

COMBINATORICS-BASED APPROACHES TO CONTROLLABILITY CHARACTERIZATION FOR BILINEAR SYSTEMS

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Abstract. The control of bilinear systems has attracted considerable attention in the field of systems and control for decades, owing to their prevalence in diverse applications across science and engineering disciplines. Although much work has been conducted on analyzing controllability properties, the mostly used tool remains the Lie algebra rank condition. In this paper, we develop alternative approaches based on theory and techniques in combinatorics to study controllability of bilinear systems. The core idea of our methodology is to represent vector fields of a bilinear system by permutations or graphs, so that Lie brackets are represented by permutation multiplications or graph operations, respectively. Following these representations, we derive combinatorial characterization of controllability for bilinear systems, which consequently provides novel applications of symmetric group and graph theory to control theory. Moreover, the developed combinatorial approaches are compatible with Lie algebra decompositions, including the Cartan and non-intertwining decomposition. This compatibility enables the exploitation of representation theory for analyzing controllability, which allows us to characterize controllability properties of bilinear systems governed by semisimple and reductive Lie algebras.

Key words. Bilinear systems, Lie groups, graph theory, symmetric groups, representation theory, Cartan decomposition

1. Introduction. Bilinear systems, a class of nonlinear systems, emerge naturally as mathematical models to describe the dynamics of numerous processes in science and engineering. Prominent examples include the Bloch system governing the dynamics of spin- $\frac{1}{2}$ nuclei immersed in a magnetic field in quantum physics [11, 19, 20], the compartmental model describing the movement of cells and molecules in biology [22, 8, 21], and the integrate-and-fire model characterizing the membrane potential of a neuron under synaptic inputs and injected current in neuroscience [7, 10]. The prevalence of bilinear systems has been actively promoting the research in control theory and engineering concerning the analysis and manipulation of such systems for decades. The initial investigation into control problems involving bilinear systems traces back to the year of 1935, when the Greek mathematician Constantin Carathéodory studied optimal control of bilinear systems presented in terms of Pfaffian forms by using calculus of variations and partial differential equations [5]. However, research in systematic analysis of fundamental properties of bilinear control systems was not prosperous until the early 1970s, when leading control theorists, such as Brockett, Jurdjevic, and Sussmann, developed geometric control theory for introducing techniques in Lie theory and differential geometry to classical control theory [4, 2, 16, 13, 3, 12].

One of the most remarkable results in geometric control theory is the Lie algebra rank condition (LARC), which establishes an equivalence between controllability of control-affine systems defined on smooth manifolds and Lie algebras generated by the vector fields governing the system dynamics [2, 14, 15]. In our recent work, based on the LARC, we developed a necessary and sufficient controllability condition for bilinear systems by using techniques in symmetric group theory [27]. In particular, we introduced a monoid structure on symmetric groups so that Lie bracket operations are compatible with monoid operations. This then resulted in a characterization of controllability in terms of elements in “symmetric monoids” for bilinear systems,

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which also offered an alternative to the LARC and further shed light on interpreting geometric control theory from an algebraic perspective.

In this paper, we propose a combinatorics-based framework to analyze controllability of bilinear systems defined on Lie groups by adopting techniques in symmetric group theory and graph theory. Specifically, the main idea is to associate such systems with permutations or graphs, so that Lie bracket operations of the vector fields governing the system dynamics can be represented by permutation multiplications and edge operations on the graphs. This combinatorics approach immediately leads to the characterizations of controllability in terms of permutation cycles and graph connectivity. In particular, we identify the classes of bilinear systems, for which controllability has equivalent symmetric group and graph representations. A prominent example is the system defined on $SO(n)$, the special orthogonal group, for which we reveal a correspondence between permutation cycles in the symmetric group and trees in the graph associated with these systems. It is worth noting that, different from our previous work on the symmetric group method [27], the correspondence between Lie bracket operations and permutation multiplications established in this paper do not require any monoid structure on symmetric groups. On the other hand, the application of graph theory in the developed combinatorics-based framework offers a distinct viewpoint to the field of control theory. Specifically, in the existing literature, graphs are naturally used in the context of networked and multi-agent systems, e.g., for describing the coupling topology and deriving structural controllability conditions [23, 24, 25], while, in this work, we establish a non-trivial relationship between graph connectivity and controllability for a single bilinear system.

Moreover, a great advantage of the developed framework is its compatibility with various Lie algebra decomposition techniques in representation theory. In particular, we illustrate the application of these methods to systems of which the underlying Lie algebras are semisimple or reductive, while in these cases, the correspondence between Lie bracket operations and permutation multiplications as well as graph operations is elusive due to their complicated algebraic structures. In this work, we exploit the Cartan and non-intertwining decompositions to decompose the system Lie algebras into simple components, so that the combinatorics-based controllability analysis is equivalently carried over to these components.

This paper is organized as follows. In Section 2, we provide the preliminaries relevant to our developments, including the LARC for systems on Lie groups and a brief review of the Lie algebra $\mathfrak{so}(n)$. In Section 3, we establish the symmetric group and graph-theoretic methods based upon the study of bilinear systems on $SO(n)$. In Section 4, we introduce the notions and tools of Cartan and non-intertwining decompositions for decomposing the system Lie algebras into simpler components, which enables and facilitates the generalization of the combinatorics-based framework to broader classes of bilinear systems. A brief review of the basics of symmetric groups and Lie algebra decompositions can be found in the appendices.

2. Preliminaries. To prepare for our development of the combinatorial controllability conditions, in this section, we briefly review the Lie algebra $\mathfrak{so}(n)$ and the LARC for right-invariant bilinear systems. Meanwhile, we introduce the notations we use throughout this paper.

2.1. The Lie Algebra Rank Condition. The LARC has been the most recognizable tool, if not unique, for analyzing controllability of bilinear systems since the 1970s. It establishes a connection between controllability and the Lie algebra generated by the vector fields governing the system dynamics. In this paper, we primarily

focus on the bilinear system evolving on a compact and connected Lie group of the form,

$$(2.1) \quad \dot{X}(t) = B_0 X(t) + \left(\sum_{i=1}^m u_i(t) B_i \right) X(t), \quad X(0) = I,$$

where $X(t) \in G$ is the state on a compact and connected Lie group G , I is the identity element of G , B_i are elements in the Lie algebra \mathfrak{g} of G , and $u_i(t) \in \mathbb{R}$ are piecewise constant control inputs. For any subset $\Gamma \subseteq \mathfrak{g}$, we use $\text{Lie}(\Gamma)$ to denote the Lie subalgebra generated by Γ , i.e., the smallest vector subspace of \mathfrak{g} containing Γ that is closed under the Lie bracket defined by $[C, D] := CD - DC$ for $C, D \in \mathfrak{g}$. With these notations, the LARC for the system in (2.1) can be stated as follows.

THEOREM 2.1 (LARC). *The system in (2.1) is controllable on G if and only if $\text{Lie}(\Gamma) = \mathfrak{g}$, where $\Gamma = \{B_0, B_1, \dots, B_m\}$.*

Proof. See [2]. □

2.2. Basics of the Lie Algebra $\mathfrak{so}(n)$. The Lie algebra $\mathfrak{so}(n)$ is a vector space of dimension $n(n-1)/2$, which consists of all n -by- n real skew-symmetric matrices. In particular, if we use Ω_{ij} to denote the skew-symmetric matrix with 1 in the (i, j) -th entry, -1 in the (j, i) -th entry, and 0 elsewhere, then the set $\mathcal{B} = \{\Omega_{ij} \in \mathbb{R}^{n \times n} : 1 \leq i < j \leq n\}$ forms a basis of $\mathfrak{so}(n)$, which we refer to as the *standard basis* of $\mathfrak{so}(n)$. The following lemma then reveals the Lie bracket relations among elements in \mathcal{B} .

LEMMA 2.2. *The Lie bracket of Ω_{ij} and Ω_{kl} satisfies the relation $[\Omega_{ij}, \Omega_{kl}] = \delta_{jk}\Omega_{il} + \delta_{il}\Omega_{jk} + \delta_{jl}\Omega_{ki} + \delta_{ik}\Omega_{lj}$, where δ is the Kronecker delta function defined by*

$$\delta_{mn} = \begin{cases} 1, & \text{if } m = n; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof follows directly from computations. □

The relations in Lemma 2.2 can also be equivalently expressed as $[\Omega_{ij}, \Omega_{kl}] \neq 0$ if and only if $i = k, i = l, j = k$, or $j = l$. This algebraic structure facilitates controllability characterization of the bilinear system governed by the vector fields represented in the standard basis \mathcal{B} , which is the main focus of the next section.

3. Combinatorics-Based Controllability Analysis for Bilinear Systems.

In this section, we introduce a combinatorics-based framework to characterize controllability of bilinear systems. Within this framework, we adopt tools in two subfields of combinatorics - the symmetric group theory and graph theory, and build connections of Lie brackets of vector fields to permutation multiplications in symmetric groups and operations on graph edges, respectively. Such connections enable us to characterize controllability in terms of permutation cycles and graph connectivity. Here, we will investigate bilinear systems defined on $\text{SO}(n)$, given by

$$(3.1) \quad \dot{X}(t) = \Omega_{i_0 j_0} X + \left(\sum_{k=1}^m u_k(t) \Omega_{i_k j_k} \right) X, \quad \Omega_{i_k j_k} \in \mathcal{B}, \quad X(0) = I.$$

as building blocks to establish this framework. Furthermore, we will show that owing to the special algebraic structure of $\mathfrak{so}(n)$ presented in Lemma 2.2, the symmetric group and the graph-theoretic approach, when applied to (3.1), give an equivalent characterization of controllability through an interconnection between symmetric groups and graphs.

3.1. The Symmetric Group Method for Controllability Analysis. In this section, we introduce the symmetric group method for analyzing controllability of the system in (3.1). In this approach, a subset of vector fields in \mathcal{B} is represented using a permutation in S_n , the symmetric group of n letters. Through this representation, we connect the Lie brackets of vector fields to permutation multiplications, so that controllability is determined by the length of permutation cycles. For a brief summary of symmetric groups and permutations, see [Appendix A](#).

3.1.1. Mapping Lie Brackets to Permutations. To establish a relation from Lie brackets to permutation multiplications, we first define a *relation* between subsets of \mathcal{B} and permutations in S_n by

$$(3.2) \quad \iota : \mathcal{P}(\mathcal{B}) \rightarrow S_n, \quad \iota(\{\Omega_{i_0 j_0}, \Omega_{i_1 j_1}, \dots, \Omega_{i_m j_m}\}) = (i_0 j_0)(i_1 j_1) \cdots (i_m j_m).$$

Because every permutation can be decomposed into a product of transpositions (2-cycles), the relation ι is surjective so that every subset of \mathcal{B} admits a permutation representation.

To see how Lie bracket operations are related to permutation multiplications by ι , we illustrate the idea using two elements $\Omega_{ij}, \Omega_{kl} \in \mathcal{B}$. On the Lie algebra level, if $[\Omega_{ij}, \Omega_{kl}] \neq 0$, then [Lemma 2.2](#) implies that $\{i, j\}$ and $\{k, l\}$ have a common index. Without loss of generality, we may assume $j = k$ and $i \neq l$, so that $[\Omega_{ij}, \Omega_{jl}] = \Omega_{il}$. Meanwhile, on the symmetric group level, we have $\iota(\{\Omega_{ij}, \Omega_{jl}\}) = (ij)(jl) = (ijl)$, so the permutation multiplication increases the cycle length by 1, from the 2-cycle factors (ij) and (jk) to a 3-cycle (ijk) . However, if $[\Omega_{ij}, \Omega_{kl}] = 0$, then $\{i, j\} \cap \{k, l\} = \emptyset$, and $\iota(\{\Omega_{ij}, \Omega_{kl}\}) = (ij)(kl)$ is a product of two disjoint cycles. The phenomenon that elements in \mathcal{B} with non-vanishing Lie brackets relating to a cycle with increased length extends inductively to larger subsets of \mathcal{B} . To be more specific, if $\Gamma \subset \mathcal{B}$ contains m elements such that the iterated Lie brackets of them are non-vanishing, then $\iota(\Gamma)$ is an $(m + 1)$ -cycle. This observation immediately motivates the use of cycle length to examine controllability of systems on $\text{SO}(n)$ in (3.1). Before we state and prove our main theorem, let us first illustrate the symmetric group method by two examples.

