Vertex Partitions into an Independent Set and a Forest with Each Component Small

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Abstract

For each integer $k \geq 2$, we determine a sharp bound on $\mathrm{mad}(G)$ such that $V(G)$ can be partitioned into sets I and F_k, where I is an independent set and $G[F_k]$ is a forest in which each component has at most k vertices. For each k we construct an infinite family of examples showing our result is best possible. Our results imply that every planar graph G of girth at least 9 (resp. 8, 7) has a partition of $V(G)$ into an independent set I and a set F such that $G[F]$ is a forest with each component of order at most 3 (resp. 4, 6).

Hendrey, Norin, and Wood asked for the largest function $g(a,b)$ such that if $\mathrm{mad}(G) < g(a,b)$ then $V(G)$ has a partition into sets A and B such that $\mathrm{mad}(G[A]) < a$ and $\mathrm{mad}(G[B]) < b$. They specifically asked for the value of $g(1,b)$, i.e., the case when A is an independent set. Previously, the only values known were $g(1,4/3)$ and $g(1,2)$. We find $g(1,b)$ whenever $4/3 < b < 2$.

1 Introduction

An (I,F_k)*-coloring* for a graph G is a partition of $V(G)$ into sets I and F such that I is an independent set and F induces a forest in which each component has at most k vertices. The *average degree of* G is $2|E(G)|/|V(G)|$. The *maximum average degree of* G, denoted *mad*(G), is the maximum, taken over all subgraphs H, of the average degree of H. In this paper, we prove a sufficient condition for a graph G to have an (I,F_k)-coloring, in terms of $\mathrm{mad}(G)$. (I,F_k)-coloring mad(G)

Theorem 1. *For each integer $k \geq 2$, let*

$$f(k) := \begin{cases} 3 - \frac{3}{3k-1} & k \text{ even} \\ 3 - \frac{3}{3k-2} & k \text{ odd} \end{cases}$$ $f(k)$

If mad$(G) \leq f(k)$, then G has an (I,F_k)-coloring.

Theorem 1 is best possible. For each positive integer k there exists an infinite family of graphs with maximum average degree approaching $f(k)$ (from above) such that none of these graphs has an (I,F_k)-coloring. Note that $f(3) = \frac{18}{7}$, $f(4) = \frac{30}{11}$, and $f(6) = \frac{48}{17}$. Each planar graph G with girth g has $\mathrm{mad}(G) < \frac{2g}{g-2}$. So Theorem 1 implies that every planar graph G of girth at least 9 (resp. 8, 7) has a partition of $V(G)$ into an independent set I and a set F where $G[F]$ is a forest with each tree of order at most 3 (resp. 4, 6); for girth 9, this is best possible,

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since [9, Corollary 4] constructs girth 9 planar graphs with no (I, F_2)-coloring. This strengthens results in [8, 10]. Theorem 1 is implied by a more general result below, our Main Theorem. Before introducing definitions and notation to state it, we briefly discuss related work.

Choi, Dross, and Ochem [6] studied a variant of (I, F_k)-colorings where they did not require the components of $G[F_k]$ to be acyclic, but only to have order at most k. They proved that G has such a coloring whenever $\mathrm{mad}(G) < \frac{8}{3}(1 - \frac{1}{3k+1})$. Theorem 1 allows a weaker hypothesis (and a stronger conclusion). Moreover, the argument on the sharpness of Theorem 1 (see Lemma 5) does not require the acyclic nature of F_k, and therefore Theorem 1 is also a sharp result for this variant of the problem. Dross, Montassier, and Pinlou [8] studied a different variant of (I, F_k)-colorings, where $G[F_k]$ has bounded maximum degree, but perhaps not bounded order (earlier related results are in [5] and [2]). Under hypotheses very similar to those in Theorem 1, they proved that G has such a coloring. These results, too, are strengthened by Theorem 1.

We can also view Theorem 1 in a more general context. Hendrey, Norin, and Wood [1, Problem #14] asked for the largest function $g(a, b)$ such that if $\mathrm{mad}(G) < g(a, b)$ then $V(G)$ has a partition into sets A and B such that $\mathrm{mad}(G[A]) < a$ and $\mathrm{mad}(G[B]) < b$. They specifically asked for the value of $g(1, b)$, which corresponds to the case that A is an independent set. Nadara and Smulewicz [10] used maximum flows to give a short proof that $g(1, b) \geq b + 1$ and $g(2, b) \geq b + 2$. However, the only exact values previously known[1] were $g(1, 4/3)$ and $g(1, 2)$ (see [4] for $g(1, 4/3)$ and see below for $g(1, 2)$). We find the value of $g(1, b)$ whenever $4/3 \leq b < 2$. $g(a,b)$

We also study a related function $\tilde{g}(a, b)$. This is the largest value for which there is a finite set $\mathcal{G}_{a,b}$ of graphs such that if $\mathrm{mad}(G) < \tilde{g}(a, b)$ and G has no graph in $\mathcal{G}_{a,b}$ as a subgraph, then $V(G)$ has a partition into sets A and B where $\mathrm{mad}(G[A]) < a$ and $\mathrm{mad}(G[B]) < b$. That is, $\tilde{g}(a, b)$ is the minimum value such that there is an infinite family of graphs G_j with $\mathrm{mad}(G_j)$ approaching $\tilde{g}(a, b)$ from above (as $j \to \infty$) and each $V(G_j)$ has no partition A, B with $\mathrm{mad}(G_j[A]) < a$ and $\mathrm{mad}(G_j[B]) < b$. Clearly, $g(a, b) \leq \tilde{g}(a, b)$, and sometimes this inequality is strict. $\tilde{g}(a,b)$

In [7] we observed that $g(1, 2) = 3$. The lower bound follows from degeneracy.[2] The upper bound $g(1, 2) \leq 3$ comes from K_4. However, K_4 is the single obstruction to strengthening this bound. In fact, we proved that $\tilde{g}(1, 2) = 3.2$. Each component of a graph G with $\mathrm{mad}(G) < 2$ is a forest. Thus, a partition of $V(G)$ into sets I and F with $\mathrm{mad}(G[I]) < 1$ and $\mathrm{mad}(G[F]) < 2 - 2/(k+1)$ is precisely an (I, F_k)-partition. In the present paper, we show that $g(1, 2 - 2/(k + 1)) = \tilde{g}(1, 2 - 2/(k + 1)) = f(k)$ for every integer $k \geq 2$ (here $f(k)$ is as defined in Theorem 1). This is particularly interesting because $\tilde{g}(1, 2) = 3.2$, but $\tilde{g}(1, b) < 3$ for every $b < 2$.

A *precoloring* of G is a partition of $V(G)$ into sets $U_0, U_1, \ldots, U_{k-1}, F_1, F_2, \ldots, F_k$, and I. Intuitively, we think of a vertex in F_j as being already colored F and having an additional $j - 1$ (fake) neighbors that are also already colored F. So, for example, if a vertex is in F_k then we cannot color any of its neighbors in $\bigcup_{j=0}^{k-1} U_j$ with F, since this would create a component colored F with at least $k + 1$ vertices. Similarly, a vertex v in U_j is uncolored, but has j fake neighbors that are colored F. So coloring v with F would create a component colored F with $j + 1$ vertices. An (I, F_k)-coloring of a precolored graph G is an (I, F_k)-coloring (I', F') of the underlying (not precolored) graph G such that $I \subseteq I'$, $\cup_{j=1}^{k} F_j \subseteq F'$ and each component of $G[F']$ has at most k vertices *including* any fake neighbors arising from the precoloring. A graph G is *precolored trivially* if $U_0 = V(G)$, so $U_1 = \cdots = U_{k-1} = F_1 = \cdots = F_k = I = \emptyset$. precoloring $U_0, \ldots, U_{k-1}$ $F_1, \ldots, F_k, I$

A precolored graph G is (I, F_k)-*critical* if G has no (I, F_k)-coloring, but every proper subgraph of G does and, furthermore, for any vertex precolored U_j or F_j, reducing j by 1 allows an (I, F_k)-coloring of G. So Theorem 1 is equivalent to saying that every (trivially precolored) (I, F_k)-critical graph G has $\mathrm{mad}(G) > f(k)$. To facilitate a proof by induction, we want to extend Theorem 1 to allow other precolorings. However, a vertex in U_j (with $j > 0$) or in F_j imposes more constraints on an (I, F_k)-coloring than one in U_0. Intuitively, a vertex in (I, F_k)-critical

[1] Borodin, Kostochka, and Yancey [3] also showed that $g(4/3, 4/3) = 14/5$.

[2] Given a vertex v of degree at most 2, by induction we partition $G - v$ into sets I and F such that I is independent and $G[F]$ is a forest. If v has no neighbor in I, then we add v to I. Otherwise, we add it to F.

$V(G) \setminus U_0$ should "count more" toward the average degree than one in U_0. This motivates weighting vertices differently, as we do below. (In Section 1.2, we explain our choice of weights.)

Definition 2. For each integer $k \geq 2$, let

- $C_E := \{3k-1 \text{ for } k \text{ even}, 3k-2 \text{ for } k \text{ odd}\}$; C_E
- $C_{U,0} := \frac{3C_E-3}{2}$;
- $C_{U,j} := C_{U,0} - 3j = \frac{3C_E-3}{2} - 3j$ for $0 < j \leq k$; $C_{U,j}$
- $C_{F,j} := C_{U,j-1} + C_I - C_E = C_E - 3j$ for $1 \leq j \leq \lfloor \frac{k+1}{2} \rfloor$; $C_{F,j}$
- $C_{F,j} := C_{U,\lfloor \frac{k-1}{2} \rfloor} + C_{U,\lceil \frac{k-1}{2} \rceil} + C_{U,j-\lfloor \frac{k+3}{2} \rfloor} - 3C_E = 3(k-j)$ for $\lfloor \frac{k+3}{2} \rfloor \leq j \leq k$; and
- $C_I := C_{U,0} + C_{F,k} - C_E = \frac{C_E-3}{2}$. C_I

Main Theorem. *Fix an integer $k \geq 2$. Let*

$$\rho_G^k(R) := \sum_{j=0}^{k-1} C_{U,j}|U_j \cap R| + \sum_{j=1}^{k} C_{F,j}|F_j \cap R| + C_I|I \cap R| - C_E|E(G[R])|,$$ ρ_G^k

for each $R \subseteq V(G)$. If a precolored graph G is (I, F_k)-critical, then $\rho_G^k(V(G)) \leq -3$.

Now is a good time to define more terminology and notation. We typically write ρ^k, rather than ρ_G^k, when there is no danger of confusion. We also write *coloring* to mean (I, F_k)-coloring. An *F-component* is a component of $G[F]$ (either for an (I, F_k)-coloring of a graph G or for a precoloring of G, where $F = \cup_{j=1}^k F_j$). We will often want to move a vertex v from U_a to U_{a+b} or from F_a to F_{a+b}, for some integers a and b. Informally, we call this "adding b F-neighbors to v". If an uncolored vertex v ever has k or more F-neighbors, then we recolor v with I (since coloring v with F would create an F-component with at least $k+1$ vertices, which is forbidden); see Lemma 6 and the comment after it. Note the following easy proposition. coloring F-component

Proposition 3. *The Main Theorem implies Theorem 1.*

Proof. Observe that $\frac{2C_{U,0}}{C_E} = f(k)$, as defined in Theorem 1. Thus, if G is precolored trivially, then the condition $\rho^k(V(G)) \geq 0$ is equivalent to $\frac{2|E(G)|}{|V(G)|} \leq f(k)$. By the Main Theorem, each (I, F_k)-critical graph G has $\rho^k(V(G)) \leq -3$. Thus, if $\text{mad}(G) \leq f(k)$, then $\rho(R) \geq 0$ for all $R \subseteq V(G)$; so G contains no (I, F_k)-critical subgraph. Hence, G has an (I, F_k)-coloring. $\square$

The proof of the Main Theorem differs somewhat depending on whether k is even or odd. However, the two cases are similar. Thus, we begin the proof (for all k) in Section 2. In Section 3 we conclude it for k even, and in Section 4 we conclude it for k odd. Before proving the Main Theorem, we discuss the sharpness examples and the gadgets that motivate our weights in Definition 2. We then conclude the introduction with a brief overview of the potential method.

1.1 Sharpness Examples

Example 4. We write *add a pendent 3-cycle at a vertex z* to mean identify z with a vertex of a new 3-cycle. *Adding ℓ pendent 3-cycles at z* means repeating this ℓ times. Similarly, *adding a 2-thread from y to z* means adding new vertices y' and z' and new edges $yy', y'z', z'z$. (Adding ℓ 2-threads is defined analogously.)

