathbb{R}^n$, then, using the nonexpansiveness of T, $\chi(T) = \lim_k T^k(0)/k = \lim_k T^k(u)/k = \lim_k (u + k\lambda)/k = (\lambda, \dots, \lambda)$. ☐

3. Inner radius of a tropical polyhedron defined by generators

For any subset $\mathcal{W}$ of $(\mathbb{R}_{\max})^n$, we define the *inner radius* of $\mathcal{W}$, denoted in-rad$(\mathcal{W})$, as the supremum of the radii of Hilbert's balls centered at a point in $\mathbb{R}^n$ and included in $\mathrm{Sp}(\mathcal{W})$. More generally, for all non-empty subsets $I \subset [n]$, we define the *relative inner radius* of $\mathcal{W}$, denoted by in-rad$_I(\mathcal{W})$, as the supremum of the radii of Hilbert's balls centered at a point in the part P_I of $(\mathbb{R}_{\max})^n$ and included in $\mathrm{Sp}(\mathcal{W})$. Thus, in particular, in-rad$_{[n]}(\mathcal{W})$ = in-rad$(\mathcal{W})$. Observe that the relative inner radius depends only on the image of $\mathcal{W} \cap P_I$ in the tropical projective space $\mathbb{P}(\mathbb{R}_{\max})^n$.

In [AGKS18], it is shown that computing the inner radius of a tropical polyhedral cone given by an external description $P = \{x \in (\mathbb{R}_{\max})^n \mid Ax \leqslant Bx\}$ reduces to computing the Collatz-Wielandt number $\mathrm{cw}(T)$ of a Shapley operator.

In this paper, we consider the somehow dual situation in which the tropical polyhedral cone is given by an internal description,

$$\mathrm{Col}(V) = \{Vx \mid x \in (\mathbb{R}_{\max})^p\} \enspace ,$$

where V is a $n \times p$ matrix with entries in the tropical semifield $\mathbb{R}_{\max}$, rather by an external description. Recall that the size of an external description of a tropical polyhedral cone can be exponential in the size of an internal description, and vice versa [AGK11]. This leads us to consider the following problem.

Problem 2. *Input: a matrix $V \in \mathbb{Z}_{\max}^{n \times p}$. Goal: Compute the inner radius of* $\mathrm{Col}(V)$.

We shall make the following assumption.

Assumption 3. The matrix V has no identically $-\infty$ row and no identically $-\infty$ column.

This is not restrictive. Indeed, let $I \subset [n]$ (resp. $J \subset [p]$) denote the set of indices of non-identically $-\infty$ rows (resp. columns) of V and V' denote the $I \times J$ submatrix of V. For $K \subset [n]$, if K is not included in I, we have in-rad$_K(V) = -\infty$, whereas if $K = I$, then in-rad$_K(V)$ = in-rad(V'). More generally, the relative inner radii of V for $K \subset I$ coincide with the ones of V' (up to permutations of rows of V).

In [AGG12], the tropical linear independence of the columns of the matrix V was studied by means of a specific Shapley operator, which will also play a key role in our approach. We set $E = \{(i,k) \in [n] \times [p] \mid V_{ik} \neq -\infty\}$. Consider the operator $T : (\mathbb{R}_{\max})^n \to (\mathbb{R}_{\max})^n$, defined by

$$T_i(x) = \inf_{k \in [p], (i,k) \in E} \Big[-V_{ik} + \max_{j \in [n], j \neq i} (V_{jk} + x_j) \Big] \enspace . \tag{9}$$

Owing to Assumption 3, the latter infimum is never taken over an empty family, so the operator does send $(\mathbb{R}_{\max})^n$ to $(\mathbb{R}_{\max})^n$. We shall sometimes write T_V instead of T to emphasize the dependence on V. Observe that T is exactly the Shapley operator of a mean payoff game defined in Section 2.2: the set of nodes belonging to Player Min is $S^{\min} := [n]$, the set of nodes belonging to Player Max is $S^{\max} := E$, with the set of allowed moves

$$\mathcal{A} = \{(i,(i,k)) \mid i \in [n], k \in [p], (i,k) \in E\} \cup \{((i,k),j) \mid (i,k) \in E, (j,k) \in E,\ i \neq j\} \enspace . \tag{10}$$

The payment associated with the arc $(i,(i,k))$ is $w_{i,(i,k)} = -V_{ik}$, whereas the payment associated with $((i,k),j)$ is $w_{(i,k),j} = V_{jk}$.

Remark 1. In this game, the mean payoff $\chi_\imath(T)$ starting from any state $\imath$ is always nonpositive. Indeed, Player Min can always play a "tit for tat" policy, moving to state (j,k) from state j, and thus, paying $-V_{jk}$ to Max, if the last move of Max was $(i,k) \to j$, so that Min paid V_{jk}. In this way, Min can cancel the last payment he made, which guarantees a nonpositive mean payoff.

Given a vector $a \in (\mathbb{R}_{\max})^n$, $a \neq \bot$, we define the *tropical hyperplane*:

$$\mathcal{H}_a := \{x \in (\mathbb{R}_{\max})^n \mid \max_{i\in[n]}(a_i + x_i) \text{ achieved at least twice}\} \ .$$

Observe that $\mathcal{H}_a$ depends only on the point in the tropical projective space represented by a. Moreover, $\mathcal{H}_a$ is stable under the additive action of scalars, so that $\mathcal{H}_a$ can be identified with the subset of the tropical projective space consisting of the equivalence classes of non-identically $-\infty$ vectors of $\mathcal{H}_a$.

For a finite vector $a \in \mathbb{R}^n$, the tropical hyperplane $\mathcal{H}_a$ divides $(\mathbb{R}_{\max})^n$ into n sectors $(S_i(a))_{i\in[n]}$, defined by

(11) $$S_i(a) := \{x \in (\mathbb{R}_{\max})^n \mid \forall j \in [n],\ x_i + a_i \geqslant x_j + a_j\} \ .$$

The vector $-a$, which is unique up to an additive constant, is called the *apex* of $\mathcal{H}_a$. Indeed, the set $\mathcal{H}_a \cap \mathbb{R}^n$ modulo the scalar additions is the support of a polyhedral complex and $-a \in \mathbb{R}^n$ is the unique vertex (cell of dimension 0) of this complex. Then, we shall say that $\mathcal{H}_a$ has a *finite apex*. See Figure 1 for an illustration.

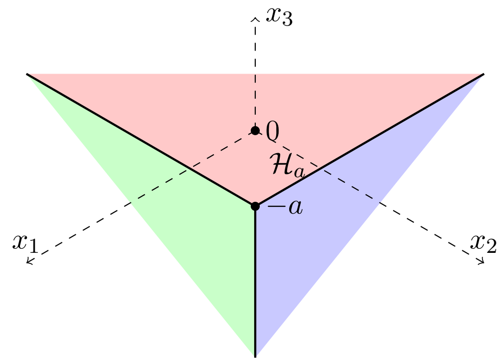


FIGURE (1) The hyperplane $\mathcal{H}_a$ with finite apex $a = (0,0,1)^\top$ and the sectors that $\mathcal{H}_a$ defines in the projective space $\mathbb{P}(\mathbb{R}_{\max})^3$.

The following result shows that verifying whether there is a tropical hyperplane containing a given collection of vectors reduces to solving a mean payoff game.

Proposition 4. [AGG12, Corollary 4.8] *For $a \in (\mathbb{R}_{\max})^n$ such that $a \neq \bot$, suppose that $V \in (\mathbb{R}_{\max})^{n\times p}$ satisfies Assumption 3, and let T be defined as above. Then, the following assertions are equivalent:*

(1) *$a \leqslant T(a)$;*
(2) *The column space* $\mathrm{Col}(V)$ *is included in $\mathcal{H}_a$.*

Corollary 5. *The columns of V are contained in a tropical hyperplane iff $\rho(T)$ is nonnegative.*

Proof. This follows from the equality $\rho(T) = \mathrm{cw}'(T)$ in Theorem 1 and the fact the supremum is achieved in (7). ☐

Theorem 6. *Let $T = T_V$ be the Shapley operator associated to the matrix $V \in (\mathbb{R}_{\max})^{n\times p}$ defined in* (9)*. Then, $\rho(T) \leqslant 0$. Moreover,*

$$-\rho(T) = \text{in-rad}(\mathrm{Col}(V)).$$

If $\rho(T)$ is finite, a maximal Hilbert's ball included in $\mathrm{Col}(V) \cap \mathbb{R}^n$ is given by $B(-a, -\rho(T))$ where a is any vector in $\mathbb{R}^n$ such that $T(a) \leqslant \rho(T) + a$.

We will deduce Theorem 6 from the following lemma:

Lemma 7. *For all $\lambda \in [-\infty, 0]$ and $a \in \mathbb{R}^n$,*

$$B(-a, -\lambda) \subset \mathrm{Col}\, V \iff T(a) \leqslant \lambda + a$$

Proof. Suppose first that λ is finite. Then, considering (9), we see that $T(a) \leqslant \lambda + a$ is equivalent to

$$\forall i \in [n],\ \exists k \in [p],\ \forall j \in [n],\ j \neq i, \quad -\lambda - a_i + a_j \leqslant V_{ik} - V_{jk} \ . \tag{12}$$

Let $x \in \mathbb{R}^n$, we have $x \in B(-a, -\lambda)$ if and only if

$$\forall i \in [n],\ \forall j \in [n], \quad x_i - x_j \leqslant -\lambda - a_i + a_j \ . \tag{13}$$

Moreover, the basic properties of residuation entail that $VV^\sharp \leqslant \mathrm{I}$, where $V^\sharp x$ is the maximal element y such that $Vy \leqslant x$. It follows that $x \in \mathrm{Col}(V)$ if and only if $x = VV^\sharp x$, or equivalently, $x \leqslant VV^\sharp x$. The latter property can be rewritten as $x_i \leqslant \max_{k\in[p]}\{V_{ik} + \min_{j\in[n]}(-V_{jk} + x_j)\}$, for all $i \in [n]$, which is equivalent to

$$\forall i \in [n],\ \exists k \in [p],\ \forall j \in [n], \quad x_i - x_j \leqslant V_{ik} - V_{jk} \ . \tag{14}$$

We can see that if eq. (12) and eq. (13) are true then eq. (14) follows, which shows the "$\Leftarrow$" direction of the lemma.

Now, we suppose that $B(-a, -\lambda) \subset \mathrm{Col}\, V$. For a given $i \in [n]$, we consider the vector $x^{(i)} \in \mathbb{R}^n$ given by $x^{(i)}_i = -\lambda - a_i$ and $x^{(i)}_j = -a_j$ for all $j \neq i$. Since $\lambda \leqslant 0$, we have $x^{(i)} \in B(-a, -\lambda)$, then $x^{(i)} \in \mathrm{Col}(V)$. Therefore by eq. (14), there exists $k \in [p]$ such that $\forall j \in [n],\ x^{(i)}_i - x^{(i)}_j \leqslant V_{ik} - V_{jk}$. Moreover, we have $\forall j \in [n], j \neq i,\ x^{(i)}_i - x^{(i)}_j = -\lambda - a_i + a_j$. Finally this yields eq. (12), which proves that $T(a) \leqslant \lambda + a$.

We finally show that the conclusion of the lemma is still true when $\lambda = -\infty$. This follows from $B(-a, +\infty) = \cup_{\mu\in(-\infty,0)} B(-a, -\mu)$ and $-\infty + a = \inf_{\mu\in(-\infty,0)} \mu + a$. □

Proof of Theorem 6. If $B(-a, -\lambda) \subset \mathrm{Col}(V)$ for some finite a, with $\lambda \leqslant 0$, by Lemma 7, we see that $T(a) \leqslant \lambda + a$, and we deduce from the Collatz-Wielandt property (Theorem 1) that $\rho(T) \leqslant \lambda$, and so, the radius of the ball, $-\lambda$, is bounded above by $-\rho(T)$.

Moreover, it follows from Assumption 3 that $\mathrm{Col}(V)$ has a finite vector a; indeed, we can take for a the supremum of the columns of V. Then, $B(-a, 0) \subset \mathrm{Col}(V)$, and by the previous observation, $0 \leqslant -\rho(T)$.

If $\rho(T) = -\infty$, then using the expression of the Collatz-Wielandt number of T, we get that for all finite $\lambda \leqslant 0$, there exists a finite vector $a \in \mathbb{R}^n$ such that $T(a) \leqslant \lambda + a$. By Lemma 7, this implies that $B(-a, -\lambda) \subset \mathrm{Col}(V)$, and so in-rad$(\mathrm{Col}(V)) \geqslant -\lambda$. Since this holds for all $\lambda \leqslant 0$, we deduce that in-rad$(\mathrm{Col}(V)) = +\infty = -\rho(T)$ is the supremum of the radius of a Hilbert's ball included in $\mathrm{Col}(V) \cap \mathbb{R}^n$.

Finally, if $\rho(T)$ is finite, since the infimum is attained in the expression of the Collatz-Wielandt number of T (see Theorem 1), there exists a finite vector $a \in \mathbb{R}^n$ such that $T(a) \leqslant \rho(T) + a$. By Lemma 7, this entails that $B(-a, -\rho(T)) \subset \mathrm{Col}(V)$.

This shows that $-\rho(T)$ is the maximum radius of a Hilbert's ball included in $\mathrm{Col}(V) \cap \mathbb{R}^n$. □

Remark 2. One can give an alternative, less direct proof, of Theorem 6 by deriving it from Theorem 16 of [AGKS18]. The latter result shows that if T is a Shapley operator which satisfies the technical condition (T must be "diagonal free"), then, the supremum of the radii of Hilbert's balls included in $S(T) := \{x \in \mathbb{R}^n \mid x \leqslant T(x)\}$ coincides with $\sup\{\mu \in \mathbb{R} \mid \exists v \in \mathbb{R}^n, \mu + v \leqslant T(v)\}$. The initial part of the proof of Lemma 7, up to (14), entails that $\mathrm{Col}(V)$ is precisely the set of vectors x such that $x \leqslant -T(-x)$.

The following is an immediate corollary of Theorem 6

Corollary 8. *The set* $\mathrm{Col}(V) \cap \mathbb{R}^n$ *is of empty interior if and only* $\rho(T) = 0$. □

By combining Corollary 8 and Corollary 5, we recover the following known result, established in [DSS05] (when the entries of the matrix V are finite).

Corollary 9 (Compare with Th. 4.2 of [DSS05])**.** *The set* $\mathrm{Col}(V) \cap \mathbb{R}^n$ *is of empty interior if and only if* $\mathrm{Col}(V)$ *is included in a tropical hyperplane.* □

The following additional corollary implies that we can check in polynomial time whether the inner radius of $\mathrm{Col}(V)$ is finite.

Corollary 10. *The following assertions are equivalent:*

(1) *The inner radius of* $\mathrm{Col}(V)$ *is infinite;*
(2) *There is no part of* $(\mathbb{R}_{\max})^n$ *that is left invariant by the operator* T*;*
(3) $T^n(0)$ *is the vector identically equal to* $-\infty$*;*
(4) $\rho(T) = -\infty$.

Proof. (3)⇒(1): Suppose that $T^n(0)$ is equal to $\perp$, the identically $-\infty$ vector. Let us take $u \in (\mathbb{R}_{\max})^n$, not identically $-\infty$, such that $T(u) = \rho(T) + u$. Then, there is a constant $\alpha \in \mathbb{R}$ such that $u_i \leqslant \alpha$, for all $i \in [n]$, and so $n\rho(T) + u = T^n(u) \leqslant T^n(0) + \alpha$ is the identically $-\infty$ vector. It follows that $\rho(T) = -\infty$. Then, by Theorem 6, the inner radius of $\mathrm{Col}(V)$ is infinite.

(1)⇒(2): Let I be a non-empty subset of $[n]$, and suppose that the part P_I consisting of vectors of $(\mathbb{R}_{\max})^n$ of support I is left invariant by T. Let u be the vector in this part such that $u_i = 0$ for all $i \in I$. Since $T(u) \in P_I$, there exists a real number α such that $T(u) \geqslant \alpha + u$. Hence, $\rho(T) = \mathrm{cw}'(T) \geqslant \alpha > -\infty$, and, by Theorem 6, $\text{in-rad}(\mathrm{Col}(V)) = -\rho(T) < +\infty$.

(2)⇒(3): Consider the map $\pi : (\mathbb{R}_{\max})^n \to \mathcal{P}([n])$, which sends a vector u to its support, $\pi(u) = \{i \in [n] \mid u_i \neq -\infty\}$, and consider the equivalence relation $\ker \pi$ on $(\mathbb{R}_{\max})^n$, such that $(x, y) \in \ker \pi$ iff $\pi(x) = \pi(y)$. The quotient set $(\mathbb{R}_{\max})^n / \ker \pi$ can be identified to $\mathcal{P}([n])$, and the order on $(\mathbb{R}_{\max})^n$ induces an order on $(\mathbb{R}_{\max})^n / \ker \pi$, corresponding to the inclusion order on $\mathcal{P}([n])$. The elements if $(\mathbb{R}_{\max})^n / \ker \pi$ are precisely the parts of $(\mathbb{R}_{\max})^n$, together with the singleton consisting of the identically $-\infty$ vector. Let $\perp = \pi^{-1}(\emptyset)$ denote this singleton, and let $\top = \pi^{-1}([n])$. Observe that $\top$ is the maximal element of $(\mathbb{R}_{\max})^n / \ker \pi$, and that $\perp$ is its minimal element.

Since the operator T is order preserving and commutes with the addition of a constant, it induces a map T_π from $(\mathbb{R}_{\max})^n / \ker \pi$ to itself, which is still order preserving. Moreover, the fixed points of T_π distinct from $\perp$ are precisely the parts of $(\mathbb{R}_{\max})^n$ that are invariant by T. We have, $T_\pi(\top) \leqslant \top$, from which we deduce that $((T_\pi)^k(\top))_{k \geqslant 0}$ is a nonincreasing sequence. If $(T_\pi)^k(\top) = (T_\pi)^{k+1}(\top) \neq \perp$, for some k, then $(T_\pi)^k(\top)$ would be an invariant part of T, contradicting the assumption. It follows that the sequence $(T_\pi^k(\top))_{k \geqslant 0}$ strictly decreases until it reaches $\perp$. Since the maximal cardinality of a chain in the lattice $\mathcal{P}(n)$ is $n + 1$, it follows that $(T_\pi)^n(\top) = \perp$. Hence, $T^n(0)$ is the identically $-\infty$ vector.

Finally, the equivalence between (1) and (4) follows from Theorem 6. □

Recall that a vector u in a tropical cone $\mathcal{V} \subset (\mathbb{R}_{\max})^n$ is *extreme* [GK07, BSS07] if $u = v \vee w$ with $v, w \in \mathcal{V}$ implies that $u = v$ or $u = w$. An *extreme direction* of $\mathcal{V}$ is of the form $\mathbb{R}_{\max} + u$, for some extreme vector of $\mathcal{V}$, i.e., it consists of the tropical scalar multiples of u. We say that a tropical cone in $(\mathbb{R}_{\max})^n$ is *simplicial* if it has precisely n extreme directions.

Proposition 11. *If a Hilbert's ball of positive radius is included in* $\mathrm{Col}(V)$*, then it is also included in a simplicial tropical cone generated by some* n *columns of* V*.*

Proof. For all maps $\sigma : [n] \to [p]$, such that $(i, \sigma(i)) \in E$, we consider the Shapley operator of the one-player game obtained when player MIN selects the action $k = \sigma(i)$ in state i, that is,

$$T^\sigma : (\mathbb{R}_{\max})^n \to (\mathbb{R}_{\max})^n, \qquad T_i^\sigma(x) = -V_{i\sigma(i)} + \max_{j\in[n], j\neq i} (V_{j\sigma(i)} + x_j) \ .$$

If $B(-a, -\lambda) \subset \mathrm{Col}(V)$, then, by Lemma 7, $T(a) \leqslant \lambda + a$. So, by choosing $k = \sigma(i)$ that achieves the minimum in the expression of $T(a)$ in (9), we get $T^\sigma(a) \leqslant \lambda + a$. Let $J := \sigma([n])$, so that $|J| \leqslant n$. Since $(i, \sigma(i)) \in E$ holds for all $i \in [n]$, the submatrix $V[J]$ of V, obtained by keeping the columns in J, cannot have a $-\infty$ row. Hence Lemma 7 can be applied to $V[J]$. We deduce that $B(-a, -\lambda) \subset \mathrm{Col}(V[J])$. Up to eliminating elements of J, we may assume that the set J is minimal to generate $\mathrm{Col}(V[J])$.