Example 3.1. Consider a system evolving on $\text{SO}(5)$, given by

$$(3.3) \quad \dot{X}(t) = \left(\sum_{i=1}^4 u_i(t) \Omega_{i, i+1} \right) X(t), \quad X(0) = I,$$

and let $\Gamma = \{\Omega_{i, i+1} : i = 1, \dots, 4\}$ denote the set of control vector fields. The correspondence between Lie brackets in Γ and permutation multiplications in S_5 follows

$$(3.4) \quad \begin{aligned} [\Omega_{12}, \Omega_{23}] &= \Omega_{13} &\leftrightarrow & (12)(23) = (123), \\ [\Omega_{23}, \Omega_{34}] &= \Omega_{24} &\leftrightarrow & (23)(34) = (234), \\ [\Omega_{34}, \Omega_{45}] &= \Omega_{35} &\leftrightarrow & (34)(45) = (345), \\ [\Omega_{12}, [\Omega_{23}, \Omega_{34}]] &= \Omega_{14} &\leftrightarrow & (12)(234) = (1234), \\ [\Omega_{23}, [\Omega_{34}, \Omega_{45}]] &= \Omega_{25} &\leftrightarrow & (23)(345) = (2345), \\ [\Omega_{12}, [\Omega_{23}, [\Omega_{34}, \Omega_{45}]]] &= \Omega_{15} &\leftrightarrow & (12)(2345) = (12345). \end{aligned}$$

Note that successively Lie bracketing elements in Γ results in $\Omega_{13}, \Omega_{14}, \Omega_{15}, \Omega_{24}, \Omega_{25}$, and Ω_{35} , together with the 4 elements in Γ , we have 10 linearly independent vector fields. Because $\mathfrak{so}(5)$ is a 10-dimensional Lie algebra, we conclude $\text{Lie}(\Gamma) = \mathfrak{so}(5)$, which implies that the system in (3.3) is controllable on $\text{SO}(5)$ by the LARC. On the

other hand, (3.4) also shows $\iota(\Gamma) = (12345)$, a cycle of maximum length in S_5 . This suggests that controllability of systems on $\text{SO}(n)$ can be characterized by cycles of *maximum* length in the corresponding symmetric group.

Example 3.2. Consider another system evolving on $\text{SO}(5)$ driven by three controls, given by

$$(3.5) \quad \dot{X}(t) = (u_1(t)\Omega_{12} + u_2(t)\Omega_{23} + u_3(t)\Omega_{45})X(t), \quad X(0) = I.$$

In this case, the single Lie brackets,

$$\begin{aligned} [\Omega_{12}, \Omega_{23}] &= \Omega_{13} &\leftrightarrow (12)(23) &= (123), \\ [\Omega_{12}, \Omega_{45}] &= 0 &\leftrightarrow (12)(45), \\ [\Omega_{23}, \Omega_{45}] &= 0 &\leftrightarrow (23)(45), \end{aligned}$$

and the double Lie brackets,

$$\begin{aligned} [\Omega_{13}, \Omega_{12}] &= [[\Omega_{12}, \Omega_{23}], \Omega_{12}] = \Omega_{23} &\leftrightarrow (12)(23)(12) &= (13), \\ [\Omega_{23}, \Omega_{13}] &= [\Omega_{23}, [\Omega_{12}, \Omega_{23}]] = \Omega_{12} &\leftrightarrow (23)(12)(23) &= (13), \\ [\Omega_{13}, \Omega_{45}] &= [[\Omega_{12}, \Omega_{23}], \Omega_{45}] = 0 &\leftrightarrow (12)(23)(45) &= (123)(45), \end{aligned}$$

result in a Lie subalgebra of dimension 4. Therefore, this system is *not* controllable on $\text{SO}(5)$. On the other hand, for $\Gamma = \{\Omega_{12}, \Omega_{23}, \Omega_{45}\}$, the computations above also show $\iota(\Gamma) = (123)(45)$, which is *not* a single cycle of maximum length in S_5 .

Examples 3.1 and *3.2* together verify the observation that cycles with the maximum length characterize controllability of bilinear systems on $\text{SO}(n)$, which we will prove in the next section.

Remark 3.3. Note that the relation ι introduced in (3.2) is *not* a well-defined function, because, for a given $\Gamma \subseteq \mathcal{B}$, $\iota(\Gamma)$ depends on the ordering of the elements in Γ . If, say, $\Gamma = \{\Omega_{12}, \Omega_{14}, \Omega_{23}, \Omega_{24}, \Omega_{34}\}$, then different element orderings,

$$\begin{aligned} \{\Omega_{12}, \Omega_{14}, \Omega_{23}, \Omega_{24}, \Omega_{34}\} &\leftrightarrow (12)(14)(23)(24)(34) = (14) \\ \{\Omega_{14}, \Omega_{12}, \Omega_{24}, \Omega_{23}, \Omega_{34}\} &\leftrightarrow (14)(12)(24)(23)(34) = (1234) \end{aligned}$$

could result in *different* permutations. Nevertheless, we can verify that for any $\Gamma \subseteq \mathcal{B}$, there always exists a subset $\Sigma \subseteq \Gamma$ such that ι relates Σ to permutations with the same (maximal) orbits, albeit different orderings of the elements in Σ . For example, for the subset $\Sigma = \{\Omega_{12}, \Omega_{23}, \Omega_{34}\}$ of Γ , $\iota(\Sigma)$ is always a 4-cycle with its orbit being $\{1, 2, 3, 4\}$, regardless of its element orderings. The existence of such a subset will be clear once we develop a graph visualization of the permutations in Section 3.2.

3.1.2. Controllability Characterization in Terms of Permutation Cycles. Leveraging the technique of mapping Lie brackets to permutations developed in the previous section, we are able to characterize controllability of systems on $\text{SO}(n)$ in terms of permutation cycles as shown in the following theorem.

THEOREM 3.4. *The control system defined on $\text{SO}(n)$ of the form*

$$(3.6) \quad \dot{X}(t) = \left(\Omega_{i_0 j_0} + \sum_{k=1}^m u_k(t) \Omega_{i_k j_k} \right) X(t), \quad X(0) = I,$$

(same system as (3.1)) where $\Gamma := \{\Omega_{i_k j_k}\} \subseteq \mathcal{B}$ for $k = 0, \dots, m$, is controllable if and only if there is a subset $\Sigma \subseteq \Gamma$ such that $\iota(\Sigma)$ is an n -cycle, where ι is the relation defined in (3.2).

Proof. By the LARC, the system in (3.6) is controllable on $\text{SO}(n)$ if and only if $\text{Lie}(\Gamma) = \mathfrak{so}(n)$. Therefore, it is equivalent to showing that $\text{Lie}(\Sigma) = \mathfrak{so}(n)$ if and only if $\iota(\Sigma)$ is an n -cycle for some $\Sigma \subseteq \Gamma$.

(Sufficiency): Suppose there exists a subset $\Sigma \subseteq \Gamma$ such that $\iota(\Sigma)$ is an n -cycle. Because an n -cycle can be decomposed into a product of at least $n - 1$ transpositions, this implies $m \geq n - 1$. Hence, it suffices to assume that the cardinality of Σ is $n - 1$, and, without loss of generality, let $\Sigma = \{\Omega_{i_1 j_1}, \dots, \Omega_{i_{n-1} j_{n-1}}\}$. Because $\iota(\Sigma)$ is an n -cycle, it follows that the index set $\{i_1, j_1, \dots, i_{n-1}, j_{n-1}\} = \{1, \dots, n\}$. Note that the set $\{i_1, j_1, \dots, i_{n-1}, j_{n-1}\}$ contains repeated elements. Next, we prove the sufficiency by induction.

When $n = 3$, suppose there exists a subset $\Sigma = \{\Omega_{ij}, \Omega_{kl}\} \subset \Gamma$ and that $\iota(\Sigma) = (ij)(kl)$ is a 3-cycle, so we must have one of the following: $i = k, j = k, i = l$, or $j = l$. Consequently, $[\Omega_{ij}, \Omega_{kl}] \in \mathcal{B} \setminus \Sigma$, so $\{\Omega_{ij}, \Omega_{kl}, [\Omega_{ij}, \Omega_{kl}]\}$ spans $\mathfrak{so}(3)$. Therefore, the system in (3.6) is controllable on $\text{SO}(3)$.

Now let us assume that for $n \geq 4$, a system defined on $\text{SO}(n - 1)$ in the form of (3.6) is controllable if there is $\Sigma \subseteq \Gamma$ such that $\iota(\Sigma)$ is an $(n - 1)$ -cycle. Let $\Sigma \subseteq \Gamma$ be a set of $n - 1$ elements such that $\iota(\Sigma) = (i_{n-1} j_{n-1})(i_{n-2} j_{n-2}) \cdots (i_1 j_1)$ is a cycle of length n , then for every integer $1 \leq k \leq n - 1$, there exists some $1 \leq l \leq n - 1$ such that $\{i_k, j_k\} \cap \{i_l, j_l\} \neq \emptyset$. Consequently, there are $n - 2$ transpositions of the form $(i_k j_k)$, $k = 1, \dots, n - 1$, such that their product is a cycle of length $n - 1$. Without loss of generality, we may assume that $\iota(\Sigma \setminus \{\Omega_{i_{n-1} j_{n-1}}\}) = (i_{n-2} j_{n-2}) \cdots (i_1 j_1)$ is a $(n - 1)$ -cycle with the nontrivial orbit $\{i_1, j_1, \dots, i_{n-2}, j_{n-2}\} = \{1, \dots, n - 1\}$. By the induction hypothesis, the system in (3.6) is controllable on $\text{SO}(n - 1) \subset \text{SO}(n)$. Equivalently, any $\Omega_{ij} \in \mathcal{B}$ such that $1 \leq i < j \leq n - 1$ can be generated by iterated Lie brackets of the elements in $\Sigma \setminus \{\Omega_{i_{n-1} j_{n-1}}\}$. Because $\iota(\Sigma) = (i_{n-1} j_{n-1}) \iota(\Sigma \setminus \{\Omega_{i_{n-1} j_{n-1}}\})$ is a n -cycle, we must have $i_{n-1} \in \{1, \dots, n - 1\}$ and $j_{n-1} = n$. Therefore, Ω_{kn} can be generated by the Lie brackets $[\Omega_{ki_{n-1}}, \Omega_{i_{n-1} j_{n-1}}]$ for any $k = 1, \dots, n - 1$. As a result, the system in (3.6) is controllable on $\text{SO}(n)$.

(Necessity): Because the system in (3.6) is controllable, $\text{Lie}(\Gamma) = \mathfrak{so}(n)$. Then, there exists a subset Σ of Γ such that $\text{Lie}(\Sigma) = \mathfrak{so}(n)$ and Σ contains no *redundant elements*, i.e., the elements that can be generated by Lie brackets of the other elements in Σ . Without loss of generality, we assume $\Sigma = \{\Omega_{i_1 j_1}, \dots, \Omega_{i_l j_l}\}$, where $l \leq m$. By Lemma 2.2, for any $\Omega_{ab}, \Omega_{cd} \in \Sigma$, if $[\Omega_{ab}, \Omega_{cd}] \neq 0$, then there must exist a bridging index, i.e., we must have one of the following cases: $a = c, a = d, b = c$, or $b = d$. This, together with $\text{Lie}(\Sigma) = \mathfrak{so}(n)$, implies that the index set J of Σ is $J = \{i_1, j_1, \dots, i_l, j_l\} = \{1, \dots, n\}$, and that for any $\Omega_{i_k j_k} \in \Sigma$, there exists some $\Omega_{i_s j_s} \in \Sigma$ with $s \neq k$ such that $\{i_k, j_k\} \cap \{i_s, j_s\} \neq \emptyset$. Moreover, because Σ contains no redundant elements, $\iota(\Sigma) = \iota(\Omega_{i_l j_l}) \cdots \iota(\Omega_{i_1 j_1})$ is a cycle whose orbit contains every element in $\{1, \dots, n\}$, namely, it is a cycle of length n . In addition, the cardinality of Σ is $n - 1$. \square

Remark 3.5. Following the above proof, it requires at least $n - 1$ controls for the system on $\text{SO}(n)$ in (3.6) to be fully controllable and, on the other hand, for $\iota(\Sigma)$, $\Sigma \subseteq \Gamma$, to reach a cycle of length n .

