INVARIANT CHAINS IN ALGEBRA AND DISCRETE GEOMETRY

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ABSTRACT. We relate finite generation of cones, monoids, and ideals in increasing chains (the *local* situation) to equivariant finite generation of the corresponding limit objects (the *global* situation). For cones and monoids there is no analog of Noetherianity as in the case of ideals and we demonstrate this in examples. As a remedy, we find local-global correspondences for finite generation. These results are derived from a more general framework that relates finite generation under closure operations to equivariant finite generation under general families of maps. We also give a new proof that non-saturated Inc-invariant chains of ideals stabilize, closing a gap in the literature.

1. INTRODUCTION

Finite generation of algebraic and geometric objects is a central necessity to efficiently work with these objects and to represent them in a computer. A well-known and important finiteness principle in algebra is Noetherianity. A commutative ring R is Noetherian if every ideal $I \subseteq R$ is finitely generated, or equivalently, every ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of ideals eventually stabilizes, that is, from some index on, all $\subseteq$ are equalities.

In some cases symmetry can augment finiteness. E.g., the polynomial ring $K[x_1,\dots,x_n]$ over a field K is Noetherian, but $K[x_1,x_2,\dots]$ is not, since $\langle x_1\rangle \subsetneq \langle x_1,x_2\rangle \subsetneq \cdots$ is an infinite ascending chain. Yet, polynomials $f \in K[x_1,x_2,\dots]$ have finitely many terms, so each is contained in some Noetherian subring $K[x_1,\dots,x_n] \subseteq K[x_1,x_2,\dots]$. This finiteness can be systematically investigated, for example, by exploiting the action of symmetric groups, which renumber the indeterminates. The chain $\langle x_1\rangle \subsetneq \langle x_1,x_2\rangle \subsetneq \cdots$ has the property that its n-th ideal $\langle x_1,\dots,x_n\rangle$ arises from the first $\langle x_1\rangle$ by an action of the symmetric group $\mathrm{Sym}(n)$ and ideal closure: $\langle x_1,\dots,x_n\rangle = \langle \mathrm{Sym}(n)(\langle x_1\rangle)\rangle$. A theorem of Cohen [7] and Aschenbrenner-Hillar [1] states that if a chain of ideals $(I_n)_n$ with $I_n \subseteq K[x_1,\dots,x_n]$ is Sym-*invariant* in the sense that $\langle \mathrm{Sym}(n+k)(I_n)\rangle \subseteq I_{n+k}$ for all $n,k \in \mathbb{N}$, then eventually the chain stabilizes, in the sense that for large enough n, $\langle \mathrm{Sym}(n+k)(I_n)\rangle = I_{n+k}$ for all $k \in \mathbb{N}$. Additionally, the union $I_\infty := \bigcup I_n \subseteq K[x_1,x_2,\dots]$ is generated by finitely many Sym-orbits. These facts, which are called Sym-*Noetherianity* or *equivariant Noetherianity* of $K[x_1,x_2,\dots]$, have interesting applications [1, 7, 13] and inspired lots of recent work. See, e.g., [9, 14, 15, 16, 17, 20, 21].

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We aim to explore such equivariant finiteness principles in the broader context of discrete geometry. For example, we are looking for a framework to formulate equivariant versions of theorems from polyhedral geometry. For this one needs to consider chains of cones $(C_n)_n$ with $C_n \subseteq \mathbb{R}^n_{\geq 0}$ with a suitable equivariance (Definition 2.11 contains the general version).

Equivariant Noetherianity cannot hold in these setting as the ambient $\mathbb{R}^n_{\geq 0}$ has no Noetherianity: There are non-polyhedral (i.e., not finitely generated) cones such as the open orthant, defined by positivity of all coordinates. A chain of open orthants stabilizes but the limit is not finitely generated. Examples 5.5–5.8 showcase these effects.

What remains of the theory is the equivalence of equivariant finite generation in the limit (the *global* situation) and stabilization of the chain (the *local* one) under appropriate assumptions. For example, the *local-global principle* for cones in Corollary 5.4 states that the limit C_∞ of a Sym-invariant family of cones is equivariantly finitely generated if and only if the family $(C_n)_n$ stabilizes and eventually all C_n are finitely generated. This is also equivalent to an eventual saturation condition $C_n = C_\infty \cap \mathbb{R}^n_{\geq 0}$ together with finite generation of C_n by rays of bounded support. Section 5 also contains a parallel development for monoids.

The similarity between the results for cones and monoids points at a generalization and unification, which we undertake in Sections 2–4. In Section 2 we abstract taking the ideal, cone, or monoid to any closure operation and the action of $\mathrm{Sym}(n)$ to any system of maps that maps objects of the chain into the later object. The generalization has many advantages. It allows to formulate a general local-global principle, i.e. the exact conditions under which the equivalence of finite generation up to symmetry and stabilization hold. This is our central Theorem 2.16. Specializing the maps to Sym and Inc (the monoid of increasing maps) yields Theorems 4.10 and 4.11. Further specializations to chains of polyhedral cones and monoids under Sym and Inc follow in Section 5. In Section 6 we return to equivariant Noetherianity in polynomial rings and use our results to fill a gap in the proof that Inc-invariant chains of ideals stabilize, a fact that is used in the literature, e.g. in [11, 20].

Our work could be phrased in the framework of FI-modules by Church, Ellenberg, and Farb (see, e.g., [4, 5, 6]). This concerns, among other things, equivariant chains of modules using symmetric groups. The fundamental difference between our developments in all but the last section is that the elements of our chains are subsets of finite-dimensional vector spaces which rarely appear with module structures, but rather take into account various closure operations. In the broader context of representation stability and twisted commutative algebras, there is, in particular, the fundamental work of Sam and Snowden (see, e.g., [22, 23]). Unlike in our situation, Noetherianity abounds and is a central tool in this theory. In summary, while it would be possible to generalize our results in Section 4 and, based on [21], phrase them in the language of FI-modules we chose a direct approach that is most suitable for applications like the algebraic and geometric situations in Sections 5 and 6.

Finite generation is essential for computation. To make our results effective, more research is needed on the concrete stability indices, for which very little is known. On the side of polyhedral geometry, there is software for dual description conversion modulo symmetry [2], but computer algebra for symmetric ideals or monoids is in its infancy.

2. Chains of sets

We begin with a general framework of chains of sets and closure operations. Our goal is to be able to transfer properties from such chains of sets to their union (or limit) and back.

Throughout this section S_∞ is any set and $\mathcal{S} = (S_n)_{n\geq 1}$ is an increasing chain of subsets

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq \cdots \text{ such that } S_\infty = \bigcup_{n\geq 1} S_n. \tag{1}$$

The set S_∞ has its notation for consistency with the limits below. One could think of S_∞ as an ambient set and $\mathcal{S}$ as an ambient chain in which the chains of interest live. A *chain of sets* $\mathcal{A} = (A_n)_{n\geq 1}$ with respect to $\mathcal{S}$ is an increasing chain

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$$

with $A_n \subseteq S_n$ for all $n \geq 1$. The *limit set* of $\mathcal{A}$ is

$$A_\infty = \bigcup_{n\geq 1} A_n \subseteq S_\infty.$$

A chain of sets $\mathcal{A}$ is *saturated* (respectively, *eventually saturated*) if

$$A_n = A_\infty \cap S_n \text{ for all } n \geq 1 \text{ (respectively, for all } n \gg 0).$$

Here and in the following "for all $n \gg 0$" means that there exists some $N \in \mathbb{N}$ such that the property holds for all $n > N$.

Any chain $\mathcal{A}$ with limit set A_∞ has a *saturation* that is a chain $\bar{\mathcal{A}} = (\overline{A}_n)_{n\geq 1}$ defined by

$$\overline{A}_n := A_\infty \cap S_n \text{ for all } n \geq 1.$$

Evidently, $\overline{\mathcal{A}}$ is the only saturated chain with limit set A_∞.

The key objects of this paper are chains of sets that possess two additional structures: First, each set is closed with respect to a closure operation, and second, the chains are invariant under a group or monoid action. In the following we describe these two structures as well as their compatibility. Our notion of closure follows the idea of E.H. Moore [19] (although it differs slightly). Let $\mathcal{P}(X)$ denote the power set of a set X.

Definition 2.1. A *closure operation* on a set X is a map $\mathrm{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ such that

(i) $A \subseteq A^{\mathrm{cl}}$ for all $A \in \mathcal{P}(X)$.
(ii) $A^{\mathrm{cl}} = (A^{\mathrm{cl}})^{\mathrm{cl}}$ for all $A \in \mathcal{P}(X)$.
(iii) If $A, B \in \mathcal{P}(X)$ with $A \subseteq B$, then $A^{\mathrm{cl}} \subseteq B^{\mathrm{cl}}$.

A set $A \in \mathcal{P}(X)$ is cl-*closed* if $A^{\mathrm{cl}} = A$.

Let cl_∞ be a closure operation on S_∞ and cl_n a closure operation on S_n for all $n \geq 1$. We call cl_∞ a *global* closure operation and $\mathcal{cl} = (\mathrm{cl}_n)_{n\geq 1}$ a chain of *local* closure operations.

Definition 2.2. A global closure operation cl_∞ is *consistent* with a chain $\mathcal{cl}$ of local closure operations (or $(\mathcal{cl}, \mathrm{cl}_\infty)$ is a *consistent system* of closure operations) if

$$(A \cap S_n)^{\mathrm{cl}_n} = A^{\mathrm{cl}_\infty} \cap S_n \quad \text{for all } n \geq 1 \text{ and } A \subseteq S_\infty.$$

Remark 2.3. Definition 2.2 formulates a *local-global* consistency of the closure operations. This implies the following *local-local* consistency:

$$(A_n \cap S_m)^{\mathrm{cl}_m} = A_n^{\mathrm{cl}_n} \cap S_m \quad \text{for all } n \geq m \geq 1 \text{ and } A_n \subseteq S_n.$$

Indeed, it follows from Definition 2.2 that $A_n^{\mathrm{cl}_n} = (A_n \cap S_n)^{\mathrm{cl}_n} = A_n^{\mathrm{cl}_\infty} \cap S_n$. Thus,

$$A_n^{\mathrm{cl}_n} \cap S_m = A_n^{\mathrm{cl}_\infty} \cap S_n \cap S_m = A_n^{\mathrm{cl}_\infty} \cap S_m = (A_n \cap S_m)^{\mathrm{cl}_m}.$$

On the other hand, local-local consistency does not imply local-global consistency. To see this, one can simply take cl_∞ to be the *trivial* closure operation (i.e. $A^{\mathrm{cl}_\infty} = S_\infty$ for any $A \subseteq S_\infty$) and $\mathcal{cl}$ the chain of identity closures considered in the next example.

Example 2.4. The following are some closure operations that appear here.

(i) Letting $A^{\mathrm{id}_n} = A$ for $A \subseteq S_n$ yields the *chain of identity closures* $(\mathrm{id}_n)_{n \geq 1}$. Obviously, this chain satisfies the local-local consistency described above.

(ii) Taking conical hulls $\mathrm{cn}_n(A) = \mathrm{cone}(A)$ for $A \in \mathcal{P}(\mathbb{R}^n)$ induces the *chain of conical closures* $\mathcal{cn} = (\mathrm{cn}_n)_{n \geq 1}$. See Section 5 for details.

(iii) Taking monoid closures $\mathrm{mn}_n(A) = \mathrm{mon}(A)$ for $A \in \mathcal{P}(\mathbb{R}^n)$ induces the *chain of monoid closures* $\mathcal{mn} = (\mathrm{mn}_n)_{n \geq 1}$. See Section 5 for details.

(iv) Let $K[X_n]$ be the polynomial ring from Section 4.1. Taking ideal closures $\langle A \rangle_n$ for $A \in \mathcal{P}(K[X_n])$ induces the *chain of ideal closures* $(\langle \cdot \rangle_n)_{n \geq 1}$. See Section 6 for details.

In what follows we fix a global closure operation cl_∞ and a chain $\mathcal{cl} = (\mathrm{cl}_n)_{n \geq 1}$ of local closure operations.

Definition 2.5. A chain of sets $\mathcal{A} = (A_n)_{n \geq 1}$ is *$\mathcal{cl}$-closed* (respectively, *eventually $\mathcal{cl}$-closed*) if A_n is cl_n-closed for all $n \geq 1$ (respectively, for all $n \gg 0$).

Given a $\mathcal{cl}$-closed chain $\mathcal{A}$, one might ask whether the limit set A_∞ is cl_∞-closed, and vice versa, if the limit set A_∞ is cl_∞-closed, what can be said about the chain $\mathcal{A}$? This type of local-global question is a frequent theme here. A partial but quite general answer is the following.

Lemma 2.6. Let $(\mathcal{cl}, \mathrm{cl}_\infty)$ be a consistent system and $\mathcal{A} = (A_n)_{n \geq 1}$ a $\mathcal{cl}$-closed chain with limit set A_∞ and saturation $\overline{\mathcal{A}}$. Then the following hold:

(i) A_∞ is cl_∞-closed if $\overline{\mathcal{A}}$ is eventually $\mathcal{cl}$-closed or $\mathcal{A}$ is eventually saturated.

(ii) If A_∞ is cl_∞-closed, then $\overline{\mathcal{A}}$ is $\mathcal{cl}$-closed.