Let u^j denote the jth column of V. Then, every u^j must be extreme in $\mathrm{Col}(V[J])$. Indeed, suppose that $u^j = v \vee w$ with $v, w \in \mathcal{V}$ with $u^j \neq v$ and $u^j \neq w$. Then, we can write $v = \vee_{k\in J}(\lambda_k + u^k)$ and $w = \vee_{k\in J}(\mu_k + u^k)$, for some $\lambda_k, \mu_k \in \mathbb{R}_{\max}$. Moreover, we must have $\lambda_j < 0$, otherwise, $v \geqslant u^j$, and since $v \leqslant v \vee w = u^j$, $v = u^j$, a contradiction. A similar result holds for w. Since $u^j = v \vee w = \vee_{k\in J}((\lambda_k \vee \mu_k) + u^k)$, and $\lambda_j \vee \mu_j < 0$, we deduce that $u^j = \vee_{k\in J\setminus\{j\}}((\lambda_k \vee \mu_k) + u^k)$ is generated by the columns $\{u^k \mid k \in J \setminus \{j\}\}$, contradicting the minimality of J. It follows that every column of $V[J]$ is extreme in $\mathrm{Col}(V[J])$.

To show that $\mathrm{Col}(V[J])$ is simplicial, it remains to check that $|J| \geqslant n$. It is known that a collection of at most $n-1$ vectors in $(\mathbb{R}_{\max})^n$ is included in a tropical hyperplane – this follows for instance from a tropical analogue of the Radon theorem, see e.g. [But03] or [AGG09, Coro. 6.13]; or this can be deduced from the characterization of the tropical rank [DSS05, IR09, AGG12]. So if $|J| < n$, then $\mathrm{Col}(V[J])$ is of empty interior, contradicting $B(-a, -\lambda) \subset \mathrm{Col}(V[J])$. ☐

We get as a corollary the following result.

Corollary 12. *We have*

$$\text{in-rad}(\mathrm{Col}(V)) = \max_J \text{in-rad}(\mathrm{Col}(V[J])) \tag{15}$$

where the maximum is taken over all subsets $J \subset [p]$ of cardinality n. Moreover, if the inner radius is positive, the maximum is achieved by J such that $\mathrm{Col}(V[J])$ is simplicial.

By convention, if $p < n$, the maximum in (15) is zero.

Proof. The inequality $\geqslant$ in (15) is trivial. If $\text{in-rad}(\mathrm{Col}(V)) = 0$, the equality trivially holds in (15). If $\text{in-rad}(\mathrm{Col}(V)) > 0$, then for all $0 \leqslant \lambda < \text{in-rad}(\mathrm{Col}(V))$, there exists a Hilbert's ball of radius λ included in $\mathrm{Col}(V)$. By Proposition 11, this ball is also included in a simplicial tropical cone generated by columns of V, which means that there exists $J \subset [p]$ of cardinality n such that $\lambda \leqslant \text{in-rad}(\mathrm{Col}(V[J])) \leqslant \max_J \text{in-rad}(\mathrm{Col}(V[J]))$. Since this holds for all $0 \leqslant \lambda < \text{in-rad}(\mathrm{Col}(V))$, we deduce the inequality $\text{in-rad}(\mathrm{Col}(V)) \leqslant \max_J \text{in-rad}(\mathrm{Col}(V[J]))$ and so the equality. ☐

Corollary 13. *Computing the inner-radius of a tropical polyhedron (Problem 2) is polynomial-time Turing equivalent to mean payoff games (Problem 1).*

Proof. We observed immediately after stating Problem 1 that the problem of computing $\chi_i(T)$, where T is the Shapley operator of a deterministic mean payoff game, satisfying Assumption 1, polynomially Turing-reduces to mean payoff games. By Theorem 6, the opposite of the inner-radius is equal to $\rho(T)$. Since, $\rho(T) = \max_{i\in[n]} \chi_i(T)$, computing the inner-radius polynomially Turing-reduces to mean payoff games.

Conversely, Corollary 3.11 of [GP13] shows in particular that mean payoff games (Problem 1) polynomially Turing-reduces to checking whether a collection of vectors $v^1, \dots, v^p$ of $(\mathbb{Z}_{\max})^n$ are included in a tropical hyperplane. By Corollary 9 and Corollary 8, the latter problem is equivalent to checking whether the inner-radius of a tropical polyhedral cone $\mathrm{Col}(V)$ vanishes. ☐

Corollary 14. *Computing the center of a Hilbert's ball of maximal radius included in* $\operatorname{Col}(V)$*, where* $V \in \mathbb{Z}_{\max}^{n\times p}$*, polynomially Turing-reduces to mean payoff games.*

Proof. We first compute the maximal radius, $-\rho(T)$, which has been noted above, polynomially Turing-reduces to mean payoff games. We can also obtain by the same type of reduction an optimal policy σ of Player Min, which satisfies $\rho(T) = \rho(T^\sigma)$. Indeed, for each move of player Min $i \to j$, we can consider a modified Shapley operator $T^{(i,j)}$ corresponding to the game in which player Min makes the move $i \to j$ when in node i (i.e., this player has no choice in node i), and all the other allowed moves are unchanged. By checking whether $\rho(T^{(i,j)}) = \rho(T)$, we can verify if the move $i \to j$ belongs to an optimal policy of Player Min. By repeatedly restricting the freedom of moves of Player Min, we arrive, after a polynomial number of evaluation of $\rho(\cdot)$, at such an optimal policy σ. We showed that the center of an optimal Hilbert's ball is of the form $-u$ where $u \in \mathbb{R}^n$ and $T(u) \leqslant \rho(T) + u$. Since $T \leqslant T^\sigma$, and $\rho(T) = \rho(T^\sigma)$, it suffices to construct a vector in $\mathbb{R}^n$ such that $T^\sigma(u) \leqslant \rho(T^\sigma) + u$. Considering the tropically linear map $B := -\rho(T^\sigma) + T^\sigma$, we see this is equivalent to $Bu \leqslant u$. A standard result of tropical spectral theory shows that one can compute such a vector u by solving a shortest path problem. Actually, a tropical generating family of the set of such vectors u is the set of columns of the so called "metric closure" or "Kleene star" B^* of the matrix B, defined as the tropical sum of the tropical powers of B, see e.g. [BCOQ92, Th. 3.101] and [But10, § 4.4]. Moreover, the tropical sum u of the columns of B^* is a finite vector. In this way, we constructed u such that $Bu \leqslant u$, and so $T(u) \leqslant \rho(T) + u$. □

Remark 3. A subset $J \subset [p]$ satisfying $\operatorname{in-rad}(\operatorname{Col}(V)) = \max_J \operatorname{in-rad}(\operatorname{Col}(V[J]))$ can be computed by using any mean payoff game algorithm that returns, together with the mean payoff $\overline{\chi}(T)$, a vector $u \in \mathbb{R}^n$ such that $T(u) \leqslant \overline{\chi}(T) + u$. Indeed, we saw in the proof of Proposition 11 that, taking any policy σ such that $T(u) = T^\sigma(u)$, and setting $J := \sigma([n])$, we have $B(-u, -\overline{\chi}(T)) \subset \operatorname{Col}(V[J])$.

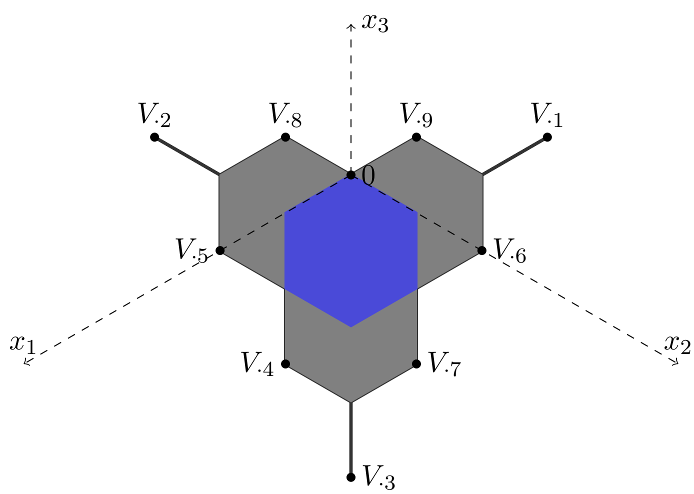


FIGURE (2) Example of an inner ball of the column space $\operatorname{Col}(V)$ in the projective space $\mathbb{P}(\mathbb{R}_{\max})^3$, where $V = \begin{pmatrix} -3 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & -3 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \\ -1 & -1 & -4 & -2 & -1 & -1 & -2 & 0 & 0 \end{pmatrix}$.

We can verify easily that $\lambda = -1$ and $a = (0,0,1)^\top$ satisfy $T(a) = \lambda + a$. Moreover, a policy σ such that $T^\sigma(a) = T(a)$ is given by $\sigma(1) = 4$, $\sigma(2) = 6$ and $\sigma(3) = 8$. Therefore, by Theorem 6 the maximal radius of a Hilbert's ball included in $\operatorname{Col} V$ is $-\lambda = 1$. Moreover, a maximal Hilbert's ball is given by $B(a,1)$, and $B(a,1)$ is included in the simplicial cone $\operatorname{Col}(V[J])$ where $J = \{4,6,8\} =$

$\sigma([3])$. This Hilbert's ball, together with the simplicial cone $\mathrm{Col}(V[J])$, are shown in Figure 2. Observe that the set J such that $\text{in-rad}(\mathrm{Col}(V[J])) = \text{in-rad}(\mathrm{Col}(V))$ is not unique, indeed, every $J' = \{i, j, k\}$ with $i \in \{4,5\}$, $j \in \{6,7\}$ and $k \in \{8,9\}$ is a candidate.

4. The strong duality theorem for tropical linear regression

In this section we will study the best approximation of a set of points in the tropical projective space by a tropical hyperplane. We will show that the best error of approximation is equal to the inner radius of the tropical module generated by this set of points.

Let $\mathcal{V} = \{v^{(1)}, \cdots, v^{(p)}\} \subset \mathbb{P}(\mathbb{R}_{\max})^n$ be a finite subset of the tropical projective space. Since we mainly focus on $\mathrm{Sp}(\mathcal{V})$, by abusing notions, we denote by $V \in (\mathbb{R}_{\max})^{n\times p}$ the matrix whose columns are given by some representatives of $v^{(1)}, \cdots, v^{(p)}$. Note that $\mathrm{Sp}(\mathcal{V}) = \mathrm{Col}(V)$, which does not depend on the choice of the representatives of $v^{(1)}, \cdots, v^{(p)}$. In the following, we use the notation $r^{\mathrm{in}}_{\mathcal{V}} = \text{in-rad}(\mathrm{Sp}(\mathcal{V}))$.

We introduce a one-sided Hausdorff distance from a set $A \subset \mathbb{P}(\mathbb{R}_{\max})^n$ to a set $B \subset \mathbb{P}(\mathbb{R}_{\max})^n$ with respect to the Hilbert's projective metric, which we shall call the *Hilbert's distance* from A to B:

$$\mathrm{dist}_H(A, B) := \sup_{a\in A} \mathrm{dist}_H(a, B) \ , \quad \text{with } \mathrm{dist}_H(a, B) := \inf_{b\in B} d(a, b) \ . \tag{16}$$

Note that $\mathrm{dist}_H(A, B) = 0$ if and only if for each part P_I of the tropical projective space, $A \cap P_I$ is included in the closure of $B \cap P_I$ with respect to the relative topology of P_I.

We are interested in the following *tropical linear regression problem*, consisting of finding a best hyperplane approximation of the set $\mathcal{V}$ in Hilbert's distance:

$$\inf_{a\in\mathbb{P}(\mathbb{R}_{\max})^n} \mathrm{dist}_H(\mathcal{V}, \mathcal{H}_a) \ . \tag{17}$$

Observe that if there is an index $i \in [n]$ such that $v^{(1)}_i = \cdots = v^{(p)}_i = -\infty$, then the tropical linear regression problem is trivially solved by setting $a_i = 0$ and $a_j = -\infty$ for $j \neq i$. Hence, in the sequel, we shall assume that the matrix V satisfies Assumption 3. In particular, considering the operator T defined in Equation (9), we know from Theorem 6 that the inner radius of $\mathrm{Sp}(\mathcal{V})$ is $-\rho(T)$.

The following lemma gives a simple formula for the Hilbert's distance from a point to a hyperplane.

Lemma 15. *For $x, a \in \mathbb{P}(\mathbb{R}_{\max})^n$, let $i^* \in \mathrm{argmax}_{i\in[n]}(x_i + a_i)$. Then the Hilbert's distance from the point x to the hyperplane $\mathcal{H}_a$ is*

$$\mathrm{dist}_H(x, \mathcal{H}_a) = x_{i^*} + a_{i^*} - \max_{i\in[n], i\neq i^*} (x_i + a_i) \ , \tag{18}$$

where we use the convention $(-\infty) - (-\infty) = 0$.

Proof. If $\max_{i\in[n]}(x_i + a_i) = -\infty$, then $x \in \mathcal{H}_a$ and Equation (18) holds with the convention $(-\infty) - (-\infty) = 0$. If $\max_{i\in[n]}(x_i + a_i) \neq -\infty$ and the maximum in the expression is attained twice, then $x \in \mathcal{H}_a$ and Equation (18) holds.

Now we focus on the case $\max_{j\in[n],j\neq i^*}(x_j + a_j) < x_{i^*} + a_{i^*} \in \mathbb{R}$, which implies $x_{i^*} \in \mathbb{R}$ and $a_{i^*} \in \mathbb{R}$. We split the argument into the following two cases.

Case 1: $\max_{j\in[n],j\neq i^*}(x_j + a_j) \in \mathbb{R}$. Then $\delta := x_{i^*} + a_{i^*} - \max_{j\in[n],j\neq i^*}(x_i + a_i) > 0$. Consider the point $\tilde{x}$ given by

$$\begin{cases} \tilde{x}_{i^*} = x_{i^*} - \delta & , \\ \tilde{x}_j = x_j & , \text{ for } j \in [n], j \neq i^* \end{cases}$$

Then $\tilde{x} \in \mathcal{H}_a$ and $d(x, \tilde{x}) = \delta$, implying $\mathrm{dist}_H(x, \mathcal{H}_a) \leqslant \delta$. Now, let $x' \in \mathcal{H}_a$, then the maximum in $\max_{j\in[n]}(x'_j + a_j)$ is achieved at least twice. So there exists $i \neq i^*$, such that $\max_{j\in[n]}(x'_j + a_j) = x'_i + a_i$. Since $i \neq i^*$, we have $\delta \leqslant x_{i^*} + a_{i^*} - (x_i + a_i)$, then $x'_i - x_i \geqslant x'_i + a_i - (x_{i^*} + a_{i^*}) + \delta$.

Since $x'_{i^*} + a_{i^*} \leqslant \max_{k\in[n]}(x'_k + a_k) = x'_i + a_i$, then $x'_{i^*} - x_{i^*} \leqslant x'_i + a_i - (x_{i^*} + a_{i^*})$. Therefore $d(x, x') \geqslant (x'_i - x_i) - (x'_{i^*} - x_{i^*}) \geqslant \delta$, which proves $\text{dist}_H(x, \mathcal{H}_a) \geqslant \delta$.

Case 2: $\max_{j\in[n], j\neq i^*}(x_j + a_j) = -\infty$. For $x' \in \mathcal{H}_a$, there exists $i \neq i^*$, such that $\max_{j\in[n]}(x'_j + a_j) = x'_i + a_i$. If $x'_i + a_i = -\infty$, then $x'_{i^*} - a_{i^*} = -\infty$. Since $a_{i^*} \in \mathbb{R}$, then $x'_{i^*} = -\infty$. Thus the fact that $x_{i^*} \in \mathbb{R}$ forces $d(x, x') = +\infty$, i.e., $\text{dist}_H(x, \mathcal{H}_a) = +\infty$ and Equation (18) holds. Now if $x'_i + a_i \in \mathbb{R}$, then $x'_i \in \mathbb{R}$ and $a_i \in \mathbb{R}$. Since the assumption $\max_{j\in[n], j\neq i^*}(x_j + a_j) = -\infty$ and $i \neq i^*$ gives us $x_i + a_i = -\infty$, we have $x_i = -\infty$, which leads to $d(x, x') = +\infty$. Therefore $\text{dist}_H(x, \mathcal{H}_a) = +\infty$ and Equation (18) holds. □

The next lemma shows that the distance from a Hilbert's ball to any tropical hyperplane is bounded below by the radius of this ball.

Lemma 16. *For $a, b \in \mathbb{P}(\mathbb{R}_{\max})^n$, suppose that the supports of a and b are not disjoint. Then, for all $r \geqslant 0$, we have*

$$\text{dist}_H(B(a, r), \mathcal{H}_b) \geqslant r \ . \tag{19}$$

Proof. Let $i^* \in \text{argmax}_{i\in[n]}(a_i + b_i)$. Since the supports of a, b are not disjoint, we have $a_{i^*} + b_{i^*} > -\infty$. Define $x \in (\mathbb{R}_{\max})^n$ by $x_{i^*} = r + a_{i^*}$ and $x_i = a_i$ for all $i \neq i^*$. Then $x \in B(a, r)$, and for all $i \neq i^*$, $x_{i^*} + b_{i^*} = r + a_{i^*} + b_{i^*} \geqslant r + a_i + b_i = r + x_i + b_i$. So by Lemma 15 we deduce that $\text{dist}_H(x, \mathcal{H}_b) \geqslant r$, which implies that $\text{dist}_H(B(a, r), \mathcal{H}_b) \geqslant r$. □

Lemma 17. *Suppose that $\mathcal{W}$ is a tropical cone in $(\mathbb{R}_{\max})^n$. Then,*

$$\text{dist}_H(\text{Sp}(\mathcal{V}), \mathcal{W}) = \text{dist}_H(\mathcal{V}, \mathcal{W}) \ . \tag{20}$$

Proof. Consider an element $x \in \text{Sp}(\mathcal{V})$, so that there exists a finite subset of points $(v^{(j)})_{j\in J}$ of $\mathcal{V}$ and $(\alpha_j)_{j\in J} \in \mathbb{R}^J$, satisfying $x = \vee_{j\in J}(\alpha_j + v^{(j)})$. Take $\lambda > \text{dist}_H(\mathcal{V}, \mathcal{W})$. Then, for any $j \in J$, there exists $w^{(j)} \in \mathcal{W}$ such that $d(v^{(j)}, w^{(j)}) \leqslant \lambda$, and so, there are real numbers γ^j, β^j such that $\gamma^j + w^{(j)} \leqslant v^{(j)} \leqslant \beta^j + w^{(j)}$, and $\beta^j - \gamma^j \leqslant \lambda$. After replacing $w^{(j)}$ by $\gamma^j + w^{(j)} \in \mathcal{W}$, we may assume that $\gamma^j = 0$. Then, $\vee_{j\in J}\alpha_j + w^{(j)} \leqslant x \leqslant \lambda + \vee_{j\in J}\alpha_j + w^{(j)}$, which entails that $\text{dist}_H(x, \mathcal{W}) \leqslant \lambda$. Since this holds for all $\lambda > \text{dist}_H(\mathcal{V}, \mathcal{W})$, we deduce that $\text{dist}_H(x, \mathcal{W}) \leqslant \text{dist}_H(\mathcal{V}, \mathcal{W})$, and so, $\text{dist}_H(\text{Sp}(\mathcal{V}), \mathcal{W}) \leqslant \text{dist}_H(\mathcal{V}, \mathcal{W})$.