Similar to the case in Theorem 3.4 for controllable systems, the controllable submanifold for an uncontrollable system also depends on the permutation related to a subset of Γ . To be more specific, the cycle decomposition of such a permutation determines the involutive distribution of the submanifold.

COROLLARY 3.6. *Given a system evolving on $\text{SO}(n)$ in the form of (3.1), let Ξ be a minimal subset of Γ , such that $\text{Lie}(\Xi) = \text{Lie}(\Gamma)$. If $\iota(\Xi) = \sigma_1 \cdot \sigma_2 \cdots \sigma_l$ so*

that each σ_k , $1 \leq k \leq l$, are pairwise disjoint cycles with the nontrivial orbits \mathcal{O}_k , then the controllable submanifold of the system is the Lie subgroup of $\text{SO}(n)$ with the Lie algebra $\text{Lie}(\Gamma) = \bigoplus_{k=1}^l \text{span}\{\Omega_{ij} : i, j \in \mathcal{O}_k\}$. Conversely, if $\text{Lie}(\Gamma) = \bigoplus_{k=1}^l \text{span}\{\Omega_{ij} : i, j \in \mathcal{O}_k\}$ for some $\mathcal{O}_k \subset \{1, 2, \dots, n\}$, then $\iota(\Xi) = \sigma_1 \cdot \sigma_2 \cdots \sigma_l$ and σ_k are that disjoint cycles with nontrivial orbits \mathcal{O}_k .

Proof. Let Ξ be a minimal subset of Γ such that $\text{Lie}(\Xi) = \text{Lie}(\Gamma)$ and Ξ does not contain redundant elements. First, let $\sigma = \iota(\Xi) \in S_n$ be a cycle with nontrivial orbit \mathcal{O} , then [Theorem 3.4](#) implies $\text{Lie}(\Xi) = \text{span}\{\Omega_{ij} : i, j \in \mathcal{O}, i < j\}$. Next, if $\sigma = \sigma_1 \cdots \sigma_l$ is a permutation as a product of disjoint cycles $\sigma_1, \dots, \sigma_l$ with $l \geq 2$, then there exists a partition $\{\Xi_1, \dots, \Xi_l\}$ of Ξ such that $\iota(\Xi_k) = \sigma_k$ for each $k = 1, \dots, l$. Let \mathcal{O}_k denotes the nontrivial orbit of σ_k for each $k = 1, \dots, l$, then $\text{Lie}(\Xi_k) = \{\Omega_{ij} : i, j \in \mathcal{O}_k, i < j\}$ and the sets $\mathcal{O}_1, \dots, \mathcal{O}_l$ are pairwise disjoint subsets of $\{1, \dots, n\}$. Hence, $\text{Lie}(\Xi_i) \cap \text{Lie}(\Xi_j) = \{0\}$ holds for all $i \neq j$, and consequently, we have $\text{Lie}(\Xi) = \text{Lie}(\Xi_1) \oplus \cdots \oplus \text{Lie}(\Xi_l)$, where \oplus denotes the direct sum of vector spaces. By the Frobenius Theorem [\[26\]](#), $\text{Lie}(\Xi)$ is completely integrable, and that the set of all its maximal integral manifolds forms a foliation \mathcal{F} of $\text{SO}(n)$. Since the initial condition of the system in [\(3.6\)](#) is the identity matrix I , the leaf of \mathcal{F} passing through I is the controllable submanifold of the system in [\(3.6\)](#). The converse is obvious following a very similar argument. \square

According to [Theorem 3.4](#) and [Corollary 3.6](#), mapping the control vector fields in Γ to permutations provides not only an alternative approach to effectively examine controllability of systems defined on $\text{SO}(n)$, but also a systematic procedure to characterize the controllable submanifold when the system is not fully controllable. Let us now revisit a previous example and see how permutations help determine system controllability.

Example 3.7 (Controllable Submanifold). Recall [Example 3.2](#), where the system in [\(3.5\)](#) is not controllable and there exist no subsets of $\Gamma = \{\Omega_{12}, \Omega_{23}, \Omega_{45}\}$ such that $\iota(\Gamma)$ is a 5-cycle. In addition, the controllable submanifold is the integral manifold of the involutive distribution $\Delta = \text{Lie}\{\Omega_{12}X, \Omega_{23}X, \Omega_{13}X, \Omega_{45}X\} = \text{span}\{\Omega_{ij}X : i, j \in \{1, 2, 3\} \text{ or } i, j \in \{4, 5\}\}$, which can be identified by the nontrivial orbits of $\iota(\Gamma) = (1, 2, 3)(4, 5)$. On the other hand, for each $X \in \text{SO}(5)$, the complement $\Delta_X^\perp = \text{span}\{\Omega_{ij}X : i = 1, 2, 3, j = 4, 5\}$ of the distribution evaluated at X contains the bridging elements required for full controllability of this system.

3.2. The Graph-Theoretic Method for Controllability Analysis. Graphs appear naturally in the research of networked systems, especially in modeling multi-agent systems and analyzing structural controllability [\[23, 24, 25\]](#). However, most graph-theoretic methods were dedicated to studying networked control systems in existing literature and were not invented and applied for understanding fundamental properties of a single bilinear system. Here, we use graphs to represent the structure of Lie algebras and then characterize controllability of bilinear systems by graph connectivity. In contrast to the symmetric group method presented in [Section 3.1](#), this graph-theoretic method establishes a correspondence between Lie bracket operations of vector fields and operations on the edges of graphs.

3.2.1. Mapping Lie Brackets to Graphs. A graph G , conventionally denoted by a 2-tuple, $G = (V, E)$, consists of a vertex set V and an edge set E . For the purpose of analyzing controllability of the system on $\text{SO}(n)$, we are particularly interested in simple graphs, i.e., undirected graphs with no loops or multiple edges, of n vertices.

Here, we denote the collection of such graphs \mathcal{G} . Without loss of generality, we further assume that every graph in \mathcal{G} has the same vertex set $V = \{v_1, \dots, v_n\}$. Following these notations, we define a map

$$(3.7) \quad \tau : \mathcal{P}(\mathcal{B}) \rightarrow \mathcal{G} \quad \text{by} \quad \tau(\Gamma) = (V, E_\Gamma) := G_\Gamma,$$

where $\mathcal{P}(\mathcal{B})$ denotes the power set of \mathcal{B} , i.e., the set consisting of all subsets of \mathcal{B} and $E_\Gamma = \{v_i v_j : \Omega_{ij} \in \Gamma\}$. Some basic properties of τ are summarized in the following proposition.

PROPOSITION 3.8 (Properties of τ).

- (i) *The map τ defined in (3.7) is bijective.*
- (ii) *For any $\Gamma \subseteq \mathcal{B}$, $|\Gamma| = |E_\Gamma|$ holds, where $|\cdot|$ denote the cardinality of a set.*
- (iii) *Let K_n denote the complete graph of n vertices, i.e., the graph whose vertices are pairwise adjacent, then $\tau(\mathcal{B}) = K_n$.*

Proof. Note that (i) and (ii) directly follow from the definition of τ . For (iii), the edge set of $\tau(\mathcal{B})$ satisfies $E_{\mathcal{B}} = \{v_i v_j : \Omega_{ij} \in \mathcal{B}\} = \{v_i v_j : 1 \leq i < j \leq n\} = \{v_i v_j : i, j = 1, \dots, n\}$, and hence we conclude $\tau(\mathcal{B}) = K_n$. \square

The property (i) in Proposition 3.8 reveals a one-to-one correspondence between the subsets of \mathcal{B} and the graphs in \mathcal{G} , which enables the representation of Lie bracket operations by graph operations as follows.

Algebraically, for any $\Omega_{ij}, \Omega_{jk} \in \mathcal{B}$, Lemma 2.2 implies $[\Omega_{ij}, \Omega_{jk}] = \Omega_{ik} \neq 0$, so that $\text{Lie}\{\Omega_{ij}, \Omega_{jk}\} = \text{span}\{\Omega_{ij}, \Omega_{jk}, \Omega_{ik}\}$. Graphically, by the definition of τ , the two edges $\tau(\Omega_{ij}) = v_i v_j$ and $\tau(\Omega_{jk}) = v_j v_k$ share a common vertex v_j , and the edge $\tau([\Omega_{ij}, \Omega_{jk}]) = \tau(\Omega_{ik}) = v_i v_k$ intersects with $\tau(\Omega_{ij})$ and $\tau(\Omega_{jk})$ at endpoints v_i and v_k , respectively. Therefore, the three edges $\tau(\Omega_{ij})$, $\tau(\Omega_{jk})$, and $\tau([\Omega_{ij}, \Omega_{jk}])$ form a triangle, or equivalently, $\tau(\{\Omega_{ij}, \Omega_{jk}, [\Omega_{ij}, \Omega_{jk}]\}) = \{v_i v_j, v_j v_k, v_i v_k\} = K_3$. This observation, as summarized in the following lemma, reveals the relationship between first-order Lie brackets and graph operations for three standard basis elements of $\mathfrak{so}(n)$, which lays the foundation for the graph-theoretic controllability analysis of bilinear systems.

LEMMA 3.9. *If $\Omega_{ij}, \Omega_{kl} \in \mathcal{B}$ satisfy $[\Omega_{ij}, \Omega_{kl}] \neq 0$, then*

- (i) *the two edges $\tau(\Omega_{ij})$ and $\tau(\Omega_{kl})$ are incident (i.e., they share a common vertex);*
- (ii) *the three edges $\tau(\Omega_{ij})$, $\tau(\Omega_{kl})$, and $\tau([\Omega_{ij}, \Omega_{kl}])$ form a triangle.*

To graphically characterize higher-order Lie brackets among arbitrary collections of standard basis elements of $\mathfrak{so}(n)$, we introduce the notion of triangular closure for graphs, which generalizes the action of “forming triangles” in Lemma 3.9.

DEFINITION 3.10 (Triangular Closure). *Let $G = (V, E)$ be a graph, and $\{G^m = (V, E^m) : m = 0, 1, \dots\}$ be an ascending chain of graphs, i.e., $G^m \subseteq G^{m+1}$ for any $m = 0, 1, \dots$, satisfying*

- (i) $G^0 = G$, i.e., $E^0 = E$.
- (ii) *For any $m \geq 0$, $v_i v_j \in E^{m+1}$ if and only if $v_i v_j \in E^m$ or there exists some vertex $v_k \in V$ such that $v_i v_k, v_k v_j \in E^m$.*

Then the union of all G^m , denoted $\bar{G} = \bigcup_{m=1}^{\infty} G^m$, or equivalently, $\bar{G} = (V, \bar{E}) = (V, \bigcup_{m=1}^{\infty} E^m)$, is called the triangular closure of G . Moreover, a graph G is called triangularly closed if $G = \bar{G}$.

Note that for a finite graph G , i.e., G has finitely many vertices and edges, the ascending chain of graphs $G = G^0 \subseteq G^1 \subseteq \dots$ in Definition 3.10 stabilizes in finite

steps, that is, there exists a nonnegative integer m such that $G^m = G^{m+1} = \dots$, which then implies $\bar{G} = G^m$. In particular, for a graph with n vertices, since it has at most $n(n-1)/2$ edges, its triangular closure can be obtained in at most $n(n-1)/2$ steps.

Remark 3.11. For readers familiar with graph theory, [Definition 3.10](#) is mathematically equivalent to the standard definition of *transitive closure*, and the equivalence will become transparent in the proof of [Theorem 3.18](#). The triangular closure we introduce here imitates the computations of graded Lie brackets/algebras in a more natural way, so that all orders of Lie brackets can be calculated in a graph.