We form an (I, F_k)-critical graph $G_{k,t}$ as follows (Figure 1 shows $G_{k,3}$). Start with vertices $v_0, \ldots, v_t$, $w_0, \ldots, w_t$, $x_0, \ldots, x_t$, where $\{v_j, w_j, x_j\}$ induces K_3 for each $j \in \{0, \ldots, t\}$. Now add $\lfloor \frac{k-2}{2} \rfloor$ pendent 3-cycles at v_0, $\lfloor \frac{k-1}{2} \rfloor$ pendent 3-cycles at w_0, and $\lfloor \frac{k}{2} \rfloor$ pendent 3-cycles at x_0. For each $j \in \{1, \ldots, t\}$, add $\lfloor \frac{k-2}{2} \rfloor$ 2-threads from v_{j-1} to v_j, $\lfloor \frac{k-1}{2} \rfloor$ 2-threads from v_{j-1} to w_j, and $\lfloor \frac{k}{2} \rfloor$ 2-threads from v_{j-1} to x_j. Finally, add a single pendent 3-cycle at v_t. $G_{k,t}$

The proof that $G_{k,t}$ is (I, F_k)-critical is a bit tedious, but we include it below for completeness. It is not needed for the proof of our Main Theorem, so the reader should feel free to skim (or skip) it. Intuitively, if we start to color $G_{k,t}$ from the left, each v_j will be in an F-component of order k; but for v_t, due to the extra pendent 3-cycle, we get an F-component of order $k+1$, a contradiction. When we delete some edge e, at some point we are able to use I on some $v_{j'}$, and we continue using I on each v_j with $j \geq j'$. The coloring of $G_{k,t} - e$ is some combination of the two colorings at the bottom of Figure 1. (It is interesting to note that the family $G_{2,t}$ is precisely those sharpness examples given by Borodin and Kostochka in [4].)

Lemma 5. *$G_{k,t}$ is (I, F_k)-critical for all integers $k \geq 2$ and $t \geq 0$.*

Proof. Let $G^j_{k,t}$ denote the subgraph of $G_{k,t}$ induced by $v_0, \dots, v_j, w_0, \dots, w_j, x_0, \dots, x_j$ along with their pendent 3-cycles and any 2-threads between them. We show by induction that $G^j_{k,t}$ has an (I, F_k)-coloring for each $j < t$; furthermore, in each such coloring v_j is in an F-component of order k. Consider $G^0_{k,t}$. Because of their pendent 3-cycles, w_0 and x_0 will have at least $\lfloor \frac{k-1}{2} \rfloor$ and $\lfloor \frac{k}{2} \rfloor$ F-neighbors (respectively) in every coloring of $G^0_{k,t}$. If both w_0 and x_0 are colored F, then they lie in an F-component of order at least $\lfloor \frac{k-1}{2} \rfloor + \lfloor \frac{k}{2} \rfloor + 2 = k+1$, a contradiction. So one of w_0 and x_0 must be colored I. Thus, v_0 is colored F; so v_0 lies in an F-component of order at least $\lfloor \frac{k-2}{2} \rfloor + \lfloor \frac{k-1}{2} \rfloor + 2 = k$. To see that $G^0_{k,t}$ has a coloring, color x_0 with I and v_0 and w_0 with F. For each 3-cycle pendent at v_0 or w_0, use I on one vertex and F on the other. For each 3-cycle pendent at x_0, use F on both vertices. This proves the base case.

Now we consider the induction step. Since v_{j-1} is in an F-component of order k in $G^{j-1}_{k,t}$, each neighbor of v_{j-1} on a 2-thread to $\{v_j, w_j, x_j\}$ must be colored I; thus, each of *their* neighbors must be colored F. Now the analysis is nearly identical that that for $j = 0$. To extend the coloring to all of $G^j_{k,t}$, color x_j with I and color v_j and w_j with F. If we instead tried to color v_j with I, then w_j and x_j must both be colored F, so they lie in an F-component of order $\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k-1}{2} \rfloor + 2 = k+1$, a contradiction.

To see that $G_{k,t}$ has no coloring, note that such a coloring would have v_t in an F-component of order k (as in the induction step above). However, due to the extra pendent 3-cycle at v_t, this creates an F-component of order $k+1$, a contradiction.

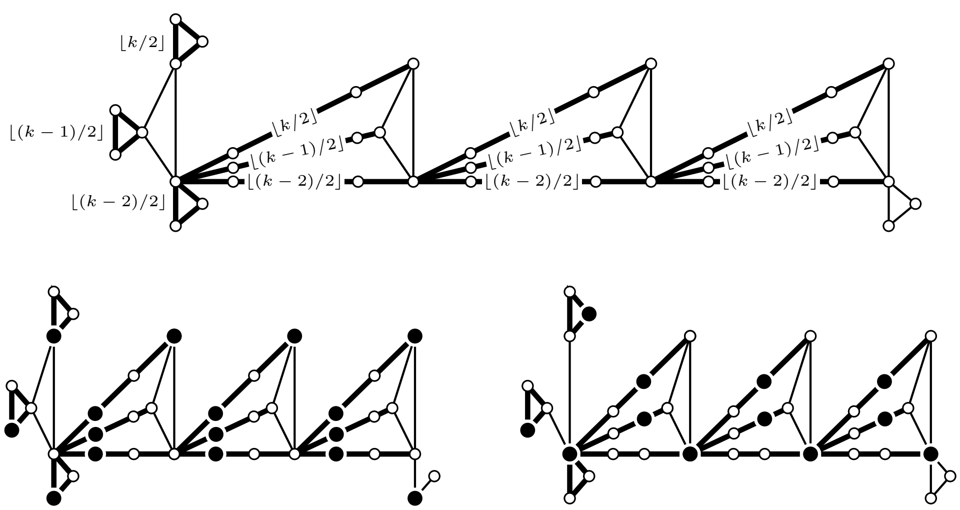

Figure 1: Top: The sharpness example $G_{k,3}$. Bold edges denote multiple pendent 3-cycles at a vertex or multiple 2-threads between two vertices. Bottom left: An (I, F_k)-coloring of $G_{k,3} - e$, where e is on the 3-cycle pendent at v_t. Bottom right: An (I, F_k)-coloring of $G_{k,3} - w_0x_0$. (Throughout, vertices in I are black and vertices in F are white.)

Finally, we show that $G_{k,t}$ is (I, F_k)-critical. That is, for each $e \in E(G_{k,t})$ subgraph $G_{k,t} - e$ has a coloring. By induction we first prove the stronger statement that if $e \in E(G^{j-1}_{k,t})$, then $G^j_{k,t} - e$ has a coloring with v_j colored I. (The intuition is that once we get this for some j', then we can ensure it for all $j' > j$, so can finish the coloring.) Afterward, we use this to prove that $G_{k,t} - e$ has an (I, F_k)-coloring for every $e \in E(G_{k,t})$.

Base case: $j = 1$. If e is not on a pendent 3-cycle at v_0, then $G^{j-1}_{k,t} - e$ has a coloring in which v_0 is colored I, as follows. Either (a) $e \in \{v_0w_0, v_0x_0\}$, so we can color two vertices in $\{v_0, w_0, x_0\}$ with I or (b) $e = w_0x_0$ or e is on a 3-cycle pendent at w_0 or x_0, so we can color both w_0 and x_0 with F. If we can color v_0 with I, then we extend to $G^1_{k,t} - e$ by using F on all neighbors of v_0 on 2-threads, using I on v_1 and neighbors of w_1 and x_1 on 2-threads, and using F on all remaining vertices. Assume instead that e is on a pendent 3-cycle at v_0. Now we color both endpoints of e with I, so that v_0 is in an F-component of order only $k - 1$. This enables us to use F on some neighbor of v_0 on a 2-thread to x_1 (and use I on its neighbor on that 2-thread). Now we use F on w_1 and x_1, and use I on v_1. This finishes the base case.

The induction step is nearly identical to the base case. Suppose $e \in E(G^{j-1}_{k,t})$. If $e \in E(G^{j-2}_{k,t})$, then $G^{j-1}_{k,t} - e$ has a coloring in which v_{j-1} uses I. We extend it to $G^j_{k,t} - e$ in exactly the same way as extending the coloring of $G^0_{k,t} - e$ to $G^1_{k,t} - e$ above. Otherwise $e \in E(G^{j-1}_{k,t}) \setminus E(G^{j-2}_{k,t})$. Recall, from above, that $G^{j-2}_{k,t}$ has a coloring, and it has v_{j-2} in an F-component of order k. Now the extension to $G^{j-1}_{k,t}$ is nearly identical to coloring $G^0_{k,t} - e$ (from the base case at the start of the proof). This proves our stronger statement by induction.

Finally, we prove that $G_{k,t} - e$ has a coloring for every $e \in E(G_{k,t})$. If e is not on the 3-cycle pendent at v_t, then we can color $G_{k,t} - e$ with I on v_t, so the extra pendent 3-cycle does not matter. If e is on the pendent 3-cycle, then we color so that v_t is in an F-component of order k without the extra 3-cycle. However, now v_t has only a single neighbor on that pendent 3-cycle, so we color that neighbor with I and the remaining vertex with F. □

1.2 Gadgets: Where the Coefficients Come From

Here we explain our choice of weights in Definition 2: C_E, $C_{U,j}$, $C_{F,j}$, C_I. Everything starts with our sharpness examples in Section 1.1. We must choose $C_{U,0}$ and C_E so that all of these examples have the same potential, i.e., $\rho^k(G_{k,t+1}) = \rho^k(G_{k,t})$ for all positive t. Note that $|V(G_{k,t+1})| - |V(G_{k,t})| = 3 + 2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k-1}{2} \rfloor + \lfloor \frac{k-2}{2} \rfloor) = C_E$ and $|E(G_{k,t+1})| - |E(G_{k,t})| = 3 + 3(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k-1}{2} \rfloor + \lfloor \frac{k-2}{2} \rfloor) = C_{U,0}$. *This* is how we chose C_E and $C_{U,0}$.

For each of I, F_j and U_j ($j > 0$) we construct a gadget, consisting of edges and vertices in U_0. Each gadget has a specified vertex v which the gadget simulates having the desired precoloring; see Figure 2. The easiest of these is U_1. The gadget is simply a 3-cycle. Suppose we add a pendent 3-cycle C at any vertex v. In any coloring of G (with C added), at least one neighbor of v on C is colored F. Further, if v is colored F, then we can color the remaining vertices of C so that exactly one is in F. Thus, this gadget precisely simulates v being in U_1. For each larger j, the gadget for U_j simply adds j pendent 3-cycles at v. Alternatively, we can define the gadgets recursively, where adding a pendent 3-cycle moves a vertex from U_j to U_{j+1}.

But how do we simulate a vertex in F_1? It is simpler (surprisingly) to start with the gadget for F_k. This is just the subgraph of $G_{k,t}$ induced by v_0, w_0, x_0 and their pendent 3-cycles. Precisely, it is formed from a K_3 by adding $\lfloor \frac{k-2}{2} \rfloor$ pendent 3-cycles at v and adding $\lfloor \frac{k-1}{2} \rfloor$ and $\lfloor \frac{k}{2} \rfloor$ pendent 3-cycles at the two other vertices of the K_3; see the left end of Figure 1. In the proof of Lemma 5, we showed that any coloring of this subgraph must have v_0 in an F-component of order k. The potential of this subgraph is 0, so $C_{F,k} = 0$. The gadget for I is simply an edge to a vertex in F_k. So $C_I = C_{U,0} - C_E + C_{F,k} = \frac{C_E - 3}{2}$. Finally, the gadget for F_1 is an edge to a vertex in I. So $C_{F,1} = C_{U,0} + C_I - C_E = C_E - 3$. Adding a pendent 3-cycle at a vertex in $C_{F,j}$ moves it to $C_{F,j+1}$. So we are tempted to say that $C_{F,j+1} = C_{F,j} - 3$ for all j; but this is not

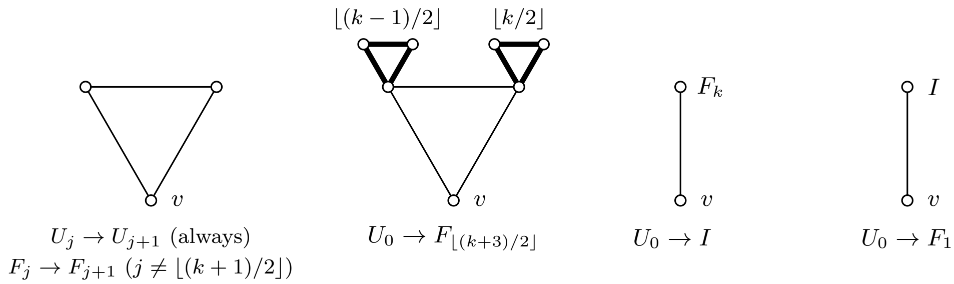

Figure 2: Gadgets to simulate precoloring.

quite right! We must simulate each F_j as efficiently as possible. We can do slightly better for $F_{j'}$ when $j' = \lfloor \frac{k+3}{2} \rfloor$. The best gadget for $F_{j'}$ is shown in Figure 2; it is formed from the gadget for F_k by *removing* $k - j'$ pendent 3-cycles at v_0. This gadget gives $C_{F,j'} = 3k - 3j'$ (rather than $C_E - 3j'$, which we get if we build up from the gadget for F_1). Now for each $j > j'$, we add $j - j'$ pendent 3-cycles at v. Thus, $C_{F,j} = 3k - 3j$ for all $j \geq j'$.