The other inequality follows from $\mathcal{V} \subset \text{Sp}(\mathcal{V})$. □

The next lemma shows that the distance from the set $\mathcal{V}$ to any tropical hyperplane is always greater than or equal to the radius of any Hilbert's ball included in the module $\text{Sp}(\mathcal{V})$.

Lemma 18 (Weak duality)**.** *We have the following inequality*

$$r^{\text{in}}_{\mathcal{V}} = \sup\{r \geqslant 0 \mid \exists a \in \mathbb{R}^n, B(a, r) \subset \text{Sp}(\mathcal{V})\} \leqslant \inf_{b\in\mathbb{P}(\mathbb{R}_{\max})^n} \text{dist}_H(\mathcal{V}, \mathcal{H}_b) \ . \tag{21}$$

Proof. Let $a \in \mathbb{R}^n$ and $r \geqslant 0$ such that $B(a, r) \subset \text{Sp}(\mathcal{V})$, and let $b \in \mathbb{P}(\mathbb{R}_{\max})^n$. Since the supports of a and b are not disjoint, by Lemma 16, we have $r \leqslant \text{dist}_H(B(a, r), \mathcal{H}_b)$. Since $B(a, r) \subset \text{Sp}(\mathcal{V})$, then $\text{dist}_H(B(a, r), \mathcal{H}_b) \leqslant \text{dist}_H(\text{Sp}(\mathcal{V}), \mathcal{H}_b)$. Therefore, by using Lemma 17, we conclude that $r \leqslant \text{dist}_H(\mathcal{V}, \mathcal{H}_b)$. □

Lemma 19. *For all $\lambda \in [-\infty, 0]$ and $b \in \mathbb{P}(\mathbb{R}_{\max})^n$, we have*

$$T(b) \geqslant \lambda + b \Leftrightarrow \text{dist}_H(\mathcal{V}, \mathcal{H}_b) \leqslant -\lambda \ . \tag{22}$$

Proof. The equivalence is trivial if $\lambda = -\infty$, so, we suppose that $\lambda \in (-\infty, 0]$. Suppose in addition that $T(b) \geqslant \lambda + b$, i.e., for any $i \in [n]$,

$$\min_{k\in[p],(i,k)\in E}[-V_{ik} + \max_{j\in[n],j\neq i}(V_{jk} + b_j)] \geqslant \lambda + b_i.$$

Then for any $i \in [n]$ and any $k \in [p]$,

$$V_{ik} + b_i \leqslant \max_{j\in[n], j\neq i} (V_{jk} + b_j) - \lambda.$$

For each $k \in [p]$, by taking $i \in \operatorname{argmax}_{j\in[n]}(V_{jk} + b_j)$ and using Lemma 15, we deduce that the distance from the column $V_{\cdot k} = v^{(k)}$ to the hyperplane $\mathcal{H}_b$ is $\leqslant -\lambda$, which implies $\operatorname{dist}_H(\mathcal{V}, \mathcal{H}_b) \leqslant -\lambda$.

Now we suppose that $\operatorname{dist}_H(\mathcal{V}, \mathcal{H}_b) \leqslant -\lambda$. For $k \in [p]$ and $i \in [n]$, if $i \notin \operatorname{argmax}_{j\in[n]}(V_{jk} + b_j)$, then

$$V_{ik} + b_i \leqslant \max_{j\in[n], j\neq i} (V_{jk} + b_j) \leqslant \max_{j\in[n], j\neq i} (V_{jk} + b_j) - \lambda.$$

Otherwise, if $i \in \operatorname{argmax}_{j\in[n]}(V_{jk}+b_j)$, then knowing that $\operatorname{dist}_H(v^{(k)}, \mathcal{H}_b) \leqslant -\lambda$ and using Lemma 15, we get $V_{ik} + b_i \leqslant \max_{j\in[n],j\neq i}(V_{jk} + b_j) - \lambda$. Therefore, we deduce that for all $i \in [n]$ and $k \in [p]$,

$$V_{ik} + b_i \leqslant \max_{j\in[n], j\neq i} (V_{jk} + b_j) - \lambda.$$

Thus for all $i \in [n]$,

$$\min_{k\in[p],(i,k)\in E} [-V_{ik} + \max_{j\in[n], j\neq i} (V_{jk} + b_j)] \geqslant \lambda + b_i,$$

namely $T(b) \geqslant \lambda + b$. □

The following theorem presents a strong duality result between a best tropical hyperplane approximation of a set $\mathcal{V}$ of points and the largest inner balls that its module $\operatorname{Sp}(\mathcal{V})$ contains.

Theorem 20 (Strong duality)**.** *We have*

$$\min_{b\in\mathbb{P}(\mathbb{R}_{\max})^n} \operatorname{dist}_H(\mathcal{V}, \mathcal{H}_b) = r^{\text{in}}_{\mathcal{V}} = \sup\{r \geqslant 0 \mid \exists a \in \mathbb{R}^n, B(a, r) \subset \operatorname{Sp}(\mathcal{V})\}. \tag{23}$$

The minimum is achieved by any vector $b \in \mathbb{P}(\mathbb{R}_{\max})^n$ such that $T(b) \geqslant \rho(T) + b$. Moreover, if $r^{\text{in}}_{\mathcal{V}}$ is finite, the supremum is achieved by a ball $B(-c, r^{\text{in}}_{\mathcal{V}})$ where $c \in \mathbb{R}^n$ is any vector such that $T(c) \leqslant \rho(T) + c$.

Proof. Theorem 6 entails that $r^{\text{in}}_{\mathcal{V}} = -\rho(T)$ and that the last assertion of the theorem holds. Moreover, the existence of a vector $b \in \mathbb{P}(\mathbb{R}_{\max})^n$ such $T(b) \geqslant \rho(T) + b$ follows from Theorem 1. Then, by Lemma 19, we have $\operatorname{dist}_H(\mathcal{V}, \mathcal{H}_b) \leqslant r^{\text{in}}_{\mathcal{V}}$, which combined with the weak duality property (21) implies that the equality holds in (21), and that b such that $T(b) \geqslant \rho(T) + b$ achieves the minimum in (23). □

The following lemma allows us to bound from below the value of the tropical linear regression problem by looking at points in the sectors of a hyperplane $\mathcal{H}_a$.

Lemma 21. *If $a \in \mathbb{R}^n$ and $r \in [0, +\infty]$ are such that*

$$\forall i \in [n], \exists k \in [p], \quad v^{(k)} \in S_i(a) \;\text{ and }\; \operatorname{dist}_H(v^{(k)}, \mathcal{H}_a) \geqslant r \;,$$

then $B(-a, r) \subset \operatorname{Col}(V)$ and $\min_{b\in\mathbb{P}(\mathbb{R}_{\max})^n} \operatorname{dist}_H(\mathcal{V}, \mathcal{H}_b) \geqslant r$.

Proof. If $r = +\infty$, then for any $i \in [n]$, there is some $\sigma_i \in [p]$ such that $v^{(\sigma_i)} \in S_i(a)$ and $\operatorname{dist}_H(v^{(\sigma_i)}, \mathcal{H}_a) = +\infty$. Since a is finite, for any $i \in [n]$, we have $v^{(\sigma_i)}_i \in \mathbb{R}$ and $v^{(\sigma(i))}_j = -\infty$ for any $j \neq i$. We deduce that $\operatorname{Sp}(v^{(\sigma_1)}, \cdots, v^{(\sigma_n)}) = \mathbb{R}_{\max}$. Then $\operatorname{Col}(V) = \mathbb{R}_{\max}$, and so $B(-a, +\infty) = \mathbb{R}_{\max} \subset \operatorname{Col}(V)$.

Now we consider $r \in [0, +\infty)$. For $i \in [n]$, by the assumption of this lemma, there exists $k \in [p]$ such that $v^{(k)} \in S_i(a)$ and $\operatorname{dist}_H(v^{(k)}, \mathcal{H}_a) \geqslant r$. Hence, by using (11) and Lemma 15, we deduce that the column $V_{\cdot k} = v^{(k)}$ satisfies $V_{ik} + a_i \geqslant r + \max_{j\neq i}(V_{jk} + a_j)$. Therefore, we have $-V_{ik} + \max_{j\neq i}(V_{jk} + a_j) \leqslant -r + a_i$, which implies for any $i \in [n]$,

$$T_i(a) = \inf_{l\in[p],(i,l)\in E} \Big[- V_{il} + \max_{j\in[n],j\neq i}(V_{jl} + a_j)\Big] \leqslant -r + a_i,$$

i.e., $T(a) \leqslant -r + a$. Therefore, by Lemma 7, we deduce that $B(-a, r) \subset \mathrm{Col}(V)$.

Finally by Theorem 20, we have

$$\min_{b\in\mathbb{P}(\mathbb{R}_{\max})^n} \mathrm{dist}_H(\mathcal{V}, \mathcal{H}_b) = r^{\mathrm{in}}_{\mathcal{V}} = \sup\{s \geqslant 0 \mid \exists x \in \mathbb{R}^n, B(x, s) \subseteq \mathrm{Sp}(\mathcal{V})\} \geqslant r.$$

□

Given a hyperplane $\mathcal{H}_b$, we call *witness point* of $\mathcal{H}_b$ any point p in $\mathcal{V}$ such that the distance from p to the hyperplane $\mathcal{H}_b$ equals the distance from the set $\mathcal{V}$ to this hyperplane.

Theorem 22 (Optimality certificates). *Let $a \in \mathbb{R}^n$, then the following assertions are equivalent:*

(1) $T(a) = \rho(T) + a$;
(2) *The hyperplane $\mathcal{H}_a$ admits a witness point in each sector, meaning that $\forall i \in [n], \exists k \in [p]$, $v^{(k)} \in S_i(a)$ and $\mathrm{dist}_H(v^{(k)}, \mathcal{H}_a) = \mathrm{dist}_H(\mathcal{V}, \mathcal{H}_a)$.*

Moreover, if these assertions hold, then, $\rho(T) = -\mathrm{dist}_H(\mathcal{V}, \mathcal{H}_a)$, $\mathcal{H}_a$ is an optimal solution of the tropical linear regression problem, and $B(-a, \mathrm{dist}_H(\mathcal{V}, \mathcal{H}_a))$ is a Hilbert's ball of maximal radius included in $\mathrm{Sp}(\mathcal{V})$.

Proof. If $a \in \mathbb{R}^n$ satisfies $T(a) = \rho(T) + a = -r^{\mathrm{in}}_{\mathcal{V}} + a$, then by Theorem 20, $\mathcal{H}_a$ is optimal in Equation (23), i.e., $\mathrm{dist}_H(\mathcal{V}, \mathcal{H}_a) = r^{\mathrm{in}}_{\mathcal{V}}$, and for all $i \in [n]$ we have

$$\min_{k\in[p],(i,k)\in E}[-V_{ik} + \max_{j\in[n],j\neq i}(V_{jk} + a_j)] = -r^{\mathrm{in}}_{\mathcal{V}} + a_i.$$

Then for all $i \in [n]$, there exists $k \in [p]$ such that

$$-V_{ik} + \max_{j\in[n],j\neq i}(V_{jk} + a_j) = -r^{\mathrm{in}}_{\mathcal{V}} + a_i,$$

i.e., $V_{ik} + a_i = r^{\mathrm{in}}_{\mathcal{V}} + \max_{j\in[n],j\neq i}(V_{jk} + a_j)$. This implies that $v^{(k)} = V_{\cdot k} \in S_i(a)$, and also by Lemma 15, that $\mathrm{dist}_H(v^{(k)}, \mathcal{H}_a) = r^{\mathrm{in}}_{\mathcal{V}} = \mathrm{dist}_H(\mathcal{V}, \mathcal{H}_a)$.

Now, we suppose that we have assertion (2). By Lemma 21, we have

$$\min_{b\in\mathbb{P}(\mathbb{R}_{\max})^n} \mathrm{dist}_H(\mathcal{V}, \mathcal{H}_b) \geqslant \mathrm{dist}_H(\mathcal{V}, \mathcal{H}_a),$$

which means that $\mathcal{H}_a$ achieves the minimum in (23), so that $\mathrm{dist}_H(\mathcal{V}, \mathcal{H}_a) = r^{\mathrm{in}}_{\mathcal{V}}$. Hence, $\forall k \in [p], \mathrm{dist}_H(v^{(k)}, \mathcal{H}_a) \leqslant r^{\mathrm{in}}_{\mathcal{V}}$, so that by Lemma 15 we have $\forall k \in [p], \forall i \in [n]$,

$$V_{ik} + a_i \leqslant r^{\mathrm{in}}_{\mathcal{V}} + \max_{j\in[n],j\neq i}(V_{jk} + a_j).$$

Therefore, we obtain

$$\forall i \in [n], \forall k \in [p], (i, k) \in E; -r^{\mathrm{in}}_{\mathcal{V}} + a_i \leqslant -V_{ik} + \max_{j\in[n],j\neq i}(V_{jk} + a_j) \ . \tag{24}$$

Assertion (2) also implies $\forall i \in [n], \exists k \in [p], V_{ik} + a_i \geqslant \max_{j\in[n],j\neq i}(V_{jk} + a_j)$ and $\mathrm{dist}_H(v^{(k)}, \mathcal{H}_a) = r^{\mathrm{in}}_{\mathcal{V}}$, with $V_{ik} \neq -\infty$ because $a \in \mathbb{R}^n$ and $V_{\cdot k} \neq \bot$. This means, by Lemma 15, that $\forall i \in [n], \exists k \in [p], (i, k) \in E, V_{ik} + a_i = r^{\mathrm{in}}_{\mathcal{V}} + \max_{j\in[n],j\neq i}(V_{jk} + a_j)$. Then

$$\forall i \in [n], \exists k \in [p], (i, k) \in E; -r^{\mathrm{in}}_{\mathcal{V}} + a_i = -V_{ik} + \max_{j\in[n],j\neq i}(V_{jk} + a_j) \ . \tag{25}$$

From (24) and (25), we conclude that

$$\forall i \in [n], -r_{\mathcal{V}}^{\text{in}} + a_i = \min_{k\in[p],(i,k)\in E} [-V_{ik} + \max_{j\in[n],j\neq i}(V_{jk} + a_j)] = T_i(a).$$

Therefore $T(a) = \rho(T) + a$.

The final part of the theorem follows from Theorem 20. □

Remark 4. When $T(a) = \rho(T) + a$ and $a \in \mathbb{R}^n$, Theorem 22 and Theorem 20 entail the following remarkable property: there is an optimal Hilbert's ball whose center coincides with the apex of an optimal regression hyperplane. This property is illustrated in Figure 4 below.

Remark 5. The situation in which $T(a) = \bot$ holds for some finite vector a (or equivalently, for all finite vectors a) is degenerate. Indeed, we observe from the proof of Theorem 22 that $T(a) = \bot$ for some finite vector a if and only if, for all $i \in [n]$, there is a vector $v^{(k)}$ such that $v_i^{(k)}$ is finite and all $v_j^{(k)}$ with $j \neq i$ are $-\infty$. Then, V contains a $n \times n$ diagonal submatrix, and so, $\text{Col}(V) = \text{Sp}(\mathcal{V}) = (\mathbb{R}_{\max})^n$.

We next exhibit a situation in which the existence of a finite eigenvector, required to apply Theorem 22, is guaranteed.

Proposition 23. *Suppose that all the vectors $v \in \mathcal{V}$ have finite entries. Then, the operator T has a finite eigenvector a.*

Proof. Theorems 9 and 13 of [GG04] imply that an order preserving and additively homogeneous map $T : \mathbb{R}^n \to \mathbb{R}^n$ has a finite eigenvector if the recession function $\hat{T}(x) := \lim_{s\to\infty} s^{-1}T(sx)$ has only fixed points on the diagonal. When the matrix V is finite, considering $T := T_V$, we have $\hat{T}_i(x) = \max_{j\in[n],j\neq i} x_j$, for all $i \in [n]$, so the latter condition is trivially satisfied. This entails that there exists a vector $a \in \mathbb{R}^n$ such that $T(a) = \rho(T) + a$. □

A more general condition, involving the notion of dominions, is given in Appendix A.

The following proposition shows that we can determine witness points from a policy $\sigma : [n] \mapsto [p]$, that satisfies $T(a) = T^{\sigma}(a)$ where a is a finite eigenvector of the operator T. For an illustration of this lemma see Figure 2.

Proposition 24. *Let $a \in \mathbb{R}^n$ such that $T(a) = -r_{\mathcal{V}}^{\text{in}} + a$, and $\sigma : [n] \mapsto [p]$ a map, such that $\forall i \in [n], (i, \sigma(i)) \in E$. We have $T(a) = T^{\sigma}(a)$ if and only if for all $i \in [n]$, $V_{\cdot\sigma(i)}$ is a witness point of $\mathcal{H}_a$ that belongs to the sector $S_i(a)$.*

Proof. If $T(a) = T^{\sigma}(a)$, then $T^{\sigma}(a) = -r_{\mathcal{V}}^{\text{in}} + a$. Therefore, we have for all $i \in [n]$, $-V_{i\sigma(i)} + \max_{j\neq i}(V_{j\sigma(i)} + a_j) = -r_{\mathcal{V}}^{\text{in}} + a_i$, i.e. $V_{i\sigma(i)} + a_i = r_{\mathcal{V}}^{\text{in}} + \max_{j\neq i}(V_{j\sigma(i)} + a_j)$, which means that $V_{\cdot\sigma(i)} \in S_i(a)$ and, by Lemma 15, that $\text{dist}_H(V_{\cdot\sigma(i)}, \mathcal{H}_a) = r_{\mathcal{V}}^{\text{in}}$, i.e for all $i \in [n]$, $V_{\cdot\sigma(i)}$ is a witness point in the sector $S_i(a)$.

Conversely, if for all $i \in [n]$, $V_{\cdot\sigma(i)}$ is a witness point in the sector $S_i(a)$. Let $i \in [n]$, we have then $V_{i\sigma(i)} + a_i = r_{\mathcal{V}}^{\text{in}} + \max_{j\neq i}(V_{j\sigma(i)} + a_j)$, i.e. $-V_{i\sigma(i)} + \max_{j\neq i}(V_{j\sigma(i)} + a_j) = -r_{\mathcal{V}}^{\text{in}} + a_i$. We know that for all $k \in [p]$, $\text{dist}_H(V_{\cdot k}, \mathcal{H}_a) \leqslant r_{\mathcal{V}}^{\text{in}}$, then by Lemma 15, $V_{ik} + a_i \leqslant r_{\mathcal{V}}^{\text{in}} + \max_{j\neq i}(V_{jk} + a_j)$, i.e. $-V_{ik} + \max_{j\neq i}(V_{jk} + a_j) \geqslant -r_{\mathcal{V}}^{\text{in}} + a_i$. Therefore, $T_i(a) = \inf_{k\in[p],(i,k)\in E} \Big[- V_{ik} + \max_{j\in[n],j\neq i}(V_{jk} + a_j)\Big] = -V_{i\sigma(i)} + \max_{j\neq i}(V_{j\sigma(i)} + a_j) = T_i^{\sigma}(a)$. □

We now formalize the tropical linear regression problem:

Problem 3 (Tropical linear regression)**.** *Input: a finite set of vectors $\mathcal{V} \subset \mathbb{Z}_{\max}^n$. Goal: compute the infimum of the one-sided Hausdorff distance of $\mathcal{V}$ to a tropical hyperplane, i.e., the value of the optimization problem* (17)*.*

Corollary 25. *The tropical linear regression problem (Problem 3) is polynomial time Turing-equivalent to mean payoff games (Problem 1).*

Proof. This follows from the strong duality theorem (Theorem 20) and Corollary 13. ☐

Corollary 26. *Computing an optimal regression hyperplane $\mathcal{H}_a$ in* (17)*, given a finite set of vectors $\mathcal{V} \subset \mathbb{Z}^n_{\max}$, polynomially Turing-reduces to mean payoff games.*

Proof. By Theorem 20, we need to find a vector a such that $T(a) \geqslant \rho(T) + a$. Arguing as in the proof of Corollary 14, but exchanging the roles of Player Max and Min, we end up with an optimal policy τ of Player Max. Then, it suffices to find a vector $a \in (\mathbb{R} \cup \{-\infty\})^n$, $a \neq \bot$, such that ${}^{\tau}T(a) \geqslant \rho({}^{\tau}T) + a$. Still arguing as in the proof of Corollary 14, we are reduced to a problem of tropical (min-plus instead max-plus) spectral theory, which again reduces to a shortest path problem. ☐

In Figure 3, we consider the same matrix V as in Figure 2. The Figure 3 shows the witness points in each of the sectors defined by the hyperplane $\mathcal{H}_a$ where $a = (0, 0, 1)^\top$ satisfies $T(a) = \lambda + a$ with $\lambda = -1$. In this example, we have two witness points in each sector: $V_{\cdot 4}$ and $V_{\cdot 5}$ are the witness points in the sector $S_1(a)$ (in green), $V_{\cdot 6}$ and $V_{\cdot 7}$ are the witness points in the sector $S_2(a)$ (in blue) and $V_{\cdot 8}$ and $V_{\cdot 9}$ are the witness points in the sector $S_3(a)$ (in red).