Recall from [Lemma 3.9](#) that given a subset $\Gamma \subseteq \mathcal{B}$ and its associated graph $G = \tau(\Gamma)$, taking first-order Lie brackets of the elements in Γ corresponds to adding edges that connect the endpoints of incident edges in G . Applying this procedure to $G = G^0$, as defined in [Definition 3.10](#), exactly results in G^1 . Inductively, successively Lie bracketing the elements in Γ up to order m will generate the graph G^m , as shown below.

THEOREM 3.12. *Given a subset $\Gamma \subseteq \mathcal{B}$, let $\Gamma^0 \subseteq \Gamma^1 \subseteq \dots$ be an ascending chain of subsets of \mathcal{B} such that $\Gamma^0 = \Gamma$, $\Gamma^1 = [\Gamma^0, \Gamma^0] \cup \Gamma^0$, \dots , $\Gamma^{m+1} = [\Gamma^m, \Gamma^m] \cup \Gamma^m$, \dots , where $[\Gamma^m, \Gamma^m] = \{[A, B] : A, B \in \Gamma^m\}$. Then $G^m = \tau(\Gamma^m)$ holds for all $m = 0, 1, \dots$*

Proof. This follows immediately from the definitions of G^m and Γ^m . \square

Recall that for any finite $G \in \mathcal{G}$, G^m stabilizes to \bar{G} in finite steps. Meanwhile, by [Theorem 3.12](#), Γ^m also stabilizes to a subset $\hat{\Gamma} \subseteq \mathcal{B}$ which must satisfy $\bar{G} = \tau(\hat{\Gamma})$. Intuitively, $\hat{\Gamma}$ is supposed to contain all the elements that can be generated by the iterated Lie brackets of the elements in Γ , because \bar{G} is the largest graph generated by G . This conclusion is then rigorously verified in the following corollary.

COROLLARY 3.13. *Let Γ be a subset of \mathcal{B} and $G = \tau(\Gamma)$ be the graph associated with Γ . If $\hat{\Gamma} \subseteq \mathcal{B}$ satisfies $\tau(\hat{\Gamma}) = \bar{G}$, then $\text{Lie}(\Gamma) = \text{span}(\hat{\Gamma})$.*

Proof. Let m be a nonnegative integer satisfying $G^m = \bar{G}$, then [Theorem 3.12](#) implies that $\hat{\Gamma} = \Gamma^m$, hence $\Gamma^r = \hat{\Gamma}$ holds for all $r \geq m$. Consequently, by the definition of $\text{Lie}(\Gamma)$, we have $\text{Lie}(\Gamma) = \text{span}(\bigcup_{i=0}^{\infty} \Gamma^i) = \text{span}(\Gamma^m) = \text{span}(\hat{\Gamma})$. \square

For the purpose of controllability analysis, the subsets of \mathcal{B} generating the whole Lie algebra $\mathfrak{so}(n)$ is of great interest. Therefore, we characterize such subsets by their associated graphs below, which is also a special case of [Corollary 3.13](#).

COROLLARY 3.14. *Consider a subset $\Gamma \subseteq \mathcal{B}$ with the associated graph $G = \tau(\Gamma)$, then $\text{Lie}(\Gamma) = \mathfrak{so}(n)$ if and only if $\bar{G} = K_n$.*

Proof. (Sufficiency): Let $\hat{\Gamma} \subseteq \mathcal{B}$ satisfy $\bar{G} = \tau(\hat{\Gamma}) = K_n$, then the properties (i) and (iii) in [Proposition 3.8](#) imply $\hat{\Gamma} = \mathcal{B}$. Consequently, $\text{Lie}(\Gamma) = \text{span}(\hat{\Gamma}) = \text{span}(\mathcal{B}) = \mathfrak{so}(n)$ by [Corollary 3.13](#).

(Necessity): If $\text{Lie}(\Gamma) = \mathfrak{so}(n)$, then there exists some nonnegative integer m such that $\Gamma^m = \mathcal{B}$. By [Theorem 3.12](#), we obtain $\bar{G} \supseteq G^m = \tau(\Gamma^m) = K_n$. On the other hand, because of $\bar{G} \subseteq K_n$, we conclude $\bar{G} = K_n$. \square

Furthermore, [Corollary 3.14](#) sheds light on a graph representation of controllability, which in turn can be characterized in terms of graph connectivity. In the following section, we will rigorously investigate this observation.

3.2.2. Controllability Characterization in Terms of Graph Connectivity. The relationship between Lie brackets and graph operations developed in [Section 3.2.1](#) enables us to employ graph theory techniques to analyze controllability of

systems on $\text{SO}(n)$ as in (3.1). In particular, motivated by the connection between a Lie subalgebra and its associated graph presented in Corollary 3.14, controllability can be analyzed through the notion of triangular closure defined in Definition 3.10.

PROPOSITION 3.15. *The bilinear system in (3.1) is controllable on $\text{SO}(n)$ if and only if $\overline{\tau(\Gamma)} = K_n$, where τ is defined as in (3.7), $\Gamma = \{\Omega_{i_0 j_0}, \dots, \Omega_{i_m j_m}\}$, and K_n is a complete graph of n vertices.*

Proof. By the LARC shown in Theorem 2.1, the system in (3.1) is controllable on $\text{SO}(n)$ if and only if $\text{Lie}(\Gamma) = \mathfrak{so}(n)$, which is equivalent to $\overline{\tau(\Gamma)} = K_n$ by Corollary 3.14. \square

Using the following two examples, we will verify Proposition 3.15 and draw a parallel between examining the LARC and generating triangular closure of the graph associated with the considered system. This comparison in turn illuminates a graphic visualization of the algebraic procedure of generating Lie algebras for the set of drift and control vector fields.

Example 3.16. Consider the system on $\text{SO}(4)$ given by

$$(3.8) \quad \dot{X}(t) = (u_1\Omega_{12} + u_2\Omega_{23} + u_3\Omega_{13} + u_4\Omega_{34})X(t), \quad X(0) = I.$$

Applying τ to the set of the control vector fields Γ results in its associated graph $G = (V, E)$ as follows,

$$\Gamma = \{\Omega_{12}, \Omega_{23}, \Omega_{13}, \Omega_{34}\} \xleftrightarrow{\tau} \{v_1v_2, v_2v_3, v_1v_3, v_3v_4\} = E.$$

Because the first order Lie brackets $[\Omega_{23}, \Omega_{34}] = \Omega_{24}$ and $[\Omega_{13}, \Omega_{34}] = \Omega_{14}$ are not in Γ , we have $\Gamma^1 = \Gamma \cup \{\Omega_{24}, \Omega_{14}\}$. Correspondingly, according to Corollary 3.13, $G^1 = (V, E^1)$ can be obtained by applying τ to Γ^1 , i.e.,

$$\Gamma^1 = \Gamma \cup \{\Omega_{24}, \Omega_{14}\} \xleftrightarrow{\tau} \{v_2v_4, v_1v_4\} \cup E = E^1.$$

Notice that $\text{span}(\Gamma^1) = \mathfrak{so}(4)$ and simultaneously $G^1 = \bar{G} = K_4$, which concludes controllability of the system in (3.8) from both algebraic and graph-theoretic perspectives. The graphs G and G^1 are shown in Figure 3.1. In particular, the two red edges in G^1 , which are not in G , correspond to the elements in $[\Gamma, \Gamma]$.

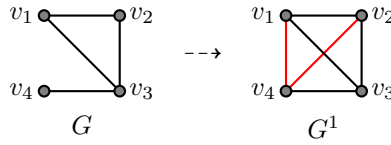


FIG. 3.1. *The graph G associated with the system (3.8) in Example 3.16 and its triangular closure G^1 . Note that the red edges in G^1 correspond to the vector fields generated by the first-order Lie brackets of the control vector fields in Γ .*

Example 3.16 presents a controllable system whose associated graph has a complete triangular closure, which in turn validates the sufficiency of Proposition 3.15. The necessity is illustrated using the following example through an uncontrollable system.

Example 3.17. Consider the system on $\text{SO}(5)$ driven by three control inputs, given by

$$(3.9) \quad \dot{X}(t) = (u_1\Omega_{12} + u_2\Omega_{23} + u_3\Omega_{34})X(t), \quad X(0) = I,$$

and let $\Gamma = \{\Omega_{12}, \Omega_{23}, \Omega_{34}\}$ denote the set of control vector fields. Some straightforward calculations yield the Lie algebra $\text{Lie}(\Gamma) = \text{span}\{\Omega_{12}, \Omega_{23}, \Omega_{34}, \Omega_{13}, \Omega_{14}, \Omega_{24}\}$, which has dimension 6. Therefore, the system in (3.9) is not controllable, since $\dim \mathfrak{so}(5) = 10$. Using the graph-theoretic approach, Figure 3.2 shows the procedure of generating \bar{H} from $H = \tau(\Gamma)$. In particular, $\bar{H} = H^2$ shown in Figure 3.2 is not complete, which verifies the necessity of Proposition 3.15.

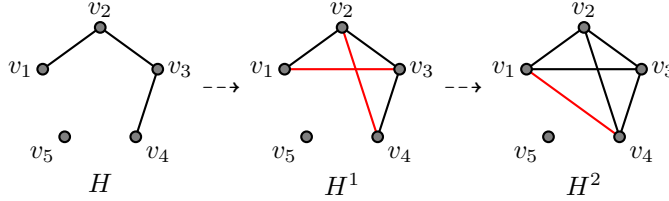


FIG. 3.2. The graph visualization of Lie bracketing control vector fields of the system in (3.9) in Example 3.17. Specifically, the graph H is associated with the set of control vector fields, H^1 visualizes the first-order Lie brackets, and H^2 is the triangular closure of H . Note that the red edges correspond to the vector fields in Γ generated by Lie brackets.

It is worth noting that the graph G in Figure 3.1 associated with the controllable system in (3.8) is connected, but the graph H in Figure 3.2 associated with the uncontrollable system in (3.9) is not. This observation inspires the characterization of controllability for systems on $\text{SO}(n)$ by graph connectivity.

THEOREM 3.18. *The system in (3.1) is controllable on $\text{SO}(n)$ if and only if $\tau(\Gamma)$ is connected, where $\Gamma = \{\Omega_{i_0j_0}, \dots, \Omega_{i_mj_m}\}$ and $\tau(\Gamma)$ is the graph associated with Γ .*

Proof. Owing to Proposition 3.15, it suffices to prove that the triangular closure of $\tau(\Gamma)$ is complete if and only if $\tau(\Gamma)$ is connected.

(Sufficiency): Suppose that $G = \tau(\Gamma) = (V, E)$ is connected, then there is a path in G from v_i to v_j for any $v_i, v_j \in V$, say $v_i w_1 w_2 \dots w_k v_j$ with $w_1, \dots, w_k \in V$. Therefore, we have $v_i w_2 \in E^1, \dots, v_i w_k \in E^{k-1}$ and $v_i v_j \in E^k \subseteq \bar{E}$. Since $v_i, v_j \in V$ are chosen arbitrarily, we conclude that the triangular closure \bar{G} contains all edges $v_i v_j$, hence $\bar{G} = K_n$. In addition, this process of generating \bar{G} is illustrated in Figure 3.3 with the case of $k = 5$.

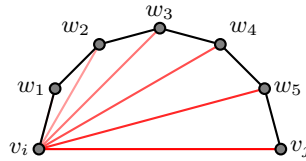


FIG. 3.3. Illustration of the proof of sufficiency of Theorem 3.18.

(Necessity): We assume that the triangular closure \bar{G} of $G = \tau(\Gamma)$ is complete. If there exists an edge $v_i v_j$ not in G , since $v_i v_j$ is in $\bar{G} = (V, \bar{E})$, we may then assume $v_i v_j \in E^k$ and $v_i v_j \notin E^{k-1}$ for some positive integer k . Hence, by Definition 3.10, there is some vertex w_1 such that $v_i w_1, w_1 v_j \in E^{k-1}$, i.e., there exists a path $v_i w_1 v_j$ in G^{k-1} connecting v_i and v_j . Repeating this procedure results in a path in G connecting v_i and v_j , which implies the connectivity of G , and hence the proof is done. \square

Remark 3.19. In addition to controllability characterization, [Theorem 3.18](#) highlights a crucial property of the map τ , that is, $\text{Lie}(\Gamma) = \mathfrak{so}(n)$ for some $\Gamma \subseteq \mathcal{B}$ if and only if $\tau(\Gamma)$ is connected, which is an equivalent formulation of [Theorem 3.18](#).