Proof. (i) Assume first that $\overline{\mathcal{A}}$ is eventually $\mathcal{cl}$-closed. Then there exists $m \geq 1$ such that

$$(A_\infty \cap S_n)^{\mathrm{cl}_n} = A_\infty \cap S_n \ \text{ for all } n \geq m.$$

Since $S_\infty = \bigcup_{n \geq 1} S_n = \bigcup_{n \geq m} S_n$, it follows from Definition 2.2 that

$$A_\infty^{\mathrm{cl}_\infty} = A_\infty^{\mathrm{cl}_\infty} \cap S_\infty = \bigcup_{n \geq m} (A_\infty^{\mathrm{cl}_\infty} \cap S_n) = \bigcup_{n \geq m} (A_\infty \cap S_n)^{\mathrm{cl}_n} = \bigcup_{n \geq m} (A_\infty \cap S_n) = A_\infty.$$

Hence, A_∞ is cl_∞-closed. If $\mathcal{A}$ is eventually saturated, then $A_\infty \cap S_n = A_n$ is cl_n-closed for all $n \gg 0$. Thus, the chain $\overline{\mathcal{A}}$ is eventually $\mathcal{cl}$-closed and we conclude by the first case.

(ii) If A_∞ is cl_∞-closed, then by Definition 2.2,

$$A_\infty \cap S_n = A_\infty^{\mathrm{cl}_\infty} \cap S_n = (A_\infty \cap S_n)^{\mathrm{cl}_n} \text{ for all } n \geq 1.$$

This means that the chain $\overline{\mathcal{A}}$ is $\mathcal{cl}$-closed. □

The following lemma is immediate given Lemma 2.6.

Lemma 2.7. Under the assumptions of Lemma 2.6 the following are equivalent:

(i) A_∞ is cl_∞-closed;
(ii) $\overline{\mathcal{A}}$ is eventually $\mathcal{cl}$-closed;
(iii) $\overline{\mathcal{A}}$ is $\mathcal{cl}$-closed.

Moreover, these equivalent statements hold if $\mathcal{A}$ is eventually saturated.

Now that the chains are described, we focus on group or monoid actions implementing symmetries within the chain. In the abstract setting we begin with just maps and specialize to actions later. Let Π be a set of maps $S_\infty \to S_\infty$, and for $m \leq n$ let $\Pi_{m,n}$ be a set of maps $S_m \to S_n$. We use the adjectives *global* to refer to the maps in Π and *local* for those in the sets $\Pi_{m,n}$. Let $\boldsymbol{\varpi}$ denote the family $\{\Pi_{m,n}\}_{m\leq n}$. We call $(\boldsymbol{\varpi},\Pi)$ a *system of maps*. For $A_m \subseteq S_m$ and $A_\infty \subseteq S_\infty$ set

$$\Pi_{m,n}(A_m) = \{\pi(v) \mid \pi \in \Pi_{m,n},\ v \in A_m\} \subseteq S_n,$$
$$\Pi(A_\infty) = \{\pi(v) \mid \pi \in \Pi,\ v \in A_\infty\} \subseteq S_\infty.$$

In most cases of interest here $\Pi_{m,n}(A_m)$ is a finite set whenever A_m is a finite subset of S_m.

Definition 2.8. A family $\boldsymbol{\varpi} = \{\Pi_{m,n}\}_{m\leq n}$ is *locally finite* if for every $n \geq m \geq 1$ and every finite subset $A_m \subseteq S_m$ the set $\Pi_{m,n}(A_m)$ is finite.

It is worth mentioning that a family $\boldsymbol{\varpi} = \{\Pi_{m,n}\}_{m\leq n}$ can be locally finite even when each $\Pi_{m,n}$ is infinite; see Lemma 4.5.

Later, when Π is a group or monoid, each set $\Pi_{m,n}$ can be derived from Π, and moreover, the system $(\boldsymbol{\varpi},\Pi)$ is (weakly) consistent in the sense of the next definition. In such cases, it is customary to use Π as a representative for the family $\boldsymbol{\varpi}$.

Definition 2.9. A system of maps $(\boldsymbol{\varpi},\Pi)$ is

(i) *weakly consistent* if

$$\Pi(A_m) = \bigcup_{n\geq m} \Pi_{m,n}(A_m) \text{ for all } m \geq 1 \text{ and } A_m \subseteq S_m;$$

(ii) *consistent* if

$$\Pi_{m,n}(A_m) = \Pi(A_m) \cap S_n \text{ for all } n \geq m \geq 1 \text{ and } A_m \subseteq S_m.$$

Remark 2.10.

(i) Consistency is stronger than weak consistency. Indeed, if $(\boldsymbol{\varpi},\Pi)$ is consistent, then for all $n \geq m \geq 1$ and $A_m \subseteq S_m$ one has

$$\Pi(A_m) = \Pi(A_m) \cap S_\infty = \Pi(A_m) \cap \Big(\bigcup_{n\geq m} S_n\Big) = \bigcup_{n\geq m} (\Pi(A_m) \cap S_n) = \bigcup_{n\geq m} \Pi_{m,n}(A_m).$$

(ii) Weak consistency is frequently applied in the form of the following easy consequence: For all $n \geq m$ and $A_m \subseteq S_m$ it holds that $\Pi_{m,n}(A_m) \subseteq \Pi(A_m)$.

(iii) The consistency of $(\boldsymbol{\varpi},\Pi)$ is a *local-global* consistency. It implies the following *local-local* consistency:

$$\Pi_{m,n}(A_m) = \Pi_{k,n}(A_m) \quad \text{for all } n \geq k \geq m \text{ and } A_m \subseteq S_m \subseteq S_k,$$

since both sides are equal to $\Pi(A_m) \cap S_n$.

We are now ready to introduce the main object of study in this paper.

Definition 2.11. Let $(\boldsymbol{\varpi},\Pi)$ be a system of maps and let $\mathcal{cl} = (\mathrm{cl}_n)_{n\geq 1}$ be a chain of closure operations.

(i) A subset $A_\infty \subseteq S_\infty$ is Π-*invariant* if

$$\Pi(A_\infty) \subseteq A_\infty.$$

(ii) A $\mathcal{cl}$-closed chain $\mathcal{A} = (A_n)_{n\geq 1}$ is $\boldsymbol{\varpi}$-*invariant* if

$$(\Pi_{m,n}(A_m))^{\mathrm{cl}_n} \subseteq A_n \text{ whenever } n \geq m.$$

A $\boldsymbol{\varpi}$-invariant chain *stabilizes* if there exists some integer $r \geq 1$ such that for all $n \geq m \geq r$ one has $(\Pi_{m,n}(A_m))^{\mathrm{cl}_n} = A_n$. The smallest such r is the $\boldsymbol{\varpi}$-*stability index* (or *stability index*) of $\mathcal{A}$ with respect to $\mathcal{cl}$ and denoted by $\mathrm{ind}^{\boldsymbol{\varpi}}_{\mathcal{cl}}(\mathcal{A})$, or $\mathrm{ind}_{\mathcal{cl}}(\mathcal{A})$, or even $\mathrm{ind}(\mathcal{A})$ if there is no danger of confusion.

Let $\mathcal{A} = (A_n)_{n\geq 1}$ be a chain of sets. We would like to describe properties of $\mathcal{A}$ (i.e. *local* properties) that can be transferred to corresponding properties of the limit A_∞ (i.e. *global* properties) and vice versa. After Lemma 2.6, here is the next example of such a property.

Lemma 2.12. Let $(\boldsymbol{\varpi},\Pi)$ be a weakly consistent system of maps and $\mathcal{A} = (A_n)_{n\geq 1}$ be a $\boldsymbol{\varpi}$-invariant $\mathcal{cl}$-closed chain with limit set A_∞. Then the following hold:

(i) A_∞ is Π-invariant.

(ii) If A_∞ is cl_∞-closed, then $\overline{\mathcal{A}}$ is a $\boldsymbol{\varpi}$-invariant $\mathcal{cl}$-closed chain.

Proof. (i) It follows from Definitions 2.9(i) and 2.11(ii) that

$$\Pi(A_\infty) = \Pi\Big(\bigcup_{m\geq 1} A_m\Big) = \bigcup_{m\geq 1} \Pi(A_m) = \bigcup_{n\geq m\geq 1} \Pi_{m,n}(A_m) \subseteq \bigcup_{n\geq 1} A_n = A_\infty.$$

Thus, A_∞ is Π-invariant.

(ii) By Lemma 2.6(ii), $\overline{\mathcal{A}}$ is $\mathcal{cl}$-closed. So it remains to show that $\overline{\mathcal{A}}$ is $\boldsymbol{\varpi}$-invariant. Let $m,n \in \mathbb{N}$ with $n \geq m$. According to Remark 2.10(ii) it holds that

$$\Pi_{m,n}(A_\infty \cap S_m) \subseteq \Pi(A_\infty \cap S_m) \subseteq \Pi(A_\infty) \subseteq A_\infty,$$

where the last inclusion follows from (i). On the other hand, one has $\Pi_{m,n}(A_\infty \cap S_m) \subseteq S_n$ by definition. Hence, $\Pi_{m,n}(A_\infty \cap S_m) \subseteq A_\infty \cap S_n$, which implies

$$(\Pi_{m,n}(A_\infty \cap S_m))^{\mathrm{cl}_n} \subseteq A_\infty \cap S_n$$

since $\overline{\mathcal{A}}$ is $\mathcal{cl}$-closed. Therefore, $\overline{\mathcal{A}}$ is $\boldsymbol{\varpi}$-invariant. □

The main focus of this work is to explore under which conditions local finite generation implies global finite generation and vice versa. We first give a definition of this property in local and global situations.

Definition 2.13. Let $(\boldsymbol{\varpi},\Pi)$ be a system of maps and $(\mathcal{cl},\mathrm{cl}_\infty)$ a system of closure operations. Let $\mathcal{A} = (A_n)_{n\geq 1}$ be a $\mathcal{cl}$-closed chain and $A \subseteq S_\infty$ a cl_∞-closed set.

(i) (local) $\mathcal{A}$ is *finitely generated* (respectively, *eventually finitely generated*) if for all $n \geq 1$ (respectively, for all $n \gg 0$) there is a finite subset $G_n \subseteq A_n$ such that $A_n = G_n^{\mathrm{cl}_n}$.
(ii) (global) A is Π-*equivariantly finitely generated* if there exists a finite subset $G \subseteq A$ such that $A = \Pi(G)^{\mathrm{cl}_\infty}$.

To discuss the relation between local and global finite generation we introduce a crucial *local-local compatibility* condition.

Definition 2.14. A chain $\mathcal{cl} = (\mathrm{cl}_n)_{n\geq 1}$ of closure operations and a family of maps $\boldsymbol{\varpi} = \{\Pi_{m,n}\}_{m\leq n}$ are *compatible* if

$$\Pi_{m,n}(A_m^{\mathrm{cl}_m}) \subseteq (\Pi_{m,n}(A_m))^{\mathrm{cl}_n} \text{ for all } n \geq m \geq 1 \text{ and } A_m \subseteq S_m.$$

Lemma 2.15. If $\mathcal{cl}$ and $\boldsymbol{\varpi}$ are compatible, then

$$(\Pi_{m,n}(A_m^{\mathrm{cl}_m}))^{\mathrm{cl}_n} = (\Pi_{m,n}(A_m))^{\mathrm{cl}_n} \text{ for all } n \geq m \geq 1 \text{ and } A_m \subseteq S_m.$$

Proof. From Definition 2.14 it follows that

$$(\Pi_{m,n}(A_m^{\mathrm{cl}_m}))^{\mathrm{cl}_n} \subseteq ((\Pi_{m,n}(A_m))^{\mathrm{cl}_n})^{\mathrm{cl}_n} = (\Pi_{m,n}(A_m))^{\mathrm{cl}_n}.$$

The reverse inclusion is obvious since $A_m \subseteq A_m^{\mathrm{cl}_m}$. □

We are now ready to describe situations where the finite generation property of a chain is inherited by its limit, and vice versa.

Theorem 2.16. Let $(\mathcal{cl},\mathrm{cl}_\infty)$ be a consistent system of closure operations and $(\boldsymbol{\varpi},\Pi)$ a system of maps. Let $\mathcal{A} = (A_n)_{n\geq 1}$ be a $\boldsymbol{\varpi}$-invariant, $\mathcal{cl}$-closed chain with limit set A_∞. Consider the statements:

(a) (local) $\mathcal{A}$ stabilizes and is eventually finitely generated.
(b) (global) A_∞ is Π-equivariantly finitely generated.

The following hold:

(i) If A_∞ is cl_∞-closed, $(\boldsymbol{\varpi},\Pi)$ is weakly consistent, and $\mathcal{cl}$ and $\boldsymbol{\varpi}$ are compatible, then (a) implies (b).

(ii) If $\boldsymbol{\varpi}$ is locally finite and $(\boldsymbol{\varpi},\Pi)$ is consistent, then (b) implies (a). Moreover, in this case $\mathcal{A}$ is eventually saturated.

Proof. Assume first (a) and the assumptions of (i). Since $\mathcal{A}$ stabilizes, its stability index $\mathrm{ind}(\mathcal{A})$ is finite. Since $\mathcal{A}$ is eventually finitely generated, we may choose an $m \geq \mathrm{ind}(\mathcal{A})$ and a finite subset $G \subseteq A_m$ such that $A_m = G^{\mathrm{cl}_m}$. So for $n \geq m$ one obtains

$$\begin{aligned}
A_n &= (\Pi_{m,n}(A_m))^{\mathrm{cl}_n} && \text{(by Definition 2.11(ii))}\\
&= (\Pi_{m,n}(G^{\mathrm{cl}_m}))^{\mathrm{cl}_n} && \text{(since } A_m = G^{\mathrm{cl}_m})\\
&= (\Pi_{m,n}(G))^{\mathrm{cl}_n} && \text{(by Lemma 2.15)}\\
&= (\Pi_{m,n}(G) \cap S_n)^{\mathrm{cl}_n} && \text{(since } \Pi_{m,n}(G) \subseteq S_n)\\
&\subseteq (\Pi(G) \cap S_n)^{\mathrm{cl}_n} && \text{(by Remark 2.10(ii))}\\
&= \Pi(G)^{\mathrm{cl}_\infty} \cap S_n && \text{(by Definition 2.2).}
\end{aligned}$$

Since $A_n \subseteq A_m$ for $n \leq m$ it follows that

$$A_\infty = \bigcup_{n\geq 1} A_n = \bigcup_{n\geq m} A_n \subseteq \bigcup_{n\geq m} (\Pi(G)^{\mathrm{cl}_\infty} \cap S_n) = \Pi(G)^{\mathrm{cl}_\infty}.$$

On the other hand, $\Pi(G)^{\mathrm{cl}_\infty} \subseteq \Pi(A_\infty)^{\mathrm{cl}_\infty} \subseteq A_\infty^{\mathrm{cl}_\infty} = A_\infty$ by Lemma 2.12(i) and the assumption that A_∞ is cl_∞-closed. Hence, $A_\infty = \Pi(G)^{\mathrm{cl}_\infty}$ is Π-equivariantly finitely generated.