It is enlightening to notice that the Main Theorem is logically equivalent to its restriction to graphs that are precolored trivially. Since this is not needed for our proof of the Main Theorem, we are content to provide only a proof sketch.

Equivalence Lemma. *The Main Theorem is true if and only if it is true when restricted to graphs with no precolored vertices.*

Proof Sketch. The case with a trivial precoloring is clearly implied by the general case. Now we show the reverse implication. Suppose the Main Theorem is false for some specific value of k. Let G be a counterexample; among all counterexamples, choose one that minimizes $|V(G)|$. We will construct another counterexample $\widehat{G}$ (for the same value of k) with $U_0 = V(\widehat{G})$.

If G has a vertex v precolored I, then we form G' from $G - v$ by coloring each neighbor of v (in G) with F. Since G is (I, F_k)-critical, so is G'. Since G' is smaller than G, we know that $\rho^k_{G'}(V(G')) \leq -3$. It is straightforward to check that $\rho^k_G(V(G)) \leq \rho^k_{G'}(V(G')) \leq -3$ (see Lemma 6 for details); so G is not a counterexample, a contradiction. Thus, $I = \emptyset$.

Now we form a graph $\widehat{G}$ from G by identifying each vertex $v \in V(G)$ colored U_j or F_j with the vertex v in the corresponding gadget (and removing the precoloring). It is easy to check that $-2 \leq \rho^k_G(V(G)) = \rho^k_{\widehat{G}}(V(\widehat{G}))$; indeed, this is exactly why we chose the values we did for $C_{U,j}$ and $C_{F,j}$. So all that remains is to show that $\widehat{G}$ is (I, F_k)-critical.

First, note that each gadget precisely simulates the precoloring. That is, every (I, F_k)-coloring of the gadget for each U_j either gives v at least j F-neighbors or it colors v with I; furthermore, some coloring of the gadget for U_j colors v with I and some other coloring of the gadget for U_j colors v with F and gives v exactly j F-neighbors. Similarly, every (I, F_k)-coloring of the gadget for each F_j colors v with F and puts it in an F-component of order at least j; and some coloring of the gadget for F_j colors v with F and puts it in a component of order exactly j. Second, note that deleting any edge from the gadget for U_j allows a coloring in which v has at most $j-1$ F-neighbors. Similarly, deleting any edge from the gadget for F_j allows a coloring in which v is in an F-component of order at most $j-1$. Thus, $\widehat{G}$ is (I, F_k)-critical. $\square$

Since the Main Theorem is equivalent to its restriction to graphs with trivial precolorings, what is the point of allowing precolorings? The point is to order the graphs in a way that is more useful for induction (note that $V(\widehat{G}) > V(G)$, so allowing precolorings enables us to simulate $\widehat{G}$ with a precolored graph G that is smaller than $\widehat{G}$). In fact, we could write the whole proof without precolorings, but the partial order on the graphs needed for that version would be much harder to understand and keep track of.

1.3 The Potential Method: A Brief Introduction

The function ρ^k is called the *potential function*, and the proof technique we employ in this paper is called the *potential method*. Here we give a brief overview. potential function potential method

The essential first step in any proof using the potential method is to find an infinite family of sharpness examples. These examples determine a sharp necessary condition on $\mathrm{mad}(G)$. So we use them to choose the coefficients $C_{U,0}$ and C_E, which define ρ. The necessary generalization (allowing precoloring and specifically all the different options U_j and F_j) varies with the problem. For some problems, we do not use precoloring at all. In one case we allowed parallel edges [7]. Whenever a generalization allows precolorings, the coefficients are all determined by the gadgets, as discussed in the previous section (so it is essential to find the gadgets with highest potential).

Behind every proof using the potential method is a typical proof using reducibility and discharging. Consider, for example, Theorem 1. Suppose we are aiming to prove that theorem and we want to show that a certain configuration H is reducible. Typically, we color $G - V(H)$ by induction and then show how to extend the coloring to $V(H)$. The reason we can color $G - V(H)$ by induction is that, by definition, $\mathrm{mad}(G - V(H)) \leq \mathrm{mad}(G)$; since $G - V(H)$ is smaller than G, the theorem holds for $G - V(H)$. *The heart of the potential method is to show that we can slightly modify $G - V(H)$ before we color it by induction.* This modification (say, adding some F-neighbors) enables us to require more of our coloring of $G - V(H)$. Since this coloring of $G - V(H)$ is more constrained, we may be able to extend it to $V(H)$, even if we could not do so for an arbitrary (I, F_k)-coloring of $G - V(H)$. To make all of this precise, we need a lower bound on $\rho^k(R)$ for all $R \subsetneq V(G)$. Such a bound is called a Gap Lemma. Our modifications may lower $\rho^k(R)$, but if we can ensure that even this lowered potential is at least -2 for all R, then we know by induction that G' cannot contain an (I, F_k)-critical subgraph, so it must have an (I, F_k)-coloring.

Once we have proved that various configurations are reducible, we use discharging to show that a (hypothetical, smallest) counterexample G to our Main Theorem cannot exist. We assign charge so that the assumption $\rho^k(V(G)) \geq -2$ implies that the sum of all initial charges is at most 4. (This is analogous, for graphs with mad $< \alpha$, to using the initial charge $\mathrm{ch}(v) := d(v) - \alpha$.) As a first step, we show that each vertex ends with nonnegative charge. With a bit more work, we show that if G has no coloring, then its total charge exceeds 4, so G is not a counterexample.

Our proof of the Main Theorem naturally translates into a polynomial-time algorithm. This is typical of proofs using the potential method. The translation is mostly straightforward. The least obvious step is efficiently finding a set of minimum potential, which can be done using a max-flow/min-cut algorithm. We discuss algorithms at length in [7, Sections 2.3 and 5].

2 Starting the Proof of the Main Theorem

Fix an integer $k \geq 2$. In what follows, we typically write ρ rather than ρ^k. We say that a graph G_1 is *smaller* than a graph G_2 if either (a) $|V(G_1)| < |V(G_2)|$ or (b) $|V(G_1)| = |V(G_2)|$ and $|E(G_1)| < |E(G_2)|$. Assume that the Main Theorem is false for k. Let G be a smallest counterexample. In this section, we prove a number of lemmas restricting the structure of G. smaller

Lemma 6. $I \cup U_k \cup F_k = \emptyset$.

Proof. Assume, to the contrary, that $I \cup U_k \cup F_k \neq \emptyset$. First, suppose there exists $v \in F_k$. Form G' from G by deleting v and adding each neighbor of v to I. For each $R' \subseteq V(G')$, subgraph $G'[R']$ has an (I, F_k)-coloring if and only if $G[R' \cup \{v\}]$ does. Since G is (I, F_k)-critical, so is G'. Since G' is smaller than G, by the minimality of G, we have $\rho_{G'}(V(G')) \leq -3$. However, now $\rho_G(V(G)) \leq \rho_{G'}(V(G')) + (C_{U,0} - C_I - C_E)d(v) = \rho_{G'}(V(G')) \leq -3$. Thus, G is not a counterexample.

Suppose instead there exists $v \in I$. Form G' from G by deleting v and adding each neighbor of v to F (we assume $d(v) \geq 1$). For each $R' \subseteq V(G')$, subgraph $G'[R']$ has an (I, F_k)-coloring if and only if $G[R' \cup \{v\}]$ does. Since G is (I, F_k)-critical, so is G'. Since G' is smaller than G, by the minimality of G we have $\rho_{G'}(V(G')) \leq -3$. Coloring a vertex in U_j with F moves it to F_{j+1}, so decreases its potential by $C_{U,j} - C_{F,j+1} \leq \frac{3C_E-3}{2} - 3j - (C_E - 3(j+1)) = \frac{C_E+3}{2}$. So $\rho_G(V(G)) \leq \rho_{G'}(V(G')) + (\frac{C_E+3}{2})d_G(v) - C_E d_G(v) + C_I = \rho_{G'}(V(G')) + (\frac{3-C_E}{2})d(v) + \frac{C_E-3}{2} \leq \rho_{G'}(V(G')) \leq -3$. Thus, G is not a counterexample.

Finally, suppose there exists $v \in U_k$. Form G' from G by coloring v with I. For each $R' \subseteq V(G')$, subgraph $G'[R']$ has an (I, F_k)-coloring if and only $G[R']$ does. Since G is (I, F_k)-critical, so is G'. Note that $\rho_{G'}(V(G')) = \rho_G(V(G)) - C_{U,k} + C_I > \rho_G(V(G))$. Now repeating the argument in the previous paragraph shows that G is not a smallest counterexample. □

At various points in our proof, we will construct a graph G' from some subgraph of G by adding F-neighbors to one or more vertices. If this ever produces an uncolored vertex v with at least k F-neighbors, then we recolor v with I, as in the final paragraph of the previous proof.

Lemma 7. *For each edge vw, at least one of v and w is in U.*

Proof. Suppose, to the contrary, that $v \in F_i$ and $w \in F_j$. Form G' from G by contracting edge vw to create a new vertex $v * w \in F_{i+j}$. Further, for each vertex x incident to both v and w, remove edges vx and wx and put x into I. Contracting edge vw decreases potential by $(C_{F,i} + C_{F,j} - C_E) - C_{F,i+j} \leq 0$. Putting a vertex x into I and deleting two incident edges decreases potential by at most $C_{U,0} - 2C_E - C_I = -C_E$; that is, it increases potential by at least C_E. Since G' is smaller than G, we have $\rho_{G'}(V(G')) \leq -3$. Thus, $\rho_G(V(G)) \leq \rho_{G'}(V(G')) \leq -3$. So G is not a counterexample. □

Lemma 8. *For each $v \in V(G)$, either $d(v) \geq 2$ or $v \in F_j$ with $j \geq \lfloor \frac{k+3}{2} \rfloor$.*

Proof. Assume, to the contrary, that $d(v) \leq 1$ and $v \notin F_j$ with $j \geq \lfloor \frac{k+3}{2} \rfloor$. Since G is critical, it is connected, so $d(v) = 1$; denote the unique neighbor of v by w. If v is uncolored, then color $G-v$ by the minimality of G. Now extend this coloring to G by coloring v with the color not used on w. So assume, by Lemma 6, that v is precolored F_j for some $j \in \{1, \ldots, \lfloor \frac{k+1}{2} \rfloor\}$. Lemma 7 implies that $w \in U_\ell$ for some ℓ. Form G' from $G - v$ by increasing the number of F-neighbors of w by j. Note that $\rho_G(V(G)) - \rho_{G'}(V(G')) \leq C_{U,\ell} + C_{F,j} - C_E - C_{U,\ell+j} = 0$. (If the new total number of F-neighbors of w is at least k, then we color w with I.) For each $R' \subseteq V(G')$, subgraph $G'[R']$ has an (I, F_k)-coloring if and only $G[R' \cup \{v\}]$ does. Since G is (I, F_k)-critical, so is G'. Since G' is smaller than G, by the minimality of G, we have $\rho_{G'}(V(G')) \leq -3$. However, now $\rho_G(V(G)) \leq \rho_{G'}(V(G') \leq -3$. Thus, G is not a counterexample. □

Recall, from Section 1.3, that the heart of any proof using the potential method is its gap lemmas. Our next definition plays a crucial role in the first of these.

Definition 9. Given $R \subsetneq V(G)$ and an (I, F_k)-coloring φ of $G[R]$, we construct $G' := H(G, R, \varphi)$ as follows; see Figure 3. Let $\overline{R} := V(G) \setminus R$. Let $\nabla(R) := \{v \in R : \exists w \in \overline{R}, vw \in E(G)\}$. To form G' from G, delete R and add two new vertices v_F, v_I, where v_F is precolored F_k and v_I is precolored I. (So $G'[\overline{R}] \cong G[\overline{R}]$.) For each $vw \in E(G)$ with $w \in \overline{R}$, $v \in R$ and $\varphi(v) = F$, add to G' the edge wv_F. For each $vw \in E(G)$ with $w \in \overline{R}$, $v \in R$ and $\varphi(v) = I$, add to G' the edge wv_I. Finally, delete v_F or v_I if it has no incident edges. So $V(G') \subseteq \overline{R} \cup \{v_F, v_I\}$. In each case, let $X := V(G') \setminus \overline{R}$.

G', $H(G, R, \varphi)$

$\overline{R}, \nabla(R)$

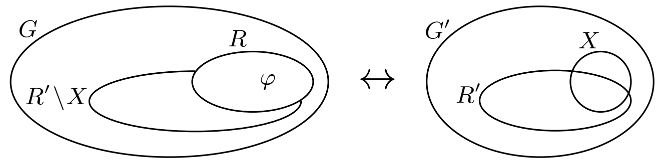

Figure 3: The construction of G' from G, R, and φ in Definition 9, and the vertex subset R' of a critical subgraph of G' in the proof of the Weak Gap Lemma. The picture is nearly identical for the proof of the Strong Gap Lemma.