Because a connected graph with n vertices contains at least $n - 1$ edges, [Theorem 3.18](#) also identifies the minimum number of control inputs for the system in (3.1) to be controllable, as identified using the symmetric group method presented in [Theorem 3.4](#) and [Remark 3.5](#).

COROLLARY 3.20. *If a system on $\text{SO}(n)$ in (3.1) is controllable, then the number of control inputs m is at least $n - 2$, i.e., $m \geq n - 2$.*

Although [Theorem 3.18](#) is developed to examine controllability, it also helps establish some general facts in graph theory from the control systems perspective. In the following, we present one such result that is related to triangular closures. This property also plays an important role in characterizing controllable submanifolds for uncontrollable systems by connected component of the graph associated with the control system.

LEMMA 3.21. *The triangular closure \bar{G} of a graph G is a disjoint union of its complete components.*

Proof. The proof is a direct application of the proof of [Theorem 3.18](#) to each connected component of G . \square

By the above [Lemma 3.21](#), we can adopt our main result in [Theorem 3.18](#) to study an uncontrollable system by taking the triangular closure of its associated graph, which is the union of the triangular closures of all connected components.

THEOREM 3.22. *The controllable submanifold of the system in (3.1) is determined by the connected components of its associated graph.*

Proof. Let $\Gamma \subseteq \mathcal{B}$ be the set of vector fields governing the dynamics of the system in (3.1), $G = \tau(\Gamma)$ be the graph representation of Γ , and \bar{G} denote the triangular closure of G . Since connected components of G determine the complete components of \bar{G} , it suffices to show that the controllable submanifold of the system is determined by the complete components of \bar{G} .

According to the Frobenius Theorem [26], the controllable submanifold of the system in (3.1) is the maximal integral submanifold of $\text{Lie}(\Gamma)$ passing through the identity matrix I . Hence, by [Lemma 3.21](#), because of the completeness of each component of \bar{G} , the set $\tau^{-1}(\bar{G}) \subseteq \mathcal{B}$ is closed under Lie bracket, which implies $\text{span } \tau^{-1}(\bar{G}) = \text{Lie}(\Gamma)$. Therefore, we conclude that $\text{Lie}(\Gamma)$, and thus its maximal integral submanifold, is determined by \bar{G} . \square

[Theorem 3.22](#) further reveals a one-to-one correspondence between the Lie algebra generated by a subset of \mathcal{B} and the triangular closure of its associated graph in \mathcal{G} . Leveraging this one-to-one correspondence, we are able to give an explicit characterization of controllable submanifolds for uncontrollable systems in terms of connected components of their associated graphs.

Example 3.23 (Controllable Submanifold). Consider two bilinear systems defined on $\text{SO}(6)$ in the form of (3.1) governed by the vector fields $\Gamma_1 = \{\Omega_{12}, \Omega_{23}, \Omega_{45}, \Omega_{46}\}$ and $\Gamma_2 = \{\Omega_{13}, \Omega_{23}, \Omega_{46}, \Omega_{56}\}$, respectively. [Figure 3.4](#) shows their associated graphs $G_1 = \tau(\Gamma_1)$ and $G_2 = \tau(\Gamma_2)$, neither of which is connected. Therefore, by [Theorem 3.12](#), both systems are not controllable on $\text{SO}(6)$. On the other hand, we notice that $\bar{G}_1 = \bar{G}_2$. So by [Theorem 3.22](#), the two systems have the same controllable

submanifold. Specifically, the controllable submanifold is the Lie subgroup of $\text{SO}(6)$ with the Lie algebra

$$\text{Lie}(\Gamma_1) = \text{Lie}(\Gamma_2) = \text{span}\{\Omega_{ij} : 1 \leq i < j \leq 3\} \oplus \text{span}\{\Omega_{ij} : 4 \leq i < j \leq 6\}.$$

Moreover, both $\overline{G_1}$ and $\overline{G_2}$ contain two complete components with the vertex sets $U = \{v_1, v_2, v_3\}$ and $W = \{v_4, v_5, v_6\}$, which are also the vertex sets of the connected components of G_1 (or G_2). It then follows that the Lie algebra of the controllable submanifold, $\text{span}\{\Omega_{ij} : v_i, v_j \in U\} \oplus \text{span}\{\Omega_{ij} : v_i, v_j \in W\}$, can be explicitly characterized by the vertex sets of the complete components of $\overline{G_1}$ and $\overline{G_2}$, as well as the connected components of G_1 and G_2 .

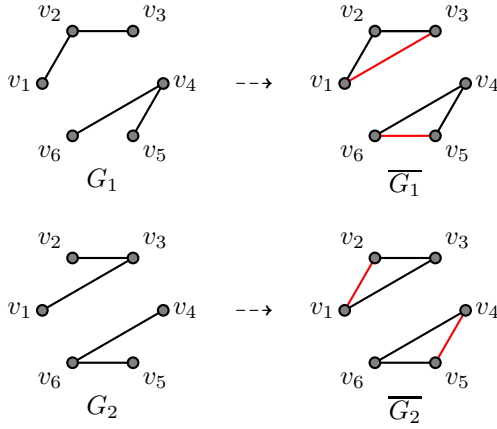


FIG. 3.4. The graphs and their triangular closures associated with the systems in [Example 3.23](#). Specifically, the graphs G_1 and G_2 on the left are associated with the systems governed by Γ_1 and Γ_2 , respectively, and their triangular closures $\overline{G_1}$ and $\overline{G_2}$ are on the right. Red edges correspond to vector fields generated by Lie brackets.

Furthermore, the developed method of characterizing controllability in terms of graph connectivity is not constrained to systems defined on Lie groups. In particular, as shown in the following example, it can be applied to study formation control of multi-agent systems defined on graphs. From an algebraic perspective, it is equivalent to using the graph-theoretic method to analyze the Lie algebra generated by *symmetric matrices*.

Example 3.24 (Formation Control). In this example, we consider formation control of a multi-agent system, which concerns with the question of coordinating the system to a consensus state. For such a purpose, the dynamics of each agent in a network of N agents with the coupling topology given by the graph $G = (V, E)$, $V = \{v_1, \dots, v_N\}$, is generally represented by

$$(3.10) \quad \dot{x}_i(t) = \sum_{j \in V(i)} u_{ij}(x_j - x_i), \quad 1 \leq i \leq N,$$

where $x_i(t) \in \mathbb{R}^n$ denotes the state of the i -th agent, $V(i) = \{1 \leq j \leq N : v_i v_j \in E\}$ denotes the set of neighboring agents of i , and $u_{ij} = u_{ji}$ are the external inputs that control the reciprocal interaction between the i -th and j -th agents [6].

We will first formulate the dynamic law in (3.10) into a matrix form, and then apply our analysis on a Lie algebra associated with it. To do this, let

$$A_{ij} := E_{ii} + E_{jj} - E_{ij} - E_{ji}$$

be an N -by- N symmetric matrix with zero row and column sums, and let $X \in \mathbb{R}^{N \times n}$ denote a matrix whose row vectors are the states of the agents:

$$X = \begin{bmatrix} x_1^\top \\ \vdots \\ x_N^\top \end{bmatrix}.$$

Then, we can rewrite (3.10) into the following matrix form,

$$(3.11) \quad \dot{X} = \sum_{v_i v_j \in E} u_{ij} A_{ij} X.$$

The formation controllability of the multi-agent system in (3.11) is determined by the LARC [6]. Thus, to study this system, we need to know the algebraic structure of matrices $\{A_{ij}\}$. Observe that for $B_{ijk} = -(\Omega_{ij} + \Omega_{jk} + \Omega_{ki})$ and distinct indices $1 \leq i, j, k, l, m \leq N$, we have

$$(3.12) \quad \begin{aligned} [A_{ij}, A_{jk}] &= B_{ijk}, \\ [B_{ijk}, A_{ij}] &= 2(A_{ik} - A_{jk}), \\ [B_{ijk}, A_{il}] &= -A_{ij} + A_{jl} + A_{ik} - A_{kl}, \\ [B_{ijk}, B_{ijl}] &= B_{ikl} + B_{jkl}, \\ [B_{ijk}, B_{ilm}] &= B_{jlm} + B_{kml} = B_{lkj} + B_{mjk}. \end{aligned}$$

Therefore, the Lie algebra $\mathfrak{g} := \text{Lie}\{A_{ij}\}$ has a decomposition, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$, with $\mathfrak{g}_1 = \text{span}\{B_{ijk}\}$ and $\mathfrak{g}_{-1} = \text{span}\{A_{ij}\}$. As a consequence, by the LARC, controllability of system (3.11) depends on whether the set $\Gamma := \{A_{ij} : (i, j) \in E\}$ generates the Lie algebra \mathfrak{g} . Similar to bilinear systems on $\text{SO}(n)$, we can adopt a graph-theoretic method for \mathfrak{g} by associating *one part of* \mathfrak{g} , i.e., \mathfrak{g}_{-1} , to a graph, which in the case of this example, *coincides* with the graph on which the system is defined. To be more specific, for a complete graph K_N and its set of edges E , we may define a map $\tau : \Gamma \rightarrow E$, which sends $A_{ij} \in \Gamma$ to $v_i v_j \in E$, so that the image of Γ is exactly the graph G . Following the correspondence τ , for two adjacent edges $v_i v_j$ and $v_j v_k$, since $\text{Lie}\{A_{ij}, A_{jk}\} = \text{Lie}\{A_{ij}, A_{jk}, A_{ki}\} = \text{span}\{A_{ij}, A_{jk}, A_{ki}, B_{ijk}\}$, the triangle with edges $v_i v_j, v_j v_k, v_k v_i$ in G represents the Lie subalgebra spanned by $\{A_{ij}, A_{jk}, A_{ki}, B_{ijk}\}$. More generally, by the algebraic relations in (3.12), any triangularly closed subgraph of K_N is associated with a subalgebra of \mathfrak{g} . Therefore, the Lie (sub)algebra generated by Γ can be represented by the triangular closure of G ; and if G is connected, then its triangular closure is complete, which suggests that $\text{Lie}(\Gamma)$ contains all A_{ij} 's, so we have $\text{Lie}(\Gamma) = \mathfrak{g}$. In conclusion, the controllability of system (3.10), and equivalently, system (3.11), is determined by graph connectivity of G .

By now, we have conducted a detailed investigation into controllability of bilinear systems on $\text{SO}(n)$ governed by the standard basis elements of $\mathfrak{so}(n)$. Before we extend the scope of our investigation to general bilinear systems, we show that, in contrast to Corollary 3.20, a driftless bilinear system on $\text{SO}(n)$ can be controllable using only two control inputs, for all $n > 0$.

Example 3.25. Recall that by [Corollary 3.20](#), driftless bilinear systems on $\text{SO}(n)$ with control vector fields in the standard basis of $\mathfrak{so}(n)$ require at least $n - 1$ inputs to be controllable. However, this conclusion may not hold for general systems governed by vector fields not in the standard basis. For example, the following system with two control inputs

$$(3.13) \quad \dot{X}(t) = [u_1(t)C_1 + u_2(t)C_2]X(t), \quad X(0) = I,$$

where $C_1 = \Omega_{12}$ and $C_2 = \sum_{i=1}^{n-1} \Omega_{i,i+1}$, is controllable on $\text{SO}(n)$. To see this, we will show $\Omega_{1k} \in \text{Lie}(\{C_1, C_2\})$ for any $2 \leq k \leq n$ by induction. At first, note that $\Omega_{13} = [C_1, C_2] \in \text{Lie}(\{C_1, C_2\})$. Next, we assume $\Omega_{12}, \Omega_{13}, \dots, \Omega_{1k} \in \text{Lie}(\{C_1, C_2\})$ for some $3 \leq k < n$, which is the induction hypothesis. Consequently, we have $[\Omega_{1k}, C_2] = \Omega_{2k} - \Omega_{1,k-1} + \Omega_{1,k+1}$ and $[\Omega_{1k}, C_1] = \Omega_{2k}$, which implies $\Omega_{1,k+1} = [\Omega_{1k}, C_2] - [\Omega_{1k}, C_1] + \Omega_{1,k-1} \in \text{Lie}(\{C_1, C_2\})$. By induction, we conclude $\Omega_{1k} \in \text{Lie}(\{C_1, C_2\})$ for any $2 \leq k \leq n$. This result implies $\text{Lie}(\Sigma) \subseteq \text{Lie}(\{C_1, C_2\})$, where $\Sigma = \{\Omega_{1k} : 1 \leq k \leq n\}$. Obviously $\tau(\Sigma)$ is a connected graph, and hence by [Theorem 3.18](#), $\text{Lie}(\Sigma) = \mathfrak{so}(n)$, and the the system in (3.13) is thus controllable.