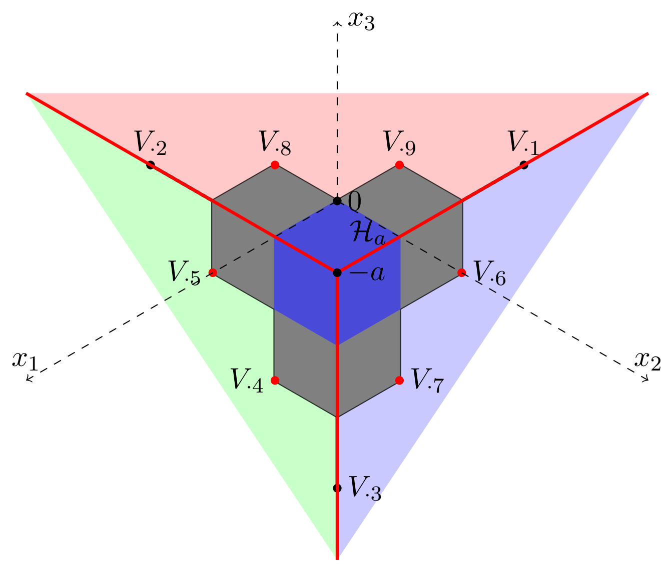


FIGURE (3) The inner ball of a column space $\mathrm{Col}(V)$ and the linear regression of the columns of V.

In Figure 4, we consider the following matrix $U \in \mathbb{R}^{3\times 4}$:

$$U = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & -2 \end{pmatrix}$$

The operator associated to U is the following map $T : (\mathbb{R}_{\max})^n \to (\mathbb{R}_{\max})^n$:

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \min[1+\max(x_2,x_3), \max(-1+x_2,x_3), -1+\max(x_2,-2+x_3), \max(1+x_2,-2+x_3)] \\ \min[\max(-1+x_1,x_3), 1+\max(x_1,x_3), \max(1+x_1,-2+x_3), -1+\max(x_1,-2+x_3)] \\ \min[\max(-1+x_1,x_2), 1+\max(x_1,-1+x_2), 2+\max(1+x_1,x_2), 2+\max(x_1,1+x_2)] \end{pmatrix} .$$

We verify easily that $\lambda = -1$ and $a = (0, 0, 1)^\top$ satisfy $T(a) = \lambda + a$, so that the inner radius of $\mathrm{Col}(U)$ is $r^{\text{in}}_{\mathcal{U}} = 1$. In this example, other hyperplanes like $\mathcal{H}_b$ and $\mathcal{H}_c$, with $b = (0, 0, -1)^\top$ and

$c = (0, 0, -\infty)^\top$, are also optimal solutions of the tropical linear regression problem, but $\mathcal{H}_a$ is the only hyperplane such that a is a finite eigenvector of the operator T and, hence, that satisfies also $B(-a, 1) \subset \mathrm{Col}(U)$.

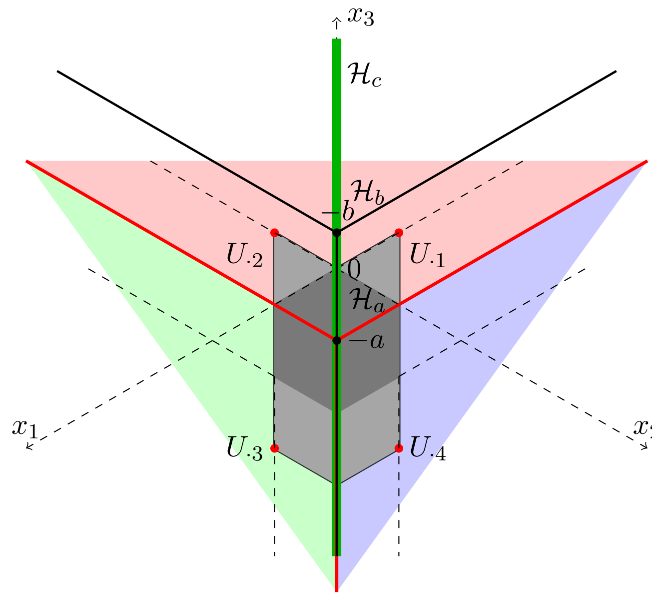


FIGURE (4) A column space $\mathrm{Col}(U)$ (light and dark gray regions) with multiple hyperplanes that are optimal solutions of the tropical linear regression problem, and multiple inner balls of maximal radius, but a unique optimal hyperplane with witness points in each sector, corresponding to the finite eigenvector $a = (0, 0, 1)^\top$ of T and to the inner ball in dark gray.

5. Tropical linear regression with sign or type patterns

Here, we study several variants of the tropical linear regression problem, which can also be solved by the present technique of reduction to a mean payoff game. The second of these variants (with "types") will arise in the economic application of Section 7.

5.1. **Tropical linear regression with signs.** Given $I, J \subset [n]$ such that $I, J \neq \emptyset$, $I \cup J = [n]$ and $I \cap J = \emptyset$ and $a \in \mathbb{P}(\mathbb{R}_{\max})^n$, we define the *signed tropical hyperplane* of type (I, J):

$$\mathcal{H}_a^{IJ} := \{x \in (\mathbb{R}_{\max})^n \mid \max_{i \in I}(a_i + x_i) = \max_{j \in J}(a_j + x_j)\} \tag{26}$$

Given a set $\mathcal{V} \subset (\mathbb{R}_{\max})^n$, of cardinality $|\mathcal{V}| = p$, the *signed tropical linear regression problem* of type (I, J) consists in finding the best approximation of $\mathcal{V}$ by a signed hyperplane of type (I, J):

$$\min_{a \in \mathbb{P}(\mathbb{R}_{\max})^n} \mathrm{dist}_H(\mathcal{V}, \mathcal{H}_a^{IJ}) \ . \tag{27}$$

Let M be a closed tropical cone of $(\mathbb{R}_{\max})^n$ and $x \in (\mathbb{R}_{\max})^n$. The projection $P_M(x)$ of the point x onto M [CGQ04] is defined by:

$$P_M(x) := \max\{z \in M \mid z \leqslant x\} \ . \tag{28}$$

The Hilbert's distance from x to M is achievd by the projection $P_M(x)$.

Theorem 27 ([CGQ04]). *Given a closed tropical semimodule $M \subset (\mathbb{R}_{\max})^n$ and $x \in (\mathbb{R}_{\max})^n$, we have:*

$$\mathrm{dist}_H(x, M) = d(x, P_M(x)) \enspace .$$

The following lemma identifies the projection of a point $x \in \mathbb{P}(\mathbb{R}_{\max})^n$ onto a signed tropical hyperplane $\mathcal{H}_a^{IJ}$.

Lemma 28. *Let $x, a \in \mathbb{P}(\mathbb{R}_{\max})^n$ and $K = \operatorname{supp} a$. The projection $P_{\mathcal{H}_a^{IJ}}(x)$ of x onto $\mathcal{H}_a^{IJ}$ is given by:*

$$[P_{\mathcal{H}_a^{IJ}}(x)]_l = \begin{cases} x_l & , \text{ for } l \in K^c \\ \min\{x_l, -a_l + \max_{j\in J}(a_j + x_j)\} & , \text{ for } l \in I \cap K \\ \min\{x_l, -a_l + \max_{i\in I}(a_i + x_i)\} & , \text{ for } l \in J \cap K \end{cases} \tag{29}$$

where K^c denotes the complementary of K in $[n]$.

Proof. Denote the right hand side vector of (29) by $\tilde{x}$. From (28), we have $P_{\mathcal{H}_a^{IJ}}(x) = \max\{z \in \mathcal{H}_a^{IJ} \mid z \leqslant x\}$. Let $z \in \mathcal{H}_a^{IJ}$ such that $z \leqslant x$. We will prove that $z \leqslant \tilde{x}$. Let $l \in I$, if $l \in I \cap K^c$, we have right away that $z_l \leqslant x_l = \tilde{x}_l$. Now if $l \in I \cap K$, knowing that $z \in \mathcal{H}_a^{IJ}$ and using (26), we have $a_l + z_l \leqslant \max_{i\in I}(a_i + z_i) = \max_{j\in J}(a_j + z_j) \leqslant \max_{j\in J}(a_j + x_j)$. Then, $z_l \leqslant -a_l + \max_{j\in J}(a_j + x_j)$. We know also that $z_l \leqslant x_l$, then $z_l \leqslant \tilde{x}_l$. Similarly the inequality $z_l \leqslant \tilde{x}_l$ can also be proved for all $l \in J$. Therefore, for all $z \in \mathcal{H}_a^{IJ}$, if $z \leqslant x$ then $z \leqslant \tilde{x}$. Using (28), it suffices now to prove that $\tilde{x} \in \mathcal{H}_a^{IJ}$. Indeed, $\max_{i\in I}(a_i+\tilde{x}_i) = \max_{i\in I\cap K}(a_i+\tilde{x}_i) = \max_{i\in I\cap K}\{\min(a_i+x_i, \max_{j\in J}(a_j+x_j))\} = \min\{\max_{i\in I\cap K}(a_i+x_i), \max_{j\in J}(a_j+x_j)\} = \min\{\max_{i\in I}(a_i+x_i), \max_{j\in J}(a_j+x_j)\}$, and by symmetry we deduce that $\max_{j\in J}(a_j + \tilde{x}_j)$ is also equal to the same quantity, and so $\tilde{x} \in \mathcal{H}_a^{IJ}$. □

Remark 6. The formula of Lemma 28 may be compared with formula for the projection of a point onto a tropical half-space $\{x \in (\mathbb{R}_{\max})^n \mid \max_{i\in I}(a_i + x_i) \leqslant \max_{j\in J}(a_j + x_j)\}$, see [AGNS11, Th. 5.1].

Proposition 29. *Let $x, a \in \mathbb{P}(\mathbb{R}_{\max})^n$. The Hilbert's distance of the point x to the signed hyperplane $\mathcal{H}_a^{IJ}$ is:*

$$\mathrm{dist}_H(x, \mathcal{H}_a^{IJ}) = |\max_{i\in I}(x_i + a_i) - \max_{j\in J}(x_j + a_j)| \enspace , \tag{30}$$

if at least one of these maxima is finite, and $\mathrm{dist}_H(x, \mathcal{H}_a^{IJ}) = 0$ otherwise.

Proof. From Theorem 27, we have $\mathrm{dist}_H(x, \mathcal{H}_a^{IJ}) = d(x, \tilde{x})$ with $\tilde{x} = P_{\mathcal{H}_a^{IJ}}(x)$. Let $K = \operatorname{supp} a$ and $O = \operatorname{supp} x$. If $I \cap K \cap O = J \cap K \cap O = \emptyset$, then $K \cap O = \emptyset$, so $x + a \equiv -\infty$, and this means that $x \in \mathcal{H}_a^{IJ}$ and so $\mathrm{dist}_H(x, \mathcal{H}_a^{IJ}) = 0$.

If $I \cap K \cap O = \emptyset$ and $J \cap K \cap O \neq \emptyset$, then $\max_{i\in I}(x_i + a_i) = -\infty$ and $\max_{j\in J}(x_j + a_j) \neq -\infty$. Let $j \in J \cap K \cap O$, we have $\tilde{x}_j = \min\{x_j, -a_j + \max_{i\in I}(a_i + x_i)\} = -\infty$ and we have $x_j \neq -\infty$, then $d(x, \tilde{x}) = +\infty = |\max_{i\in I}(x_i + a_i) - \max_{j\in J}(x_j + a_j)|$. By symmetry we treat the case when $I \cap K \cap O \neq \emptyset$ and $J \cap K \cap O = \emptyset$.

Now, we suppose that $I \cap K \cap O \neq \emptyset$ and $J \cap K \cap O \neq \emptyset$. Let $i \in I \cap K \cap O$, we have $x_i - \tilde{x}_i = x_i + \max\{-x_i, a_i - \max_{j\in J}(a_j + x_j)\} = \max\{0, x_i + a_i - \max_{j\in J}(a_j + x_j)\}$. Then, we have $\max_{i\in I}(x_i - \tilde{x}_i) = \max_{i\in I\cap K\cap O}(x_i - \tilde{x}_i) = \max\{0, \max_{i\in I\cap K\cap O}(x_i + a_i) - \max_{j\in J}(a_j + x_j)\} = \max\{0, \max_{i\in I}(x_i + a_i) - \max_{j\in J}(a_j + x_j)\}$, and symmetrically, we have $\max_{j\in J}(x_j - \tilde{x}_j) = \max\{0, \max_{j\in J}(x_j + a_j) - \max_{i\in I}(a_i + x_i)\}$. Therefore, we deduce that $\max_{l\in[n]}(x_l - \tilde{x}_l) = |\max_{i\in I}(x_i + a_i) - \max_{j\in J}(x_j + a_j)|$.

To finish the proof we need now to show that $\min_{l\in[n]}(x_l - \tilde{x}_l) = 0$. This is a general property of the projection $\tilde{x} = P_M(x)$ of a vector on a closed tropical cone: since $\tilde{x} \leqslant x$, the minimum is nonnegative, and if the minimum is positive, adding a small constant ϵ to every entry of $\tilde{x}$, we get a vector $\tilde{x}^\epsilon$ which still belongs to M and satisfies $\tilde{x}^\epsilon \leqslant x$, contradicting $P_M(x) = \max\{z \in M \mid z \leqslant x\}$. □

In the sequel, we suppose that the following Assumption 4 holds.

Assumption 4. We suppose that for each $l \in [n]$, there exists $v \in \mathcal{V}$, such that $v_l \neq -\infty$.

We now introduce the operator $T^{IJ} : (\mathbb{R}_{\max})^n \mapsto (\mathbb{R}_{\max})^n$, defined by:

$$(31) \qquad T_l^{IJ}(x) := \begin{cases} \inf_{v\in\mathcal{V}, v_l \neq -\infty}\{-v_l + \max_{j\in J}(v_j + x_j)\}, \text{ if } l \in I \ , \\ \inf_{v\in\mathcal{V}, v_l \neq -\infty}\{-v_l + \max_{i\in I}(v_i + x_i)\}, \text{ if } l \in J \ . \end{cases}$$

The following result, analogous to Lemma 19, gives a metric interpretation of the sub-eigenspace of the operator T^{IJ}.

Lemma 30. *Let $\lambda \in [-\infty, 0]$ and $a \in \mathbb{P}(\mathbb{R}_{\max})^n$, we have*

$$T^{IJ}(a) \geqslant \lambda + a \Leftrightarrow \mathrm{dist}_H(\mathcal{V}, \mathcal{H}_a^{IJ}) \leqslant -\lambda \ .$$

Proof. The equivalence is trivial if $\lambda = -\infty$, so, we suppose that $\lambda \in (-\infty, 0]$. We have

$$(32) \qquad T^{IJ}(a) \geqslant \lambda + a \Leftrightarrow \begin{cases} \forall l \in I, \forall v \in \mathcal{V}, v_l \neq -\infty; -v_l + \max_{j\in J}(v_j + a_j) \geqslant \lambda + a_l \\ \forall l \in J, \forall v \in \mathcal{V}, v_l \neq -\infty; -v_l + \max_{i\in I}(v_i + a_i) \geqslant \lambda + a_l \end{cases}$$

$$(33) \qquad \Leftrightarrow \begin{cases} \forall v \in \mathcal{V}, \max_{l\in I}(v_l + a_l) \leqslant \max_{j\in J}(v_j + a_j) - \lambda \\ \forall v \in \mathcal{V}, \max_{l\in J}(v_l + a_l) \leqslant \max_{i\in I}(v_i + a_i) - \lambda \end{cases}$$

Let $\mathcal{V}'$ denote the set of vectors $v \in \mathcal{V}$ for which at least one of the latter maxima are finite, and observe that the vectors of $\mathcal{V} \setminus \mathcal{V}'$ trivially belong to $\mathcal{H}_a^{I,J}$. Then, using Proposition 29, we see that the last condition in (33) is equivalent to

$$\forall v \in \mathcal{V}', d(v, \mathcal{H}_a^{IJ}) = |\max_{i\in I}(v_i + a_i) - \max_{j\in J}(v_j + a_j)| \leqslant -\lambda$$

i.e., $\mathrm{dist}_H(\mathcal{V}, \mathcal{H}_a^{IJ}) \leqslant -\lambda$. □

Let $w \in \mathbb{R}^n$ and $r \geqslant 0$, we define the *vertical interval* of type I, J centered at point w and of radius r,

$$B_{IJ}(w, r) = \{\lambda + w + \mu e^J \mid \mu \in [-r, r], \lambda \in \mathbb{R}\} \ ,$$

where e^J is the vector of $\mathbb{R}^n$ such that $e^J_l = 0$ for $l \in I$ and $e^J_l = 1$ for $l \in J$. Using the identity $-\mu + \mu e^J = -\mu e^I$, we see

$$B_{IJ}(w, r) = \{\lambda + w + \mu e^I \mid \mu \in [-r, r], \lambda \in \mathbb{R}\} \ .$$

Lemma 31. *Let $\lambda \in [-\infty, 0]$ and $a \in \mathbb{R}^n$, we have*

$$B_{IJ}(-a, -\lambda) \subset \mathrm{Sp}(\mathcal{V}) \Rightarrow T^{IJ}(a) \leqslant \lambda + a \ .$$

Proof. Suppose first that λ is finite. If $B_{IJ}(-a, -\lambda) \subset \mathrm{Sp}(\mathcal{V})$, then

$$\forall \mu \in [-\lambda, \lambda], \exists (\alpha_v)_{v\in\mathcal{V}} \in \mathbb{R}^p, -a + \mu e^J = \max_{v\in\mathcal{V}}(\alpha_v + v).$$

Let $\mu \in [-\lambda, \lambda]$, we have

$$(34) \qquad \forall i \in I, (\forall v \in \mathcal{V}, -a_i \geqslant \alpha_v + v_i \text{ and } \exists v^{(i)} \in \mathcal{V}, -a_i = \alpha_{v^{(i)}} + v_i^{(i)}) \ ,$$

and also

$$(35) \qquad \forall j \in J, (\forall v \in \mathcal{V}, -a_j + \mu \geqslant \alpha_v + v_j \text{ and } \exists v^{(j)} \in \mathcal{V}, -a_j + \mu = \alpha_{v^{(j)}} + v_j^{(j)}) \ .$$

From (35), we have $\forall v \in \mathcal{V}, \sup_{j\in J}(v_j + a_j) \leqslant -\alpha_v + \mu$, and from (34), we have $\forall i \in I, v_i^{(i)} \neq -\infty$ because $\alpha_{v^{(i)}} + v_i^{(i)} = -a_i \in \mathbb{R}$. Then, for all $i \in I$, we have $T_i^{IJ}(a) = \inf_{v\in\mathcal{V}, v_i\neq-\infty}\{-v_i + \sup_{j\in J}(v_j + a_j)\} \leqslant \inf_{v\in\mathcal{V}, v_i\neq-\infty}\{-v_i - \alpha_v + \mu\} \leqslant -v_i^{(i)} - \alpha_{v^{(i)}} + \mu = \mu + a_i$. This being true for all $\mu \in [-\lambda, \lambda]$, we take here $\mu = \lambda$ and we get that $\forall i \in I, T_i^{IJ}(a) \leqslant \lambda + a_i$.