Now assume (b) and the assumptions of (ii). Since A_∞ is Π-equivariantly finitely generated, there exists a finite subset $G \subseteq A_\infty$ such that $A_\infty = \Pi(G)^{\mathrm{cl}_\infty}$. Since G is finite, we may assume $G \subseteq A_m$ for some large enough m. Thus, for $n \geq m$ one has

$$\begin{aligned}
(\Pi_{m,n}(A_m))^{\mathrm{cl}_n} &\subseteq A_n \subseteq A_\infty \cap S_n\\
&= \Pi(G)^{\mathrm{cl}_\infty} \cap S_n && \text{(since } A_\infty = \Pi(G)^{\mathrm{cl}_\infty})\\
&= (\Pi(G) \cap S_n)^{\mathrm{cl}_n} && \text{(by Definition 2.2)}\\
&= (\Pi_{m,n}(G))^{\mathrm{cl}_n} && \text{(by Definition 2.9(ii))}\\
&\subseteq (\Pi_{m,n}(A_m))^{\mathrm{cl}_n} && \text{(since } G \subseteq A_m),
\end{aligned}$$

hence equalities hold throughout. That is,

$$A_\infty \cap S_n = A_n = (\Pi_{m,n}(A_m))^{\mathrm{cl}_n} = (\Pi_{m,n}(G))^{\mathrm{cl}_n}.$$

This shows that $\mathcal{A}$ stabilizes and is eventually saturated. Moreover, $\mathcal{A}$ is eventually finitely generated because G is finite and $\boldsymbol{\varpi}$ is locally finite. ☐

Now we formulate assumptions that guarantee equivalence of (a) and (b) in Theorem 2.16.

Corollary 2.17. Let $(\mathcal{cl}, \mathrm{cl}_\infty)$ and $(\boldsymbol{\varpi}, \Pi)$ be consistent systems and $\boldsymbol{\varpi}$ be locally finite and compatible with $\mathcal{cl}$. If $\mathcal{A} = (A_n)_{n\geq 1}$ is a $\boldsymbol{\varpi}$-invariant, $\mathcal{cl}$-closed chain such that the limit A_∞ is cl_∞-closed, then the following statements are equivalent:

(a) (local) $\mathcal{A}$ stabilizes and is eventually finitely generated;
(b) (global) A_∞ is Π-equivariantly finitely generated.

Moreover, if either of the above equivalent statements holds, then $\mathcal{A}$ is eventually saturated.

For later applications it is useful to relax the assumption that $(\boldsymbol{\varpi}, \Pi)$ is consistent in Theorem 2.16(ii). Analyzing the proof of this part, we see that its conclusion still holds true if the consistency of $(\boldsymbol{\varpi}, \Pi)$ is replaced by the existence of a finite subset $G \subseteq A_\infty$ such that

$$A_\infty = \Pi(G)^{\mathrm{cl}_\infty} \quad \text{and} \quad \Pi(G) \cap S_n \subseteq \Pi_{m,n}(A_m) \ \text{ for all } n \geq m \gg 0. \tag{2}$$

Hence, we obtain the following.

Proposition 2.18. Let $(\mathcal{cl}, \mathrm{cl}_\infty)$ be a consistent system of closure operations and $(\boldsymbol{\varpi}, \Pi)$ be a system of maps. Assume that $\boldsymbol{\varpi}$ is locally finite. Let $\mathcal{A} = (A_n)_{n\geq 1}$ be a $\boldsymbol{\varpi}$-invariant, $\mathcal{cl}$-closed chain with limit set A_∞. If there exists a finite set $G \subseteq A_\infty$ such that condition (2) is satisfied, then $\mathcal{A}$ stabilizes and is eventually finitely generated.

3. Symmetric groups and a related monoid

We are mainly interested in systems of maps that are induced by actions of symmetric groups or the monoid of increasing functions. We discuss here some properties of these objects that will be used later.

Let $\mathbb{N} = \{1, 2, \dots\}$ denote the set of positive integers and, for $n \in \mathbb{N}$, set $[n] = \{1, \dots, n\}$. We adopt the convention that $[\infty] = \mathbb{N}$. Let $\mathrm{Sym}(n)$ denote the symmetric group on $[n]$ for any $n \in \mathbb{N}$. Since $\mathrm{Sym}(n)$ can be naturally regarded as the stabilizer subgroup of $n+1$ in $\mathrm{Sym}(n+1)$, we have an increasing chain of finite symmetric groups $\mathrm{Sym}(1) \subseteq \mathrm{Sym}(2) \subseteq \cdots \subseteq \mathrm{Sym}(n) \subseteq \cdots$. The limit of this chain is

$$\mathrm{Sym}(\infty) := \bigcup_{n\geq 1} \mathrm{Sym}(n).$$

We often use Sym as an abbreviation for $\mathrm{Sym}(\infty)$. For $m \leq n$ let $\iota_{m,n} \colon [m] \to [n]$ denote the canonical embedding, i.e. $\iota_{m,n}(k) = k$ for all $k \in [m]$. We define

$$\mathrm{Sym}_{m,n} := \mathrm{Sym}(n) \circ \iota_{m,n} = \{\sigma \circ \iota_{m,n} \mid \sigma \in \mathrm{Sym}(n)\}.$$

Let $\mathcal{Sym}$ denote the family $\{\mathrm{Sym}_{m,n}\}_{m\leq n}$. In the cases of interest, $(\mathcal{Sym}, \mathrm{Sym})$ is consistent by Lemma 4.4(ii).

Consider next the *monoid of strictly increasing maps on* $\mathbb{N}$, defined as

$$\mathrm{Inc} := \{\pi \colon \mathbb{N} \to \mathbb{N} \mid \pi(n) < \pi(n+1) \text{ for all } n \geq 1\}.$$

When $m \leq n$ we set

$$\mathrm{Inc}_{m,n} := \{\pi \in \mathrm{Inc} \mid \pi(m) \leq n\}.$$

Thus, each $\mathrm{Inc}_{m,n}$ is a subset of Inc and for any $m \geq 1$ there is an increasing chain

$$\mathrm{Inc}_{m,m} \subseteq \mathrm{Inc}_{m,m+1} \subseteq \cdots \subseteq \mathrm{Inc}_{m,n} \subseteq \cdots \quad \text{with limit} \quad \mathrm{Inc} = \bigcup_{n \geq m} \mathrm{Inc}_{m,n}\,. \tag{3}$$

Let $\mathcal{I}nc$ denote the family $\{\mathrm{Inc}_{m,n}\}_{m \leq n}$. Then in the setting of Section 4, $(\mathcal{I}nc, \mathrm{Inc})$ is a weakly consistent system of maps by Lemma 4.4(i). Since $\pi([m]) \subseteq [n]$ for any $\pi \in \mathrm{Inc}_{m,n}$, we may view the restriction $\pi|_{[m]}$ as a map $[m] \to [n]$. It is also useful to consider the set

$$\overline{\mathrm{Inc}}_{m,n} := \{\overline{\pi} \colon [m] \to [n] \mid \text{ there exists } \pi \in \mathrm{Inc}_{m,n} \text{ with } \overline{\pi} = \pi|_{[m]}\},$$

which can be regarded as the quotient of $\mathrm{Inc}_{m,n}$ by the equivalence relation: $\pi_1 \sim \pi_2$ if $\pi_1|_{[m]} = \pi_2|_{[m]}$. The next lemma records a useful relationship between Sym and Inc.

Lemma 3.1. Let $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$ with $m \leq n$. Then the following hold:

(i) $\mathrm{Sym}_{m,n}$ equals the set of injective maps from $[m] \to [n]$ and $\overline{\mathrm{Inc}}_{m,n}$ equals the set of injective order-preserving maps $[m] \to [n]$.
(ii) $\overline{\mathrm{Inc}}_{m,n} \subseteq \mathrm{Sym}_{m,n}$. In particular, $\pi|_{[m]} \in \mathrm{Sym}_{m,n}$ for every $\pi \in \mathrm{Inc}_{m,n}$.
(iii) $\mathrm{Sym}_{m,n} = \overline{\mathrm{Inc}}_{m,n} \circ \mathrm{Sym}(m) := \{\pi \circ \sigma \mid \pi \in \overline{\mathrm{Inc}}_{m,n}, \sigma \in \mathrm{Sym}(m)\}$.

Proof. (i) and (ii) are easy to check directly from the definitions. Using (i), a proof of (iii) amounts to checking that each injective map $[m] \to [n]$ factors into a reordering of $[m]$ followed by an order-preserving injective map $[m] \to [n]$. □

We also need the following decomposition, which can be shown to also hold for Sym.

Lemma 3.2. For any $m, n \in \mathbb{N}$ with $n > m$ it holds that

$$\mathrm{Inc}_{m,n} = \mathrm{Inc}_{m+1,n} \circ \mathrm{Inc}_{m,m+1} = \mathrm{Inc}_{n-1,n} \circ \mathrm{Inc}_{m,n-1}\,.$$

Proof. From [20, Proposition 4.6] it follows that $\mathrm{Inc}_{m,n} = \mathrm{Inc}_{m+1,n} \circ \mathrm{Inc}_{m,m+1}$. Based on this decomposition, one can show by induction that

$$\mathrm{Inc}_{m,n} = \mathrm{Inc}_{n-1,n} \circ \mathrm{Inc}_{n-2,n-1} \circ \cdots \circ \mathrm{Inc}_{m,m+1} = \mathrm{Inc}_{n-1,n} \circ \mathrm{Inc}_{m,n-1}\,.$$ □

4. Sets up to symmetry

In this section we specialize the theory in Section 2 to Sym- and Inc-invariant chains. For simplicity, we restrict the discussion to a setup which is compatible with the ones in the forthcoming sections so that the results in this section can be applied in a direct and effective way. To this end, we only consider chains of sets with respect to an ambient chain $\mathcal{R} = (R_n)_{n \geq 1}$ in which each R_n is either a polynomial ring or a subset of $\mathbb{R}^n$.

4.1. **Ambient chains.** We describe two types of ambient chains $\mathcal{R} = (R_n)_{n\geq 1}$ for the rest of this paper. The first is a chain of polynomial rings. Let K be a field and c a positive integer. For any $n \in \mathbb{N}$ let R_n be the polynomial ring $K[X_n] := K[x_{i,j} \mid i \in [c],\ j \in [n]]$.

Then, for $m \leq n$, the embedding $\iota_{m,n} \colon [m] \to [n]$ induces a canonical embedding $R_m \to R_n$, also denoted by $\iota_{m,n}$, that maps each polynomial in R_m to the same polynomial considered as an element of R_n. Thus, we obtain the chain

$$R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n \subseteq \cdots \text{ with limit } R_\infty = \bigcup_{n\geq 1} R_n = K[X] := K[x_{i,j} \mid i \in [c],\ j \in \mathbb{N}].$$

The second type of the ambient chain $\mathcal{R} = (R_n)_{n\geq 1}$ are chains with $R_n \subseteq \mathbb{R}^n$ for all $n \geq 1$ that are $\mathcal{S}ym$*-invariant*, meaning that

$$\mathrm{Sym}_{m,n}(R_m) \subseteq R_n \ \text{ for all } n \geq m \geq 1. \tag{4}$$

The inclusion is to be understood as follows. Every element $\pi = \sigma \circ \iota_{m,n} \in \mathrm{Sym}_{m,n}$ with $\sigma \in \mathrm{Sym}(n)$ gives rise to a map $\mathbb{R}^m \to \mathbb{R}^n$. First, the embedding $\iota_{m,n} \colon [m] \to [n]$ induces a canonical inclusion $\iota_{m,n} \colon \mathbb{R}^m \to \mathbb{R}^n$ that embeds $\mathbb{R}^m$ as the first m coordinates in $\mathbb{R}^n$:

$$\iota_{m,n}(v) = (v, 0, \ldots, 0) \in \mathbb{R}^n \ \text{ for any } v \in \mathbb{R}^m. \tag{5}$$

Then $\sigma \in \mathrm{Sym}(n)$ acts on $\mathbb{R}^n$ by permuting coordinates, that is

$$\sigma(v_1, \ldots, v_n) = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)}) \quad \text{for any } (v_1, \ldots, v_n) \in \mathbb{R}^n. \tag{6}$$

Thus, each $\pi = \sigma \circ \iota_{m,n} \in \mathrm{Sym}_{m,n}$ defines a map $\pi \colon \mathbb{R}^m \to \mathbb{R}^n$, and we understand (4) as

$$\mathrm{Sym}_{m,n}(R_m) := \{\pi(R_m) \mid \pi \in \mathrm{Sym}_{m,n}\} \subseteq R_n \ \text{ for all } n \geq m \geq 1.$$

Via the embedding $\iota_{m,n}$ we regard R_m as a subset of R_n for $m \leq n$ and thereby get a chain with limit $R_\infty = \bigcup_{n\geq 1} R_n$ contained in the infinite dimensional vector space $\mathbb{R}^{(\mathbb{N})} := \bigcup_{n\geq 1} \mathbb{R}^n$. The canonical basis of $\mathbb{R}^{(\mathbb{N})}$ consists of the vectors ε_i, $i \in \mathbb{N}$, with

$$(\varepsilon_i)_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

With respect to the identity closures in Example 2.4(i), an ambient chain $\mathcal{R} = (R_n)_{n\geq 1}$ of the second type is exactly a $\mathcal{S}ym$-invariant chain in the sense of Definition 2.11(ii). Chains of this type include the cases where R_n equals $\mathbb{R}^n$, $\mathbb{R}^n_{\geq 0}$, $\mathbb{Z}^n$, or $\mathbb{N}^n$ for all $n \geq 1$.