3.3. Equivalence Between the Symmetric Group and Graph-Theoretic Methods. In Sections 3.1 and 3.2, we developed two combinatorics-based methods to analyze controllability of bilinear systems. Both methods connect the Lie brackets of vector fields to operations on combinatorial objects. We will show next that an equivalence exists between the symmetric group and the graph-theoretic method when systems on $\text{SO}(n)$ are concerned. We first illustrate this equivalence through a controllable system on $\text{SO}(4)$.

Example 3.26. Let us revisit the system in (3.8) in [Example 3.16](#), governed by the set of vector fields $\Gamma = \{\Omega_{12}, \Omega_{23}, \Omega_{13}, \Omega_{34}\}$. We have shown therein that this system is controllable on $\text{SO}(4)$ by using the graph-theoretic method; and for the symmetric group method, we may choose $\Sigma_1 = \{\Omega_{12}, \Omega_{13}, \Omega_{34}\} \subset \Gamma$ so that $\iota(\Sigma_1) = (1342)$ is a 4-cycle. However, Σ_1 is not the only subset that is related to a 4-cycle, and, for example, one can easily verify that ι also relates the subsets $\Sigma_2 = \{\Omega_{13}, \Omega_{23}, \Omega_{34}\}$ and $\Sigma_3 = \{\Omega_{12}, \Omega_{23}, \Omega_{34}\}$ to 4-cycles as $\iota(\Sigma_2) = (13)(23)(34) = (1342)$ and $\iota(\Sigma_3) = (12)(23)(34) = (1234)$. Moreover, it is worth noting that Σ_1, Σ_2 , and Σ_3 are the only subsets of Γ that are related to 4-cycles. Meanwhile, and more importantly, their graph representations $\tau(\Sigma_1), \tau(\Sigma_2)$, and $\tau(\Sigma_3)$ coincide with all three spanning trees of the graph $\tau(\Gamma)$ associated with the system (see [Table 3.1](#)). On the other hand, from the aspect of Lie algebra, we observe that Σ_i is a *minimal* subset of Γ generating $\text{Lie}(\Gamma)$ for each $i = 1, 2, 3$, that is, $\Sigma' = \Sigma_i$ for any $\Sigma' \subseteq \Sigma_i$ satisfying $\text{Lie}(\Sigma') = \text{Lie}(\Gamma)$. This observation sheds light on the general result: given a system on $\text{SO}(n)$ governed by the set of vector fields Γ , if Σ is a minimal subset of Γ with $\text{Lie}(\Sigma) = \text{Lie}(\Gamma)$, then $\iota(\Sigma)$ is an n -cycle if and only if $\tau(\Sigma)$ is a spanning tree of $\tau(\Gamma)$.

THEOREM 3.27. *Consider a bilinear system on $\text{SO}(n)$ as in (3.1) and let $\Gamma \subseteq \mathcal{B}$ denote the set of vector fields governing the system dynamics. Suppose $\Sigma \subseteq \Gamma$ is a minimal subset such that $\iota(\Sigma) = \sigma \in S_n$ is an n -cycle (i.e., Σ has no proper subset that is also related to an n -cycle via ι), then its associated graph $\tau(\Sigma)$ is a spanning tree of $\tau(\Gamma)$, and the system is therefore controllable. Conversely, for a controllable system, any spanning tree T of the connected graph $\tau(\Gamma)$ corresponds to a subset $\Sigma' = \tau^{-1}(T)$, such that $\Sigma' \subseteq \Gamma$ is minimal and that $\iota(\Sigma')$ is an n -cycle in S_n .*

Proof. From group theory we know that a *minimal* Σ with $\iota(\Sigma)$ being an n -cycle should consist of $n - 1$ transpositions, and that the union of the orbits of all $n - 1$

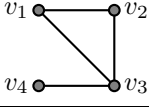
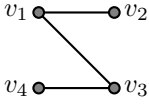
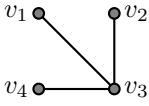
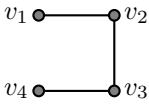
Set of control vector fields	Graph	Permutation in S_4
$\Gamma = \{\Omega_{12}, \Omega_{23}, \Omega_{13}, \Omega_{34}\}$		$\iota(\Gamma) = (12)(23)(13)(34) = (234)$
$\Sigma_1 = \{\Omega_{12}, \Omega_{13}, \Omega_{34}\}$		$\iota(\Sigma_1) = (12)(13)(34) = (1342)$
$\Sigma_2 = \{\Omega_{13}, \Omega_{23}, \Omega_{34}\}$		$\iota(\Sigma_2) = (13)(23)(34) = (1342)$
$\Sigma_3 = \{\Omega_{12}, \Omega_{23}, \Omega_{34}\}$		$\iota(\Sigma_3) = (12)(23)(34) = (1234)$

TABLE 3.1

A comparison between two methods analyzing controllability: the symmetric groups method and the graph-theoretic method. Note that the graphs associated with Σ_1 , Σ_2 and Σ_3 are spanning trees of the associated graph of Γ , and that any tree is related to a 4-cycle in the symmetric group S_4 .

transpositions is the orbit of σ . This means the graph $\tau(\Sigma)$ has n vertices and $n - 1$ edges. Since a graph with n vertices and $n - 1$ edges is both *connected* and *acyclic*, and since $\tau(\Sigma)$ covers all n vertices of $\tau(\Gamma)$, we conclude that $\tau(\Sigma)$ is the spanning tree of $\tau(\Gamma)$.

On the other hand, for a subset $\Sigma' \subseteq \Gamma$ satisfying that $\tau(\Sigma')$ is a spanning tree of $\tau(\Gamma)$, we must have $|\Sigma'| = n - 1$. Since a decomposition of an n -cycle needs at least $n - 1$ transpositions, if $\iota(\Sigma')$ is an n -cycle, then Σ' is obviously minimal. The following claim shows that $\iota(\Sigma')$ is indeed an n -cycle, regardless of the ordering of elements in Σ' .

CLAIM. A tree consisting of k edges in the connected graph $\tau(\Gamma)$ in [Theorem 3.27](#) is related to a $(k + 1)$ -cycle via ι , regardless of the ordering of transpositions.

Proof of Claim. Let us consider a tree T with k edges in $\tau(\Gamma)$, and prove the claim by induction. It is trivial for $k = 1$; and for $k = 2$, say $T = v_{j_1}v_{j_2}v_{j_3}$, then ι sends T to either $(j_1j_2j_3)$ or $(j_1j_3j_2)$, depending on the orderings of (j_1j_2) and (j_2j_3) . Assume the claim is true for $k = l - 1$ for some $l \in \mathbb{Z}_+$; and for a tree T with $k = l$ edges, we can choose a subtree T' of T that consists of $l - 1$ edges. Without loss of generality, we may assume T' has vertices $\{v_1, \dots, v_l\}$ and that T has an additional vertex v_{l+1} and an additional edge v_1v_{l+1} . Let $\{\sigma_1, \dots, \sigma_{l-1}\}$ be a set of transpositions such that each σ_i is related to a distinct edge in T' by ι , and let $\rho = (1, l + 1)$ denote the transposition related to the additional edge v_1v_{l+1} in T . Our goal is to show that the permutation

$$(3.14) \quad \sigma_{i_1} \cdots \sigma_{i_t} \rho \sigma_{i_{t+1}} \cdots \sigma_{i_{l-1}}$$

is an $(l + 1)$ -cycle. Note that for any $1 \leq i, j_1, j_2 \leq l$, we have the following law of

commutation:

$$(3.15) \quad (i, l+1)(j_1 j_2) = \begin{cases} (j_1 j_2)(i, l+1) & \text{if neither } j_1 \text{ or } j_2 \text{ equals to } i; \\ (j_1 j_2)(j_2, l+1) & \text{if } j_1 = i. \end{cases}$$

Therefore, we can rewrite the permutation (3.14) as $\sigma_{i_1} \cdots \sigma_{i_{l-1}} \rho'$, where $\rho' = (j_p, l+1)$ for some $1 \leq j_p \leq l$. By our assumption, $\sigma_{i_1} \cdots \sigma_{i_{l-1}}$ is an l -cycle: $\sigma_{i_1} \cdots \sigma_{i_{l-1}} = (j_1 j_2 \cdots j_l)$, so finally we have

$$\begin{aligned} \sigma_{i_1} \cdots \sigma_{i_t} \rho \sigma_{i_{t+1}} \cdots \sigma_{i_{l-1}} &= \sigma_{i_1} \cdots \sigma_{i_{l-1}} \rho' = (j_1 j_2 \cdots j_l)(j_p, l+1) \\ &= (j_1 \cdots j_p, l+1, j_{p+1} \cdots j_l), \end{aligned}$$

which is an $(l+1)$ -cycle. It is clear that the ordering of transpositions is irrelevant in our proof. \blacklozenge

Therefore, a spanning tree $\tau(\Sigma')$ consisting of $n-1$ edges is related to an n -cycle in S_n via ι , which finishes our proof. \square

Given a controllable system on $\text{SO}(n)$, [Theorem 3.27](#) reveals the relation between n -cycles and spanning trees of the associated graph. In particular, for such a system governed by the set $\Gamma \subseteq \mathcal{B}$ of vector fields, this theorem supplements [Theorem 3.18](#) by explicitly describing the subsets of Γ that are related to n -cycles using graphs. The following corollary then summarizes all the symmetric group and graph-theoretic characterizations of controllability for systems on $\text{SO}(n)$.

COROLLARY 3.28. *Consider a bilinear system defined on $\text{SO}(n)$ as in (3.1), and let Γ denote the set of vector fields governing the system dynamics. The following are equivalent:*

- (1) *The system is controllable on $\text{SO}(n)$.*
- (2) *$\tau(\Gamma)$ is a connected graph.*
- (3) *For any minimal subset $\Sigma \subseteq \Gamma$ generating $\mathfrak{so}(n)$, $\iota(\Sigma)$ is an n -cycle and $\tau(\Sigma)$ is a spanning tree of $\tau(\Gamma)$.*

In the remainder of this section, we will focus on uncontrollable systems. Recall [Theorem 3.22](#) that the controllable submanifold for an uncontrollable system on $\text{SO}(n)$ is determined by the connected components of its associated graph. Meanwhile, according to [Corollary 3.6](#), by applying the method of symmetric groups, the controllable submanifold can also be characterized by the nontrivial orbits of $\iota(\Xi)$ for a minimal subset $\Xi \subseteq \Gamma$ generating $\text{Lie}(\Gamma)$. To see that the two methods are equivalent and to extend [Theorem 3.27](#) to uncontrollable cases, we first introduce the concept of *spanning forests*, which generalizes the notion of spanning trees to disconnected graphs. Given a (disconnected) graph, its *spanning forest* is a maximal acyclic subgraph, or equivalently, a subgraph consisting of a spanning tree in each connected component of the graph [1]. Following this definition, we will show that the minimal subset $\Xi \subseteq \Gamma$ in [Corollary 3.6](#) corresponds to a spanning forest of $\tau(\Gamma)$, so that the controllable submanifold can also be equivalently described by the connected components of the spanning forest. This result is illuminated in the following example.