Lemma 10 (Weak Gap Lemma). *If $R \subsetneq V(G)$ and $|R| \geq 1$, then $\rho(R) \geq 1$.*

Proof. Suppose, to the contrary, that there exists such an R with $\rho(R) \leq 0$. Choose R to minimize $\rho(R)$. By Lemma 6, $F_k = \emptyset$. So each vertex has positive potential. Thus, $|R| \geq 2$ and R induces at least one edge. Since G is critical, $G[R]$ has an (I, F_k)-coloring φ. Let $G' := H(G, R, \varphi)$. If G' has an (I, F_k)-coloring φ', then the union of φ and φ' is an (I, F_k)-coloring of G (since each edge from R to $\overline{R}$ has endpoints with opposite colors). So G' has a critical subgraph G''; let $R' := V(G'')$ (it is possible that some vertices in R' have fewer F-neighbors in G'' than in G'). Note that $|V(G')| \leq |V(G)|$ and $|E(G')| < |E(G)|$; thus, G' is smaller than G. As a result, G'' is smaller than G. Thus, $\rho_{G'}(R') \leq \rho_{G''}(R') \leq -3$. Since $G'[X]$ is edgeless, $\rho_{G'}(X') \geq 0$ for every $X' \subseteq X$. Now

$$\begin{aligned} \rho_G((R' \setminus X) \cup R) &\leq \rho_{G'}(R') - \rho_{G'}(R' \cap X) + \rho_G(R) \\ &\leq -3 + \rho_G(R) \\ &< \rho_G(R). \end{aligned} \tag{1}$$

Since $\rho_G((R' \setminus X) \cup R) < \rho_G(R)$ and we chose R to minimize $\rho_G(R)$, this implies that $(R' \setminus X) \cup R = V(G)$. But now $\rho(V(G)) \leq -3$, so G is not a counterexample. □

The Strong Gap Lemma, which we prove next, is one of the most important lemmas in the paper. Very roughly, the proof mirrors that of the Weak Gap Lemma, but it is much more nuanced, which allows us to prove a far stronger lower bound (one that grows linearly with k).

Lemma 11 (Strong Gap Lemma). *If $R \subsetneq V(G)$ and $G[R]$ contains an edge, then $\rho(R) \geq \frac{C_E - 3}{2}$.*

Before proving the lemma formally, we give a proof sketch. Choose R to minimize $\rho(R)$ R
among $R \subsetneq V(G)$ such that $G[R]$ contains an edge. For the sake of contradiction, assume that $\rho(R) < \frac{C_E-3}{2}$; by integrality, $\rho(R) \leq \frac{C_E-5}{2}$. Let $t := \left\lfloor \frac{\rho(R)+2}{3} \right\rfloor$. Again, by integrality, $3t \geq \rho(R)$. t
By the Weak Gap Lemma, $t \geq 1$.

We essentially repeat the proof of the Weak Gap Lemma, but more carefully. In that proof it was crucial that $\rho_{G'}(V(G') \setminus \overline{R}) \geq \rho_G(R)$. To ensure this now, we will show that $\rho_{G'}(V(G') \setminus \overline{R}) \geq \frac{C_E-5}{2}$. To do this, before using induction to get an (I, F_k)-coloring φ of $G[R]$, we modify $G[R]$ slightly, to get a graph G_R. Denote $\nabla(R)$ by $v_1, \ldots, v_s$. We must ensure that in the coloring φ of $G[R]$ the components colored F containing $v_1, \ldots, v_s$ do not each contain k vertices. Specifically, if $F^1, ..., F^m$ are the F-components of φ containing vertices $v_1, \ldots, v_s$, then we want to maximize $\sum_{j=1}^m (k - |F^j|)$. When constructing G', this will allow us to create vertices v_j that are precolored $F_{|F^j|}$, rather than F_k. When $j \leq \lfloor \frac{k-2}{2} \rfloor$, recall that $C_{F,k-j} = 3j$. Thus, to ensure that $\rho(X) \geq \rho(R)$, it suffices to have $\sum_{j=1}^m (k - |F^j|) \geq t$, since then $\rho(X) \geq \sum_{j=1}^m 3(k - |F^j|) \geq 3t \geq \rho(R)$, as desired.

We construct G_R from $G[R]$ by adding "fake" neighbors precolored F to vertices in $\nabla(R)$; in total, we must add at least t such fake F-neighbors. More formally, we move vertices from F_{a_j} to F_{b_j} where $\sum b_j = t + \sum a_j$. The reason that we can color the resulting graph G_R is that we chose R to minimize $\rho(R)$. In particular, $\rho_G(Y) \ge \rho_G(R)$ for all $Y \subseteq R$ (that induces at least one edge). Thus, $\rho_{G_R}(Y) \ge \rho_G(Y) - 3t \ge \rho_G(R) - 3\lfloor\frac{\rho_G(R)+2}{3}\rfloor \ge -2$. Thus, Y cannot induce a critical graph in G_R or some subgraph of it; so, G_R is colorable. Making all this precise requires more details, which we give below in Case 2.

Proof. We exactly repeat the first paragraph above; in particular, we define R and t as above. Before proceeding to the main case, we handle the easy case that $\rho(\nabla(R)) < \rho(R)$.

Case 1: $\boldsymbol{\rho(\nabla(R)) < \rho(R)}$. By our choice of R, we know that $G[\nabla(R)]$ is edgeless; also $R \setminus \nabla(R) \neq \emptyset$. That is, $\nabla(R)$ is an independent separating set. Moreover, each vertex of $\nabla(R)$ is colored F, since $\min\{C_{U,k-1}, C_I\} \ge \min\{\frac{3k-3}{2}, \frac{C_E-3}{2}\} > \frac{C_E-5}{2} \ge \rho(R)$. Form $\tilde{G}$ from G by moving each vertex of $\nabla(R)$ into F_k. For each $S \subseteq V(G)$ such that $G[S]$ contains an edge, we have $\rho_{\tilde{G}}(S) \ge \rho_G(S) - \rho_G(\nabla(R)) > \rho_G(S) - \rho_G(R) \ge 0$. Furthermore, $\rho_{\tilde{G}}(S) \ge 0$ for each $S \subseteq V(G)$ such that $G[S]$ is edgeless, since each vertex has nonnegative potential. Thus, every proper induced subgraph of $\tilde{G}$ has an (I, F_k)-coloring. Denote the components of $G - \nabla(R)$ by $C^1, C^2, \ldots, C^r$. For each j, by induction we have an (I, F_k)-coloring of $\tilde{G}[C^j \cup \nabla(R)]$. The union of these colorings is a coloring of G, which contradicts that G is a counterexample.

Case 2: $\boldsymbol{\rho(\nabla(R)) \ge \rho(R)}$. Now we show how to form G_R from $G[R]$ so that our (I, F_k)-coloring φ of G_R ensures $\rho_{G'}(V(G') \setminus \overline{R}) \ge \rho_G(R)$. Denote $\nabla(R)$ by $v_1, \ldots, v_s$. First suppose $v_1, \ldots, v_s$
that some v_ℓ is uncolored; say $v_\ell \in U_{p_\ell}$. To form G_R from $G[R]$, we move v_ℓ to $U_{p_\ell+t}$; if $p_\ell + t > k-1$, then we instead move v_ℓ to I. (We leave all other vertices in $\nabla(R)$ unchanged.) Now assume that each $v_j \in \nabla(R)$ is colored F. Say $v_j \in F_{p_j}$ for each $v_j \in \nabla(R)$. We pick nonnegative integers ℓ_j iteratively as follows. Let $\ell_j := \min\{k - p_j, t - \sum_{j''<j} \ell_{j''}\}$. Note that $\rho(\{v_j\}) \le 3(k-p_j)$ for all j. So, if $\sum \ell_j \le t-1$, then $\rho(\nabla(R)) \le 3(t-1) < \rho(R)$; this contradicts the case we are in. Thus, $\sum \ell_j = t$ (also, $\ell_j \ge 0$ for all j). Form G_R from $G[R]$ by moving each v_j into $F_{p_j+\ell_j}$.

We claim G_R has an (I, F_k)-coloring. Since G_R is smaller than G, this will hold by induction once we show that $\rho_{G_R}(R') \ge -2$ for each $R' \subseteq R$. Assume, to the contrary, that $\rho_{G_R}(R') \le -3$, for some R'. Now

$$\rho_G(R') \le \rho_{G_R}(R') + 3t \le -3 + 3t = 3\left(\left\lfloor\frac{\rho_G(R)+2}{3}\right\rfloor - 1\right) < \rho_G(R).$$

By our choice of R, this implies that R' is edgeless. But this contradicts $\rho_{G_R}(R') \le -3$, since each vertex contributes nonnegative potential. Thus, G_R has the desired (I, F_k)-coloring φ.

We construct G' from G, R, and φ as follows. As described above, G' contains $G[\overline{R}]$, to which we add new vertices that we call X. Let $F^1, F^2, \ldots, F^m$ denote the components of F in φ that contain at least one vertex of $\nabla(R)$. For each F^j, let $(k - \ell'_j)$ be the number of vertices in F^j when φ is viewed as a coloring of $G[R]$ (not G_R); when constructing G', add to X a vertex $v_{F,j} \in F_{k-\ell'_j}$. If φ uses I on one or more vertices in $\nabla(R)$, then add to G' a single vertex $v_I \in I$.

Next, we must show that $\rho_{G'}(X) \ge \rho_G(R)$. Recall that X denotes the vertices in G' that are not in G. By construction, $G'[X]$ is edgeless, so $\rho_{G'}(X) = \sum_{v_j \in X} \rho_{G'}(v_j)$. If $v_I \in X$, then $\rho_{G'}(X) \ge \rho_{G'}(\{v_I\}) = C_I = \frac{C_E-3}{2} > \rho_G(R)$, so we are done. Thus, we assume that $v_I \notin X$. Essentially, we want to show that each $v_j \in X \cap F_{k-\ell'_j}$ adds $3\ell'_j$ to $\rho_{G'}(X)$. Since $\sum \ell'_j \ge t$, we get $\rho_{G'}(X) = \sum \rho_{G'}(\{v_j\}) = \sum 3\ell'_j \ge 3t \ge \rho_G(R)$. But there is a small complication.

We only have $\rho_{G'}(\{v_j\}) = 3\ell'_j$ when $\ell'_j \le \lceil\frac{k-3}{2}\rceil$; otherwise $\rho_{G'}(\{v_j\}) = C_E - 3(k - \ell'_j)$, which is $3\ell'_j - 1$ when k is even and $3\ell'_j - 2$ when k is odd. If $\ell'_j \ge \lceil\frac{k-1}{2}\rceil$ for at least two values of j, then $\rho_{G'}(X) \ge 2(C_E - 3(k - \lceil\frac{k-1}{2}\rceil)) \ge \frac{C_E-5}{2} \ge \rho_G(R)$, as desired. So assume that $\ell'_j \ge \lceil\frac{k-1}{2}\rceil$ for at most one value of j. If k is even, then $\rho_G(R) \le \frac{C_E-5}{2} = \frac{3k-6}{2}$, so $t = \lfloor\frac{3k-2}{6}\rfloor = \lfloor\frac{k-2}{2}\rfloor$. Thus, either $\ell'_j \le \lceil\frac{k-1}{2}\rceil$ for each j, or $\sum \ell'_j > t$. In both cases, $\rho_{G'}(X) \ge \rho_G(R)$. Assume instead that k is odd. If $\rho_G(R) < \frac{C_E-5}{2}$, then $t \le \lfloor\frac{k-2}{2}\rfloor$, and the analysis is similar to that above for k even. So we instead assume that $\rho_G(R) = \frac{C_E-5}{2}$ and $\ell'_i = \frac{k-1}{2} = t$ for some i (with

$\ell'_j = 0$ for all other j). But in this case, $\rho_{G'}(X) = 3t-2$ and $\rho_G(R) = \frac{3k-7}{2} = \frac{3k-3}{2} - 2 = 3t-2$. So, again $\rho_{G'}(X) \geq \rho_G(R)$, as desired.

The graph G' is smaller than G, since by construction $|V(G')| \leq |V(G)|$ (equality may be possible if $G[R] \cong K_{1,s-1}$) and $|E(G')| < |E(G)|$, since $G[R]$ contains an edge. Each vertex $v \in \overline{R}$ has at most one neighbor in R since otherwise

$$\rho(R \cup \{v\}) \leq \rho(R) + C_{U,0} - 2C_E \leq \rho(R) - \frac{C_E+3}{2} \leq \frac{C_E-5}{2} - \frac{C_E+3}{2} = -4.$$

If $R \cup \{v\} = V(G)$, then $\rho(V(G)) \leq -4$, which contradicts that G is a counterexample. Otherwise, $R \cup \{v\} \subsetneq V(G)$ and $\rho(R \cup \{v\}) < \rho(R)$, which contradicts our choice of R. So each $v \in \overline{R}$ has at most one neighbor in R. This means that G' does not have an (I, F_k)-coloring, since such a coloring could be combined with φ to produce an (I, F_k)-coloring of G. So G' contains an (I, F_k)-critical subgraph G''. Let $W'' := V(G'')$, and by induction $\rho_{G''}(W'') \leq -3$.