Similarly, we have $\forall j \in J, T^{IJ}_j(a) = \inf_{v\in\mathcal{V}, v_j\neq -\infty}\{-v_j + \sup_{i\in I}(v_i + a_i)\} \leqslant \inf_{v\in\mathcal{V}, v_j\neq -\infty}\{-v_j - \alpha_v\} \leqslant -v^{(j)}_j - \alpha_{v^{(j)}} = -\mu + a_j$. By taking here $\mu = -\lambda$, we get that $\forall j \in J, T^{IJ}_j(a) \leqslant \lambda + a_j$. Therefore, we get that $T^{IJ}(a) \leqslant \lambda + a$.

The conclusion of the lemma is still true when $\lambda = -\infty$. This follows from $B_{IJ}(-a, +\infty) = \cup_{\mu\in(-\infty,0)} B_{IJ}(-a, -\mu)$ and $-\infty + a = \inf_{\mu\in(-\infty,0)} \mu + a$. □

Lemma 32. *Let $\lambda \in [-\infty, 0]$, we have*

$$\exists u \in \mathbb{R}^n; T^{IJ}(u) \leqslant \lambda + u \Rightarrow \exists w \in \mathbb{R}^n; B_{IJ}(w, -\lambda) \subset \mathrm{Sp}(\mathcal{V}) \enspace .$$

Proof. Suppose first that λ is finite. For simplicity of notation, we shall assume that $u = 0$. The general case reduces to this one by replacing every vector $v \in \mathcal{V}$ by the vector $v + u$. We denote by V the matrix whose columns are the elements of $\mathcal{V}$. Since $T^{IJ}(u) \leqslant \lambda + u$, denoting by σ a map $[n] \to [p]$ such that for all $l \in [n]$, $v = V_{\cdot\sigma(l)}$ achieves the minimum in (31), we get:

$$\forall l \in I,\ \forall j \in J, \qquad V_{j\sigma(l)} \leqslant V_{l\sigma(l)} + \lambda \enspace , \tag{36}$$

$$\forall l \in J,\ \forall i \in I, \qquad V_{i\sigma(l)} \leqslant V_{l\sigma(l)} + \lambda \enspace . \tag{37}$$

Consider the vectors

$$w^I := \vee_{i\in I} - V_{i\sigma(i)} + V_{\cdot\sigma(i)}, \qquad w^J := \vee_{j\in J} - V_{j\sigma(j)} + V_{\cdot\sigma(j)},$$

so that $w^I, w^J \in \mathrm{Sp}(\mathcal{V})$. By considering the values $i = l$ or $j = l$ in the suprema above, we get

$$w^I_l \geqslant 0, \forall l \in I, \qquad w^J_l \geqslant 0, \forall l \in J \enspace . \tag{38}$$

Moreover, using (36), we get

$$w^I_j = \vee_{i\in I} - V_{i\sigma(i)} + V_{j\sigma(i)} \leqslant \lambda, \qquad \text{for all } j \in J \enspace , \tag{39}$$

and similarly, using (37),

$$w^J_i \leqslant \lambda, \text{ for all } i \in I \enspace . \tag{40}$$

Define the vector w by

$$w_l = \begin{cases} w^I_l, \text{ if } l \in I \enspace , \\ w^J_l, \text{ if } l \in J \enspace . \end{cases}$$

Using (38)–(40), we deduce that for all $\mu \in [\lambda, -\lambda]$,

$$w + \mu e^I = (w^I + \mu) \vee w^J \in \mathrm{Sp}(\mathcal{V}) \enspace ,$$

and so $B_{IJ}(w, -\lambda) \subset \mathrm{Sp}(\mathcal{V})$.

We finally show that the conclusion of the lemma is still true when $\lambda = -\infty$. This follows from the fact that the above center w depends only on the vectors of $\mathcal{V}$ and does not depend on λ, and also from the facts that $B(w, +\infty) = \cup_{\mu\in(-\infty,0)} B(w, -\mu)$ and $-\infty + w = \inf_{\mu\in(-\infty,0)} \mu + w$. □

The next result is immediate from Lemmas 31 and 32. It is analogous to Lemma 7. It shows that the existence of a super-eigenvector of $T^{I,J}$ is equivalent to the existence of a *vertical interval* included in the module $\mathrm{Sp}(\mathcal{V})$.

Proposition 33. *Let $\lambda \in [-\infty, 0]$, and $a \in \mathbb{R}^n$, we have*

$$\exists u \in \mathbb{R}^n; T^{IJ}(u) \leqslant \lambda + u \Leftrightarrow \exists w \in \mathbb{R}^n; B_{IJ}(w, -\lambda) \subset \mathrm{Sp}(\mathcal{V}) \enspace .$$ □

We now derive a strong duality theorem for signed tropical regression.

Theorem 34. *We have*

$$\min_{a\in\mathbb{P}(\mathbb{R}_{\max})^n} \operatorname{dist}_H(\mathcal{V},\mathcal{H}_a^{IJ}) = -\rho(T^{IJ}) = \sup\{r \geqslant 0 \mid \exists w\in\mathbb{R}^n, B_{IJ}(w,r)\subset \operatorname{Sp}(\mathcal{V})\} \ . \tag{41}$$

The minimum is achieved by any vector $b\in\mathbb{P}(\mathbb{R}_{\max})^n$ such that $T^{IJ}(b)\geqslant \rho(T^{IJ})+b$. Moreover, if $\rho(T^{IJ})$ is finite, the supremum is achieved by a ball $B(c,\rho(T^{IJ}))$ where $c\in\mathbb{R}^n$ can be deduced from any vector u such that $T^{IJ}(u)\leqslant \rho(T^{IJ})+u$.

Proof. From Proposition 33, Lemma 30 and the Collatz-Wielandt property (Theorem 1), we deduce the strong duality property (41). Moreover, the existence of a vector $b\in\mathbb{P}(\mathbb{R}_{\max})^n$ such $T^{IJ}(b)\geqslant \rho(T^{IJ})+b$ follows from Theorem 1. Then, by Lemma 30, we have $\operatorname{dist}_H(\mathcal{V},\mathcal{H}_b)\leqslant -\rho(T^{IJ})$, which implies that b such that $T^{IJ}(b)\geqslant\rho(T^{IJ})+b$ achieves the minimum in (41).

Finally, if $\rho(T^{IJ})$ is finite, since the infimum is attained in the expression of the Collatz-Wielandt number of T^{IJ} (see Theorem 1), there exists a finite vector $u\in\mathbb{R}^n$ such that $T^{IJ}(u)\leqslant\rho(T^{IJ})+u$. By the proof of Lemma 32, we can then construct a vector c such that $B_{IJ}(c,-\rho(T^{IJ}))\subset\operatorname{Sp}(\mathcal{V})$. □

Remark 7. When the set $I=\{i\}$ is of cardinality one, the regression problem for the signed hyperplane (26) has the following special form:

$$\operatorname*{Min}_{a\in\mathbb{R}^n}\max_{v\in\mathcal{V}} |v_i - (\max_{j\neq i} a_j - a_i + v_j)| \ . \tag{42}$$

This can be solved in a direct way [MCT21], avoiding the recourse to mean payoff games.Indeed, (42) reduces to the following "one-sided" tropical linear regression problem. Given sample points $(x^{(k)},y^{(k)})$ in $\mathbb{R}^n\times\mathbb{R}^m$, for $k\in[p]$, compute

$$\operatorname*{Min}_{A}\max_{k\in[p]} \|y^{(k)}-Ax^{(k)}\|_\infty \ , \tag{43}$$

where the minimum is taken over tropical matrices A of size $m\times n$, and the product $Ax^{(k)}$ is understood tropically. Up to a straightforward duality, this problem was solved in [But10, Theorem 3.5.2], the result being attributed there to Cuninghame-Green [CG79]. Alternatively, this solution may be recovered by combining [CF00, Coro. 1] with the explicit formula of the tropical projection [CGQ04, Th. 5]. More precisely, define the matrix $\bar{A}\in\mathbb{R}^{m\times n}$ by $\bar{A}_{ij}:=\min_{k\in[p]} y_i^{(k)}-x_j^{(k)}$, so that $\bar{A}$ is the maximal matrix such that $Ax^{(k)}\leqslant y^{(k)}$ for all $k\in[p]$. Let $\delta:=\max_{k\in p}\|y^{(k)}-\bar{A}x^{(k)}\|_\infty$, and $A^{\rm opt}_{ij}=\bar{A}_{ij}+\delta/2$. Then, $A^{\rm opt}$ is the greatest optimal solution. It can be computed in $O(mnp)$ arithmetic operations. By specializing this formula, one can solve (42) in $O(np)$ arithmetic operations. We refer the reader to [MCT21] for more information, and for the solution of further problems of this category.

Remark 8. In contrast, when I,J are part of the input, the signed linear tropical regression problem is polytime Turing equivalent to mean payoff games. This can be seen as follows. The reduction to mean payoff games is a consequence of Theorem 34. Conversely, observe that finding a signed tropical hyperplane $\mathcal{H}_a^{I,J}$ containing a set $\mathcal{V}=\{v^{(1)},\dots,v^{(p)}\}$ in $\mathbb{R}^n$ is equivalent to solving a tropical linear system of the form $Bx=Cy$, where $x\in(\mathbb{R}_{\max})^I$, $y\in(\mathbb{R}_{\max})^J$, $B\in(\mathbb{R}_{\max})^{p\times I}$, $C\in(\mathbb{R}_{\max})^{p\times J}$, $B_{ki}=v_i^{(k)}$ for $i\in I$ and $C_{kj}=v_j^{(k)}$ for $j\in I$. Indeed, the vector a defining $\mathcal{H}_a^{I,J}$ is given by $a_i=x_i$ for $i\in I$ and $a_j=y_j$ for $j\in J$. We know from [AGG12] that deciding whether a mean payoff game has an initial winning position is equivalent to the existence of a non-identically $-\infty$ solution $z\in(\mathbb{R}_{\max})^s$ of a system of tropical linear inequalities $Fz\leqslant Gz$, where $F,G\in(\mathbb{Z}_{\max})^{r\times s}$ are given. Such a system $Fz\leqslant Gz$ can be rewritten as $Bx=Cy$ by introducing lift variables $u,v\in(\mathbb{R}_{\max})^r$, so that $v=Fz$ and u is a slack variable. Setting $y:=(u,v)$, identified to a column vector, $B:=\left(\begin{smallmatrix}\mathbf{0} & \mathrm{I}\\ \mathrm{I} & \mathrm{I}\end{smallmatrix}\right)$ and $C:=\left(\begin{smallmatrix}F\\ G\end{smallmatrix}\right)$, where $\mathbf{0}$ is a zero tropical matrix, and I the identity matrix, we see that $Fz\leqslant Gz$ has a non-identically $-\infty$ solution iff $By=Cz$ has a non-identically

$-\infty$ solution. It follows that mean payoff games reduce to checking whether there is a solution of a signed tropical linear regression problem with zero error.

5.2. **Tropical linear regression with type information.** The following variant will be relevant to the application to economy considered below, to measure the "distance to equilibria" of a market. We suppose the set of points $\mathcal{V}$ is the disjoint union $\mathcal{V} = \cup_{i\in[n]}\mathcal{V}_i$, where each $\mathcal{V}_i$ is non-empty. We shall say that the points of $\mathcal{V}_i$ are of *type* $i \in [n]$. Note that the set of types is the same as the set of indices of vectors. For each type $i \in [n]$, we consider the signed hyperplane:

$$\mathcal{H}_a^i := \mathcal{H}_a^{\{i\}\{i\}^c} = \{x \in (\mathbb{R}_{\max})^n \mid a_i + x_i = \max_{j\neq i}(a_j + x_j)\} \ .$$

The *typed tropical linear regression* problem associated to the partition $\mathcal{V}_1, \dots, \mathcal{V}_n$ of $\mathcal{V}$, is defined as:

$$\text{(44)} \qquad \operatorname*{Min}_{a\in\mathbb{P}(\mathbb{R}_{\max})^n} \max_{i\in[n]} \operatorname{dist}_H(\mathcal{V}_i, \mathcal{H}_a^i) \ .$$

The value of this problem is small if and only if for each $i \in [n]$, the points of $\mathcal{V}_i$ are close to the signed tropical hyperplane $\mathcal{H}_a^i$.

From Proposition 29, we know that $\operatorname{dist}_H(v, \mathcal{H}_a^i) = |v_i + a_i - \max_{j\neq i}(v_j + a_j)|$.

We suppose in the sequel that Assumption 4 holds. For each type $i \in [n]$, we consider the Shapley operator $T^{\text{ty},i} : (\mathbb{R}_{\max})^n \mapsto (\mathbb{R}_{\max})^n$, given by (31) where the type considered is $(I, J) = (\{i\}, \{i\}^c)$ and to the set of points is $\mathcal{V}_i$:

$$\text{(45)} \qquad T_l^{\text{ty},i}(x) := \begin{cases} \inf_{v\in\mathcal{V}_i, v_i\neq-\infty}\{-v_i + \max_{j\neq i}(v_j + x_j)\}, \text{ if } \ l = i \ , \\ \inf_{v\in\mathcal{V}_i, v_l\neq-\infty}\{-v_l + v_i\} + x_i, \text{ if } \ l \neq i \ . \end{cases}$$

We consider now the Shapley operator $T^{\text{ty}} : (\mathbb{R}_{\max})^n \mapsto (\mathbb{R}_{\max})^n$ given by the infimum of the operators $T^{\text{ty},i}, i \in [n]$. It is given by:

$$\text{(46)} \qquad T_l^{\text{ty}}(x) := \min_{i\in[n]} T_l^{\text{ty},i}(x) \ .$$

The following lemma, analogous to Lemma 19, gives a metric interpretation of the sub-eigenspace of the operator T.

Lemma 35. *Let $\lambda \in [-\infty, 0]$ and $a \in \mathbb{P}(\mathbb{R}_{\max})^n$, we have*

$$T^{\text{ty}}(a) \geqslant \lambda + a \Leftrightarrow \max_{i\in[n]} \operatorname{dist}_H(\mathcal{V}_i, \mathcal{H}_a^i) \leqslant -\lambda \ .$$

Proof. Let $\lambda \in [-\infty, 0]$ and $a \in \mathbb{P}(\mathbb{R}_{\max})^n$. From (46) and Lemma 30, we deduce the equivalence:

$$\begin{aligned} T^{\text{ty}}(a) \geqslant \lambda + a &\Leftrightarrow \forall i \in [n], T^{\text{ty},i}(x) \geqslant \lambda + a \\ &\Leftrightarrow \forall i \in [n], \operatorname{dist}_H(\mathcal{V}_i, \mathcal{H}_a^i) \leqslant -\lambda \ , \\ &\Leftrightarrow \max_{i\in[n]} \operatorname{dist}_H(\mathcal{V}_i, \mathcal{H}_a^i) \leqslant -\lambda \ . \end{aligned}$$

□

From Lemma 35 and theorem 1, we deduce the following result, showing that the tropical linear regression problem with types, associated to the sets $\mathcal{V}_1, \dots, \mathcal{V}_n$, also reduces to a mean payoff game.

Theorem 36. *We have,*

$$\min_{a\in\mathbb{P}(\mathbb{R}_{\max})^n} \max_{i\in[n]} \operatorname{dist}_H(\mathcal{V}_i, \mathcal{H}_a^i) = -\rho(T^{\text{ty}}) \ .$$

Moreover, the minimum is achieved by any vector $a \in \mathbb{P}(\mathbb{R}_{\max})^n$ such that $T^{\text{ty}}(a) \geqslant \rho(T^{\text{ty}}) + a$.

Remark 9. Typed tropical linear regression should be compared with the tropical SVM problem introduced in [GJ08]. In the tropical SVM setting, we have a partition of the set of points in n *color classes*, $\mathcal{V}_{c_1}, \dots, \mathcal{V}_{c_n}$, and we are looking for a tropical hyperplane $\mathcal{H}_a$, and for a permutation σ of $\{1, \dots, n\}$ such that for all $i \in [n]$, all the points of color c_i are in the same *sector* $S_{\sigma(i)}(a)$. In other words, we want the tropical hyperplane to separate the n color classes. This is not possible in general, so one needs to consider metric versions, modelling the minimization of classification errors [TWY20]. A possible metric formulation, in the spirit of the present approach, would be to consider

$$\min_{\sigma \in \mathfrak{S}_n} \min_{a \in \mathbb{R}^n} \max_{i \in [n]} \mathrm{dist}_H(\mathcal{V}_i, S_{\sigma(i)}(a)) \qquad \text{(Metric Tropical SVM)} \tag{47}$$

where $\mathfrak{S}_n$ denotes the symmetric group on n letters. By comparison with (44), we see that we have in addition a minimization over the symmetric group, but the subproblem with a fixed permutation σ arising in the SVM problem is simpler than the analogous problem of typed tropical linear regression, since the sector $S_{\sigma(i)}$ is convex, whereas the set $\mathcal{H}_a^i$ arising in (44) is not a convex one. In the application described below, it is the set $\mathcal{H}_a^i$ that is relevant to measure the “distance to equilibrium”.

In Figure 5(a), we consider the following matrix $V \in \mathbb{R}^{3\times 11}$:

$$V = \begin{pmatrix} 1 & 1 & 2 & 0 & 0 & 0 & -3 & -1 & 0 & 0 & -2 \\ 0 & -2 & 0 & 1 & 1 & 2 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & -2 & -2 & -1 & -2 & 0 & 2 & 3 & 1 & 1 \end{pmatrix}, \tag{48}$$

and the types are given by the subsets of $\mathcal{V} = [11]$ as follows $\mathcal{V}_1 = \{1, 2, 3, 4\}$, $\mathcal{V}_2 = \{5, 6, 7, 8\}$ and $\mathcal{V}_3 = \{9, 10, 11\}$.