Example 3.29. Consider a bilinear system on $\text{SO}(6)$ in the form of (3.1) governed by the set of vector fields $\Gamma = \{\Omega_{12}, \Omega_{14}, \Omega_{23}, \Omega_{24}, \Omega_{34}, \Omega_{56}\}$. As shown in [Table 3.2](#), $\iota(\Gamma)$ is disconnected with two components, and hence this system is not controllable on $\text{SO}(6)$. To describe its controllable submanifold, we choose a spanning forest $\tau(\Xi_1)$ of

Set of control vector fields	Graph	Permutation in S_6 and its nontrivial orbits
$\Gamma = \left\{ \begin{array}{l} \Omega_{12}, \Omega_{14}, \Omega_{23}, \\ \Omega_{24}, \Omega_{34}, \Omega_{56} \end{array} \right\}$		$\iota(\Gamma) = (12)(14)(23)(24)(34)(56)$ $= (14)(56)$ Orbits = $\{1, 4\}, \{5, 6\}$
$\Xi_1 = \{\Omega_{14}, \Omega_{24}, \Omega_{34}, \Omega_{56}\}$		$\iota(\Xi_1) = (14)(24)(34)(56)$ $= (1432)(56)$ Orbits = $\{1, 2, 3, 4\}, \{5, 6\}$
$\Xi_2 = \{\Omega_{12}, \Omega_{24}, \Omega_{34}, \Omega_{56}\}$		$\iota(\Xi_2) = (12)(24)(34)(56)$ $= (1243)(56)$ Orbits = $\{1, 2, 3, 4\}, \{5, 6\}$

TABLE 3.2

A comparison between the symmetric group method and the graph-theoretic method for an uncontrollable system on $SO(6)$. Both graphs associated with subsets Ξ_1 and Ξ_2 are spanning forest of the associated graph of Γ .

the associated graph $\tau(\Gamma)$ with $\Xi_1 = \{\Omega_{14}, \Omega_{24}, \Omega_{34}, \Omega_{56}\}$. Note that the permutation $\iota(\Xi_1) = (14)(24)(34)(56) = (1432)(56)$ has two nontrivial orbits: $\mathcal{O}_1 = \{1, 2, 3, 4\}$ and $\mathcal{O}_2 = \{5, 6\}$, each corresponds to a connected component of the graph $\tau(\Gamma)$, or equivalently, a summand in the decomposition of the Lie algebra of the controllable submanifold:

$$\text{Lie}(\Gamma) = \text{span}\{\Omega_{ij} : i, j \in \mathcal{O}_1\} \oplus \text{span}\{\Omega_{ij} : i, j \in \mathcal{O}_2\}.$$

Now suppose we choose a different spanning forest $\tau(\Xi_2)$ which corresponds to another subset $\Xi_2 = \{\Omega_{12}, \Omega_{24}, \Omega_{34}, \Omega_{56}\} \subseteq \Gamma$. Note that the permutation $\iota(\Xi_2) = (1243)(56)$ is different from $\iota(\Xi_1)$, but both have the *same* orbits. The graphs and permutations associated with Γ and its subsets Ξ_1 and Ξ_2 are also listed in [Table 3.2](#).

In general, for a spanning forest F of $\tau(\Gamma)$, we know by [Theorem 3.27](#) that each tree T_i consisting of n_i vertices in F is related to an n_i -cycle via ι , which characterizes a summand of the decomposition of $\text{Lie}(\Gamma)$. So by applying [Theorem 3.27](#) to each (maximal) tree in the forest F , we have the following [Corollary 3.30](#), which describes the relation between the associated graphs and permutations for an uncontrollable bilinear system.

COROLLARY 3.30. *Given an uncontrollable bilinear system defined on $SO(n)$ in the form of (3.1) governed by the set of vector fields Γ . Let F be a spanning forest of $\tau(\Gamma)$ and if we denote $\Xi = \tau^{-1}(F)$, then Ξ is a minimal subset of Γ with the same generating Lie algebra and the controllable submanifold of the system is determined by the nontrivial orbits of $\iota(\Xi)$.*

Proof. For a spanning forest F of $\tau(\Gamma)$, let T_1, \dots, T_l be (maximal) trees in F s.t. $F = T_1 \sqcup \dots \sqcup T_l$, where $T_i = (V_i, E_i)$ with $|V_i| = n_i$, $|E_i| = n_i - 1$. By [Theorem 3.27](#), each T_i is related to an n_i -cycle $\iota(\Xi_i) \in S_{n_i}$ for $\Xi_i = \tau^{-1}(T_i)$, and the orbit of $\iota(\Xi_i)$ determines the Lie (sub)algebra $\mathfrak{g}_i := \text{Lie}(\Xi_i)$. Therefore, distinct orbits of $\iota(\Xi)$ consist of the orbits of each $\iota(\Xi_1), \dots, \iota(\Xi_l)$, which determines the Lie algebra generated by Γ : $\text{Lie}(\Gamma) = \text{Lie}(\Xi) = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_l$. Since the controllable submanifold is determined by the Lie subalgebra $\text{Lie}(\Gamma) = \text{Lie}(\Xi)$, we conclude that it is also determined by distinct orbits of $\iota(\Xi)$, for $\Xi = \tau^{-1}(F)$. \square

4. Combinatorics-Based Controllability Analysis via Lie Algebra Decompositions. Utilizing the algebraic structure of $\mathfrak{so}(n)$, we have developed combinatorial methods that identified vector fields in the standard basis of $\mathfrak{so}(n)$, as well as vector fields generating structured Lie algebras, e.g., the multi-agent system described in [Example 3.24](#), with transpositions in S_n and edges of n -vertices graphs. It was also shown that such identifications lead to an equivalence between the two methods for analyzing controllability of systems on $\text{SO}(n)$ as defined in [\(3.1\)](#).

However, in many cases, the system Lie algebra may be too complicated to associate each of its elements to a permutation or a graph edge, so that the combinatorial methods cannot be directly applied. This dilemma can be resolved through the decomposition of the Lie algebra into components with simpler algebraic structures such that the combinatorial methods can be applied to each component. This idea allows us to generalize the combinatorial framework to bilinear systems defined on boarder classes of Lie groups. To this end, we adopt techniques in representation theory, including the Cartan and non-intertwining decomposition. Some basics of representation theory can be found in [Appendix B](#).

4.1. Cartan Decomposition in Symmetric Group Method. The Cartan decomposition, named after the influential French mathematician Élie Cartan, provides a major tool for understanding the algebraic structures of semisimple Lie groups and Lie algebras. Its generalized form, the root space decomposition, decomposes a Lie algebra into a direct sum of vector subspaces, called the root spaces, as introduced in [Appendix B](#). However, each root space is not necessarily a Lie subalgebra, i.e., Lie bracket operations may not be closed in the root spaces. This nature of the Cartan (root space) decomposition then disables the use of the graph-theoretic method since it violates the ‘‘triangle rule’’ shown in [Lemma 3.9](#) (ii). As a result, here we pursue and generalize the symmetric group method to analyze controllability of systems with its vector fields living in the root spaces of semisimple Lie algebras.

In representation theory, the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$, which consists of 3×3 complex matrices with vanishing trace, serves as a primary example to illustrate the Cartan decomposition of semisimple Lie algebras. Therefore, to illustrate our idea, we consider the driftless bilinear system evolving on the Lie group $\text{SL}(3, \mathbb{C})$ consisting of 3×3 complex matrices with determinant 1, given by

$$(4.1) \quad \dot{Z}(t) = \left(\sum_{j=1}^m u_j(t) B_j \right) Z(t),$$

where the state $Z(t) \in \text{SL}(3, \mathbb{C})$, the control vector fields $B_j \in \Gamma \subseteq \mathcal{B}'' := \{H_k, X_l, Y_l : k = 1, 2; l = 1, 2, 3\}$, the basis of $\mathfrak{sl}(3, \mathbb{C})$ with

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\begin{aligned}
X_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & X_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, & X_3 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
Y_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & Y_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, & Y_3 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},
\end{aligned}$$

and the control inputs $u_j(t) \in \mathbb{C}$.

One can easily check that the two Lie subalgebras $\mathfrak{k}_1 = \text{Lie}\{H_1, X_1, Y_1\}$ and $\mathfrak{k}_2 = \text{Lie}\{H_2, X_2, Y_2\}$, when considered as Lie algebras over \mathbb{R} , are isomorphic to $\mathfrak{so}(3)$. As discussed in Section 3.1, controllability of systems on $\text{SO}(3)$ can be characterized by permutation cycles in S_3 . This suggests that we can characterize controllability of systems on $\text{SL}(3, \mathbb{C})$ by two copies of S_3 . Formally, we want to establish a map $\iota: \mathcal{P}(\mathcal{B}'') \rightarrow S_3 \oplus S_3$, where \oplus denotes the direct sum of groups, so that non-vanishing Lie brackets correspond to cycles with increased length. In this case, we define an element $\sigma = (\sigma_1, \sigma_2)$ in $S_3 \oplus S_3$ to be a cycle if both σ_1 and σ_2 are cycles in S_3 , and the length of σ is defined to be the sum of the length of σ_1 and σ_2 . Here is one possible definition of ι :

$$\begin{aligned}
H_1 &\mapsto (e, e), & H_2 &\mapsto (e, e), \\
X_1 &\mapsto ((12), e), & X_2 &\mapsto (e, (12)), & X_3 &\mapsto ((12), (12)), \\
Y_1 &\mapsto ((23), e), & Y_2 &\mapsto (e, (23)), & Y_3 &\mapsto ((23), (23)),
\end{aligned}$$

where e denotes the identity of S_3 . Following this definition of ι , we can check that if $B_1, B_2 \in \mathcal{B}''$ satisfy $[B_1, B_2] \neq 0$, then the length of $\iota([B_1, B_2])$ is greater than or equal to the length of both $\iota(B_1)$ and $\iota(B_2)$. Moreover, if neither B_1 nor B_2 is equal to H_1 or H_2 , then the length of $\iota([B_1, B_2])$ is strictly greater than the length of both $\iota(B_1)$ and $\iota(B_2)$. This relation between Lie brackets of elements in \mathcal{B}'' and length of cycles in $S_3 \oplus S_3$ allows us to draw the following conclusion:

PROPOSITION 4.1. *The system in (4.1) is controllable on $\text{SL}(3, \mathbb{C})$ if and only if there exists a subset Σ of $\Gamma = \{B_1, \dots, B_m\}$ such that $\iota(\Sigma)$ is a 6-cycle in $S_3 \oplus S_3$.*

From the perspective of representation theory, the basis \mathcal{B}'' induces the Cartan decomposition of the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$, in which the 2-dimensional Cartan subalgebra is spanned by H_1 and H_2 . Moreover, the Weyl group of $\mathfrak{sl}(3, \mathbb{C})$ is S_3 . The above facts provide another explanation for requiring two copies of S_3 in the characterization of controllability for systems on $\text{SL}(3, \mathbb{C})$ governed by vector fields in \mathcal{B}'' . Notice that the concepts of Cartan subalgebras and Weyl groups are well-defined for all semisimple Lie algebras, not only for $\text{SL}(3, \mathbb{C})$. Also, Weyl groups are all finite groups and thus subgroups of some symmetric groups. As a result, it is possible to extend the symmetric-group characterization of controllability to systems defined on general semisimple Lie groups. To be more specific, consider the bilinear system defined on a semisimple Lie group G of the form,

$$(4.2) \quad \dot{X} = \left(\sum_{i=1}^m u_i B_i \right) X, \quad X(0) = I,$$

where B_i are elements in the Lie algebra \mathfrak{g} of G . Moreover, let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ be the Cartan decomposition of \mathfrak{g} with \mathfrak{h} being the Cartan subalgebra and W be the Weyl group of \mathfrak{g} . We further assume that $B_i \in \mathfrak{h}$ or $B_i \in \mathfrak{k}$ for every $i = 1, \dots, m$, then the

above discussion leads to the following conjecture for systems defined on semisimple Lie groups.

CONJECTURE 4.2. *The system in (4.2) is controllable on G if and only if there exists $\Sigma \subseteq \Gamma$ such that $\iota(\Sigma)$ is a cycle of maximal length in W^h , where $\Gamma = \{B_1, \dots, B_m\}$ is the set of control vector fields, $h = \dim \mathfrak{h}$, and W^h denotes the direct sum of h copies of W .*

Recall that the central idea of the symmetric group approach to controllability analysis is to map elements with non-vanishing Lie brackets to cycles with increased length. However, all elements in the Cartan subalgebra have vanishing Lie brackets. The intuition behind the above conjecture comes from the need of appropriately representing these elements using permutations by mapping elements in different root spaces to permutation cycles in different components of the direct sum of h copies of symmetric groups, where h denotes the dimension of the Cartan subalgebra. Moreover, because the interaction between elements in and outside the Cartan subalgebra is characterized by the Weyl group, which is a subgroup of a symmetric group, the symmetric group method applies directly.