Because G is (I, F_k)-critical (and thus does not contain proper (I, F_k)-critical subgraphs) $W'' \cap X \neq \emptyset$. Since $G'[X]$ is edgeless, $\rho_{G'}(X') \geq 0$ for all $X' \subseteq X$. Let $W := (W'' \setminus X) \cup R$. By submodularity,

$$\rho_G(W) \leq \rho_{G'}(W'') - \rho_{G'}(X \cap W'') + \rho_G(R) \leq (-3) - (0) + \rho_G(R). \tag{2}$$

By our choice of R, this implies that $W = V(G)$. We are then in one of two cases, each of which improves the bound in (2). If $X \subset W''$, then $X \cap W'' = X$, so we use the prior result that $\rho_{G'}(X) \geq \rho_G(R)$ to strengthen (2) and conclude that $\rho_G(V(G)) = \rho_G(W) \leq \rho_{G'}(W'') \leq -3$, which is a contradiction. So assume that $X \setminus W'' \neq \emptyset$. Because $W = V(G)$, we have $\overline{R} \subset W''$. By construction, every vertex in X has a neighbor in $\overline{R}$ in G', and therefore at least one edge with an endpoint in R and the other endpoint in $\overline{R}$ was not accounted for in (2). Thus, (2) improves to $\rho_G(W) \leq \rho_G(R) - 3 - C_E \leq -\frac{C_E+11}{2} < -3$, which is a contradiction. This finishes Case 2, which completes the proof. □

It will be convenient to write U_j^i for the set of vertices with degree i in U_j; similarly for F_j^i. U_j^i, F_j^i
When we do discharging, vertices in U_j^2 will need lots of charge, particularly when j is small. This motivates our next lemma. It says that when j is small enough, such vertices do not exist.

Lemma 12. *If $U_j^2 \neq \emptyset$, then $j \geq \frac{C_E-7}{6}$.*

Proof. Assume, to the contrary, that there exists $j \leq \frac{C_E-9}{6}$ and $v \in U_j^2$. Denote the neighbors of v by v_1 and v_2. Our basic plan is to delete v and add $j+1$ F-neighbors to each of v_1 and v_2; call this new graph G'. We show that G' has an (I, F_k)-coloring φ', and extend φ' to G as follows. If both v_1 and v_2 are colored with F, then color v with I. Otherwise, color v with F. It is easy to see this yields an (I, F_k)-coloring of G, a contradiction. Mainly, we need to show that $\rho_{G'}(R') \geq -2$ for all $R' \subseteq V(G')$, which we do by the Strong Gap Lemma. This proves that G' has the desired (I, F_k)-coloring. We also need to handle the possibility that our construction of G' creates a component of F with more than k vertices.

Case 1: For each $v_i \in N(v)$ either $v_i \in U$ or else $v_i \in F_{\ell_i}$ and $\ell_i + j + 1 \leq k$. We follow the outline above, but need to clarify a few details. If adding $j+1$ F-neighbors to some $v_i \in U$ results in v_i having at least k F-neighbors, then we instead color v_i with I. By design, we do not create any vertices in U with more than $k-1$ F-neighbors or vertices in F-components of order more than k. We also need to check that we do not create any edges with both endpoints colored I. By Lemma 6, no vertex of G is colored I. So we only need to check that we do not use I on both v_1 and v_2 when $v_1v_2 \in E(G)$. Suppose that we do. Assume that $v_1 \in U_{\ell_1}$ and $v_2 \in U_{\ell_2}$. So $\ell_1 + j + 1 \geq k$ and $\ell_2 + j + 1 \geq k$. Now $\rho_G(\{v, v_1, v_2\}) = C_{U,\ell_1} + C_{U,\ell_2} + C_{U,j} - 3C_E = \frac{9C_E-9}{2} - 3(j + \ell_1 + \ell_2) - 3C_E = \frac{3C_E-9}{2} - 3(j + \ell_1 + 1) - 3(\ell_2 - 1) \leq \frac{C_E-9}{2} - 3(\ell_2 - 1) \leq \frac{C_E-9}{2} - 3(k - 2 - \frac{C_E-9}{6}) = C_E - 3 - 3k < -3$. This contradicts the Weak Gap Lemma. Thus, G' has a valid precoloring.

Now we must show that $\rho_{G'}(R') \geq -2$ for all $R' \subseteq V(G')$. If $G[R']$ is edgeless, then clearly $\rho(R') \geq 0$. So assume $G[R']$ has at least one edge. If $R' \cap N(v) = \emptyset$, then $\rho_{G'}(R') = \rho_G(R') \geq 1$, by the Weak Gap Lemma. Instead suppose that $|R' \cap N(v)| = 1$. By the Strong Gap Lemma, $\rho_{G'}(R') \geq \rho_G(R') - 3(j+1) \geq \frac{C_E-3}{2} - 3(j+1) \geq \frac{C_E-3}{2} - 3\frac{C_E-3}{6} = 0$. Finally, suppose that $|R' \cap N(v)| = 2$. Now the Weak Gap Lemma (and the fact that $\rho_G(V(G)) \geq -2$) gives

$$\begin{aligned}
\rho_{G'}(R') &\geq \rho_G(R' \cup \{v\}) + 2C_E - C_{U,j} - 3(j+1)2 \\
&= \rho_G(R' \cup \{v\}) + 2C_E - (\frac{3C_E-3}{2} - 3j) - 6(j+1) \\
&= \rho_G(R' \cup \{v\}) + \frac{C_E+3}{2} - 3j - 6 \\
&\geq \rho_G(R' \cup \{v\}) + \frac{C_E}{2} + \frac{3}{2} - \frac{C_E-9}{2} - 6 \\
&= \rho_G(R' \cup \{v\}) \\
&\geq -2.
\end{aligned}$$

Case 2: There exists $v_i \in N(v)$ such that $v_i \in F_{\ell_i}$ and $j + \ell_i \geq k$. If v_1 and v_2 are both precolored F, then we simply delete v (since we can extend φ' to G by coloring v with I). So, we assume that $v_1 \in F_{\ell_1}$ with $j + \ell_1 \geq k$ and $v_2 \in U_{\ell_2}$. Now we simply delete v and color v_2 with F. We must again ensure that $\rho_{G'}(R') \geq -2$ for all $R' \subseteq V(G')$. If $v_2 \notin R'$, then $\rho_{G'}(R') = \rho_G(R') \geq 1$. So, assume that $v_2 \in R'$. If $G'[R']$ is edgeless, then clearly $\rho_{G'}(R') \geq 0$. So assume that $G'[R']$ has at least one edge. Now, similar to above:

$$\begin{aligned}
\rho_{G'}(R') &\geq \rho_G(R' \cup \{v, v_1\}) + 2C_E - C_{F,\ell_1} - C_{U,j} - C_{U,\ell_2} + C_{F,\ell_2+1} \\
&\geq \rho_G(R' \cup \{v, v_1\}) + 2C_E - 3(k - \ell_1) - (3C_E - 3 - 3(j + \ell_2)) + (C_E - 3(\ell_2 + 1)) \\
&= \rho_G(R' \cup \{v, v_1\}) - 3k + 3\ell_1 + 3j + 3\ell_2 - 3\ell_2 \\
&= \rho_G(R' \cup \{v, v_1\}) - 3k + 3(j + \ell_1) \\
&\geq \rho(G' \cup \{v, v_1\}) \\
&\geq -2.
\end{aligned}$$

□

It will turn out that when $j > \frac{C_E-5}{6}$ vertices in U_j^2 will have nonnegative initial charge. By Lemma 12, we know that $U_j^2 = \emptyset$ when $j < \frac{C_E-7}{6}$. Thus, to finish the proof we focus on the vertices in U_j^2 when $j = \frac{C_E-5}{6}$ (in Section 3, where k is even) and when $j = \frac{C_E-7}{6}$ (in Section 4, where k is odd).

3 Finishing the Proof when k is Even

Throughout this section, k is always even. Recall that when k is even $C_E = 3k - 1$. We let C_E
$\ell := \frac{C_E-5}{6} = \frac{3k-6}{6} = \frac{k}{2} - 1$. ℓ

Lemma 13. *G does not contain adjacent vertices v and w with $v, w \in U_\ell^2$.*

Proof. Assume the lemma is false. Let v' and w' denote the remaining neighbors of v and w, respectively (possibly $v' = w'$). By symmetry between v' and w', we assume that $v' \notin F_j$ with $j \geq k - \ell$ (otherwise $\rho(\{v, w, v', w'\}) \leq 2C_{F,k-\ell} + 2C_{U,\ell} - 3C_E = 6\ell + 2(\frac{3C_E-3}{2} - 3\ell) - 3C_E = -3$, which contradicts the Weak Gap Lemma). Form G' from $G \setminus \{v, w\}$ by adding $\ell + 1$ F-neighbors to v'. If v' now has at least k F-neighbors, then move v' to I. (By our assumption on v', we know that v' is not in an F-component of order at least $k + 1$.)

Fix $R' \subseteq V(G')$. If $G'[R']$ has no edges, then $\rho_{G'}(R') \geq 0$, since each individual vertex has nonnegative potential. If $v' \notin R'$, then $\rho_{G'}(R') = \rho_G(R') \geq 1$, by the Weak Gap Lemma.

Assume instead that $v' \in R'$ and $G[R']$ contains at least one edge. By the Strong Gap Lemma, $\rho_{G'}(R') \geq \rho_G(R') - 3(\ell+1) \geq \frac{C_E-3}{2} - 3(\ell+1) = \frac{C_E-3}{2} - \frac{C_E-5+6}{2} = -2$. Thus, by minimality, G' has an (I, F_k)-coloring φ'.

We extend φ' to v and w as follows. If $\varphi'(v') = I$, then color v with F and color w with the color unused on w'. Similarly, if $\varphi'(w') = I$, then color w with F and color v with the color unused on v'. (If $\varphi'(v') = \varphi'(w') = I$, then v and w lie in an F-component with order $2(\ell) + 2 = \frac{C_E-5}{3} + 2 = 2(\frac{k}{2} - 1) + 2 = k$.) Suppose instead that $\varphi'(v') = \varphi'(w') = F$. Now color w with I and v with F. Note that this is an (I, F_k)-coloring of G, because of the extra F-neighbors of v' in G'. □

Now we use discharging to show that G cannot exist. We define our initial charge function so that our assumption $\rho(V(G)) \geq -2$ gives an upper bound on the sum of the initial charges. (Recall the values of $C_{U,j}$ and $C_{F,j}$ from Definition 2. By Lemma 6, $I = \emptyset$.) Precisely, let

- $\text{ch}(v) := C_E d(v) - 2C_{U,j} = C_E d(v) - 2(\frac{3C_E-3}{2} - 3j)$ $\quad$ ch(v)
  $= C_E(d(v) - 3) + 3 + 6j$ for each $v \in U_j$; and
- $\text{ch}(v) := C_E d(v) - 2C_{F,j} = C_E d(v) - 2(C_E - 3j)$
  $= C_E(d(v) - 2) + 6j$ for each $v \in F_j$ with $j \leq \ell + 1$; and
- $\text{ch}(v) := C_E d(v) - 2C_{F,j} \geq C_E d(v) - 2(3k - 3(\ell+2))$
  $= C_E d(v) - 3k + 6 = C_E(d(v) - 1) + 5$ for each $v \in F_j$ with $j \geq \ell + 2$
  (and this inequality is strict when $j > \ell + 2$).

This definition of ch(v) yields the inequality

$$\sum_{v \in V(G)} \text{ch}(v) = -2\rho(V(G)) \leq 4. \tag{3}$$

$d(v)$	U_0	U_1	F_1	U_ℓ	$U_{\ell+1}$	$U_{\ell+2}$	$F_{\ell+2}$
1							4
2			4	0	2	8	
3	0	6	$C_E + 3$				
4	$C_E - 1$	$C_E + 5$	$2C_E + 2$				

Table 1: Lower bounds on the final charges (when k is even).

We use a single discharging rule, and let $ch^*(v)$ denote the charge at v after discharging. $\quad$ ch$^*(v)$

(R1) Each vertex in U_ℓ^2 takes 1 from each neighbor.