From now on, we always assume the ambient chain $\mathcal{R} = (R_n)_{n\geq 1}$ is one of the two types described above. For $m \in \mathbb{N}$ the canonical embedding $\iota_{m,\infty} \colon R_m \to R_\infty$ allows to identify R_m with a subset of R_∞. Thus, for any $v \in R_m$, the elements v, $\iota_{m,n}(v)$, $\iota_{m,\infty}(v)$ are all identical for every $n \geq m$. By definition, each $w \in R_\infty$ is contained in some R_m. The smallest such m is the *width* of w:

$$\mathrm{width}(w) := \min\{m \in \mathbb{N} \mid w \in R_m\}.$$

When $R_\infty = K[X]$, let $\operatorname{var}(w)$ denote the set of variables appearing in w. If $R_\infty \subseteq \mathbb{R}^{(\mathbb{N})}$, then we write $w = (w_1, w_2, \dots)$ with $w_j \in \mathbb{R}$. The *support* of w is

$$\operatorname{supp}(w) := \begin{cases} \{j \in \mathbb{N} \mid x_{i,j} \in \operatorname{var}(w) \text{ for some } i \in [c]\} & \text{if } R_\infty = K[X], \\ \{j \in \mathbb{N} \mid w_j \neq 0\} & \text{if } R_\infty \subseteq \mathbb{R}^{(\mathbb{N})}. \end{cases}$$

In the polynomial ring case our definition of support is coarser than the usual definition, in which $\operatorname{supp}(w)$ is the set of monomials of w. However, the above definition is more useful for our purposes. We call $|\operatorname{supp}(w)|$ the *support size* of w. Evidently, $|\operatorname{supp}(w)| \leq \operatorname{width}(w)$.

4.2. **Sym and Inc actions.** Given an ambient chain $\mathcal{R} = (R_n)_{n\geq 1}$, the systems $(\mathcal{Sym}, \mathrm{Sym})$ and $(\mathcal{Inc}, \mathrm{Inc})$ give rise to systems of maps on $(\mathcal{R}, R_\infty)$. In particular, this yields actions of $\mathrm{Sym}(\infty)$ and Inc on the ambient set R_∞. We now collect basic properties of these actions and discuss the consistency and local finiteness of $(\mathcal{Sym}, \mathrm{Sym})$ and $(\mathcal{Inc}, \mathrm{Inc})$.

Consider $(\mathcal{Sym}, \mathrm{Sym})$ first. Let $m \leq n$ and $\pi = \sigma \circ \iota_{m,n} \in \mathrm{Sym}_{m,n}$ with $\sigma \in \mathrm{Sym}(n)$. Then π induces a map $\pi\colon R_m \to R_n$ given by

$$\pi(v) = \begin{cases} \sum_{k\geq 1} v_k \varepsilon_{\pi(k)} & \text{if } v = \sum_{k\geq 1} v_k \varepsilon_k \in R_m \subseteq \mathbb{R}^m, \\ x_{i,\pi(j)} & \text{if } v = x_{i,j} \in R_m = K[X_m], \end{cases} \tag{7}$$

where in the first case, π is precisely the map induced by the composition of the maps in (5) and (6), which is well-defined because the chain $\mathcal{R}$ is $\mathcal{Sym}$-invariant. By this definition, $\mathrm{Sym}_{m,n}$ is identified with a set of maps $R_m \to R_n$. In particular, the group $\mathrm{Sym}_{n,n} = \mathrm{Sym}(n)$ acts on R_n, and moreover, the action of $\mathrm{Sym}(n+1)$ on R_{n+1} extends that of $\mathrm{Sym}(n)$ on R_n for all $n \geq 1$. Since $\mathrm{Sym}(\infty) = \bigcup_{n\geq 1} \mathrm{Sym}(n)$ and $R_\infty = \bigcup_{n\geq 1} R_n$, these actions together build an action of $\mathrm{Sym}(\infty)$ on R_∞. We have thus defined the system of maps $(\mathcal{Sym}, \mathrm{Sym})$ on $(\mathcal{R}, R_\infty)$.

Some useful observations from the above definition are recorded in the next lemma.

Lemma 4.1. Let $m, n \in \mathbb{N}$ with $m \leq n$ and $v \in R_m$. Then the following hold:

(i) $|\operatorname{supp}(v)| = |\operatorname{supp}(\pi(v))|$ for every $\pi \in \mathrm{Sym}_{m,n}$.
(ii) $\mathrm{Sym}_{m,n}(v) = \mathrm{Sym}(n)(v)$.

Proof. (i) follows easily from (7). To see (ii), recall that v and $\iota_{m,n}(v)$ are identified in R_n. Hence, $\mathrm{Sym}_{m,n}(v) = (\mathrm{Sym}(n) \circ \iota_{m,n})(v) = \mathrm{Sym}(n)(\iota_{m,n}(v)) = \mathrm{Sym}(n)(v)$. □

Next, we construct the system of maps $(\mathcal{Inc}, \mathrm{Inc})$, based on the system $(\mathcal{Sym}, \mathrm{Sym})$. Let $\pi \in \mathrm{Inc}_{m,n}$. Then $\overline{\pi} := \pi|_{[m]} \in \overline{\mathrm{Inc}}_{m,n} \subseteq \mathrm{Sym}_{m,n}$ by Lemma 3.1(ii). Hence, $\overline{\pi}$ induces a map $R_m \to R_n$, explicitly described in (7). So we can define a map $\pi\colon R_m \to R_n$ by letting $\pi(v) = \overline{\pi}(v)$ for all $v \in R_m$. Now let $\pi' \in \mathrm{Inc}$ and $w \in R_\infty$. Then $w \in R_m$ for some m and there exists $n \geq m$ such that $\pi' \in \mathrm{Inc}_{m,n}$. Evidently, $\pi'(w)$ does not depend on the choice of m and n. This implies that π' induces a map $\pi'\colon R_\infty \to R_\infty$. Thus, we obtain the system of maps $(\mathcal{Inc}, \mathrm{Inc})$ on $(\mathcal{R}, R_\infty)$.

From the construction it is clear that (7) still holds if Sym is replaced by Inc. In particular, Inc acts as a monoid on R_∞. The next result compares the actions of Sym and Inc.

Lemma 4.2. Let $m,n \in \mathbb{N}$ with $m \leq n$. Then for any $v \in R_m$ and $w \in R_\infty$ the following hold:

(i) $\mathrm{Inc}_{m,n}(v) = \overline{\mathrm{Inc}}_{m,n}(v) \subseteq \mathrm{Sym}_{m,n}(v)$.

(ii) $\mathrm{Inc}(w) \subseteq \mathrm{Sym}(w)$.

Proof. (i) follows from viewing $\pi \in \mathrm{Inc}_{m,n}$ as $\pi|_{[m]} \in \overline{\mathrm{Inc}}_{m,n}$. To prove (ii), let $\pi \in \mathrm{Inc}$. Then $w \in R_m$ and $\pi \in \mathrm{Inc}_{m,n}$ for some $n \geq m$. So it follows from (i) and Lemma 4.1(ii) that

$$\pi(w) \in \mathrm{Sym}_{m,n}(w) = \mathrm{Sym}(n)(w) \subseteq \mathrm{Sym}(w). \qquad \square$$

Sometimes it is necessary to describe the truncated orbits $\mathrm{Sym}(w) \cap R_n$ and $\mathrm{Inc}(w) \cap R_n$ of an element $w \in R_\infty$. Truncating at the width suffices to not lose information.

Lemma 4.3. Let $w \in R_\infty$ and $m,n \in \mathbb{N}$ with $n \geq m$.

(i) If $\mathrm{width}(w) = m$, then $\mathrm{Inc}(w) \cap R_n = \mathrm{Inc}_{m,n}(w)$.

(ii) If $w \in R_m$ (i.e. $\mathrm{width}(w) \leq m$), then $\mathrm{Sym}(w) \cap R_n = \mathrm{Sym}(n)(w) = \mathrm{Sym}_{m,n}(w)$.

Proof. (i) We may assume that $w \neq 0$. Let $n \geq m$. It suffices to prove the inclusion

$$\mathrm{Inc}(w) \cap R_n \subseteq \mathrm{Inc}_{m,n}(w).$$

Take $u \in \mathrm{Inc}(w) \cap R_n$. Then $\mathrm{width}(u) \leq n$ and $u = \pi(w)$ for some $\pi \in \mathrm{Inc}$.

Consider first the case that $R_\infty \subseteq \mathbb{R}^{(\mathbb{N})}$. Using $\mathrm{width}(w) = m$, we write $w = \sum_{k=1}^m w_k \varepsilon_k$ with $w_m \neq 0$. Then $u = \pi(w) = \sum_{k=1}^m w_k \varepsilon_{\pi(k)}$. This implies that $\pi(m) \leq n$ since $w_m \neq 0$ and $\mathrm{width}(u) \leq n$. Hence, $\pi \in \mathrm{Inc}_{m,n}$ and $u = \pi(w) \in \mathrm{Inc}_{m,n}(v)$.

In the case $R_\infty = K[X]$, w involves a variable $x_{i,m}$ for some $i \in [c]$ since $\mathrm{width}(w) = m$. Thus $u = \pi(w)$ involves $x_{i,\pi(m)}$. Since $\mathrm{width}(u) \leq n$, one again gets $\pi \in \mathrm{Inc}_{m,n}$ as above.

(ii) The first equality is easy to see, while the second follows from Lemma 4.1(ii). $\square$

We are now ready to discuss the consistency and local finiteness of the systems of maps $(\mathcal{Sym}, \mathrm{Sym})$ and $(\mathcal{Inc}, \mathrm{Inc})$.

Lemma 4.4.

(i) The system $(\mathcal{Inc}, \mathrm{Inc})$ is weakly consistent.

(ii) The system $(\mathcal{Sym}, \mathrm{Sym})$ is consistent.

Proof. (i) follows from (3) and (ii) from Lemma 4.3(ii). $\square$

The system $(\mathcal{Inc}, \mathrm{Inc})$ is not consistent as Example 5.10 shows.

Because of the preceding result and the way that the family $\mathcal{Sym}$ (respectively, $\mathcal{Inc}$) is derived from Sym (respectively, Inc), one usually uses Sym (respectively, Inc) as a representative for $\mathcal{Sym}$ (respectively, $\mathcal{Inc}$). So, for example, the statement of the next result is another way of saying that the families $\mathcal{Sym}$ and $\mathcal{Inc}$ are locally finite in the sense of Definition 2.8. For simplicity we also use the terms Sym- or Inc-*invariant chain* instead of $\mathcal{Sym}$- or $\mathcal{Inc}$-*invariant chain*, and so forth.

Lemma 4.5. Sym and Inc are locally finite.

Proof. Let $n \geq m \geq 1$ and $A_m \subseteq R_m$ be a finite subset. We have to show that $\mathrm{Sym}_{m,n}(A_m)$ and $\mathrm{Inc}_{m,n}(A_m)$ are finite sets. The first set is finite because $\mathrm{Sym}_{m,n}$ finite. The second one is also finite since $\mathrm{Inc}_{m,n}(A_m) = \overline{\mathrm{Inc}}_{m,n}(A_m)$ by Lemma 4.2(i) and $\overline{\mathrm{Inc}}_{m,n}$ is finite. □

4.3. **Sym- and Inc-invariant chains.** As a consequence of Lemma 4.2 there are more Inc-invariant chains than Sym-invariant chains. Generalizing [20, Lemma 7.6] we have:

Lemma 4.6. Let $\mathcal{cl} = (\mathrm{cl}_n)_{n\geq 1}$ be a chain of closure operations.

(i) If a $\mathcal{cl}$-closed chain $(A_n)_{n\geq 1}$ is Sym-invariant, then it is Inc-invariant.
(ii) If $A \subseteq R_\infty$ is Sym-invariant, then it is Inc-invariant.

Example 4.7. The converses of Lemma 4.6(i) and (ii) are not true. Consider the chain of identity closures as in Example 2.4(i). Let $A_1 = \{0\}$ and $A_n = \{0\} \times \mathbb{R}^{n-1}_{\geq 0} \subseteq \mathbb{R}^n$ for $n \geq 2$. Then $\mathcal{A} = (A_n)_{n\geq 1}$ is an Inc-invariant chain of sets. However, it is not Sym-invariant since, e.g., $\mathrm{Sym}_{2,n}(A_2)$ contains vectors $v \in \mathbb{R}^n$ with $v_1 \neq 0$.

Remark 4.8.

(i) By Lemma 4.6(i), one can view a Sym-invariant chain as an Inc-invariant chain. This was used for chains of ideals in [13] and elsewhere. Also in our general setup, with some more effort, it can be shown that if $\mathcal{A} = (A_n)_{n\geq 1}$ is a Sym-invariant, $\mathcal{cl}$-closed chain of sets, then $\mathcal{A}$ stabilizes as a Sym-invariant chain if and only if it stabilizes as an Inc-invariant chain. In this case $\mathrm{ind}^{\mathrm{Sym}}(\mathcal{A}) = \mathrm{ind}^{\mathrm{Inc}}(\mathcal{A})$.
(ii) By Lemma 4.6(ii), any Sym-invariant subset $A \subseteq R_\infty$ is also Inc-invariant. Then it can be shown with some more effort that equivariant finite generation of a Sym-invariant, cl_∞-closed set $A \subseteq R_\infty$ holds for Sym if and only if it holds for Inc.