The operators $T^{\mathrm{ty},i} : (\mathbb{R}_{\max})^n \mapsto (\mathbb{R}_{\max})^n$ given by (45) and associated to the above matrix V and partition $(\mathcal{V}_i)_{i\in[3]}$ are given by:

$$T^{\mathrm{ty},1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \min[-1+\max(-2+x_2, x_3), -2+\max(x_2, -2+x_3), \max(1+x_2, -2+x_3)] \\ -1+x_1 \\ 1+x_1 \end{pmatrix},$$

$$T^{\mathrm{ty},2}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1+x_2 \\ \min[-2+\max(x_1, -2+x_3), -1+\max(-3+x_1, x_3), \max(-1+x_1, 2+x_3)] \\ -2+x_2 \end{pmatrix},$$

$$T^{\mathrm{ty},3}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1+x_3 \\ 1+x_3 \\ \min[-3+\max(x_1, x_2), -1+\max(x_1, -3+x_2), -1+\max(-2+x_1, x_2)] \end{pmatrix}.$$

Then the operator $T^{\mathrm{ty}} : (\mathbb{R}_{\max})^n \mapsto (\mathbb{R}_{\max})^n$ given by (46) is in this example:

$$T^{\mathrm{ty}}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \min[-1+\max(-2+x_2, x_3), -2+\max(x_2, -2+x_3), 1+x_2, 1+x_3,] \\ \min[-2+\max(x_1, -2+x_3), -1+\max(-3+x_1, x_3), -1+x_1, 1+x_3] \\ \min[-3+\max(x_1, x_2), -1+\max(x_1, -3+x_2), 1+x_1, -2+x_2] \end{pmatrix}.$$

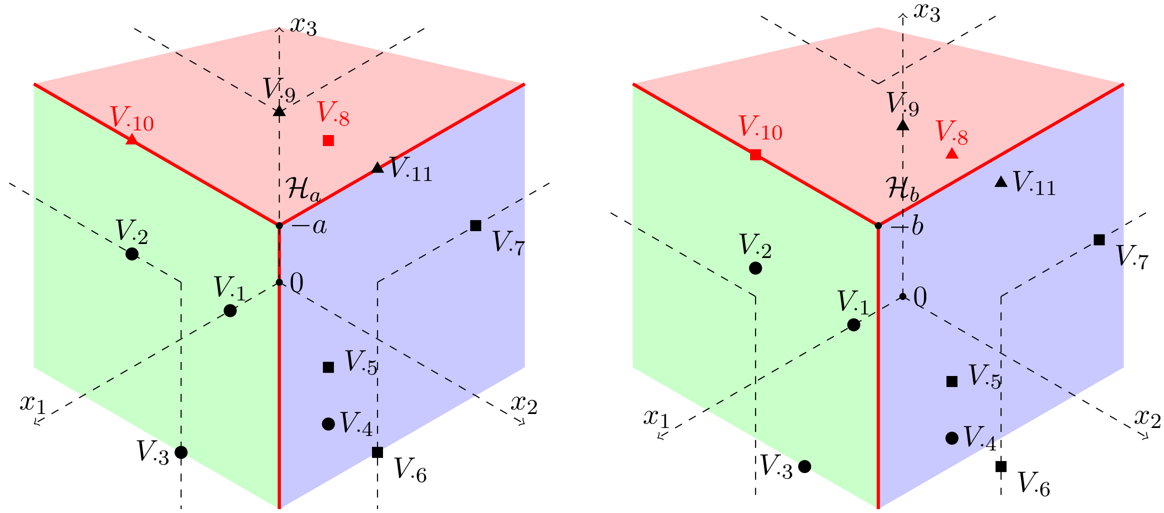
We verify easily that $\lambda = -2$ and $a = (0, 0, -1)^\top$ satisfy $T^{\mathrm{ty}}(a) = \lambda + a$, so that by Theorem 36 the apex a is optimal for the typed tropical linear regression problem (44).

We notice that in this case, the tropical hyperplane $\mathcal{H}_a$ has at least one witness point in each sector, which means, by Theorem 22, that $\mathcal{H}_a$ is also an optimal hyperplane in the sense of the usual tropical linear regression studied in Section 4.

Now, if we consider the same matrix V in (48), but we exchange the types of the points $V_{\cdot 8}$ and $V_{\cdot 10}$, i.e. we consider the partition $\widetilde{\mathcal{V}}_1 = \{1,2,3,4\}$, $\widetilde{\mathcal{V}}_2 = \{5,6,7,10\}$ and $\widetilde{\mathcal{V}}_3 = \{8,9,11\}$, then the new typed Shapley operator $\widetilde{T^{\mathrm{ty}}}$ is given by:

$$\widetilde{T^{\mathrm{ty}}}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \min[-3+x_2, 3+x_3] \\ \min[-2+\max(x_1,-2+x_3), -1+\max(-3+x_1,x_3), -1+x_1, 1+x_3] \\ \min[1+x_1, -4+x_2] \end{pmatrix} .$$

We verify easily that $\mu = -5/2$ and $b = (0, 1/2, -1)^\top$ satisfy $\widetilde{T^{\mathrm{ty}}}(b) = \mu + b$. This example is presented in Figure 5(b). Here, we notice that the hyperplane $\mathcal{H}_b$ that is optimal in the typed tropical linear regression sense (Section 5.2) does not have witness points in each sector, which means that it is not optimal in the usual tropical linear regression framework (Section 4).



(b)

FIGURE (5) Figure 5(a): A set of typed points $\mathcal{V}$ with three types in $\mathbb{P}(\mathbb{R}_{\max})^3$ with an optimal tropical hyperplane $\mathcal{H}_a$ in the sense of the typed tropical regression, where $a = (0,0,-1)^\top$ satisfies $T^{\mathrm{ty}}(a) = -2 + a$. Figure 5(b): The same set of typed points $\mathcal{V}$ as Figure 5(a) but with the types of the two points $V_{\cdot 8}$ and $V_{\cdot 10}$ being exchanged, and an optimal tropical hyperplane $\mathcal{H}_b$ in the sense of the typed tropical regression, where $b = (0, 1/2, -1)^\top$ satisfies $\widetilde{T^{\mathrm{ty}}}(b) = -5/2 + b$.

6. Algorithmic aspects

In this section, we explain how the tropical linear regression problem can be effectively solved by using mean-payoff games algorithms. Throughout the section, we assume that the set of points $\mathcal{V}$ is given by as the set of columns the matrix V. By Corollary 14, in theory, any algorithm solving mean payoff games in the weakest sense (deciding the inequality $\chi_i(T) \geqslant 0$) can be used. However, some game algorithms lead to more direct approaches, we next discuss some of these.

Considering the strong duality result, Theorem 20, and the result on the existence of witness points Theorem 22, the key algorithmic issues are:

(i) to compute the upper mean payoff, $\rho(T)$ (which is the opposite of the value of the tropical linear regression problem);

(ii) to decide whether there is a finite eigenvector $u \in \mathbb{R}^n$ such that $T(u) = \rho(T) + u$, and to compute such an eigenvector (when this is so, $-u$ is the center an an optimal ball included in $\mathrm{Sp}(\mathcal{V})$ and the apex of an optimal regression hyperplane, see Remark 4);
(iii) to find a sub-eigenvector $b \in (\mathbb{R}_{\max})^n \setminus \{\bot\}$, satisfying $T(b) \geqslant \rho(T) + b$ (then, $\mathcal{H}_b$ is an optimal regression hyperplane);
(iv) to find a super-eigenvector $c \in \mathbb{R}^n$ satisfying $T(c) \leqslant \rho(T) + c$ (then, $-c$ is the center of an optimal ball included in $\mathrm{Sp}(\mathcal{V})$.

For simplicity of the discussion, we assume that T sends $\mathbb{R}^n \to \mathbb{R}^n$. The case in which T sends $\mathbb{R}^n$ to $(\mathbb{R}_{\max})^n$ reduces to this one by considering the action of T on the parts of $(\mathbb{R}_{\max})^n$ and looking for invariant parts.

Then, problems (i)–(iv) are solved, simultaneously, as soon as we know an invariant half-line of T. Indeed, we observed after stating Theorem 1 that if (u, η) is an invariant half-line, then $\chi(T) = \eta$. In this way, $\rho(T) = \max_{i\in[n]} \chi_i(T)$ is determined, and this solves issue (i). Moreover, by Proposition 3 T admits a finite eigenvector if and only if η is a constant vector, i.e., $\eta = (\lambda, \dots, \lambda)$ for some $\lambda \in \mathbb{R}$, and u is an eigenvector. This solves issue (ii). We observed in the proof of Theorem 1 that u satisfies $T(u) \leqslant \rho(T) + u$, and so, this solves issue (iii). Finally, setting $I := \{i \in [n] \mid \chi_i(T) = \rho(T)\}$, and defining the vector $\bar{u}$ such that $\bar{u}_i = u_i$ for $i \in I$ and $\bar{u}_i = -\infty$ otherwise, it can be checked that $T(\bar{u}) \geqslant \bar{u} + \rho(T)$, which solves issue (iv).

More generally, the reduction in the second part of the proof of Corollary 14 shows that algorithm which returns an optimal policy σ of Player Min, i.e., a policy such that $\chi(T) = \chi(T^\sigma)$, can be used to produce a finite vector $c \in \mathbb{R}^n$ such that $T(c) \leqslant \overline{\chi}(T) + c$, by reduction to a tropical eigenvalue problem. Moreover, any algorithm which returns an optimal policy τ of Player Max, i.e., a policy such that $\chi(T) = \chi({}^\tau T)$, can be used to produce a vector $b \in (\mathbb{R}_{\max})^n \setminus \{\bot\}$, satisfying $T(b) \geqslant \rho(T) + b$, see the second part of the reduction in Corollary 26.

We refer the reader to [Cha09] for a comparative discussion of mean payoff game algorithms. The main known algorithms include the pumping algorithm of [GKK88], value iteration [ZP96], and different algorithms based on the idea of policy iteration [BV07, Sch08, DG06]. In particular, the algorithm of [DG06] returns an invariant half-line. The policy iterations algorithms [BV07, DG06] were reported in [Cha09] to have the best experimental behavior, although policy iteration is are generally exponential [Fri09].

For the present application to tropical linear regression, we often know in advance that the operator T has a finite eigenvector; this occurs in particular if the entries of the matrix V are finite, and more generally, under the dominion condition of Theorem 41. Then, one can use another algorithm, projective Krasnoselkii-Mann value iteration [GS20], which is straightforward to implement and still effective. Starting from a vector $v^0 = (0, \cdots, 0)^\top$, this algorithm computes the following sequence:

$$\tilde{v}^{k+1} = T(v^k) - (\max_{i\in[n]} T(v^k)_i)e, \tag{49}$$

$$v^{k+1} = (1-\gamma)v^k + \gamma\tilde{v}^{k+1}. \tag{50}$$

where $e = (1, \cdots, 1)^\top \in \mathbb{R}^n$, and $\gamma \in (0, 1)$ is fixed, $1-\gamma$ being interpreted as a *damping parameter*. In the original Krasnoselskii-Mann algorithm, one writes simply $v^{k+1} = (1-\gamma)v^k + \gamma T(v^k)$. It follows from [GS20, Coro. 13], based on a general result of Baillon and Bruck [BB92] on the convergence of the original Krasnoselskii-Mann algorithm in normed spaces, see also [CSV14], that v^k does converge to an eigenvector of T as soon as such a (finite) eigenvector u exists. Moreover, $\|T(v^k) - v^k\|_H \leqslant 2\|u\|_H/\sqrt{\pi\gamma(1-\gamma)k}$. In practice, we fix a desired precision $\epsilon > 0$, and stop the computation of the sequence v^k when $\|T(v^k) - v^k\|_H \leqslant \epsilon$.

We now analyze the complexity of the projective Krasnoselskii-Mann algorithm in our special setting. The following observation, shows that, notwithstanding the quadratic size of $S^{\max}$ in the game associated with T (see the discussion after (9)), the operator T can be evaluated in linear time.

Proposition 37. *The operator T can be evaluated in $O(|E|)$ arithmetic operations.*

Proof. We write $T_i(x) = \min_{k\in[p],(i,k)\in E}(-V_{ik}+y_{ik})$ where $y_{ik} = \max_{j\in[n],j\neq i,(j,k)\in E}(V_{jk}+x_k)$. First, for each column k of the matrix V, we compute the column maximum $M_k := \max_{j\in[n],(j,k)\in E}(V_{jk} + x_k)$ together with an arbitrary index j_k that achieves this maximum, and also the second column maximum, $m_k := \max_{j\in[n],j\neq j_k,(j,k)\in E}(V_{jk} + x_k)$. This preprocessing requires $O(|E|)$ arithmetic operations. We observe that $y_{ik} = m_k$ if $i = j_k$ and $y_{i_k} = M_k$ otherwise. Hence, all the y_{ik} with $(i, k) \in E$ can be computed in $O(|E|)$ arithmetic operations. Finally, the $T_i(x)$ are obtained from the y_{ik} in $O(|E|)$ arithmetic operations. □

We set:

$$W := \max_{v\in\mathcal{V}} \|v\|_H \ .$$

Lemma 38. *Suppose that $\mathcal{V}$ is finite, then any finite eigenvector u of T satisfies $\|u\|_H \leqslant W$.*

Proof. By definition of W, we have $v \in B_H(0, W)$ for all $v \in \mathcal{V}$, and since $B_H(0, W)$ is stable by tropical linear combinations, we get $\mathrm{Sp}(\mathcal{V}) \subset B_H(0, W)$. Moreover, by Lemma 7, $B_H(u, -\rho(T)) \subset \mathrm{Sp}(\mathcal{V})$. Hence $u \in B_H(0, W)$, meaning that $\|u\|_H \leqslant W$. □

Remark 10. There are situations (Appendix A) in which although some vectors of $\mathcal{V}$ have infinite entries, it is still the case that T has a finite eigenvector. Then, we may still show that there exists a finite eigenvector with not too large entries. To see this, we need to replace W by $W' := \max_{k\in[p]} \delta(V_{\cdot,k})$, where $\delta(V_{\cdot,k}) = \max_{i\in[n],(i,k)\in E} V_{ik} - \min_{j\in[n](j,k)\in E} V_{jk}$. We can always choose such an eigenvector u in such a way that $\|u\|_H = O(nW')$, by appealing to a Blackwell optimality argument, using the proof method of [Sko18, Lemma 8.51] (details are left to the reader). Note that in the special case in which V has finite entries, the bound on $\|u\|_H$ is improved by a factor n.

Corollary 39 (Approximate optimality certificate)**.** *Suppose that $\mathcal{V} \subset \mathbb{R}^n$ is of cardinality p. Then, the projective Krasnoselskii-Mann iteration returns in a number of arithmetic operations $O(npW/\epsilon^2)$ a vector $u \in \mathbb{R}^n$ such that $-u$ is both the center of a ball of radius $-\rho(T) - \epsilon$ included in $\mathrm{Sp}(\mathcal{V})$ and the apex of a regression hyperplane, $\mathcal{H}_u$, such that $\mathrm{dist}_H(\mathrm{Sp}(\mathcal{V}), \mathcal{H}_u) \leqslant -\rho(T) + \epsilon$.*

Proof. By [GS20, Coro. 13] and Lemma 38, after $k = O(\lceil W/\epsilon^2\rceil)$ iterations, we end up with a vector $u := v^k$ which satisfies $\|T(u) - u\|_H \leqslant \epsilon$. Moreover, by Proposition 37, each iteration requires $O(np)$ arithmetic operations. Setting $\underline{\lambda} := \mathrm{bot}(T(u) - u)$, where $\mathrm{bot}(x) := \min_i x_i$, we deduce that $\underline{\lambda} + u \leqslant T(u) \leqslant \underline{\lambda}(T) + \epsilon + u$, which, by Theorem 1, entails that $\rho(T) \leqslant \underline{\lambda}(T) + \epsilon$. Then, by Theorem 20, $B(-u, -\rho(T) - \epsilon) \subset \mathrm{Col}(V)$. The proof that $\mathrm{dist}_H(\mathrm{Col}(V), \mathcal{H}_u) \leqslant -\rho(T) + \epsilon$ is dual. □

The following result shows that the factor in $1/\epsilon^2$ can be replaced by $1/\epsilon$ if we look separately for the center of a Hilbert's ball included in $\mathrm{Sp}(\mathcal{V})$ and for the apex of an approximate tropical linear regression hyperplane (in Corollary 39, the apex and the center coincide).

Corollary 40. *Suppose that $\mathcal{V} \subset \mathbb{R}^n$ is of cardinality p. Then, an ϵ-approximation of the inner radius of* $\mathrm{Col}(V)$*, as well as vectors $v, z \in \mathbb{R}^n$ satisfying $B_H(v, \text{in-rad}(\mathcal{V})-\epsilon) \subset \mathcal{V}$ and $\mathrm{dist}_H(\mathcal{V}, \mathcal{H}_z) \leqslant$* in-rad$(\mathcal{V}) + \epsilon$ *can be obtained in $O(npW/\epsilon)$ arithmetic operations.*

Proof of Corollary 40. We now rely on the value iteration approach of [AGKS18, Sko18]. The latter computes the sequence given by $v^0 = 0$, $v^k := T(v^{k-1})$, together with the numbers $\bar{\lambda}^k := \max_{i\in[n]} v_i^k$,

$\underline{\lambda}^k := \min_{i\in[n]} v_i^k$. The sequence v^k generally does not converge, even up to an additive constant. So, we rely on the following "regularized" sequence [GG04],

$$w^k := \inf(v^0, v^1 - \bar{\lambda}^k/k, \dots, v^{k-1} - \bar{\lambda}^k(k-1)/k) \ . \tag{51}$$

Lemma 8.18 of [Sko18] entails that $\rho(T)$ satisfies $\underline{\lambda}^k/k \leqslant \rho(T) \leqslant \bar{\lambda}^k/k$ with $\bar{\lambda}^k/k - \rho(T) \leqslant \|u\|_H/k$ and $\rho(T) - \underline{\lambda}^k/k \leqslant \|u\|_H/k$, where $u \in \mathbb{R}^n$ is an arbitrary finite eigenvector of T. Hence, it suffices to execute the algorithm up to the iteration $k := \lceil W/\epsilon \rceil$ to make sure that $\bar{\lambda}^k \leqslant \rho(T) + \epsilon$ and $\underline{\lambda}^k \geqslant \rho(T) - \epsilon$. Moreover, Lemma 2 of [GG04] entails that $T(w^k) \leqslant \bar{\lambda}^k + w^k$. Hence, by Lemma 7, $-w^k$ is the center of a Hilbert's ball of radius $-\bar{\lambda}^k$ included in $\mathrm{Sp}(\mathcal{V})$. The construction of the apex of an approximate optimal regression hyperplane uses a dual argument, replacing inf by sup in (51). □

Remark 11. The conclusions of Corollary 39 and Corollary 40 can be extended to the situation in which some vectors of $\mathcal{V}$ have infinite entries, provided T has a finite eigenvector. Using Remark 10, we need to replace W by $W'n$ in the bounds of Corollary 39 and Corollary 40.

7. Illustration: inferring hidden information from equilibria in repeated invitations to tenders

We now illustrate our results on an example from auction theory, in which tropical linear regression allows one to identify secret information from the observation of prices offered in repeated invitations to tenders (ITT).

7.1. **Auction model with hidden preference factors.**

We suppose a public decision maker chooses the best offer made by the firms responding to ITT. In accordance with market regulations, see e.g. [cod21, Art. R.2152-7], the best offer is not nessarily the one with the lowest price: other factors, like technical quality, respect of environment, of social impact, can also be taken into account. In the presence of corruption, decisions may be also influenced by bribes.

We assume that this ITT is done repeatedly for a similar service or product each time and in front of the same local firms. We label the firms by $1, 2, \cdots, n$, and we suppose that we have a history of q ITTs with the prices offered by each firm, that are revealed by the decision maker, after having made her choice.

More precisely, we denote the price offered by firm $i \in [n]$ for the ITT number $j \in [q]$ by p_{ij}. We assume that the decision maker has a non public preference factor $f_i > 0$ for each firm i, and that she selects the firm of index i minimizing the expression:

$$\min_{i\in[n]} p_{ij} f_i^{-1} \ . \tag{52}$$

In this way, the decision maker considers that for a requested price of p_{ij}, the final cost to be taken into account is $p_{ij} f_i^{-1}$, where $f_i^{-1} \geqslant 1$ is a proportional penalty depending on her estimate f_i of the technical, environmental, or social quality of the firm (the larger f_i, the better its quality).

The same model applies to the situation in which $f_i^{-1} = 1 - \alpha_i \beta$ for some $0 \leqslant \alpha_i \leqslant 1$ and $0 \leqslant \beta < 1$. Now, α_i may be interpreted as a proportional bribe: the firm promises to secretly give back $\alpha_i p_{ij}$ to the decision maker if its offer is accepted, and the parameter β measures how sensitive is the decision maker to bribery ($\beta = 0$ corresponds to a totally honest decision maker, and $\beta = 1^-$ to a totally dishonest one). This is a variant of the classical *first-price sealed-bid auction* [Kri02], incorporating the secret preference.