Lemma 14. *After discharging by (R1) above, each vertex v with an entry in Table 1 has $ch^*(v)$ at least as large charge as shown. Each other vertex v has $ch^*(v) \geq 5$.*

Proof. Note that $\text{ch}^*(v) \geq \text{ch}(v) - d(v)$ for all $v \in V(G)$. If $v \in U_j$, then $\text{ch}^*(v) \geq C_E(d(v) - 3) + 3 + 6j - d(v) = (C_E - 1)(d(v) - 3) + 6j$. If $v \in F_j$ and $j \leq \ell + 1$, then $\text{ch}^*(v) \geq C_E(d(v) - 2) + 6j - d(v) = (C_E - 1)(d(v) - 2) + 6j - 2$. If $v \in F_j$ and $j \geq \ell + 2$, then $\text{ch}^*(v) \geq C_E(d(v) - 1) + 5 - d(v) = (C_E - 1)(d(v) - 1) + 4$ (and this inequality is strict when $j > \ell+2$). By Lemma 6, $I = \emptyset$; by Lemma 12, $U_j^2 = \emptyset$ when $j < \ell$. By Lemma 8, each $v \in V(G)$ has $d(v) \geq 2$ unless $v \in F_j^1$ with $j \geq \ell+2$. If $v \in U_{\ell+1}^2$, then $\text{ch}^*(v) \geq -C_E+1+(C_E-5+6) = 2$. Thus, if $v \notin U_\ell^2$, then the lemma follows from what is above.

By Lemma 13, if $v \in U_\ell^2$, then v does not give away any charge. So v finishes with $\text{ch}(v) + 2(1) = -C_E + 3 + 6\ell + 2(1) = -C_E + 5 + (C_E - 5) = 0$. □

Corollary 15. *$V(G) \subseteq U^2_\ell \cup U^2_{\ell+1} \cup U^3_0 \cup U^4_0 \cup F^1_{\ell+2} \cup F^2_1$ (with $U^4_0 = \emptyset$ when $k \geq 4$) and $2|U^2_{\ell+1}| + 4|U^4_0| + 4|F^1_{\ell+2}| + 4|F^2_1| \leq 4$.*

Proof. This follows directly from Lemma 14 and (3). □

Lemma 16. *G has an (I, F_k)-coloring, and is thus not a counterexample.*

Proof. We now construct an (I, F_k)-coloring of G. We color each $v \in U^2_\ell$ with I and each $v \notin U^2_\ell$ with F. By Lemma 13, we know that U^2_ℓ is an independent set. So we only must check that $G - U^2_\ell$ is a forest in which each component has order at most k.

Suppose that $G - U^2_\ell$ contains a cycle, C. Clearly C has no vertex in $U^4_0 \cup F^2_1$, since such a vertex would end with charge at least 6, a contradiction. (Also, C has no vertex in $F^1_{\ell+2}$.) Furthermore, each vertex in $U^2_{\ell+1} \cup U^3_0$ on such a cycle would end with charge at least 2. Since G is simple, C has length at least 3, so its vertices end with charge at least 6, a contradiction. Thus, $G - U^2_\ell$ is acyclic. If $U^4_0 \cup F^1_{\ell+2} \cup F^2_1 \neq \emptyset$, then $U^2_{\ell+1} = \emptyset$ and $|U^4_0 \cup F^1_{\ell+2} \cup F^2_1| = 1$. Furthermore, G is a bipartite graph with U^2_ℓ as one part and $U^3_0 \cup U^4_0 \cup F^1_{\ell+2} \cup F^2_1$ as another (otherwise G has total charge at least 5, a contradiction). So G has an (I, F_k)-coloring using I on U^2_ℓ and F on $U^3_0 \cup U^4_0 \cup F^1_{\ell+2} \cup F^2_1$.

Assume instead that $U^4_0 \cup F^1_{\ell+2} \cup F^2_1 = \emptyset$. Recall that $G - U^2_\ell$ is a forest. Let T denote a component of this forest, let $n_2 := |U^2_{\ell+1} \cap V(T)|$, and let $n_3 := |U^3_0 \cap V(T)|$. The number of edges incident to T is $(\sum_{v \in V(T)} d(v)) - 2|E(T)| = 2n_2 + 3n_3 - 2(n_2 + n_3 - 1) = n_3 + 2$. Recall that T gives away 1 along each such edge. Each vertex counted by n_3 begins with 3, and each vertex counted by n_2 begins with 4. Thus the total final charge of vertices of T is $4n_2 + 3n_3 - (n_3 + 2) = 4n_2 + 2n_3 - 2$. Since G has total charge at most 4, either $n_2 = 1$ and $n_3 \leq 1$ or else $n_2 = 0$ and $n_3 \leq 3$. Now color all vertices of T with F, except when $n_2 = 0$, $n_3 = 3$, and $k = 2$. In that case, the total final charge of T is 4, so every other component of $G - U^2_\ell$ is an isolated vertex in U^3_0. Now color the leaves of T with F and the center vertex, say v, with I. Also recolor the neighbor of v outside of T with F. □

4 Finishing the Proof when k is Odd

4.1 Reducible Configurations when k is Odd

Throughout this section, k is always odd. Recall that when k is odd $C_E = 3k - 2$. Further, C_E
let $\ell := \frac{C_E - 7}{6} = \frac{3k-9}{6} = \frac{k-3}{2}$. (Note that C_E and ℓ are defined differently from the previous ℓ
section.) We will frequently use the fact that $2\ell + 3 = k$.

Lemma 17. *G does not contain adjacent vertices v and w with $v \in U^2_\ell$ and $w \in U^2_\ell \cup U^2_{\ell+1}$.*

Proof. Assume the lemma is false. Let v' and w' denote the remaining neighbors of v and w, respectively (possibly $v' = w'$). Form G' from $G \setminus \{v, w\}$ by adding $\ell + 1$ F-neighbors to v'. (Suppose this puts v' in an F-component of order at least $k+1$. In this case, $\rho(\{v', v, w\}) \leq C_{F,k-\ell} + C_{U,\ell} + C_{U,\ell+1} - 2C_E = 3\ell + (\frac{3C_E-3}{2} - 3\ell) + (\frac{3C_E-3}{2} - 3(\ell+1)) - 2C_E = 3C_E - 3 - 2C_E - 3(\ell+1) = C_E - 3 - 3(\frac{C_E-7}{6} + 1) = \frac{C_E-5}{2}$, which contradicts the Strong Gap Lemma. So v' is not in an F-component of order at least $k+1$.)

Now we show that $\rho_{G'}(R') \geq -2$ for all $R' \subseteq V(G')$. Fix some $R' \subseteq V(G')$. If $v' \notin R'$, then $\rho_{G'}(R') = \rho_G(R') \geq 1$, by the Weak Gap Lemma. If $G'[R']$ has no edges, then $\rho_{G'}(R') \geq 0$, since each coefficient in Definition 2 is nonnegative. Assume instead that $v' \in R'$ and $G[R']$ has at least one edge. By the Strong Gap Lemma, $\rho_{G'}(R') \geq \rho_G(R') - 3(\ell+1) \geq \frac{C_E-3}{2} - 3(\ell+1) = \frac{C_E-3}{2} - \frac{C_E-7+6}{2} = -1$. Thus, G' has an (I, F_k)-coloring φ'.

We extend φ' to v and w as follows. If $\varphi'(v') = I$, then color v with F and color w with the color unused on w'. Similarly, if $\varphi'(w') = I$, then color w with F and color v with the color unused on v'. (If $\varphi'(v') = \varphi'(w') = I$, then v and w lie in an F-component with order at most $2(\ell) + 3 = \frac{C_E-7}{3} + 3 = \frac{3k-9}{3} + 3 = k$.) Suppose instead that $\varphi'(v') = \varphi'(w') = F$. Now color w with I and v with F. Note that this is an (I, F_k)-coloring of G, because of the extra F-neighbors of v' in G'. $\square$

Lemma 18. *G does not contain a vertex $v \in U_0^3$ with all three neighbors in U_ℓ^2.*

Proof. Suppose the lemma is false. Form G' from G by deleting v and its three 2-neighbors. Since G is critical, G' has an (I, F_k)-coloring φ'. Now we extend φ' to all of G. Color each 2-neighbor of v with the color unused on its neighbor in G'. If all three 2-neighbors of v are colored F, then color v with I. Otherwise, color v with F. This produces an (I, F_k)-coloring of G (because $2\ell + 3 = k$). $\square$

Lemma 19. *G does not contain adjacent vertices $v, w \in U_0^3$ such that v has two neighbors in U_ℓ^2 and w has at least one neighbor in U_ℓ^2.*

Proof. Suppose the lemma is false. Denote the 2-neighbors of v by x and y, and denote a 2-neighbor of w in U_ℓ^2 by z. Denote by w', x', y', and z' the remaining neighbors of w, x, y, and z (other than v, w, and z); see Figure 4. We want to form G' from G by deleting y and contracting both edges incident to z; however, this creates parallel edges when $w'z' \in E(G)$, so we consider two cases. Before doing that, we briefly consider the possibility that $y = z$.

If $y = z$, then by criticality we color $G - \{v, w, x, y/z\}$. To extend the coloring to G, we color w with the color unused on w' and color x with the color unused on x'. If both w and x are colored F, then we color v with I; otherwise, we color v with F. Finally, if both v and w are colored F, then we color y/z with I; otherwise, we color y/z with F. It is easy to check that this coloring has no cycle colored F and no edge with both endpoints colored I. It also has no F-component of size larger than $2\ell + 3 = k$. Thus, we assume $y \neq z$.

Case 1: $\boldsymbol{w'z' \notin E(G)}$. Form G' from G by deleting y and contracting both edges incident to z; the new vertex $w{*}z'$ formed from w and z' inherits the precoloring of z'.

Consider $R' \subseteq V(G')$. If $w * z' \notin R'$, then $\rho_{G'}(R') = \rho_G(R') \geq 1$, by the Weak Gap Lemma. If $G'[R']$ has no edges, then $\rho_{G'}(R') \geq 0$, since each individual vertex has nonnegative potential. So assume that $w * z' \in R'$ and $G'[R']$ has at least one edge. Now

$$\begin{aligned}
\rho_{G'}(R') &= \rho_G((R' \setminus \{w * z'\}) \cup \{w, z, z'\}) - C_{U,\ell} - C_{U,0} + 2C_E \\
&= \rho_G((R' \setminus \{w * z'\}) \cup \{w, z, z'\}) - (3C_E - 3 - 3\ell) + 2C_E \\
&= \rho_G((R' \setminus \{w * z'\}) \cup \{w, z, z'\}) - C_E + 3 + \frac{C_E - 7}{2} \\
&= \rho_G((R' \setminus \{w * z'\}) \cup \{w, z, z'\}) - \frac{C_E + 1}{2} \\
&\geq -2,
\end{aligned}$$

where the final inequality holds because the Strong Gap Lemma gives $\rho_G(R' \setminus \{w * z'\}) \cup \{w, z, z'\}) \geq \frac{C_E-3}{2}$. Thus, G' has an (I, F_k)-coloring φ'.

Case 2: $\boldsymbol{w'z' \in E(G)}$. Again form G' from G by deleting y and contracting both edges incident to z; the new vertex $w * z'$ formed from w and z' inherits the precoloring of z'. Since $w'z' \in E(G)$, this creates parallel edges between w' and $w * z'$. If one of w' and $w * z'$ is colored with F, then delete both of the parallel edges and color the other endpoint with I. (By Lemma 7, at least one of w' and z' is not colored F.) If neither w' nor $w * z'$ is colored with F, then we delete one edge between w' and $w * z'$ and add $\frac{k-1}{2}$ F-neighbors to each of them. (It is not possible that each of w' and $w * z'$ ends with at least k F-neighbors, so gets recolored I, since in that case $\rho_G(\{w, z, w', z'\})$ violates the Strong Gap Lemma.)

Now we must show that $\rho_{G'}(R') \geq -2$ for all $R' \subseteq V(G')$. If $R' \cap \{w', w*z'\} = \emptyset$, then $\rho_{G'}(R') = \rho_G(R') \geq 1$ by the Weak Gap Lemma. So, we assume that $R' \cap \{w', w*z'\} \neq \emptyset$. We will compute $\rho_{G'}(R') - \rho_G((R' \setminus \{w*z'\}) \cup \{w, z, w', z'\})$. For convenience, let $\alpha := -2C_{U,0} - C_{U,\ell} + C_I + 3C_E$. We have 5 cases to consider.

1. We added F-neighbors to both w' and $w*z'$ and $|R' \cap \{w', w*z'\}| = 2$. Now $\rho_{G'}(R') - \rho_G((R' \setminus \{w*z'\}) \cup \{w, z, w', z'\}) = -C_{U,0} - C_{U,\ell} + 3C_E - 3(k-1) = \alpha + C_{U,0} - C_I - 3(k-1) = \alpha + \frac{3C_E-3}{2} - \frac{C_E-3}{2} - (C_E - 1) = \alpha + 1$.
2. We added F-neighbors to both w' and $w*z'$ and $|R' \cap \{w', w*z'\}| = 1$. Now $\rho_{G'}(R') - \rho_G((R' \setminus \{w*z'\}) \cup \{w, z, w', z'\}) \geq -2C_{U,0} - C_{U,\ell} + 4C_E - \frac{3k-3}{2} = \alpha + C_E - C_I - \frac{3k-3}{2} = \alpha + C_E - \frac{C_E-3}{2} - \frac{C_E-1}{2} = \alpha + 2$.
3. We moved w' or $w*z'$ to I and $|R' \cap \{w', w*z'\}| = 2$. Now $\rho_{G'}(R') - \rho_G((R' \setminus \{w*z'\}) \cup \{w, z, w', z'\}) \geq -2C_{U,0} - C_{U,\ell} + C_I + 3C_E = \alpha$.
4. We moved w' or $w*z'$ to I and R' contains the one we moved to I, but not the other. Now $\rho_{G'}(R') - \rho_G((R' \setminus \{w*z'\}) \cup \{w, z, w', z'\}) \geq -2C_{U,0} - C_{U,\ell} - C_{F,1} + C_I + 4C_E = \alpha - C_{F,1} + C_E \geq \alpha + 3$.
5. We moved w' or $w*z'$ to I and R' contains the one we did not move to I, but not the other. Now $\rho_{G'}(R') - \rho_G((R' \setminus \{w*z'\}) \cup \{w, z, w', z'\}) \geq -2C_{U,0} - C_{U,\ell} + 4C_E = \alpha - C_I + C_E > \alpha$.