The following characterization of stabilization, which generalizes [20, Lemma 5.2], could be of independent interest.

Proposition 4.9. Let $\mathcal{cl} = (\mathrm{cl}_n)_{n\geq 1}$ be a chain of closure operations that is compatible with Inc. Let $\mathcal{A} = (A_n)_{n\geq 1}$ be an Inc-invariant, $\mathcal{cl}$-closed chain of sets. Then for $r \in \mathbb{N}$ the following statements are equivalent:

(i) $\mathcal{A}$ stabilizes and its stability index is at most r;
(ii) $\mathrm{Inc}_{n,n+1}(A_n)^{\mathrm{cl}_{n+1}} = A_{n+1}$ whenever $n \geq r$;
(iii) $\mathrm{Inc}_{r,n}(A_r)^{\mathrm{cl}_n} = A_n$ whenever $n \geq r$.

Proof. The implication (i) ⇒ (ii) follows directly from the definition. We prove (iii) from (ii) by induction on n. First consider the case $n = r$. Since $\mathrm{Inc}_{r,r}$ contains the identity map, one has $A_r \subseteq \mathrm{Inc}_{r,r}(A_r) \subseteq \mathrm{Inc}_{r,r}(A_r)^{\mathrm{cl}_r} \subseteq A_r$, hence $A_r = \mathrm{Inc}_{r,r}(A_r)^{\mathrm{cl}_r}$.

Assume we have shown that $A_n = \mathrm{Inc}_{r,n}(A_r)^{\mathrm{cl}_n}$ for some $n \geq r$. Then A_{n+1} equals

$$\mathrm{Inc}_{n,n+1}(A_n)^{\mathrm{cl}_{n+1}} = \mathrm{Inc}_{n,n+1}(\mathrm{Inc}_{r,n}(A_r)^{\mathrm{cl}_n})^{\mathrm{cl}_{n+1}} = \mathrm{Inc}_{n,n+1}(\mathrm{Inc}_{r,n}(A_r))^{\mathrm{cl}_{n+1}} = \mathrm{Inc}_{r,n+1}(A_r)^{\mathrm{cl}_{n+1}},$$

by (ii), the induction hypothesis, Lemma 2.15, and Lemma 3.2.

To prove (iii) $\Rightarrow$ (i), take any $n > r$. We show that $A_n = (\mathrm{Inc}_{m,n}(A_m))^{\mathrm{cl}_n}$ for all $n \geq m \geq r$ by induction on m. The case $m = r$ follows from (iii). Assume $A_n = (\mathrm{Inc}_{m,n}(A_m))^{\mathrm{cl}_n}$ for some m with $n > m \geq r$. By Lemma 3.2, $\mathrm{Inc}_{m,n} = \mathrm{Inc}_{m+1,n} \circ \mathrm{Inc}_{m,m+1}$. It follows that

$$A_n = (\mathrm{Inc}_{m,n}(A_m))^{\mathrm{cl}_n} = (\mathrm{Inc}_{m+1,n} \circ \mathrm{Inc}_{m,m+1}(A_m))^{\mathrm{cl}_n} \subseteq (\mathrm{Inc}_{m+1,n}(A_{m+1}))^{\mathrm{cl}_n} \subseteq A_n.$$

Hence, $A_n = (\mathrm{Inc}_{m+1,n}(A_{m+1}))^{\mathrm{cl}_n}$, which concludes the induction argument. □

4.4. **Finite generation up to symmetry.** We apply and refine the results on local-global finite generation in Section 2 to Sym- and Inc-invariant chains. For this we consider a general system of closure operations $(\mathcal{cl}, \mathrm{cl}_\infty)$, leaving more specific discussions until later sections.

For Sym-invariant chains, apart from the local-global principle in Corollary 2.17, there is one additional local characterization. The following result generalizes [1, Theorem 4.7] and [13, Corollary 3.7]; see Section 6 for more details.

Theorem 4.10. Let $(\mathcal{cl}, \mathrm{cl}_\infty)$ be a consistent system of closure operations that is compatible with Sym. Let $\mathcal{A} = (A_n)_{n\geq 1}$ be a Sym-invariant, $\mathcal{cl}$-closed chain for which A_∞ is cl_∞-closed. Then the following statements are equivalent:

(a) (local) $\mathcal{A}$ stabilizes and is eventually finitely generated;
(b) (local) There exists an $r \in \mathbb{N}$ such that for all $n \geq r$ the following hold:
  (i) (saturation) $A_\infty \cap R_n = A_n$,
  (ii) (support size) A_n is finitely generated by elements of support size at most r;
(c) (global) A_∞ is Sym-equivariantly finitely generated.

Proof. From Lemmas 4.4(ii) and 4.5 we know that the system of maps $(\mathcal{Sym}, \mathrm{Sym})$ satisfies the remaining assumptions of Corollary 2.17. So (a) and (c) are equivalent. We show the implications (a)+(c) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a).

(a)+(c) $\Rightarrow$ (b): Since A_∞ is Sym-equivariantly finitely generated (by (c)), it follows from Corollary 2.17 that $\mathcal{A}$ is eventually saturated. Thus, there exists $r \in \mathbb{N}$ such that $A_\infty \cap R_n = A_n$ for all $n \geq r$, which is (b)(i). By (a), $\mathrm{ind}(\mathcal{A})$ is finite and A_n is finitely generated for $n \gg 0$. So we may assume that $r \geq \mathrm{ind}(\mathcal{A})$ and A_r is finitely generated by, say, $v_1, \dots, v_s \in A_r$. Then for all $n \geq r$ we have $A_n = (\mathrm{Sym}_{r,n}(v_1) \cup \cdots \cup \mathrm{Sym}_{r,n}(v_s))^{\mathrm{cl}_n}$. Each $v_t \in A_r \subseteq R_r$ has support size at most r. Since $\mathrm{Sym}_{r,n}$ is finite and its elements do not change the support size of v_t by Lemma 4.1(i), we conclude (b)(ii).

(b) $\Rightarrow$ (a): Assuming r as in (b), it suffices to show that for all $n \geq r$, $(\mathrm{Sym}_{r,n}(A_r))^{\mathrm{cl}_n} = A_n$. This is trivially true for $n = r$. Let $n > r$. By (b)(ii), A_n is generated by some $u_1, \dots, u_p \in R_n$ of support size at most r. We want to show that $u_1, \dots, u_p \in (\mathrm{Sym}_{r,n}(A_r))^{\mathrm{cl}_n}$. For any $t \in [p]$, $|\mathrm{supp}(u_t)| \leq r < n$ and thus there exists a $\sigma \in \mathrm{Sym}(n)$ such that $\sigma(u_t) \in R_r$. Hence, $\sigma(u_t) \in R_r \cap A_\infty = A_r$, using (b)(i). Then by Lemma 4.1(ii),

$$u_t = \sigma^{-1}(\sigma(u_t)) \in \mathrm{Sym}(n)(A_r) = \mathrm{Sym}_{r,n}(A_r) \subseteq (\mathrm{Sym}_{r,n}(A_r))^{\mathrm{cl}_n}.$$ □

For Inc-invariant chains we obtain a weaker version of Theorem 4.10. The polynomial ring case of this result and its relation to [13, Theorem 3.6] are discussed in Section 6.

Theorem 4.11. Let $(\mathcal{cl}, \mathrm{cl}_\infty)$ be a consistent system of closure operations so that $\mathcal{cl}$ is compatible with Inc. Let $\mathcal{A} = (A_n)_{n\geq 1}$ be an Inc-invariant, $\mathcal{cl}$-closed chain such that the limit set A_∞ is cl_∞-closed. Consider the following statements:

(a) (local) $\mathcal{A}$ stabilizes and is eventually finitely generated.
(b) (global) A_∞ is Inc-equivariantly finitely generated.

Then (a) implies (b) and if $\mathcal{A}$ is eventually saturated, then (a) and (b) are equivalent.

Proof. That (a) implies (b) follows immediately from Theorem 2.16(i) and Lemma 4.4(i).

To prove the equivalence, we can assume that $\mathcal{A}$ is saturated. To see this, replace $\mathcal{A}$ by its saturation $\overline{\mathcal{A}}$. Then (b) and the assumptions of the theorem are still satisfied by Lemma 2.12(ii); (a) remains unchanged since $\mathcal{A}$ and $\overline{\mathcal{A}}$ coincide eventually (because $\mathcal{A}$ is eventually saturated). Therefore, it is harmless to assume that $\mathcal{A}$ is saturated.

We prove (b) $\Rightarrow$ (a) using Proposition 2.18. To verify its assumptions, use Lemma 4.5 for the local finiteness of Inc. Then by (b), there exist $w_1,\dots,w_s \in A_\infty$ such that

$$A_\infty = (\mathrm{Inc}(w_1)\cup\cdots\cup\mathrm{Inc}(w_s))^{\mathrm{cl}_\infty}.$$

Choose m so that $w_t \in A_m$ for all $t \in [s]$. To apply Proposition 2.18, it remains to show that

$$\mathrm{Inc}(w_t)\cap R_n \subseteq \mathrm{Inc}_{m,n}(A_m) \quad \text{for all } n \geq m \text{ and } t \in [s]. \tag{8}$$

Let $k = k_t := \mathrm{width}(w_t)$. It is evident that $k \leq m$ and $w_t \in A_\infty \cap R_k = A_k$, where we used that $\mathcal{A}$ is saturated. By Lemma 4.3(i), $\mathrm{Inc}(w_t)\cap R_n = \mathrm{Inc}_{k,n}(w_t) \subseteq \mathrm{Inc}_{k,n}(A_k)$. To prove (8) it suffices to show that

$$\mathrm{Inc}_{k,n}(A_k) \subseteq \mathrm{Inc}_{m,n}(A_m) \quad \text{for all } k \leq m.$$

By Lemma 3.2, $\mathrm{Inc}_{k,n} = \mathrm{Inc}_{k+1,n}\circ\mathrm{Inc}_{k,k+1}$, and hence

$$\mathrm{Inc}_{k,n}(A_k) = (\mathrm{Inc}_{k+1,n}\circ\mathrm{Inc}_{k,k+1})(A_k) \subseteq \mathrm{Inc}_{k+1,n}(A_{k+1}).$$

The proof finishes with a finite induction. ☐

Remark 4.12. Comparing Theorems 4.10 and 4.11, the following questions are natural:

(1) Can the assumption that $\mathcal{A}$ is eventually saturated in Theorem 4.11 be omitted?
(2) Does there exist a characterization for Inc-invariant chains similar to Theorem 4.10(b)? Precisely, when $\mathcal{A}$ is eventually saturated, is it true that A_∞ is Inc-equivariantly finitely generated if $\mathcal{A}$ is eventually finitely generated by elements of bounded support size?

These questions can be affirmatively answered for chains of ideals (Theorem 6.3), but in general both have negative answers as we show in Examples 5.10 and 5.11.

5. Cones and monoids up to symmetry

In this section we specialize the results of the previous section to invariant chains of convex cones and monoids. We provide various examples to demonstrate that some assumptions of our results are indispensable and give counterexamples to potential strengthenings.

The results of this section have recently been employed to extend foundational results in polyhedral geometry to the equivariant setting [18]. For standard terminology on cones and monoids the reader is referred to [3] and [24].

5.1. **Cones up to symmetry.** We consider convex cones in the nonnegative orthant $\mathbb{R}^{(\mathbb{N})}_{\geq 0}$ of $\mathbb{R}^{(\mathbb{N})}$. That is, the ambient chain is $\mathcal{R} = (R_n)_{n\geq 1}$ with $R_n = \mathbb{R}^n_{\geq 0}$ for all $n \geq 1$ and $R_\infty = \bigcup_{n\geq 1} \mathbb{R}^n_{\geq 0} = \mathbb{R}^{(\mathbb{N})}_{\geq 0}$. To consider convex cones, the closure operations are $\mathrm{cn}_n = \mathrm{cone}(\cdot)$ in $\mathbb{R}^n_{\geq 0}$ for $n \geq 1$ and $\mathrm{cn}_\infty = \mathrm{cone}(\cdot)$ in R_∞, where $\mathrm{cone}(A)$ consists of finite nonnegative linear combinations from A, that is, $\mathrm{cone}(A) = \{\sum_{i=1}^k \lambda_i a_i \mid k \in \mathbb{N},\ a_i \in A,\ \lambda_i \in \mathbb{R}_{\geq 0}\}$. We restrict our attention to cones in R_∞ instead of the whole space $\mathbb{R}^{(\mathbb{N})}$, since the system of conical hulls $(\mathcal{cn}, \mathrm{cn}_\infty)$ is consistent on R_∞, but not on $\mathbb{R}^{(\mathbb{N})}$.

Lemma 5.1. Consider the ambient chain $\mathcal{R} = (\mathbb{R}^n_{\geq 0})_{n\geq 1}$ of nonnegative orthants as above. Then the system of conical hulls $(\mathcal{cn}, \mathrm{cn}_\infty)$ is consistent.

Proof. For any $A \subseteq \mathbb{R}^{(\mathbb{N})}_{\geq 0}$ and $n \in \mathbb{N}$ we need to show that $\mathrm{cn}_n(A \cap \mathbb{R}^n_{\geq 0}) = \mathrm{cn}_\infty(A) \cap \mathbb{R}^n_{\geq 0}$. The inclusion "$\subseteq$" is obviously true. For the reverse inclusion, take $v \in \mathrm{cn}_\infty(A) \cap \mathbb{R}^n_{\geq 0}$. Then $\mathrm{width}(v) \leq n$ and there exist $a_1, \ldots, a_k \in A$ and $\lambda_1, \ldots, \lambda_k > 0$ such that $v = \sum_{i=1}^k \lambda_i a_i$. By positivity of the λ_i, and since $a_i \in \mathbb{R}^{(\mathbb{N})}_{\geq 0}$ for all $i \in [k]$, we have $\mathrm{width}(a_i) \leq \mathrm{width}(v) \leq n$. Thus $a_i \in A \cap \mathbb{R}^n_{\geq 0}$ for all $i \in [k]$, and hence $v = \sum_{i=1}^k \lambda_i a_i \in \mathrm{cn}_n(A \cap \mathbb{R}^n_{\geq 0})$. □

It is easy to give examples showing that $(\mathcal{cn}, \mathrm{cn}_\infty)$ is not consistent on $\mathbb{R}^{(\mathbb{N})}$.