We suppose that the same firms answer in a recurrent manner to invitations from the same decision maker, and that the factors f_i secretly attached to each firm are kept constant. Then we expect that the prices to be offered to constitute an *equilibrium*, meaning that for each invitation $j \in [q]$, the minimum $\min_{i\in[q]} p_{ij} f_i^{-1}$ is achieved twice at least. Indeed, if the firm i that wins the invitation offers a price p_{ij} such that $p_{ij}f_i^{-1}$ is strictly smaller than $p_{kj}f_k^{-1}$ for all $k \in [n] \setminus \{i\}$, it may offer a higher price and still win the offer, so, in the long run, if an invitation of the same type is made recurrently, the firm will adapt its offer.

This can be modeled in terms of membership to a tropical hyperplane. We put $V_{ij} = -\log(p_{ij})$ and $a_i = \log f_i$, so that the decision maker selects the firm of index achieving the maximum in

$$\max_{i\in[n]}(V_{ij} + a_i) \ . \tag{53}$$

Assuming the prices p_{ij} are observed, our goal is to infer the secret information f_i, i.e. the preference factor for firm i, or the bribe offered by this firm.

We first suppose that for each invitation, the identity of the firm that wins the contract is not known to us. We want to infer the hidden information $f = (f_i)_{i\in[n]}$. So, we look for a tropical hyperplane $\mathcal{H}_b$ that is the best regression of the set $\mathcal{V}$ formed by the points $(V_{\cdot j})_{j\in[q]}$ following the analysis of Section 4, i..e, we solve a problem of the form (17). Following Theorem 20, we solve this problem by computing a super-eigenvector $b \in \mathbb{R}^n$ of T, i.e. such that $T(b) \geqslant \rho(T) + b$, where the operator T is given by (9).

We note that the decision maker cares only about the *relative* preference factors between the firms, in the sense that if all the preference factors f_i, $i \in [n]$ are multiplied by the same positive constant, the choices of the decision maker will not change. Therefore, we can suppose without loss of generality that $\max_{j\in[n]} f_j = 1$, or equivalently, $\max_{j\in[n]} a_j = 0$.

7.2. **Numerical instance and experiments.** In the following toy example, we take $n = 3$ firms, and a history of $q = 6$ ITTs. We suppose that the decision maker attributes to the firms the preference factors $f = (1, 0.8, 0.6)$, and we take $\forall i \in [3], a_i = \log(f_i)$.

We generated the matrix V_{ij} and the prices $p_{ij} = \exp(-V_{ij})$ by the following structured probabilistic model. We consider six types of products with prices of different order of magnitude. In Table 1 the reference prices of these products are $P = (1, 3, 9, 25, 70, 130)$. For each $j \in [6]$, we draw entries $A_{ij}, i \in [3]$ randomly in the interval $K_{ij} = [0.9\times P_j f_i, 1.1\times P_j f_i]$ following a log-uniform law, i.e. equal to the exponential of a variable generated uniformly on the logarithm of the interval K_{ij}. We choose the log-uniform law because it's in adequacy with Benford's law that is observed in real-life price instances. Then, we take $B_{ij} = -\log(A_{ij})$, and we project each column $B_{\cdot j}$ into the tropical hyperplane $\mathcal{H}_a$, to get a vector $C_{\cdot j}$, such that for a given $i \in \operatorname{argmax}_{k\in[n]}(B_{kj} + a_k)$, we take $C_{ij} = \max_{k\neq i}(B_{kj} + a_k) - a_i$ and we take for all $k \neq i$, $C_{kj} = B_{kj}$. Now the columns $C_{\cdot j}$ belong to the tropical hyperplane $\mathcal{H}_a$. To model the inefficiency of the market, we perturb these columns by taking $V_{ij} = C_{ij} + \delta_{ij}$, with δ_{ij} generated randomly uniformly in $[-\delta, \delta]$, with $\delta = 0.05$. Then the prices are given by $p_{ij} = \exp(-V_{ij})$.

To solve our example, we used the projective Krasnoselskii-Mann iteration described in Section 6, with a damping parameter $\gamma = 1/2$. We take $b = v^N$ that gives the approximation of the preference factors by tropical linear regression: $f_i^{\text{reg}} = \exp(b_i)$, $i \in [n]$.

We define the error of the approximation e as the ratio between the Hilbert's distance of the set $\mathcal{V}$ to the hyperplane $\mathcal{H}_b$, which measures the "distance to equilibrium" in this market, and the maximal absolute value of the logarithm of the Hilbert's seminorms of the price vectors $(p_{\cdot j})_{j\in[q]}$:

$$e := \frac{\operatorname{dist}_H(\mathcal{V}, \mathcal{H}_b)}{\max_{j\in[q]} |\log(\|p_{\cdot j}\|_H)|} \ .$$

The following Table 1 shows the preference factors f_i, the prices p_{ij} generated with this model and for each invitation we underlined the price of the firm wining that invitation in the sense of achieving the minimum in (52). Table 1 shows also the prediction f^{reg} of the preference factors that we find by tropical linear regression.

In this example, we set a target accuracy of $\epsilon = 10^{-8}$, and we get that the number of iterations N needed to get $\|T(v^N) - v^N\|_H \leqslant \epsilon$ is $N = 25$. By setting $b = v^N$, we have $\mathrm{dist}_H(\mathcal{V}, \mathcal{H}_b) = 4.21 \times 10^{-2}$ and $\max_{j\in[q]} |\log(\|p_{\cdot j}\|_H)| = 3.84$, and this leads to an error equal to $e = 1.09 \times 10^{-2}$. Figure 6 shows the points $(V_{\cdot j})_{j\in[6]}$ in the projective space $\mathbb{P}(\mathbb{R}_{\max})^3$, with the tropical hyperplane $\mathcal{H}_b$ (in blue solid lines) and the points of the space that are at distance equal to $\mathrm{dist}_H(\mathcal{V}, \mathcal{H}_b)$ from $\mathcal{H}_b$ (in blue dashed lines). Figure 6 shows in particular the existence of a witness point in each on the three sectors associated to the tropical hyperplane $\mathcal{H}_b$.

	individual houses	social housing	school	road	stadium	bridge	f	f^{reg}
Firm 1	1.02	3.21	<u>8.72</u>	26.2	69.8	<u>123</u>	1	1
Firm 2	0.81	2.65	7.49	20.3	<u>53.8</u>	106	0.8	0.81
Firm 3	<u>0.6</u>	<u>1.86</u>	5.5	<u>14.7</u>	41.8	76	0.6	0.605

TABLE (1) Prices proposed by firms in million euros, the vector of preference factors f and its estimation by tropical linear regression f^{reg} based on the observation of the prices.

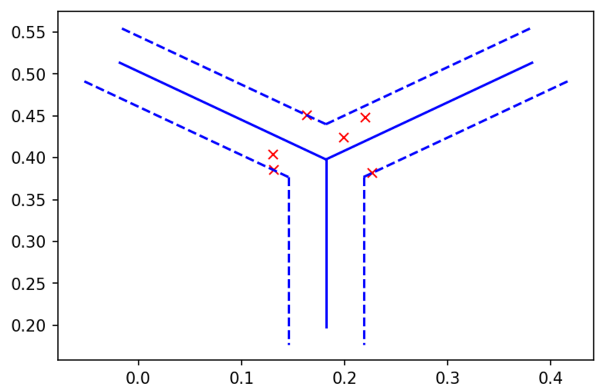


FIGURE (6) The points $(V_{\cdot j})_{j\in[6]}$ in the projective space $\mathbb{P}(\mathbb{R}_{\max})^3$, with the tropical hyperplane $\mathcal{H}_b$ (in blue solid lines) and the points of the space that are at distance equal to $\mathrm{dist}_H(\mathcal{V}, \mathcal{H}_b)$ from $\mathcal{H}_b$ (in blue dashed lines).

Now we consider a similar example still with $n = 3$ firms, but with $q = 100$ invitations to tenders. We use the same generation model, the reference prices $P_j, j \in [100]$, being generated randomly following a log-uniform law on the interval $[1, 100]$. We set a target accuracy of $\epsilon = 10^{-8}$, and we get that the number of iterations N needed to get $\|T(v^N) - v^N\|_H \leqslant \epsilon$ is $N = 24$. By setting $b = v^N$, we have $\mathrm{dist}_H(\mathcal{V}, \mathcal{H}_b) = 7.69 \times 10^{-2}$ and $\max_{j\in[q]} |\log(\|p_{\cdot j}\|_H)| = 3.72$, and this leads to an error equal to $e = 2.06 \times 10^{-2}$, and the approximation of the preference factors that we obtain is $f^{\mathrm{reg}} = (1, 0.7994, 0.6018)$. Figure 7 shows the points $(V_{\cdot j})_{j\in[100]}$ and the approximation hyperplane $\mathcal{H}_b$ obtained in this case with a history of $q = 100$ invitations. We observe also that we have at least a witness point in each sector defined by the tropical hyperplane $\mathcal{H}_b$.

7.3. **Example of regression with types – in which the identities of the winners of the invitations are known.** We now suppose the decision maker makes public not only the bid prices that were offered to her, but also the identities of the firms that won the different invitations $j \in [q]$. Then, we can write the set of points $\mathcal{V}$ as a disjoint union $\mathcal{V} = \cup_{\ell\in[n]} \mathcal{V}_\ell$, where $\mathcal{V}_\ell$ is the set of invitations won by firm ℓ. This information can be exploited through the *typed* tropical linear regression of Section 5.2. Indeed, if $v = V_{\cdot j} \in \mathcal{V}_\ell$, and if the market is "at equilibria", we know not

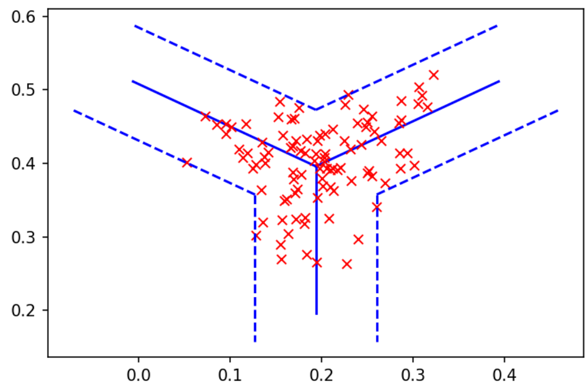


FIGURE (7) The points $(V_{\cdot j})_{j\in[100]}$ in the projective space $\mathbb{P}(\mathbb{R}_{\max})^3$, with the tropical hyperplane $\mathcal{H}_b$ (in blue solid lines) and the points of the space that are at distance equal to $\mathrm{dist}_H(\mathcal{V},\mathcal{H}_b)$ from $\mathcal{H}_b$ (in blue dashed lines).

only that the maximum $\max_{i\in[n]}(V_{ij}+a_i)$ is achieved twice, but that it must be achieved by the firm that won the invitation, i.e., $i=\ell$. Thus, the vector $v\in\mathcal{V}_\ell$ should be close to the *signed* tropical hyperplane $\mathcal{H}_a^\ell$, a finer condition than being close to $\mathcal{H}_a\supset\mathcal{H}_a^\ell$. So, to infer the vector a, we now solve the typed regression problem (44), instead of the untyped problem (17). Following Theorem 36, we are looking for a super-eigenvector b such that $T^{\mathrm{ty}}(b)\geqslant\rho(T^{\mathrm{ty}})+b$, where the operator T^{ty} is given by (46).

We use the same two examples above, and we generate the information of the firm winning each contract $j\in[q]$ by using the information f known by the decision maker. We construct the sets $\mathcal{V}_i$ and the operator T^{ty}, and we find a super-eigenvector of T^{ty} by using the projective Krasnoselkii-Mann value iteration algorithm described in Section 6.

After doing the numerical experiments, we find that, the apex b found by typed tropical linear regression, taking advantage of the knowledge of which firm won each invitation, is the same as the one found above by tropical linear regression, for both examples with $q=6$ and $q=100$. Hence, here, the additional information provided by the identity of the winners did not help to improve the inference of hidden preferences, by comparison with the basic model in which only the history of the bid prices is used.

8. Concluding remarks

We solved the tropical linear regression problem, when the metric is of *sup-norm* type, and for tropical linear spaces of codimension 1 (tropical hyperplanes), but for a configuration of points of arbitrary cardinality.

Several open problems related to the present work arise when changing either the class of metrics or of tropical spaces.

For instance, we may replace Hilbert's metric by the L_p-projective metric, i.e., the metric obtained by modding out the L_p normed space $\mathbb{R}^n$ by the action of additive constants, or by replacing the Hausdorff distance in (16) by a L_p type distance, for $p\in[1,\infty)$. Approaches based on mixed linear programming, or on local descent, have been proposed in [YZZ19, PYZ20, Hoo17] in some specific cases.

Another generalization consists in replacing hyperplanes by tropical linear spaces of a codimension not necessarily 1. Recall that the *tropical Grassmannian* $\mathrm{Gr}_{k,n}^{\mathrm{trop}}$ can be defined as the image by a non-archimedean valuation of the Grassmannian $\mathrm{Gr}_{k,n}(\mathbb{K})$ over an (algebraically closed) non-archimedean field, under the Plücker embedding, see [SS04, FR15]. In this way, an element of

$\mathrm{Gr}^{\mathrm{trop}}_{k,n}$ is represented by its *tropical Plücker coordinates* $p = (p_I) \in (\mathbb{R} \cup \{-\infty\})^{\binom{n}{k}}$. This vector yields a *tropical linear space* $L(p)$, defined by

$$L(p) = \bigcap_I \{x \in (\mathbb{R}_{\max})^n \mid \max_{i \in I}(p_{I\setminus\{i\}}) + x_j) \text{ is achieved at least twice}\} \ ,$$

where the minimum is taken over all subsets of $[n]$ of cardinality $k+1$. When $k = n-1$, $V(p)$ is a tropical hyperplane. Hence, a general version of tropical linear regression problem can be written as

$$\min_{p \in \mathrm{Gr}^{\mathrm{trop}}_{k,n}} \max_{v \in \mathcal{V}} \min_{x \in L(v)} \|v - x\|_H \ . \tag{54}$$

We solved here this problem when $k = n-1$. When $k = 1$, $L(p)$ is reduced to a single point, and it is not difficult to see that (54) reduces to a linear program. We leave it as an open question to solve this problem when $1 < k < n-1$. The same problem may be considered when p is a valuated matroid, or when it is inside the image of the Stiefel map [FR15], meaning that p is given by the maximal tropical minors of a matrix. A version of the latter problem (with a L_1-type error) is considered in [YZZ19]. One may also replace the linear space $L(p)$ by the *column space* of a tropical matrix A, which boils down to finding a best approximation by a tropical polyhedral cone with a fixed number of vertices, see [Hoo17, PYZ20].

Acknowledgment

We thank Quentin Jacquet and Marie Laclau for helpful comments and references, and also Nicolas Vieille for helpful references.

Appendix A. Dominions of the two players and existence of a finite eigenvector

The strongest form of strong duality (Theorem 22), with the existence of *witness points*, is valid whenever the Shapley operator T in Equation (9) has a finite eigenvector. In this appendix, we provide a sufficient condition for the existence of this eigenvector, which is less demanding than the condition of Proposition 23 (requiring V to have only finite entries).

We recall that the operator T represents a game Γ with two players Min and Max, such that when we are at state i, player Min plays first by choosing a column $k \in [p]$ such that $(i,k) \in E$, then player Max chooses a state $j \in [n]$ such that $j \neq i$ and $(j,k) \in E$. Moreover, policies can be defined using (10). The game Γ is played repeatedly starting from a given initial position.

We call *dominion* of one player a nonempty subset of states $I \subset [n]$ such that from any initial position in I, that player can force the state to remain in I at each stage of the repeated game, whatever actions the other player chooses. This means that there exist a policy of that player such that for any strategy of the other player, a trajectory of the game starting in I is such that the states visited by Min are all contained in I. The next result, which follows from a more general result (which applies to arbitrary Shapley operators) relates the lack of disjoint dominions of the two players with the existence of a finite eigenvector of a polyhedral Shapley operator.

Theorem 41 (Corollary of Thm. 1.2 of [AGH20])**.** *The following assertions are equivalent:*

(1) *The two players do not have disjoint dominions in the game Γ;*
(2) *For all $r \in \mathbb{R}^n$, the operator $r + T$ has a finite eigenvector.*

Deciding the existence of disjoint dominions for (deterministic) mean payoff games is equivalent to deciding the existence of a non-trivial fixed point of a monotone Boolean function, which is a NP-complete problem, see the discussion in [AGH15]. However, we next show that for the restricted class of games associated to the Shapley operator T_V, this problem can be solved in polynomial time.

We make the following assumption, which is required for the operator T to send $\mathbb{R}^n$ to $\mathbb{R}^n$, and a fortiori, to have a finite eigenvector.

Assumption 5. Each column of the matrix V contains at least two finite entries.

Proposition 42. *Suppose that Assumption 3 and Assumption 5 hold. Then, the following assertions are equivalent:*

1. *There are disjoint dominions for the two players in the game Γ;*
2. *There exist nonempty subsets I, J of $[n]$, such that $I \cup J = [n]$, $I \cap J = \emptyset$, some columns of V have support included in I, and the other columns of V have at least two finite entries in J.*
3. *There exists a subset K of $[p]$, such that $K \neq \emptyset$ and $K \neq [p]$, and such that if we denote by I_K the union of supports of the columns of V in K, then all the columns not in K have at least two finite entries that are outside I_K. In this case, I_K together with its complement $[n] \setminus I_K$ constitute disjoint dominions of players Min and Max, respectively.*

Proof. We verify first that the assertion (2) and the first part of assertion (3) are equivalent. Indeed, it is straightforward that (3) implies (2) by taking $I = I_K$ and $J = [n] \setminus I_K$. Now, if (2) is true, we take $K = \{k \in [p] \mid \operatorname{supp} V_{\cdot k} \subset I\}$, so $I_K = \cup_{k \in K} \operatorname{supp} V_{\cdot k} \subset I$, and each column $k \notin K$ has at least two finite entries in J, i.e. outside I_K.

Now we suppose that assertion (1) is satisfied, i.e. there are disjoint dominions I and J respectively for Player Min and Player Max. Let us show that this implies (2). The set J is a dominion for Player Max, then there exists a policy τ for Max, such that, for all $i \in J$, for any possible action $(i, k) \in E$ of Player Min, the policy τ sends the state in J, that is $\tau((i, k)) \in J$. Since a policy for Max is a map from E to $[n]$ such that $j = \tau((i, k))$ satisfies $j \neq i$ and $(j, k) \in E$, this implies that, for all $(i, k) \in E$ with $i \in J$, there exists $(j, k) \in E$ with $j \in J$ such that $j \neq i$. Therefore, for all $k \in [p]$, $\operatorname{supp} V_{\cdot k} \cap J$ is either empty or it contains at least two elements. We take $I' = [n] \setminus J \supset I \neq \emptyset$, then the sets I', J satisfy the assertion (2).

Now, we suppose that the first part of assertion (3) is true, and show that I_K and $J = [n] \setminus I_K$ are disjoint dominions of players Min and Max respectively, which will imply (1). Indeed, if $i \in I_K$, then there exists $k \in K$, such that $i \in \operatorname{supp} V_{\cdot k} \subset I_K$. Let us consider a policy σ of Min such that if $i \in I_K$ then $\sigma(i) = (i, k)$ with $k \in K$. Then, if $i \in I_K$, and if Min plays the action $(i, k) = \sigma(i)$, for any possible action of player Max (which exists by Assumption 5), that is a choice of $j \in \operatorname{supp} V_{\cdot k}$ such that $j \neq i$, we have $j \in I_K$. This shows that I_K is a dominion of Player Min. Now, let $i \in J$, for any action $(i, k) \in E$ of Min (which exists by Assumption 3), we have $k \in K$, since $i \in \operatorname{supp} V_{\cdot k}$ and $i \notin I_K$, so by (3), there exits $j \in \operatorname{supp} V_{\cdot k} \setminus I_K$, with $j \neq i$. So $j \in J$ and j is a possible action of Max when the game is in state (i, k). Considering the policy τ for Max, such that $\tau(i, k) = j$ for i, k, j as before, we get that the set J is a dominion of Player Max. □

From the proof of Proposition 42, we deduce in a straightforward manner the following observation, which will be used in Algorithm 1. Note that in the present setting (deterministic mean payoff games), if I, J are disjoint dominions of the two players, then $[n] \setminus J$ and J are also dominions of the two players, hence we shall restrict our search to disjoint dominions that constitute partitions of $[n]$.