Note that $\alpha = -2C_{U,0} - C_{U,\ell} + 3C_E + C_I = -3C_E + 3 - (C_E + 2) + 3C_E + \frac{C_E-3}{2} = -\frac{C_E+1}{2}$. Now, by the Strong Gap Lemma, $\rho_{G'}(R') \geq \rho_G((R' \setminus \{w*z'\}) \cup \{w, z, w', z'\}) - \frac{C_E+1}{2} = \frac{C_E-3}{2} - \frac{C_E+1}{2} = -2$. Thus, G' again has an (I, F_k)-coloring φ'.

We will show how to extend φ' to G (after possibly modifying it a bit). We first extend φ' to an (I, F_k)-coloring φ of $G - y$ by uncontracting the two edges incident to z, coloring both w and z' with $\varphi'(w*z')$, and coloring z with the opposite color.

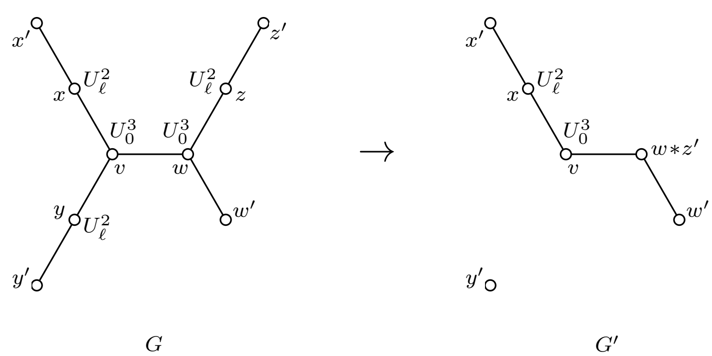

Figure 4: Forming G' from G in the proof of Lemma 19.

Suppose that $\varphi(y') = I$. If $\varphi(v) = I$, then we color y with F and are done. So assume $\varphi(v) = F$. If $\varphi(w) = \varphi(x) = I$, then we again color y with F and are done. If $\varphi(w) = \varphi(x) = F$, then we recolor v with I and are done as above. So assume that exactly one of w and x uses I in φ and the other uses F. First suppose that $\varphi(w) = I$ and $\varphi(x) = F$. If $\varphi(x') = I$, then we color y with F and are done. Instead assume that $\varphi(x') = F$. Now we recolor x with I and color y with F. Thus, we assume instead that $\varphi(w) = F$ and $\varphi(x) = I$. If both neighbors of w other than v are colored I, then we color y with F and are done. So assume that z is the only neighbor of w colored I. Let s_1 and s_2 denote the orders of the F-components of φ that contain w and z', respectively. If $s_1 \leq k - (\ell + 1)$, then we color y with F. If $s_2 \leq k - (\ell + 1)$, then we recolor z with F, recolor w with I, and color y with F. The key observation is that one of these two inequalities must hold. Suppose not. The F-component in φ' containing $w * z$ shows that $k \geq s_1 + s_2 - 1$. If both inequalities above fail, then $k \geq s_1+s_2-1 \geq (k-(\ell+1)+1)+(k-(\ell+1)+1)-1 = 2k-2\ell-1 = 2k-(k-3)-1 = k+2$, which is a contradiction.

Suppose instead that $\varphi(y') = F$. If $\varphi(v) = F$, then we color y with I and are done. Assume instead that $\varphi(v) = I$. First suppose w' and z are colored I. Now recolor v with F and color y with I; finally, if x' is colored F, then recolor x with I. This gives an (I, F_k)-coloring of G. Suppose instead that w' is colored F. Let s_1 and s_2 denote the orders of the F-components of φ that contain w and z', respectively. Suppose that $s_1 \leq k-(\ell+2)$. Color y with I, recolor v with F, and if $\varphi(x') = F$, then recolor x with I. This gives an (I, F_k)-coloring of G. Suppose instead that $s_2 \leq k-(\ell+1)$. Again color y with I, recolor v with F, and if $\varphi(x') = F$, then recolor x with I. Finally, recolor w with I and recolor z with F. Again, this gives an (I, F_k)-coloring of G. The key observation is that one of these two inequalities must hold; the proof is identical to that in the previous paragraph, except that the first inequality is tighter by 1. □

4.2 Discharging when k is Odd

Now we use discharging to show that G cannot exist. It is helpful to remember that $I = \emptyset$, by Lemma 6, and $U_j^2 = \emptyset$ when $j < \ell$, by Lemma 12. Furthermore, by Lemma 8, each $v \in V(G)$ satisfies $d(v) \geq 2$ unless $v \in F_j$ with $j \geq \frac{k+3}{2}$. We define our initial charge function so that our assumption $\rho(V(G)) \geq -2$ gives an upper bound on the sum of the initial charges. (Recall the values of $C_{U,j}$ and $C_{F,j}$ from Definition 2.) Precisely, let

- $\mathrm{ch}(v) := C_E d(v) - 2C_{U,j} = C_E d(v) - 2(\frac{3C_E-3}{2} - 3j)$ ch(v)
  $= C_E(d(v)-3) + 3 + 6j$ for each $v \in U_j$; and
- $\mathrm{ch}(v) := C_E d(v) - 2C_{F,j} = C_E d(v) - 2(C_E - 3j)$
  $= C_E(d(v)-2) + 6j$ for each $v \in F_j$ with $j \leq \frac{k+1}{2}$; and
- $\mathrm{ch}(v) := C_E d(v) - 2C_{F,j} \geq C_E d(v) - 2(3k - \frac{3k+9}{2})$
  $= C_E d(v) - 3k + 9 = C_E(d(v)-1) + 7$ for each $v \in F_j$ with $j \geq \frac{k+3}{2}$.

This definition of $\mathrm{ch}(v)$ yields the inequality

$$\sum_{v \in V(G)} \mathrm{ch}(v) = -2\rho(V(G)) \leq 4. \tag{4}$$

$d(v)$	U_0	U_1	U_2	F_1	F_2	U_ℓ	$U_{\ell+1}$	$U_{\ell+2}$
2				2	8	0	0	4
3	0	3	9	C_E	C_E+6			
4	C_E-5	C_E+1	C_E+7	$2C_E-2$				

Table 2: Lower bounds on the final charges (when k is odd).

We use two discharging rules, and let $ch^*(v)$ denote the charge at v after discharging. ch$^*(v)$

(R1) Each $v \in U_\ell^2$ (2-vertex) takes 2 from each neighbor.

(R2) Each $v \in U_0^3$ (3-vertex) with two neighbors in U_ℓ^2 takes 1 from its other neighbor.

Lemma 20. *After discharging with rules (R1) and (R2) above, each vertex v with an entry in Table 2 has $ch^*(v)$ at least as large charge as shown. Each other vertex v has $ch^*(v) \geq 5$.*

Proof. Note that $\mathrm{ch}^*(v) \geq \mathrm{ch}(v) - 2d(v)$ for all $v \in V(G)$. If $v \in U_j$, then $\mathrm{ch}^*(v) \geq C_E(d(v)-3) + 3 + 6j - 2d(v) = (C_E-2)(d(v)-3) + 6j - 3$. If $v \in F_j$ and $j \leq \frac{k+1}{2}$, then $\mathrm{ch}^*(v) \geq$

$C_E(d(v)-2)+6j-2d(v) = (C_E-2)(d(v)-2)+6j-4$. If $v \in F_j$ and $j \geq \frac{k+3}{2}$, then $\mathrm{ch}^*(v) \geq C_E(d(v)-1)+7-2d(v) = (C_E-2)(d(v)-1)+5$. If $v \notin U^2_\ell \cup U^2_{\ell+1} \cup U^3_0$, then the lemma follows from what is above.

If $v \in U^2_\ell$, then v has no neighbors in $U^2_\ell \cup U^2_{\ell+1}$, by Lemma 17. Thus, $\mathrm{ch}^*(v) = -4+2(2) = 0$. If $v \in U^2_{\ell+1}$, then v has no neighbors in U^2_ℓ, by Lemma 17. Thus, $\mathrm{ch}^*(v) \geq 2-2(1) = 0$. Finally, suppose that $v \in U^3_0$. By Lemma 18, v does not have three neighbors in U^2_ℓ. A vertex in U^3_0 is *needy* if it has two neighbors in U^2_ℓ. By Lemma 19, a vertex in U^3_0 cannot have both a neighbor needy in U^2_ℓ and a needy 3-neighbor. Thus, we have $\mathrm{ch}^*(v) \geq \min\{3-2, 3-2(2)+1, 3-3(1)\} = 0$. □

Corollary 21. *$V(G) = U^2_\ell \cup U^2_{\ell+1} \cup U^2_{\ell+2} \cup F^2_1 \cup U^3_0 \cup U^3_1 \cup U^4_0$. Furthermore $4|U^2_{\ell+2}|+2|F^2_1|+3|U^3_1|+(C_E-5)|U^4_0| \leq 4$. (In particular, $U^4_0 = \emptyset$ when $k \geq 5$.)*

Proof. This corollary follows directly from Lemma 20 and (4). □

If we knew that $\sum_{v\in V(G)} \mathrm{ch}(v) < 0$, then Lemma 20 would yield a contradiction. However, we only know that $\sum_{v\in V(G)} \mathrm{ch}(v) \leq 4$, so we are not done yet. We will now try to construct the desired coloring. We show that we can do this unless $\sum_{v\in V(G)} \mathrm{ch}(v) > 4$, which gives the desired contradiction. Our basic plan is to color all of U^2_ℓ with I. This will force all neighbors of U^2_ℓ into F. Furthermore, all but a constant number of vertices in $V(G) \setminus U^2_\ell$ will go into F. To do this, we consider the components of $G \setminus U^2_\ell$. All but a constant number of these have size at most 4, and all have size at most 8.

Lemma 22. *Each component of $G \setminus U^2_\ell$ is one of the 30 shown below in Figures 5-9, and has final charge as shown. (The coloring of vertices as black and white can be ignored for now.)*

Proof. Let J be a component of $G \setminus U^2_\ell$. Let $\mathrm{ch}^*(J) := \sum_{v\in V(J)} \mathrm{ch}^*(v)$. We will prove that if J is some component other than one of those shown, then either G contains a reducible configuration or $\mathrm{ch}^*(J) > 4$; both possibilities yield a contradiction.

Case 1: $V(J) \cap U^4_0 \neq \emptyset$. (By Corollary 21, this is possible only when $k = 3$.) Assume $v \in V(J) \cap U^4_0$. If $V(J) = \{v\}$, then we are done. Otherwise, $\mathrm{ch}^*(v) \geq 3$. So, by Table 2, we know $V(J) \setminus \{v\} \subseteq U^3_0 \cup U^2_{\ell+1}$. Let w be a neighbor of v in J. If $w \in U^2_{\ell+1}$, then $\mathrm{ch}^*(J) \geq \mathrm{ch}^*(v)+\mathrm{ch}^*(w) \geq 4+1$, a contradiction. The same is true if $w \in U^3_0$ unless w is needy (recall that w cannot have both a neighbor in U^2_ℓ and a needy 3-neighbor, by Lemma 19). If v has at most two needy 3-neighbors, then we are done. Otherwise, $\mathrm{ch}^*(v) \geq 5$, a contradiction.

Case 2: $V(J) \cap U^2_{\ell+2} \neq \emptyset$. Assume $v \in V(J) \cap U^2_{\ell+2}$. If $V(J) = \{v\}$, then we are done. Otherwise, $\mathrm{ch}^*(v) \geq 5$, a contradiction.

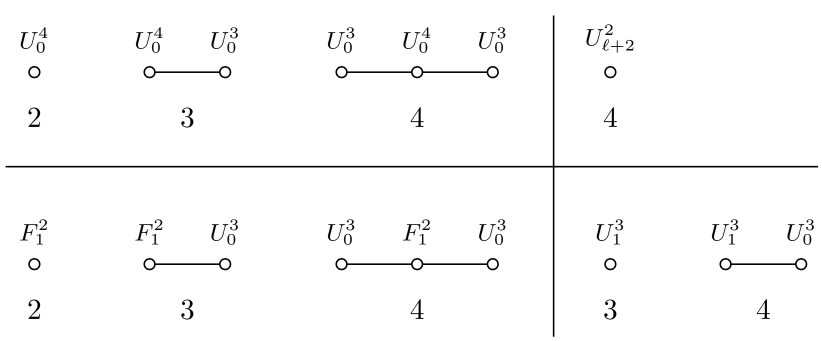

Figure 5: The 9 possible components of $G \setminus U^2_0$ in Cases 1–4.