Example 5.2. Let $A = \{(1,1), (1,-1)\} \subset \mathbb{R}^2$. Identifying $\mathbb{R}$ with $\mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$, one has

$$\mathrm{cn}_1(A \cap \mathbb{R}) = \mathrm{cn}_1(\emptyset) = \{0\} \subsetneq \mathbb{R}_{\geq 0} = \mathrm{cn}_2(A) \cap \mathbb{R}.$$

By Remark 2.3, this means that the system $(\mathcal{cn}, \mathrm{cn}_\infty)$ is not consistent if we choose the ambient chain with $R_n = \mathbb{R}^n$ for all $n \geq 1$.

The next result shows that the chain $\mathcal{cn}$ of conical hulls is compatible with $\mathrm{Sym}(\infty)$ and Inc, even when one considers the whole space $\mathbb{R}^{(\mathbb{N})}$.

Lemma 5.3. Let $\Pi = \mathrm{Sym}(\infty)$ or $\Pi = \mathrm{Inc}$. Then for any $m \leq n$ and $A_m \subseteq \mathbb{R}^m$ one has $\Pi_{m,n}(\mathrm{cn}_m(A_m)) \subseteq \mathrm{cn}_n(\Pi_{m,n}(A_m))$. Thus, with $\mathcal{R} = (\mathbb{R}^n_{\geq 0})_{n\geq 1}$, $\mathcal{cn}$ is compatible with Π.

Proof. Let $\pi \in \Pi_{m,n}$ and $v = \sum_{i=1}^k \lambda_i a_i \in \mathrm{cn}_m(A_m)$ with $a_i \in A_m$ and $\lambda_i \geq 0$. We expand $a_i = \sum_{j=1}^m a_{ij}\varepsilon_j$ in the canonical basis. Then by (7),

$$\pi(v) = \pi\Big(\sum_{j=1}^m \big(\sum_{i=1}^k \lambda_i a_{ij}\big)\varepsilon_j\Big) = \sum_{j=1}^m \big(\sum_{i=1}^k \lambda_i a_{ij}\big)\varepsilon_{\pi(j)} = \sum_{i=1}^k \lambda_i \pi(a_i).$$

This yields $\pi(v) \in \mathrm{cn}_n(\Pi_{m,n}(A_m))$, as desired. □

It is obvious that for an increasing chain $\mathcal{C} = (C_n)_{n\geq 1}$ of convex cones with $C_n \subseteq \mathbb{R}^n$ for all $n \geq 1$, the limit set C_∞ is a convex cone. With Lemmas 5.1 and 5.3 in place, we are ready to apply the results of the previous section to chains of cones. At first we formulate the following version of Theorem 4.10.

Corollary 5.4. Let $\mathcal{C} = (C_n)_{n\geq 1}$ be a Sym-invariant chain of convex cones $C_n \subseteq \mathbb{R}^n_{\geq 0}$ with limit cone C_∞. Then the following statements are equivalent:

(a) $\mathcal{C}$ stabilizes and is eventually finitely generated;
(b) There exists an $r \in \mathbb{N}$ such that for all $n \geq r$ the following hold:
  (i) $C_\infty \cap \mathbb{R}^n_{\geq 0} = C_n$,
  (ii) C_n is finitely generated by elements of support size at most r;
(c) C_∞ is Sym-equivariantly finitely generated.

We now give several examples that demonstrate the necessity of the conditions in Corollary 5.4. The first shows that in (a) and (b) the word "eventually" cannot be omitted.

Example 5.5. Consider the chain $\mathcal{C} = (C_n)_{n\geq 1}$ with $C_1 = \{0\}$,

$$C_2 = \{(x,y) \in \mathbb{R}^2_{\geq 0} \mid x > 0,\ y > 0\} \cup \{(0,0)\} \ \text{ and } \ C_n = \mathbb{R}^n_{\geq 0} \ \text{ for } n \geq 3.$$

Then $\mathcal{C}$ stabilizes as a Sym-invariant chain of convex cones. The limit cone $C_\infty = \mathbb{R}^{(\mathbb{N})}_{\geq 0} = \mathrm{cn}_\infty(\mathrm{Sym}(\varepsilon_1))$ is Sym-equivariantly generated by the first basis vector. However, this does not imply that all cones C_n are finitely generated, since C_2 is not. Additionally $C_\infty \cap \mathbb{R}^2_{\geq 0} = \mathbb{R}^2_{\geq 0} \neq C_2$. Thus, $\mathcal{C}$ is not saturated.

The only insight in this example is that finite subsequences can often be changed rather wildly without changing the limit C_∞. This suggests that properties of C_∞ can only be related to the tail of the chain, or alternatively, to the unique saturated chain defining it.

The following example shows that for the implication "(a) $\Rightarrow$ (c)" in Corollary 5.4 it is necessary to assume that the cones are eventually finitely generated.

Example 5.6. Consider the chain $\mathcal{C} = (C_n)_{n\geq 1}$ with $C_1 = \{0\}$,

$$C_2 = \{(x,y) \in \mathbb{R}^2_{\geq 0} \mid x > 0,\ y > 0\} \cup \{(0,0)\} \ \text{ and } \ C_n = \mathrm{cn}_n(\mathrm{Sym}_{2,n}(C_2)) \ \text{ for } n \geq 3.$$

By construction, $\mathcal{C}$ stabilizes as a Sym-invariant chain of convex cones. However, C_∞ is not Sym-equivariantly finitely generated, since then $C_\infty \cap \mathbb{R}^2_{\geq 0}$ would be finitely generated. But $C_\infty \cap \mathbb{R}^2_{\geq 0} = C_2$ because C_2 contains exactly the elements of C_∞ of width at most two, and C_2 is evidently not finitely generated.

We now show that in Corollary 5.4(b) both (i) and (ii) are necessary and independent.

Example 5.7. For $n \geq 2$ let $v_n = (n,1,0,\dots,0) \in \mathbb{R}^n_{\geq 0}$. Consider the chain $\mathcal{C} = (C_n)_{n\geq 1}$ with $C_1 = \{0\}$, $C_2 = \mathrm{cn}_2(\mathrm{Sym}(2)(v_2))$ and

$$C_n = \mathrm{cn}_n\left(\mathrm{Sym}(n)\big(\iota_{n-1,n}(C_{n-1}) \cup \{v_n\}\big)\right) \subseteq \mathbb{R}^n_{\geq 0} \ \text{ for } \ n \geq 3.$$

Then $\mathcal{C}$ is a Sym-invariant chain of convex cones and the following hold:

(i) All cones C_n are pointed, rational and finitely generated (as a cone) by one element of support size two up to symmetry. More precisely, for $n \geq 2$ we have $C_n = \mathrm{cn}_n(\mathrm{Sym}(n)(v_n))$. In particular, the orbits of all generators of C_m in C_n are redundant whenever $m < n$.
(ii) C_∞ is not Sym-equivariantly finitely generated.
(iii) $C_\infty \cap \mathbb{R}^n_{\geq 0} \neq C_n$ for all $n \geq 2$.

Proof. (i) By definition, $\mathrm{cn}_n(\mathrm{Sym}_{n-1,n}(C_{n-1})) \subseteq C_n$ for $n \geq 2$. Thus, $\mathcal{C}$ is indeed a Sym-invariant chain of pointed and rational convex cones. Let $\widetilde{C}_n = \mathrm{cn}_n(\mathrm{Sym}(n)(v_n))$ for $n \geq 2$. Clearly, $\widetilde{C}_n \subseteq C_n$ for every $n \geq 2$. We show by induction that equality holds, which is trivially true for $n = 2$. Let $n > 3$ and assume that $C_{n-1} = \widetilde{C}_{n-1}$. It suffices to prove that

$$\mathrm{cn}_n(\mathrm{Sym}_{n-1,n}(C_{n-1})) \subseteq \widetilde{C}_n. \tag{9}$$

Using Lemma 5.3 and the fact that $\mathrm{Sym}_{n-1,n} = \mathrm{Sym}_{n-1,n} \circ \mathrm{Sym}(n-1)$ one has

$$\begin{aligned}\mathrm{cn}_n(\mathrm{Sym}_{n-1,n}(C_{n-1})) &= \mathrm{cn}_n(\mathrm{Sym}_{n-1,n}(\mathrm{cn}_{n-1}(\mathrm{Sym}(n-1)(v_{n-1}))))\\ &\subseteq \mathrm{cn}_n(\mathrm{Sym}_{n-1,n}(\mathrm{Sym}(n-1)(v_{n-1})))\\ &= \mathrm{cn}_n(\mathrm{Sym}_{n-1,n}(v_{n-1})) = \mathrm{cn}_n(\mathrm{Sym}(n)(\iota_{n-1,n}(v_{n-1}))).\end{aligned}$$

Since $v_{n-1} \in C_{n-1}$, this implies

$$\mathrm{cn}_n(\mathrm{Sym}_{n-1,n}(C_{n-1})) = \mathrm{cn}_n(\mathrm{Sym}(n)(\iota_{n-1,n}(v_{n-1}))). \tag{10}$$

So for (9) we only need to show that $\iota_{n-1,n}(v_{n-1}) \in \widetilde{C}_n$. Applying $\iota_{2,n}$ to

$$(n-1,1) = \frac{n^2-n-1}{n^2-1}(n,1) + \frac{1}{n^2-1}(1,n)$$

we obtain

$$\iota_{n-1,n}(v_{n-1}) = \frac{n^2-n-1}{n^2-1} v_n + \frac{1}{n^2-1}\sigma(v_n),$$

where $\sigma \in \mathrm{Sym}(n)$ is the transposition $(1\,2)$. Hence, $\iota_{n-1,n}(v_{n-1}) \in \mathrm{cn}_n(\mathrm{Sym}(n)(v_n)) = \widetilde{C}_n$, as desired. So $C_n = \widetilde{C}_n$ for $n \geq 2$ and the orbits of all generators of C_{n-1} in C_n are redundant. It follows that the same is true for the cones C_m and C_n, whenever $m < n$.

(ii) We claim that $\mathcal{C}$ does not stabilize. Indeed, it suffices to show that

$$v_n \in C_n \setminus \mathrm{cn}_n(\mathrm{Sym}_{n-1,n}(C_{n-1})) \ \text{ for all } \ n \geq 2.$$

Assume the contrary. Then $v_n \in \mathrm{cn}_n(\mathrm{Sym}_{n-1,n}(C_{n-1}))$ for some $n \geq 2$. So by (10), there exist $\lambda_1, \ldots, \lambda_k > 0$ and $\sigma_1, \ldots, \sigma_k \in \mathrm{Sym}(n)$ such that

$$v_n = \sum_{i=1}^{k} \lambda_i \sigma_i(\iota_{n-1,n}(v_{n-1})). \tag{11}$$

By the positivity of the λ_i, $\mathrm{width}(\sigma_i(\iota_{n-1,n}(v_{n-1}))) \leq 2$ for all $i \in [k]$. It follows that

$$\sigma_i(\iota_{n-1,n}(v_{n-1})) = (n-1,1,0,\ldots,0) \ \text{ or } \ \sigma_i(\iota_{n-1,n}(v_{n-1})) = (1,n-1,0,\ldots,0)$$

for all $i \in [k]$. This together with (11) yields the existence of $\lambda, \mu \geq 0$ such that

$$(n,1) = \lambda(n-1,1) + \mu(1,n-1).$$

But this is impossible. Hence, the chain C does not stabilize.

By Corollary 5.4, C_∞ is not Sym-equivariantly finitely generated. One can also show that C_∞ is the subset of $\mathbb{R}^{(\mathbb{N})}_{\geq 0}$ consisting of 0 and all elements of support size at least two, leading to the following explicit description:

$$C_\infty = \mathbb{R}^{(\mathbb{N})}_{\geq 0} \setminus \Big(\bigcup_{n \geq 1} \mathbb{R}_{>0}\, \varepsilon_n \Big).$$

(iii) Since $\mathrm{width}(v_{n+1}) = 2$, we have $v_{n+1} \in C_\infty \cap \mathbb{R}^2_{\geq 0} \subseteq C_\infty \cap \mathbb{R}^n_{\geq 0}$ for $n \geq 2$. On the other hand, it has been shown in (ii) that $v_{n+1} \notin C_n$. Hence, $C_\infty \cap \mathbb{R}^n_{\geq 0} \neq C_n$ for all $n \geq 2$. □

The next example gives a chain with no global bound for the support sizes of generators.

Example 5.8. For $n \geq 2$, let $w_n = (n-1, 1, \dots, 1) \in \mathbb{R}^n_{\geq 0}$. Consider the chain $C = (C_n)_{n \geq 1}$ with $C_1 = \{0\}$, $C_2 = \mathrm{cn}_2(\mathrm{Sym}(2)(w_2))$ and

$$C_n = \mathrm{cn}_n\left(\mathrm{Sym}(n)\big(\iota_{n-1,n}(C_{n-1}) \cup \{w_n\}\big)\right) \subseteq \mathbb{R}^n_{\geq 0} \ \text{ for } \ n \geq 3.$$

Then C is a Sym-invariant chain of convex cones and the following hold:

(i) All cones C_n are pointed, rational and finitely generated (as cones), but there exists no global bound for the support sizes of generators of the cones.
(ii) C_∞ is not Sym-equivariantly finitely generated.
(iii) $C_\infty \cap \mathbb{R}^n_{\geq 0} = C_n$ for all $n \geq 2$.