Lemma 43. *If $D^{\mathrm{Min}}, D^{\mathrm{Max}} \subset [n]$ are disjoint dominions of players Min and Max respectively, that constitute a partition of $[n]$, and K is a subset of columns of V such that the set $S = \cup_{k \in K} \operatorname{supp} V_{\cdot k}$ satisfies $S \subset D^{\mathrm{Min}}$, then for each column $k \notin K$ that has only one finite entry i outside of S, we have $S \cup \{i\} \subset D^{\mathrm{Min}}$.* □

Theorem 44. *Algorithm 1, which decides the existence of disjoint dominions in the game Γ associated to a matrix $V \in (\mathbb{R}_{\max})^{n \times p}$, is correct, and it makes $O(n^2 p^2)$ arithmetic operations.*

Algorithm 1 Detecting dominions in the game arising from the tropical linear regression problem, for an input matrix $V \in (\mathbb{R}_{\max})^{n\times p}$.

1: **for** $k \in [p]$ **do**
2: $K \leftarrow \{k\}$
3: $S \leftarrow$ the support of column k of V
4: Declare S to be AUGMENTED (Boolean flag)
5: **while** S is declared as AUGMENTED **do**
6: Declare S not to be AUGMENTED
7: Declare all the elements of $[p] \setminus K$ to be UNSCANNED (Boolean flags)
8: **while** $([p] \setminus K)$ contains an UNSCANNED element **do**
9: $\ell \leftarrow$ smallest UNSCANNED element of $[p] \setminus K$, declare ℓ to be SCANNED
10: $S_\ell \leftarrow \{i \in [n] \setminus S \mid V_{i\ell} \text{ is finite}\}$
11: **if** $|S_\ell| = 1$ **then** ▷ *column ℓ of V has precisely one finite entry outside S*
12: $K \leftarrow K \cup \{\ell\}$, $S \leftarrow S \cup S_\ell$
13: Declare S to be AUGMENTED
14: **end if**
15: **if** $|S_\ell| = 0$ **then** ▷ *column ℓ of V has no finite entries outside S*
16: $K \leftarrow K \cup \{\ell\}$
17: **end if**
18: ▷ S is the union of supports of the columns of K
19: **end while**
20: **end while**
21: **if** $S \neq [n]$ **then return** S and $[n] \setminus S$ which are disjoint dominions of Players Min and Max respectively
22: **end if**
23: **end for**
24: There are no disjoint dominions

Proof. The algorithm looks for a set of columns $\bar{K}$ satisfying the last statement of Proposition 42. Since the set $\bar{K}$ is required to be nonempty, it suffices for each $k \in [p]$, to verify whether there is such a set $\bar{K} \ni k$ (for loop of the algorithm).

We next show that the algorithm admits the following invariants.

(1) At Line 18, S is the union of supports of the columns of K.
(2) If there is a subset $\bar{K} \ni k$ satisfying the last statement of Proposition 42, with associated then at line 18 of the algorithm, the set K satisfies $K \subset \bar{K}$ and the set S satisfies $S \subset D^{\text{Min}}$.

The first invariant is enforced by lines 10, 12 and 16. We prove that the loop invariant at line 18 holds by induction on the cardinality of S. Let us assume that the condition of the first "if", i.e., $|S_\ell| = 1$ is satisfied. Then, by Lemma 43, and by the induction assumption, we must have $S \cup \{i\} \subset D^{\text{Min}}$. Moreover, the last statement of Proposition 42 entails that $K \cup \{\ell\} \subset \bar{K}$, and so, the loop invariant is valid in this case. Moreover, if the condition of the second "if", i.e., $|S_\ell| = 0$ is satisfied, then, the second invariant is still valid. This shows that the loop invariant is always valid.

At the exit of the outer while loop, at line 21, we have by construction that every column of V with index outside K has at least two finite entries outside S. Then, by the last statement of Proposition 42, if $S \neq [n]$, S and $[n] \setminus S$ provide disjoint dominions of Players Min and Max, whereas if $S = [n]$, there are no dominions arising from a set $\bar{K} \ni k$. This shows the correctness of the algorithm.

Each iteration of the inner "while" loop makes $O(n)$ arithmetic operations, and every outer "while loop" executes the inner while loop $O(p)$ times. Moreover, the number of outer "while loop" iterations is at most $n-1$. Finally, we have at most p iterations in the "for" loop, which leads to a complexity bound of $O(n^2p^2)$ arithmetic operations for the algorithm. $\square$

We call *Boolean pattern* of the matrix $V \in (\mathbb{R}_{\max})^{n\times p}$ the matrix with entries in $\{0,-\infty\}$, obtained by replacing each finite entry of V by 0. Theorem 41 provides a sufficient condition involving the Boolean pattern on V, which guarantees that for all matrices V with this pattern, the operator T admits a finite eigenvector. This condition is not necessary. Consider the following Boolean pattern:

$$\begin{pmatrix} 0_{3,2} & 0_{3,2} \\ (-\infty)_{3,2} & 0_{3,2} \end{pmatrix} , \tag{55}$$

where for $\alpha \in \{0,-\infty\}$, $\alpha_{p,q}$ denotes the $p\times q$ matrix with entries identically equal to α.

Proposition 45. *If V is a matrix with Boolean pattern* (55)*, then, the operator T has a finite eigenvector, but the associated game admits disjoint dominions.*

Proof. First we have that the set $K = \{1,2\}$ satisfies the condition (3) of Proposition 42, and from the proof of Proposition 42, we have that the sets $I = \{1,2,3\}$ and $J = \{4,5,6\}$ are disjoint dominions of Player Min and Player Max respectively.

To show that T has a finite eigenvector, by Proposition 3, it suffices to check that $\chi(T) = 0$. The inequality $\chi(T) \leqslant 0$ follows from Remark 1. We next show that $\chi(T) \geqslant 0$.

If the game starts from a state $i \in \{4,5,6\}$, Player Min must choose the next state to be a pair (i,k) with $k \in \{3,4\}$, and Player Max can respond by choosing the next state j to belong to $\{4,5,6\}$. So, Player Max can force Min to play the same game as the one defined by the submatrix $X := (V_{ij})_{i\in\{4,5,6\},j\in\{3,4\}}$. Since the matrix X consists of only 2 columns of $(\mathbb{R}_{\max})^3$, it follows from Corollary 12 that the inner radius of $\mathrm{Col}(X)$ is equal to 0. Then by Theorem 1, $\rho(T_X) = 0$, and this entails that Player Max can ensure a payment equal to 0 in the original game, so that $\forall i \in \{4,5,6\}, \chi_i(T) = 0$.

Suppose now that the initial state $i \in \{1,2,3\}$. Since $\chi(T) = \min_\sigma \chi(T^\sigma)$ where the minimum is taken over the stationary policies of Player Min, it suffices to show that for any such policy, and for $i \in \{1,2,3\}$, $\chi_i(T^\sigma) \geqslant 0$. If this policy of Player Min chooses the column 3 or 4, Player Max can again enforce Player Min to play the game associated to the submatrix X, and then Player Max can ensure a payment 0 as before. Now, if the policy of Player Min does not choose the columns 3 and 4, Player Max is forced to play a subgame correspinding to the the submatrix $Y := (V_{ij})_{i\in\{1,2,3\},j\in\{1,2\}}$, and by the same reasoning as before, we know that the value of this game is equal to 0. Then $\forall i \in \{1,2,3\}, \chi_i(T) = 0$. $\square$

We leave it as an open question to characterize the Boolean patterns of V which guarantee that the operator T has a finite eigenvector.

References

[AGG09] M. Akian, S. Gaubert, and A. Guterman. Linear independence over tropical semirings and beyond. In G.L. Litvinov and S.N. Sergeev, editors, *Proceedings of the International Conference on Tropical and Idempotent Mathematics*, volume 495 of *Contemporary Mathematics*, pages 1–38. American Mathematical Society, 2009.

[AGG12] M. Akian, S. Gaubert, and A. Guterman. Tropical polyhedra are equivalent to mean payoff games. *International Journal of Algebra and Computation*, 22(1):125001 (43 pages), 2012.

[AGH15] M. Akian, S. Gaubert, and A. Hochart. Ergodicity conditions for zero-sum games. *Discrete & Continuous Dynamical Systems*, 35(9):3901–3931, 2015.

[AGH20] M. Akian, S. Gaubert, and A. Hochart. A game theory approach to the existence and uniqueness of nonlinear perron-frobenius eigenvectors. *Discrete & Continuous Dynamical Systems - A*, 40:207–231, 2020.

[AGK11] X. Allamigeon, S. Gaubert, and R. Katz. The number of extreme points of tropical polyhedra. *J. Comb. Theory Series A*, 118(1):162–189, 2011.

[AGKS18] X. Allamigeon, S. Gaubert, R. Katz, and M. Skomra. Condition numbers of stochastic mean payoff games and what they say about nonarchimedean semidefinite programming. In *Proceedings of the 23rd International Symposium on Mathematical Theory of Networks and Systems (MTNS2018)*, 2018.

[AGNS11] M. Akian, S. Gaubert, V. Niţică, and I. Singer. Best approximation in max-plus semimodules. *Linear Algebra and its Applications*, 435(12):3261–3296, 2011.

[Ard04] F. Ardila. Subdominant matroid ultrametrics. *Annals of Combinatorics*, 8:379–389, 2004.

[BB92] J. B. Baillon and R. E. Bruck. Optimal rates of asymptotic regularity for averaged nonexpansive mappings. In K. K. Tan, editor, *Proceedings of the Second International Conference on Fixed Point Theory and Applications*, pages 27–66. World Scientific Press, 1992.

[BCOQ92] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity*. Wiley, 1992.

[Ber20] D. I. Bernstein. L-infinity optimization to Bergman fans of matroids with an application to phylogenetics. *Siam J. Discrete Math.*, 34(1):701–720, 2020.

[BK19] E. Baldwin and P. Klemperer. Understanding preferences: "demand types", and the existence of equilibrium with indivisibilities. *Econometrica*, 87(3):867–932, 2019.

[BP07] R. Burguet and Martin K. Perry. Bribery and favoritism by auctioneers in sealed-bid auctions. *The B.E. Journal of Theoretical Economics*, 7(1):1–27, June 2007.

[BSS07] P. Butkovič, H. Schneider, and S. Sergeev. Generators, extremals and bases of max cones. *Linear Algebra Appl.*, 421(2-3):394–406, 2007.

[But03] P. Butkovič. Max-algebra: the linear algebra of combinatorics? *Linear Algebra Appl.*, 367:313–335, 2003.

[But10] P. Butkovič. *Max-linear Systems: Theory and Algorithms*. Springer Monogr. Math. Springer, London, 2010.

[BV07] H. Bjorklund and S. Vorobyov. A combinatorial strongly subexponential strategy improvement algorithm for mean payoff games. *Discrete Appl. Math.*, 155:210229, 2007.

[CF00] V. Chepoi and B. Fichet. ℓ_∞-approximation via subdominants. *Journal of Mathematical Psychology*, 44:600–616, 2000.

[CG79] R. A. Cuninghame-Green. *Minimax algebra*, volume 166 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag, Berlin, 1979.

[CGQ04] G. Cohen, S. Gaubert, and J. P. Quadrat. Duality and separation theorems in idempotent semimodules. *Linear Algebra and Appl.*, 379:395–422, 2004.

[Cha09] J. Chaloupka. Parallel algorithms for mean-payoff games: an experimental evaluation. In *Algorithms—ESA 2009*, volume 5757 of *Lecture Notes in Comput. Sci.*, pages 599–610. Springer, Berlin, 2009.

[CLMV05] O. Compte, A. Lambert-Mogiliansky, and T. Verdier. Corruption and competition in procurement auctions. *The RAND Journal of Economics*, 36(1):1–15, 2005.

[cod21] *Code de la commande publique*. Journal officiel, 2021. Available from legifrance.gouv.fr.

[CSV14] R. Cominetti, J. A. Soto, and J. Vaisman. On the rate of convergence of Krasnosel'skii-Mann iterations and their connection with sums of Bernoullis. *Israel Journal of Mathematics*, 199(2):757–772, 2014.

[DG06] V. Dhingra and S. Gaubert. How to solve large scale deterministic games with mean payoff by policy iteration. In *Proceedings of the 1st international conference on Performance evaluation methodolgies and tools (VALUETOOLS)*, volume 180, Pisa, Italy, 2006. article No. 12.

[DGJ17] J. Depersin, S. Gaubert, and M. Joswig. A tropical isoperimetric inequality. *Séminaire Lotharingien de Combinatoire*, 78B, 2017. Article #27, 12 pp., Proceedings of FPSAC 2017 (29th Conference on Formal Power Series and Algebraic Combinatorics, London).

[DKM01] V. Danilov, G. Koshevoy, and K. Murota. Discrete convexity and equilibria in economies with indivisible goods and money. *Mathematical Social Sciences*, 41(3):251–273, May 2001.

[DS04] M. Develin and B. Sturmfels. Tropical convexity. *Doc. Math.*, 9:1–27, 2004. (Erratum pp. 205–206).

[DSS05] M. Develin, F. Santos, and B. Sturmfels. On the rank of a tropical matrix. In *Combinatorial and computational geometry*, volume 52 of *Math. Sci. Res. Inst. Publ.*, pages 213–242. Cambridge Univ. Press, Cambridge, 2005.

[EKL06] M. Einsiedler, M. Kapranov, and D. Lind. Non-Archimedean amoebas and tropical varieties. *J. Reine Angew. Math.*, 601:139–157, 2006.

[EM79] A. Ehrenfeucht and J. Mycielski. Positional strategies for mean payoff games. *Internat. J. Game Theory*, 8(2):109–113, 1979.

[FR15] A. Fink and F. Rincón. Stiefel tropical linear spaces. *Journal of Combinatorial Theory, Series A*, 135:291 – 331, 2015.

[Fri09] O. Friedmann. An exponential lower bound for the parity game strategy improvement algorithm as we know it. In *Proceedings of the 24th Annual IEEE Symposium on Logic in Computer Science (LICS 2009)*, pages 145–156. IEEE Computer Society, 2009.

[GG04] S. Gaubert and J. Gunawardena. The Perron-Frobenius theorem for homogeneous, monotone functions. *Trans. of AMS*, 356(12):4931–4950, 2004.

[GJ08] B. Gärtner and M. Jaggi. Tropical support vector machines. Technical report, ACSTR-362502-01, 2008.

[GK07] S. Gaubert and R. Katz. The Minkowski theorem for max-plus convex sets. *Linear Algebra and Appl.*, 421:356–369, 2007.

[GK11] S. Gaubert and R. Katz. Minimal half-spaces and external representation of tropical polyhedra. *Journal of Algebraic Combinatorics*, 33(3):325–348, 2011.

[GKK88] V. A. Gurvich, A. V. Karzanov, and L. G. Khachiyan. Cyclic games and finding minimax mean cycles in digraphs. *Zh. Vychisl. Mat. i Mat. Fiz.*, 28(9):1407–1417, 1439, 1988.

[GMQ11] S. Gaubert, W. McEneaney, and Z. Qu. Curse of dimensionality reduction in max-plus based approximation methods: theoretical estimates and improved pruning algorithms. In *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC 11)*, pages 1054–1061, Orlando, FL, USA, December 2011.

[GP13] D. Grigoriev and V. V. Podolskii. Complexity of tropical and min-plus linear prevarieties. *Computational Complexity*, pages 1–34, 2013.

[GS20] S. Gaubert and N. Stott. A convergent hierarchy of non-linear eigenproblems to compute the joint spectral radius of nonnegative matrices. *Mathematical Control and Related Fields*, 10(3):573–590, 2020.

[Hoo17] J. Hook. Linear regression over the max-plus semiring: algorithms and applications, 2017. arXiv:1712.03499.

[IR09] Z. Izhakian and L. Rowen. The tropical rank of a tropical matrix. *Communications in Algebra*, 37(11):3912–3927, 2009.

[JSY07] M. Joswig, B. Sturmfels, and J. Yu. Affine buildings and tropical convexity. *Alban. J. Math.*, 1:187–211, 2007.

[Koh80] E. Kohlberg. Invariant half-lines of nonexpansive piecewise-linear transformations. *Math. Oper. Res.*, 5(3):366–372, 1980.

[Kri02] V. Krishna. *Auction theory*. Academic Press, 2002.

[LL69] T. M. Liggett and S. A. Lippman. Stochastic games with perfect information and time average payoff. *SIAM Rev.*, 11(4):604–607, 1969.

[LSTY17] B. Lin, B. Sturmfels, X. Tang, and R. Yoshida. Convexity in tree spaces. *SIAM Journal on Discrete Mathematics*, 31(3):2015–2038, January 2017.

[MCT21] P. Maragos, V. Charisopoulos, and E. Theodosis. Tropical geometry and machine learning. *Proceedings of the IEEE*, 109(5):728–755, May 2021.

[MN81] J.-F. Mertens and A. Neyman. Stochastic games. *Internat. J. Game Theory*, 10(2):53–66, 1981.

[MS15] D. Maclagan and B. Sturmfels. *Introduction to Tropical Geometry*, volume 161 of *Grad. Stud. Math.* AMS, Providence, RI, 2015.

[MSZ15] J.-F. Mertens, S. Sorin, and S. Zamir. *Repeated games*. Cambridge University Press, 2015.

[Nus86] R. D. Nussbaum. Convexity and log convexity for the spectral radius. *Linear Algebra Appl.*, 73:59–122, 1986.

[PYZ20] R. Page, R. Yoshida, and L. Zhang. Tropical principal component analysis on the space of phylogenetic trees. *Bioinformatics*, 36(17):4590–4598, 06 2020.

[Rac13] S. Rachmilevitch. Bribing in first-price auctions. *Games and Economic Behavior*, 77(1):214–228, January 2013.

[Sch08] S. Schewe. An optimal strategy improvement algorithm for solving parity and payoff games. In M. Kaminski and S. Martini, editors, *Computer Science Logic*, pages 369–384, Berlin, Heidelberg, 2008. Springer Berlin Heidelberg.

[Ser07] S. Sergeev. Max-plus definite matrix closures and their eigenspaces. *Linear Algebra Appl.*, 421(2):182–201, 2007.

[Sko18] M. Skomra. *Spectraèdres tropicaux : application à la programmation semi-définie et aux jeux à paiement moyen*. Theses, Université Paris-Saclay, December 2018.

[SS04] D. Speyer and B. Sturmfels. The tropical Grassmannian. *Adv. Geom.*, 4(3):389–411, 2004.

[TM19] A. Tsiamis and P. Maragos. Sparsity in max-plus algebra and systems. *Discrete Event Dynamic Systems*, 29:163–189, 2019.

[TWY20] X. Tang, H. Wang, and R. Yoshida. Tropical support vector machine and its applications to phylogenomics, 2020. arXiv:2003.00677.

[TY19] N. M. Tran and J. Yu. Product-mix auctions and tropical geometry. *Mathematics of Operations Research*, 44(4):1396–1411, November 2019.

[YZZ19] R. Yoshida, L. Zhang, and X. Zhang. Tropical principal component analysis and its application to phylogenetics. *Bulletin of Mathematical Biology*, 81:568–597, 2019.

[ZP96] U. Zwick and M. Paterson. The complexity of mean payoff games on graphs. *Theoret. Comput. Sci.*, 158(1-2):343–359, 1996.

INRIA and CMAP, École polytechnique, IP Paris, CNRS
Email address: Marianne.Akian@inria.fr
Email address: Stephane.Gaubert@inria.fr
Email address: Yang.Qi@inria.fr
Email address: Omar.Saadi@polytechnique.edu