Case 3: $V(J) \cap F^2_1 \neq \emptyset$. Assume $v \in V(J) \cap F^2_1$. If $V(J) = \{v\}$, then we are done. Otherwise, let w be a neighbor of v in J. If $w \in U^2_{\ell+1}$, then $\mathrm{ch}^*(J) \geq \mathrm{ch}^*(v)+\mathrm{ch}^*(w) \geq 4+1$, a contradiction. Thus, we must have $w \in U_0^3$. If w is not needy, then $\mathrm{ch}^*(v) + \mathrm{ch}^*(w) \geq 4 + 1$, a contradiction. Thus, v has one or two needy neighbors (and this is all of J).

Case 4: $V(J) \cap U_1^3 \neq \emptyset$. Assume $v \in V(J) \cap U_1^3$. If $V(J) = \{v\}$, then we are done. Otherwise, let w be a neighbor of v. If w is not a needy 3-neighbor of v, then $\mathrm{ch}^*(v) \geq 5$, a contradiction. Further, v has at most one needy 3-neighbor. Thus, we are done.

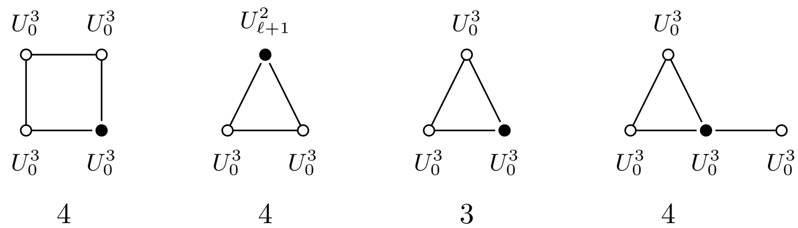

Figure 6: The 4 possible components of $G \setminus U_\ell^2$ in Case 5, those that have a cycle.

Case 5: $V(J) \subseteq U_{\ell+1}^2 \cup U_0^3$ and J contains a cycle. Let C be a cycle in J; see Figure 6. It is easy to check that each cycle vertex finishes with charge at least 1; thus $|C| \leq 4$. If $|C| = 4$, then each cycle vertex is in U_0^3 and has a neighbor in U_ℓ^2. Thus, $J \cong C_4$. Now suppose $|C| = 3$. If C contains a vertex in $U_{\ell+1}^2$, then it contains exactly one such vertex, and its other two vertices are in U_0^3, each with a neighbor in U_ℓ^2. So $J \cong C_3$ (with a single vertex in $U_{\ell+1}^2$). So assume C is a 3-cycle with all vertices in U_0^3. If no vertex on C has a neighbor in $J \setminus C$, then we are done. Otherwise, exactly one cycle vertex does, and it is a needy 3-neighbor.

Case 6: $V(J) \subseteq U_{\ell+1}^2 \cup U_0^3$ and J is a tree. Let $T := J$. Let $n_2 := |U_{\ell+1}^2 \cap V(T)|$ and $n_3 := |U_0^3 \cap V(T)|$. Recall that no vertex in $U_{\ell+1}^2$ has a neighbor in U_ℓ^2, by Lemma 17. So each leaf of T is in U_0^3. Form T' from T by replacing paths with internal vertices in $U_{\ell+1}^2$ by edges. So $|V(T')| = n_3$. Let $\mathrm{ch}^*(T') := \mathrm{ch}^*(T) - 2|U_{\ell+1}^2 \cap V(T)|$. Note that $\mathrm{ch}^*(T')$ is precisely the sum of charges that would have ended on T' if it had appeared in G when we did the discharging. Since each vertex of T' has degree 3 in G, the number of edges (externally) incident to T is $3|T'| - \sum_{v \in T'} d_{T'}(v) = 3|T'| - 2(|T'| - 1) = |T'| + 2$. Since $\mathrm{ch}(T') = 3|T'|$, and T' sends 2 along each incident edge, we have $\mathrm{ch}^*(T') = 3|T'| - 2(|T'| + 2) = |T'| - 4 = n_3 - 4$. Since $\mathrm{ch}^*(T) \leq 4$, we get that $n_3 \leq 8$. Recall that a vertex in U_0^3 with three neighbors in U_ℓ^2 is reducible, by Lemma 18. So $n_3 \geq 2$. Note that $\mathrm{ch}^*(T) = n_3 - 4 + 2n_2 \leq 4$. So $n_2 \leq \frac{8-n_3}{2}$. For brevity, we henceforth denote $|V(T')|$ by $|T'|$. We consider the seven possibilities when $|T'| \in \{2, \ldots, 8\}$.

Suppose $|T'| = 2$. By Lemma 19, the edge of T' must be subdivided in T by one or more vertices of $U_{\ell+1}^2$. We have $n_2 \leq 3$, which gives the 3 possibilities in Figure 7.

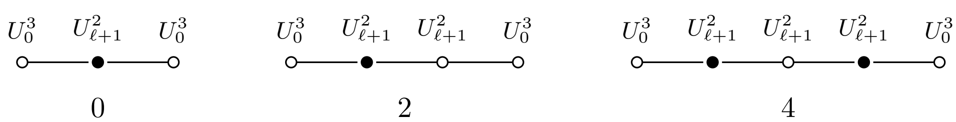

Figure 7: The 3 possible components of $G \setminus U_0^2$ in Case 6 when $|T'| = 2$.

Suppose $|T'| = 3$. By Lemma 19, no vertex in U_0^3 has both a needy 3-neighbor and a neighbor in U_ℓ^2. Thus, each edge of T' must be subdivided in T by a vertex in $U_{\ell+1}^2$. Recall that $n_2 \leq \frac{8-n_3}{2}$. So $n_3 = 3$ and $n_2 = 2$.

Suppose $|T'| = 4$. The only 4-vertex trees are $K_{1,3}$ and P_4. Recall that $n_2 \leq \frac{8-n_3}{2} = 2$. If $T' \cong P_4$, then T must contain a vertex in $U_{\ell+1}^2$ incident to each leaf. There is a unique such tree, a 6-vertex path with each neighbor of a leaf in $U_{\ell+1}^2$ (and the four other vertices in U_0^3). So assume $T' \cong K_{1,3}$. Now $n_2 \in \{0, 1, 2\}$. This results in 1, 1, and 2 possibilities with orders 4, 5, and 6.

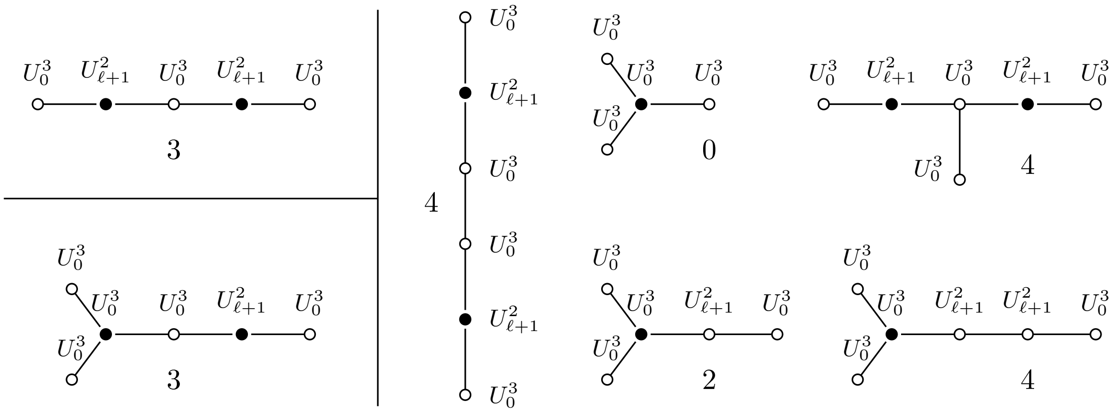

Figure 8: The 7 possible components of $G \setminus U_\ell^2$ in Case 6 when $|T'| \in \{3, 4, 5\}$.

Suppose $|T'| = 5$. The only 5-vertex subcubic trees are P_5 and $K_{1,3}$ with an edge subdivided. Now $n_2 \le 1$. Thus, we cannot have $T' \cong P_5$, since then T would have a vertex in U_0^3 with both a needy 3-vertex and a neighbor in U_ℓ^2, which contradicts Lemma 19. So T' is formed from $K_{1,3}$ by subdividing a single edge. Now we have a single possibility for T, which is formed from $K_{1,3}$ by subdividing a single edge twice.

Suppose $|T'| = 6$. Now $n_2 \le 1$. There are 4 subcubic trees on 6 vertices. However, two of them contain two copies of a leaf adjacent to a vertex of degree 2 (in the tree). Neither of these are valid options for T', by Lemma 19. Thus, either T' is formed by subdividing a single edge of $K_{1,3}$ twice or else T' is a double-star (adjacent 3-vertices, with 4 leaves). The first option yields one case, and the second yields 3 cases (since we might not add a vertex of $U_{\ell+1}^2$).

Suppose $|T'| = 7$. Now $n_2 = 0$; that is, $T' = T$. Thus, each leaf of T' must be adjacent to a vertex of degree 3 in T'. Since T' has a 3-vertex, it has at least 3 leaves. Since each leaf has a neighbor of degree 3 in T, tree T has at least two 3-vertices. There is a single possibility.

Suppose $|T'| = 8$. The analysis is nearly the same as when $|T'| = 7$. Now T' must contain at least 4 leaves and at least two 3-vertices. Either T' has 5 leaves and three 3-vertices or else T' has 4 leaves, two 2-vertices, and two 3-vertices. Each case gives a single possibility. $\square$

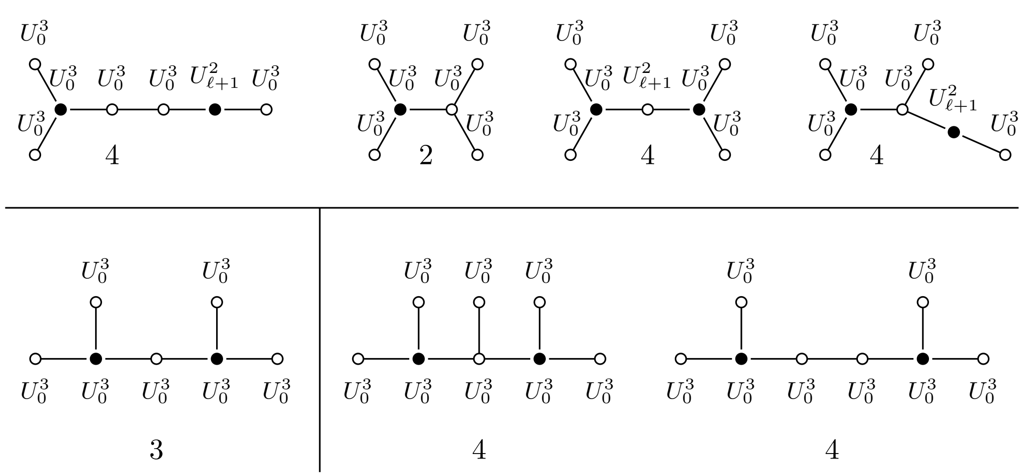

Figure 9: The 7 possible components of $G \setminus U_\ell^2$ in Case 6 when $|T'| \in \{6, 7, 8\}$.

Lemma 23. *G has an (I, F_k)-coloring, and is thus not a counterexample.*

Proof. We now construct an (I, F_k)-coloring of G. As we described above, our plan is to color all vertices of U_ℓ^2 with I (since they form an independent set, by Lemma 17). For each possible acyclic component J of $G \setminus U_\ell^2$, shown in Figures 5, 7, 8, and 9, we show how to extend this coloring to J. Those vertices drawn as white are colored with F and those drawn as black are colored with I. Doing this preserves that I is an independent set and $G[F]$ is a forest with at most k vertices in each component. The only complication is the four possible components J that contain a cycle, shown in Figure 6. In fact, the second and fourth of these are fine. Suppose instead that $J \in \{C_3, C_4\}$ with all vertices in U_0^3. Now we color one vertex v of J with I (and the rest with F). To preserve that I is an independent set, we recolor the neighbor w of v in U_ℓ^2 with F. We must ensure that w does not become part of a tree on $k+1$ vertices. Since $\text{ch}^*(J) \geq 3$, every other component J' of $G \setminus U_\ell^2$ has $\text{ch}^*(J') \leq 1$; in particular, this is true of the component containing the neighbor of w other than v. So J' is either $K_{1,3}$ (with all vertices in U_0^3) or else P_3 (with its center vertex in $U_{\ell+1}^2$ and leaves in U_0^3). In each case for J', the subgraph induced by its vertices colored F is an independent set. Thus, recoloring w with F creates a tree colored F with at most 2 vertices. □

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