Proof. (i) As for Example 5.7(i) we see that C is a Sym-invariant chain of pointed, rational and finitely generated convex cones. An easy induction shows that C_n is generated by

$$G_n = \{\mathrm{Sym}_{m,n}(w_m) \mid 2 \leq m \leq n\}.$$

In this generating set, w_n is irredundant for C_n whenever $n \geq 3$. To see this, consider the supporting hyperplane $H = \{(z_1, \dots, z_n) \in \mathbb{R}^n \mid z_1 = \sum_{l=2}^n z_l\}$ of C_n. If w_n were redundant, there would exist $\lambda_1, \dots, \lambda_k > 0$ and $u_1, \dots, u_k \in G_n \setminus \{w_n\}$ such that $w_n = \sum_{i=1}^k \lambda_i u_i$. Since w_n lies on the supporting hyperplane H, it follows that $u_i \in H$ for all $i = 1, \dots, n$. Thus, each u_i must have $m-1$ as its first entry and exactly $(m-1)$ entries 1 as the remaining nonzero entries for some $m < n$. Comparing the second entry in $w_n = \sum_{i=1}^k \lambda_i u_i$, we find $\sum_{i=1}^k \lambda_i \leq 1$. Now comparing the first entry yields $n-1 = \sum_{i=1}^k \lambda_i u_{i,1} < \sum_{i=1}^k \lambda_i (n-1) \leq n-1$, which is impossible. Hence, C_n has an irredundant generator of support size n.

(ii) By (i) and Corollary 5.4, C_∞ is not Sym-equivariantly finitely generated.

(iii) According to Remark 2.3, the consistency of $(\mathcal{cn}, \mathrm{cn}_\infty)$ (by Lemma 5.1) gives

$$C_k \cap \mathbb{R}^n_{\geq 0} = \mathrm{cn}_k(G_k) \cap \mathbb{R}^n_{\geq 0} = \mathrm{cn}_n\left(G_k \cap \mathbb{R}^n_{\geq 0}\right) = \mathrm{cn}_n(G_n) = C_n$$

for all $k \geq n \geq 2$. It follows that

$$C_\infty \cap \mathbb{R}^n_{\geq 0} = \bigcup_{k \geq n} C_k \cap \mathbb{R}^n_{\geq 0} = \bigcup_{k \geq n} \left(C_k \cap \mathbb{R}^n_{\geq 0}\right) = C_n \quad \text{for all} \quad n \geq 2. \qquad \square$$

For the reader's convenience, we briefly summarize the above examples in Table 1.

Example	(a)		(b)		(c)
	C stabilizes	C evtl. f. g.	C evtl. satur.	C evtl. f. g. by elem. of bound. supp. size	C_∞ Sym-equiv. f. g.
5.5	✓	✓ (C_2 not f. g.)	✓ (C not satur.)	✓	✓
5.6	✓	✗	✓	✗	✗
5.7	✗	✓	✗	✓	✗
5.8	✗	✓	✓	✗	✗

TABLE 1. Summary of Examples 5.5–5.8

We consider Inc-invariant chains next. By Lemmas 5.1 and 5.3, we obtain the following cone version of Theorem 4.11.

Corollary 5.9. Let $C = (C_n)_{n \geq 1}$ be an Inc-invariant chain of convex cones $C_n \subseteq \mathbb{R}^n_{\geq 0}$ with limit C_∞. Consider the following statements:

(a) C stabilizes and is eventually finitely generated.
(b) C_∞ is Inc-equivariantly finitely generated.

Then (a) implies (b) and if C is eventually saturated, then (a) and (b) are equivalent.

We again give counterexamples to potential strengthenings of the corollary. The first shows that, in contrast to Corollary 5.4(b)(i), the equivariant finite generation of the limit cone C_∞ does not imply that C is eventually saturated, and moreover, that the equivalence in Corollary 5.9 does not hold without the assumption that C is eventually saturated. By Theorem 2.16(ii), this also implies that the system $(\mathcal{I}nc, \mathrm{Inc})$ is not consistent.

Example 5.10. For $n \geq 2$ let

$$A_n = \{\varepsilon_1, \ldots, \varepsilon_{n-1}\} \text{ and } B_n = \{\varepsilon_i + n\varepsilon_n \mid 1 \leq i \leq n-1\}.$$

Consider the chain $C = (C_n)_{n \geq 1}$ with $C_1 = \{0\}$ and $C_n = \mathrm{cn}_n(A_n \cup B_n) \subseteq \mathbb{R}^n_{\geq 0}$ for $n \geq 2$. Then the following hold:

(i) C is an Inc-invariant chain of convex cones and each C_n is finitely generated by vectors of support size at most two.
(ii) C_∞ is Inc-equivariantly finitely generated.
(iii) C does not stabilize.
(iv) $C_\infty \cap \mathbb{R}^n_{\geq 0} \neq C_n$ for all $n \geq 2$.

Proof. (i) For $n \geq 2$ let $\widetilde{B}_{n+1} = \{\varepsilon_i + n\varepsilon_{n+1} \mid 1 \leq i \leq n\}$. We claim that

$$\mathrm{cn}_{n+1}(\mathrm{Inc}_{n,n+1}(C_n)) = \mathrm{cn}_{n+1}(A_{n+1} \cup \widetilde{B}_{n+1}) \text{ for } n \geq 2.$$

One has $\mathrm{Inc}_{n,n+1}(A_n) = A_{n+1}$ and $\mathrm{Inc}_{n,n+1}(B_n) = B_n \cup \widetilde{B}_{n+1}$. Evidently, $B_n \subseteq \mathrm{cn}_{n+1}(A_{n+1})$ and therefore,

$$\begin{aligned}\mathrm{cn}_{n+1}(\mathrm{Inc}_{n,n+1}(C_n)) &= \mathrm{cn}_{n+1}(\mathrm{Inc}_{n,n+1}(\mathrm{cn}_n(A_n \cup B_n))) = \mathrm{cn}_{n+1}(\mathrm{Inc}_{n,n+1}(A_n \cup B_n)) \\ &= \mathrm{cn}_{n+1}(A_{n+1} \cup B_n \cup \widetilde{B}_{n+1}) = \mathrm{cn}_{n+1}(A_{n+1} \cup \widetilde{B}_{n+1}),\end{aligned}$$

where we used Lemma 2.15 in the second equality and also that conical hulls are compatible with Inc by Lemma 5.3. Now since $(n+1)(\varepsilon_i + n\varepsilon_{n+1}) = \varepsilon_i + n(\varepsilon_i + (n+1)\varepsilon_{n+1})$, it follows that $\widetilde{B}_{n+1} \subseteq \mathrm{cn}_{n+1}(A_{n+1} \cup B_{n+1})$. Hence

$$\mathrm{cn}_{n+1}(\mathrm{Inc}_{n,n+1}(C_n)) \subseteq \mathrm{cn}_{n+1}(A_{n+1} \cup B_{n+1}) = C_{n+1} \text{ for } n \geq 2,$$

i.e. $\mathcal{C}$ is an Inc-invariant chain of convex cones. The remaining assertion is obvious.

(ii) Since

$$\mathbb{R}^{(\mathbb{N})}_{\geq 0} = \bigcup_{n \geq 2} \mathrm{cn}_n(A_n) \subseteq \bigcup_{n \geq 2} C_n = C_\infty \subseteq \mathbb{R}^{(\mathbb{N})}_{\geq 0},$$

one obtains $C_\infty = \mathbb{R}^{(\mathbb{N})}_{\geq 0} = \mathrm{cn}_\infty(\mathrm{Inc}(\varepsilon_1))$. Thus, C_∞ is Inc-equivariantly generated by ε_1.

(iii) For all $n \geq 2$ and $1 \leq i \leq n$ one has

$$\varepsilon_i + (n+1)\varepsilon_{n+1} \notin \mathrm{cn}_{n+1}(A_{n+1} \cup \widetilde{B}_{n+1}) = \mathrm{cn}_{n+1}(\mathrm{Inc}_{n,n+1}(C_n)).$$

This implies $\mathrm{cn}_{n+1}(\mathrm{Inc}_{n,n+1}(C_n)) \subsetneq C_{n+1}$ for $n \geq 2$, which means that $\mathcal{C}$ does not stabilize.

(iv) The assertion follows since $\varepsilon_n \in \mathbb{R}^n_{\geq 0} = C_\infty \cap \mathbb{R}^n_{\geq 0}$, but $\varepsilon_n \notin C_n$ for $n \geq 2$. □

As announced in Remark 4.12, the equivalence "(b) ⇔ (c)" in Corollary 5.4 cannot be extended to Inc-invariant chains. The following example shows this and also that the equivalence "(a) ⇔ (b)" in Corollary 5.4 does not hold for Inc-invariant chains.

Example 5.11. For $n \geq 2$ let

$$E_n = \{\varepsilon_2, \dots, \varepsilon_n\} \text{ and } F_n = \{i\varepsilon_1 + \varepsilon_i \mid 2 \leq i \leq n\}.$$

Consider the chain $\mathcal{C} = (C_n)_{n \geq 1}$ with $C_1 = \{0\}$ and $C_n = \mathrm{cn}_n(E_n \cup F_n) \subseteq \mathbb{R}^n_{\geq 0}$ for $n \geq 2$. Then the following hold:

(i) $\mathcal{C}$ is Inc-invariant and each C_n is finitely generated in support size at most two.
(ii) $C_\infty \cap \mathbb{R}^n_{\geq 0} = C_n$ for all $n \geq 2$.
(iii) $\mathcal{C}$ does not stabilize.
(iv) C_∞ is not Inc-equivariantly finitely generated.

Proof. (i) For $n \geq 2$ let $\widetilde{F}_{n+1} = \{i\varepsilon_1 + \varepsilon_{i+1},\ i\varepsilon_2 + \varepsilon_{i+1} \mid 2 \leq i \leq n\}$. It is evident that

$$\mathrm{Inc}_{n,n+1}(E_n) = E_{n+1} \text{ and } \mathrm{Inc}_{n,n+1}(F_n) = F_n \cup \widetilde{F}_{n+1}.$$

Similarly to Example 5.10 one can show that

$$\mathrm{cn}_{n+1}(\mathrm{Inc}_{n,n+1}(C_n)) = \mathrm{cn}_{n+1}(E_{n+1} \cup F_n \cup \widetilde{F}_{n+1}) \text{ for } n \geq 2.$$

Since $F_n \subseteq F_{n+1}$ and $\widetilde{F}_{n+1} \subseteq \mathrm{cn}_{n+1}(E_{n+1} \cup F_{n+1})$, it follows that

$$\mathrm{cn}_{n+1}(\mathrm{Inc}_{n,n+1}(C_n)) \subseteq \mathrm{cn}_{n+1}(E_{n+1} \cup F_{n+1}) = C_{n+1}.$$

Hence, $\mathcal{C}$ is an Inc-invariant chain of convex cones. The remaining assertion is obvious.

(ii) Similar to Example 5.8(iii) one has

$$C_k \cap \mathbb{R}^n_{\geq 0} = \mathrm{cn}_k(E_k \cup F_k) \cap \mathbb{R}^n_{\geq 0} = \mathrm{cn}_n\left((E_k \cup F_k) \cap \mathbb{R}^n_{\geq 0}\right) = \mathrm{cn}_n(E_n \cup F_n) = C_n$$

for all $k \geq n \geq 2$. Therefore,

$$C_\infty \cap \mathbb{R}^n_{\geq 0} = \bigcup_{k \geq n} C_k \cap \mathbb{R}^n_{\geq 0} = \bigcup_{k \geq n} \left(C_k \cap \mathbb{R}^n_{\geq 0}\right) = C_n \text{ for all } n \geq 2.$$

(iii) $\mathcal{C}$ does not stabilize since $(n+1)\varepsilon_1 + \varepsilon_{n+1} \in C_{n+1} \setminus \mathrm{cn}_{n+1}(\mathrm{Inc}_{n,n+1}(C_n))$ for all $n \geq 2$.

(iv) C_∞ is not Inc-equivariantly finitely generated by Corollary 5.9. □

All cones considered here are given in their *V-representation*, i.e., generated by vectors. This is suitable to consider finite generation of cones. By the Minkowski–Weyl theorem [24, Theorem 1.3], finitely generated cones have a finite H-representation as an intersection of linear halfspaces. It would be interesting to understand the interaction of this duality with Π-equivariance. To work on the following problem, it might be necessary to replace the ambient space, which is a direct limit of $(\mathbb{R}^n_{\geq 0})_{n \geq 1}$, with an inverse limit. In general, inverse limits as ambient spaces provide many interesting directions for the future.

Problem 5.12. Let a chain of convex cones be given in their H-representations. Find H-versions of Corollaries 5.4 and 5.9 and relate them to the V-versions provided there. [1]

5.2. **Monoids up to symmetry.** In almost complete analogy to the cones discussed above, one can also consider monoids in $\mathbb{Z}^{(\mathbb{N})}_{\geq 0}$. Then the ambient chain is $\mathcal{R} = (\mathbb{Z}^n_{\geq 0})_{n \geq 1}$ with ambient set $R_\infty = \bigcup_{n \geq 1} \mathbb{Z}^n_{\geq 0} = \mathbb{Z}^{(\mathbb{N})}_{\geq 0}$. The closure operations are $\mathrm{mn}_n = \mathrm{mon}(\cdot)$ in R_n for $n \geq 1$ and $\mathrm{mn}_\infty = \mathrm{mon}(\cdot)$ in R_∞, where $\mathrm{mon}(\cdot)$ denotes the monoid closure. For $A \subseteq \mathbb{Z}^{(\mathbb{N})}$ it is defined as $\mathrm{mon}(A) = \{\sum_{i=1}^k m_i a_i \mid k \in \mathbb{N},\ a_i \in A,\ m_i \in \mathbb{Z}_{\geq 0}\}$. Now the theory develops in complete analogy to the case of cones, so we omit the details. One first shows that the system of monoid closures $(\mathcal{mn}, \mathrm{mn}_\infty)$ is consistent. An easy modification of Example 5.2 shows that $(\mathcal{mn}, \mathrm{mn}_\infty)$ is not consistent if one works in the ambient chain $(\mathbb{Z}^n)_{n \geq 1}$. As in Lemma 5.3, $\mathcal{mn}$ is compatible with Π for $\Pi = \mathrm{Sym}(\infty)$ and $\Pi = \mathrm{Inc}$. Again, if $\mathcal{M} = (M_n)_{n \geq 1}$ is an increasing chain of monoids with $M_n \subseteq \mathbb{Z}^n$ for all $n \geq 1$, then the limit M_∞ is a monoid in R_∞.

With these ingredients one finds the monoid versions of Theorems 4.10 and 4.11.

[1] This problem has been resolved for Sym-invariant chains of cones in [18].

Corollary 5.13. Let $\mathcal{M} = (M_n)_{n\geq 1}$ be a Sym-invariant chain of monoids $M_n \subseteq \mathbb{Z}^n_{\geq 0}$ with limit M_∞. Then the following statements are equivalent:

(a) $\mathcal{M}$ stabilizes and is eventually finitely generated;
(b) There exists an $r \in \mathbb{N}$ such that for all $n \geq r$ the following hold:
  (i) $M_\infty \cap \mathbb{Z}^n_{\geq 0} = M_n$,
  (ii) M_n is finitely generated by elements of support size at most r;
(c) M_∞ is Sym-equivariantly finitely generated.

Corollary 5.14. Let $\mathcal{M} = (M_n)_{n\geq 1}$ be an Inc-invariant chain of monoids $M_n \subseteq \mathbb{Z}^n_{\geq 0}$ with limit M_∞. Consider the following statements:

(a) $\mathcal{M}$ stabilizes and is eventually finitely generated.
(b) M_∞ is Inc-equivariantly finitely generated.

Then (a) implies (b) and if $\mathcal{M}$ is eventually saturated, then (a) and (b) are equivalent.

Various possible relaxations of the above results are false. We briefly discuss two.

Example 5.15. Let $M_1 = \{0\}$, $M_2 = \mathrm{mn}_2(e_1 + ke_2 \mid k \in \mathbb{N})$ and $M_n = \mathrm{mn}_n(\mathrm{Sym}_{2,n}(M_2))$ for $n \geq 3$. Then $\mathcal{M} = (M_n)_{n\geq 1}$ is a Sym-invariant chain of monoids. One checks that M_n is not finitely generated for $n \geq 2$ and M_∞ is not Sym-equivariantly finitely generated. By definition, $\mathcal{M}$ stabilizes. Thus, in Corollary 5.13(a) the second assumption cannot be omitted.

Example 5.16. Let $M_1 = \{0\}$, $M_2 = \mathrm{mn}_2(e_1 + 2e_2)$ and $M_n = \mathrm{mn}_n(\mathrm{Inc}_{n-1,n}(M_{n-1}) \cup \{e_1 + ne_n\})$ for $n \geq 3$. This Inc-invariant chain of finitely generated monoids does not stabilize and M_∞ is not Inc-equivariantly finitely generated, since for $i < k \leq j$, the generator $e_i + ke_j$ is irredundant. Thus, the first assumption in Corollary 5.14(a) is necessary for (a)$\Rightarrow$(b).

Remark 5.17. As for the cone and monoid situations studied in this section, one can also consider invariant chains of polytopes in $\mathbb{R}^{(\mathbb{N})}_{\geq 0}$. It is easy to derive the polytope version of Theorems 4.10 and 4.11 that are analogous to what we have done for cones and monoids.

6. Chains of ideals

We apply Theorem 4.10 and strengthen Theorem 4.11 when the equivariant chain consists of ideals in polynomial rings. These results are known in the literature. However, the proof in [13] of Theorem 6.3 contains a gap. We fill this gap.

Consider the ambient chain $\mathcal{R} = (R_n)_{n\geq 1}$ of polynomial rings $R_n = K[X_n]$. Then $R_\infty = K[x_{i,j} \mid i \in [c],\ j \in \mathbb{N}]$. For $n \in \mathbb{N} \cup \{\infty\}$ let $\langle \cdot \rangle_n$ denote the ideal closure operation in R_n, that is, if $A \subseteq R_n$ then $\langle A \rangle_n = \{\sum_{i=1}^k f_i a_i \mid k \in \mathbb{N},\ f_i \in R_n,\ a_i \in A\}$. One checks that these closure operations satisfy the assumptions of Theorems 4.10 and 4.11: $\langle \cdot \rangle_\infty$ is consistent with the chain $(\langle \cdot \rangle_n)_{n\geq 1}$; the chain $(\langle \cdot \rangle_n)_{n\geq 1}$ is compatible with both Sym and Inc; and for any chain $\mathcal{I} = (I_n)_{n\geq 1}$ of increasing ideals $I_n \subseteq R_n$ the limit set $I_\infty = \bigcup_{n\geq 1} I_n$ is an ideal in R_∞.

Hilbert's basis theorem (see, e.g. [10, Theorem 1.2]) says that R_n is a Noetherian ring for all $n \in \mathbb{N}$, that is, every ideal in R_n is finitely generated. The ring R_∞ is not Noetherian, but it is Π-Noetherian for $\Pi = \mathrm{Sym}$ and $\Pi = \mathrm{Inc}$, in the sense that every Π-invariant ideal in R_∞ is

Π-equivariantly finitely generated. This result was first proved by Cohen [7, Proposition 2], [8, Theorem 7] and later rediscovered by Aschenbrenner and Hillar [1, Theorem 1.1] and Hillar and Sullivant [13, Theorem 3.1, Corollary 3.5]. Moreover, [13, Theorem 3.1] shows that every Π-invariant ideal in R_∞ has a finite Π-Gröbner basis with respect to the lexicographic order $\preccurlyeq$ on R_∞ induced by

$$x_{i,j} \preccurlyeq x_{k,l} \Leftrightarrow j < l \;\text{ or }\; j = l \;\text{ and }\; i \leq k.$$

Here a subset G of a Π-invariant ideal $I \subseteq R_\infty$ is a Π-*Gröbner basis* for I with respect to $\preccurlyeq$, if for any $f \in I$ there exist $g \in G$ and $\pi \in \Pi$ such that $\mathrm{in}_\preccurlyeq(f)$ is divisible by $\mathrm{in}_\preccurlyeq(\pi(g))$.

The above results show that chains of ideals are better behaved than chains of cones and monoids. In particular, the statements of Theorems 4.10(c) and 4.11(b) on the equivariant finite generation of the limit ideal are *always true* for any Sym- or Inc-invariant chain of ideals. Thus, for Sym-invariant chains of ideals Theorem 4.10 immediately yields the following consequence, the first two statements of which are [13, Corollary 3.7] (see also [1, Theorem 4.7]).

Corollary 6.1. Let $\mathcal{I} = (I_n)_{n\geq 1}$ be a Sym-invariant chain of ideals $I_n \subseteq R_n$. Then

(i) $\mathcal{I}$ stabilizes.
(ii) There exists an $r \in \mathbb{N}$ such that for all $n \geq 1$, I_n is finitely generated by elements of support size at most r.
(iii) $\mathcal{I}$ is eventually saturated.

For Inc-invariant chains, using that Theorem 4.11(b) always holds, one obtains:

Corollary 6.2. Let $\mathcal{I} = (I_n)_{n\geq 1}$ be an Inc-invariant chain of ideals with $I_n \subseteq R_n$ for all $n \geq 1$. If $\mathcal{I}$ is eventually saturated, then it stabilizes.

Surprisingly, compared to cones and monoids, this result can be much improved.

Theorem 6.3. Every Inc-invariant chain of ideals stabilizes.

This result is the second part of [13, Theorem 3.6], but its proof contains a gap as we now explain. In order to show that an Inc-invariant chain of ideals $\mathcal{I} = (I_n)_{n\geq 1}$ stabilizes, the idea in [13] is to consider the global ideal $I_\infty = \bigcup_{n\geq 1} I_n$, show that this ideal has a finite Inc-Gröbner basis based on Higman's lemma (see [13, Theorem 3.1]), and then apply [13, Lemma 2.18] to get the desired conclusion (see [13, Theorem 2.19]). However, the problem here is that [13, Lemma 2.18] is applicable only to saturated chains, and so this argument works only for such chains.

To fix the gap, one can still follow the idea of using Higman's lemma as in [13], but instead of passing to the global ideal I_∞, one needs to work directly on the chain $\mathcal{I}$. The following proof was suggested by an anonymous referee and is much simpler than our original one.

Proof of Theorem 6.3. Let $\mathcal{I} = (I_n)_{n\geq 1}$ be an arbitrary Inc-invariant chain of ideals. Using the lexicographic order and a Gröbner basis argument, we may assume that the chain $\mathcal{I}$ consists of monomial ideals (see, e.g. [20, Lemma 7.1]). For every $n \in \mathbb{N}$ we have a bijection between the set $\operatorname{Mon}(R_n)$ of monomials of R_n and $(\mathbb{N}^c)^n$, in which each $u \in \operatorname{Mon}(R_n)$ is mapped to its exponent vector. Thus, there is a bijection between the disjoint unions

$$\operatorname{Mon}(\mathcal{R}) := \biguplus_{n\geq 1} \operatorname{Mon}(R_n) \quad \text{and} \quad (\mathbb{N}^c)^* := \biguplus_{n\geq 1} (\mathbb{N}^c)^n.$$

Endowing $\operatorname{Mon}(\mathcal{R})$ and $(\mathbb{N}^c)^*$ with the Higman partial order (see, e.g. [13, Definition 3.2]) the above bijection is in fact a poset isomorphism. By Higman's lemma [12, Theorem 4.3], the Higman order is a well-partial-order on $(\mathbb{N}^c)^*$. Hence, it is so on $\operatorname{Mon}(\mathcal{R})$ as well, and the chain $\mathcal{I}$ stabilizes. □

Remark 6.4. In the context of FI- and OI-modules, a similar idea to the one used in the proof of Theorem 6.3 was employed in [21] to show that certain FI- and OI-modules have finite Gröbner bases (see, in particular, [21, Propositions 5.3, 6.2, Theorem 6.14]).

We conclude this section with a discussion of the stability index of Inc-invariant chains.

Example 6.5. Although any non-saturated Inc-invariant chain of ideals stabilizes by Theorem 6.3, its stability index can behave very badly, unlike that of a saturated chain. Let $\mathcal{I} = (I_n)_{n\geq 1}$ be an Inc-invariant chain of ideals and let G be a finite Inc-Gröbner basis for the limit ideal I_∞ with respect to the lexicographic order $\preccurlyeq$. If $\mathcal{I}$ is saturated and $G \subseteq R_r$ for some $r \in \mathbb{N}$, then $\operatorname{ind}(\mathcal{I}) \leq r$ by [13, Lemma 2.18]. However, such a bound does not hold if $\mathcal{I}$ is not saturated. Indeed, let $m \in \mathbb{N}$ and consider the chain $\mathcal{I} = (I_n)_{n\geq 1}$ with

$$I_n = \begin{cases} \langle x_1^2, x_2^2, \dots, x_n^2 \rangle_n & \text{if } n \leq m-1, \\ \langle x_1, x_2^2, \dots, x_m^2 \rangle_n & \text{if } n = m, \\ \langle \operatorname{Inc}_{m,n}(I_m) \rangle_n & \text{if } n \geq m+1. \end{cases}$$

Then $\operatorname{ind}(\mathcal{I}) = m$, while $G = \{x_1\} \subseteq R_1$ is an Inc-Gröbner basis for the limit ideal $I_\infty = \bigcup_{n\geq 1} I_n = \langle x_1, x_2, x_3, \dots \rangle_\infty = \langle \operatorname{Inc}(x_1) \rangle_\infty$. This shows that there is no bound for $\operatorname{ind}(\mathcal{I})$ in terms of the smallest index r with $G \subseteq R_r$.

For non-saturated chains one may try to replace the condition "$G \subseteq R_r$" by "$G \subseteq I_s$" and ask for a bound for $\operatorname{ind}(\mathcal{I})$ in terms of the smallest index s with $G \subseteq I_s$. This is a reasonable question because in Example 6.5, $G = \{x_1\} \subseteq I_m$ and $s = m = \operatorname{ind}(\mathcal{I})$. Unfortunately, this also does not work. To build an example, note that in the chain $\mathcal{I}$ above one has

$$I_{m+j} = \langle \operatorname{Inc}_{m,m+j}(I_m) \rangle_{m+j} = \langle x_1, \dots, x_{j+1}, x_{j+2}^2, \dots, x_{m+j}^2 \rangle_{m+j} \quad \text{for } j \geq 0.$$

Now take $l \in \mathbb{N}$ and consider the chain $\mathcal{I}' = (I'_n)_{n\geq 1}$ with

$$I'_n = \begin{cases} I_n & \text{if } n \leq m+l-1, \\ I_{m+l} + \langle x_{l+2} \rangle_n & \text{if } n = m+l, \\ \langle \operatorname{Inc}_{m+l,n}(I'_{m+l}) \rangle_n & \text{if } n \geq m+l+1. \end{cases}$$

It is clear that $I'_\infty = \bigcup_{n\geq 1} I'_n = \langle x_1, x_2, x_3, \dots \rangle_\infty = I_\infty$. Thus, $G' = G = \{x_1\} \subseteq I'_m$ is an Inc-Gröbner basis for I'. However, $\operatorname{ind}(\mathcal{I}') = m+l$ is not bounded by m as l is arbitrary.

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