4.2. Non-Intertwining Decomposition in Graph-Theoretic Method.

In the case that the Lie algebra generated by drift and control vector fields of a bilinear system can be decomposed into components that are Lie subalgebras, we will see that the graph-theoretic method applies more naturally for controllability analysis. One decomposition of this type is the *non-intertwining decomposition*, through which a Lie algebra is decomposed into a direct sum of Lie subalgebras so that elements from different Lie subalgebras have vanishing Lie brackets. The non-intertwining decomposition generalizes the notion of block diagonalization for matrices.

DEFINITION 4.3. *For a given Lie algebra \mathfrak{g} , we call a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_m$ non-intertwining if $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for any Lie subalgebras $\mathfrak{g}_i, \mathfrak{g}_j$, $1 \leq i \neq j \leq m$.*

For example, every reductive Lie algebra admits a non-intertwining decomposition, and many familiar Lie algebras are reductive, such as the algebra of $n \times n$ complex matrices $\mathfrak{gl}(n, \mathbb{C})$ and the algebra of $n \times n$ skew-symmetric complex matrices $\mathfrak{so}(n, \mathbb{C})$ [17]. If a Lie algebra admits a non-intertwining decomposition, then we will be able to associate each of its components with a graph. The subsequent question is whether graph representation developed in Section 3.2 remains valid to characterize controllability. The answer to this question can be illustrated by a system defined on $\text{SO}(4)$ whose Lie algebra $\mathfrak{so}(4)$ can be decomposed into a direct sum of two non-intertwining copies of $\mathfrak{so}(3)$, as shown in the following example.

Example 4.4. Let $\mathcal{B}' = \{A_1, A_2, A_3, B_1, B_2, B_3\}$ be a non-standard basis of $\mathfrak{so}(4)$, where

$$(4.3) \quad \begin{aligned} A_1 &= \frac{\Omega_{23} + \Omega_{14}}{2}, & A_2 &= \frac{\Omega_{13} - \Omega_{24}}{2}, & A_3 &= \frac{\Omega_{12} + \Omega_{34}}{2}, \\ B_1 &= \frac{\Omega_{13} + \Omega_{24}}{2}, & B_2 &= \frac{\Omega_{14} - \Omega_{23}}{2}, & B_3 &= \frac{\Omega_{12} - \Omega_{34}}{2}. \end{aligned}$$

The Lie brackets of the elements in \mathcal{B}' satisfy $[A_i, A_j] = A_k$, $[B_i, B_j] = B_k$ for any ordered 3-tuple $(i, j, k) = (1, 2, 3)$, $(2, 3, 1)$ or $(3, 1, 2)$, and $[A_i, B_j] = 0$ for any $1 \leq i, j \leq 3$. As a result, $\mathfrak{so}(4)$ admits a non-intertwining decomposition as $\mathfrak{so}(4) = \text{Lie}\{A_1, A_2, A_3\} \oplus \text{Lie}\{B_1, B_2, B_3\}$.

We note that the Lie bracket relations among elements in $\{A_1, A_2, A_3\}$, as well as $\{B_1, B_2, B_3\}$, are the same as the Lie bracket relations among elements in the

standard basis of $\mathfrak{so}(3)$. In other words, both $\text{Lie}\{A_1, A_2, A_3\}$ and $\text{Lie}\{B_1, B_2, B_3\}$ are isomorphic to $\mathfrak{so}(3)$, so K_3 becomes the suitable graph representation for each set. Moreover, because $[A_i, B_j] = 0$ for any $i, j = 1, 2, 3$, the graph representation for the non-standard basis $\mathcal{B}' = \{A_1, A_2, A_3\} \sqcup \{B_1, B_2, B_3\}$ is a *disjoint union* of two copies of K_3 , as shown in [Figure 4.1](#), instead of the complete graph K_4 associated with the standard basis of $\mathfrak{so}(4)$.

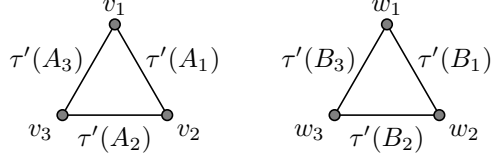


FIG. 4.1. The graphs associated with the sets $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$ in [Example 4.4](#).

This example illuminates how the graph representation of controllability developed in [Section 3.2](#) can be extended to the bilinear system governed by vector fields generating a non-intertwining Lie algebra, after modifying the definition of τ in [\(3.7\)](#) accordingly.

PROPOSITION 4.5. *Consider a bilinear system on $\text{SO}(4)$ governed by the vector fields in \mathcal{B}' , given by*

$$(4.4) \quad \dot{X}(t) = \left(\sum_{i=1}^m u_i C_i \right) X(t), \quad X(0) = I,$$

with $\Gamma = \{C_1, \dots, C_m\} \subseteq \mathcal{B}'$. Given a graph map $\tau' : \mathcal{P}(\mathcal{B}') \rightarrow \mathcal{G}'$, where \mathcal{G}' denotes the collection of subgraphs of $K_3 \sqcup K_3$, satisfying

$$\tau'(A_i) = v_i v_{i+1} \quad \text{and} \quad \tau'(B_i) = w_i w_{i+1},$$

with the index taken modulo 3, the system in [\(4.4\)](#) is controllable if and only if $\overline{\tau'(\Gamma)} = K_3 \sqcup K_3$, or equivalently, if and only if each component of $\tau'(\Gamma)$ is connected in K_3 .

Proof. The above result becomes obvious once we verify the following properties of τ' (c.f. [Lemma 3.9](#)), which are straightforward.

- (1) $\tau'(\mathcal{B}') = K_3 \sqcup K_3$;
- (2) For distinct $C_1, C_2 \in \mathcal{B}'$, their Lie bracket $[C_1, C_2] \neq 0$ if and only if the two edges $\tau'(C_1)$ and $\tau'(C_2)$ have a common vertex;
- (3) The edges $\tau'(C_1), \tau'(C_2)$ and $\tau'([C_1, C_2])$ form a triangle if $[C_1, C_2] \neq 0$, or equivalently,

$$\tau'(\{C_1, C_2, [C_1, C_2]\}) = K_3,$$

for any $C_1, C_2 \in \mathcal{B}'$ such that $[C_1, C_2] \neq 0$. □

In addition, recall from [Corollary 3.20](#) that three control inputs are enough to have a controllable driftless system on $\text{SO}(4)$ governed by the vector fields in the standard basis; or equivalently, three edges can form a connected graph with four vertices. However, for systems in the form of [\(4.4\)](#), they require at least four control inputs to be controllable on $\text{SO}(4)$. From the graph aspect, this is because both components of $\tau(\Gamma)$ require at least two edges to be connected.

Example 4.4 further illustrates that for bilinear systems evolving on $SO(n)$ governed by non-standard basis vector fields, i.e., vector fields that are not in the form of standard basis elements, in $\mathfrak{so}(n)$, controllability may not be characterized by using one complete graph K_n . Taking the system in (4.4) as an example, because the Lie algebra of its state-space can be decomposed into a direct sum of two non-intertwining components, its graph representation also requires two components. This finding elucidates that the number of components of the graph associated with a bilinear system is determined by the number of summands in the non-intertwining decomposition of the underlying Lie algebra of the system.

THEOREM 4.6. *Given a bilinear system*

$$(4.5) \quad \dot{X}(t) = \left(\sum_{i=1}^m \sum_{j=1}^{n_i} u_{ij} B_{ij} \right) X(t), \quad X(0) = I,$$

defined on a Lie group G whose Lie algebra \mathfrak{g} admits a non-intertwining decomposition as $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$, where $B_{ij} \in \mathcal{B}_i$ and \mathcal{B}_i is a basis of \mathfrak{g}_i for each i . Suppose each \mathcal{B}_i is associated with a connected graph G_i such that a subset $\Sigma_i \subseteq \mathcal{B}_i$ generates \mathfrak{g}_i if and only if its associated graph $\tau(\Sigma_i)$ is a connected subgraph of G_i , then the system in (4.5) is controllable on G if and only if $\tau(\Gamma_i)$ is connected for every $i = 1, \dots, m$, where $\Gamma_i = \{B_{ij} : j = 1, \dots, n_i\}$.

Proof. By the assumption, $\tau(\Gamma_i)$ is connected if and only if $\text{Lie}(\Gamma_i) = \mathfrak{g}_i$ for each $i = 1, \dots, m$. Together with the non-intertwining property between each pair of \mathfrak{g}_i and \mathfrak{g}_j , the connectivity of $\tau(\Gamma_i)$ for all i is equivalent to

$$\text{Lie}(\Gamma) = \bigoplus_{i=1}^m \text{Lie}(\Gamma_i) = \bigoplus_{i=1}^m \mathfrak{g}_i = \mathfrak{g},$$

where $\Gamma = \bigcup_{i=1}^m \Gamma_i$. The proof is then concluded by applying the LARC. \square

Remark 4.7 (Symmetric Group Method for Systems Governed by Non-intertwining Lie Algebras). We find it worthwhile to mention that the symmetric group method also applies to bilinear systems with their underlying Lie algebras admitting a non-intertwining decomposition, through a properly defined ι . For instance, in **Example 4.4**, since both $\{A_i\}$ and $\{B_i\}$ in (4.3) are isomorphic to the standard basis in $\mathfrak{so}(3)$, the symmetric group method extends to the systems in (4.4) as well, by associating each component in the decomposition to a copy of S_3 and defining $\iota(A_i, B_j) = ((i, i+1), (j, j+1))$, with the index taken modulo 3. Consequently, the system in (4.4) is controllable if and only if ι relates Γ to two disjoint 3-cycles in $S_3 \oplus S_3$.

5. Summary. In this paper, we develop a combinatorics-based framework to characterize controllability of bilinear systems evolving on Lie groups, in which Lie bracket operations of vector fields are represented by operations on permutations in a symmetric group and edges in a graph. Through such representations, we obtain the tractable and transparent combinatorial characterizations of controllability in terms of permutation cycles and graph connectivity. This framework is established by first considering bilinear systems on $SO(n)$, and we show that, in this case, the permutation and graph representations are equivalent. Then, by exploiting techniques in representation theory, we extend our investigation into a more general category of bilinear systems via proper decompositions of the underlying Lie algebras of the systems. In particular, we illustrate the application of the developed combinatorial methods to

bilinear systems whose underlying Lie algebras admit the Cartan or non-intertwining decomposition. The presented methodology not only provides an alternative to the LARC, but also advances geometric control theory by integrating it with techniques in combinatorics and representation theory. As a final remark, compared to known graph-theoretic methods mostly developed for networked or multi-agent systems, our framework proposes novel applications of graphs to the study of bilinear control systems.

Appendix A. Symmetric Groups and Permutations.

In this appendix, we give a brief review of the symmetric group theory. For a thorough discussion on symmetric groups, the reader can refer to any standard algebra textbook, for example [18]. Let X_n be a finite set of n elements, and without loss of generality, we may assume $X_n = \{1, \dots, n\}$. A *permutation* σ of X_n is a bijection from X_n onto itself, and is denoted by

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$$

if $\sigma(1) = i_1, \dots, \sigma(n) = i_n$ for distinct $i_1, \dots, i_n \in X_n$. A permutation that switches only two elements is called a *transposition*, and is denoted by $\sigma = (i_1 i_2)$ if $i_1 \neq i_2$ and σ fixes all other indices except for $\sigma(i_1) = i_2$ and $\sigma(i_2) = i_1$. More generally, an *r -cycle* denoted by $\sigma = (i_1 i_2 \cdots i_r)$ is a permutation that satisfies $\sigma(i_1) = i_2, \sigma(i_2) = i_3, \dots, \sigma(i_r) = i_1$ and fixes all other indices. It can be shown that any permutation can be decomposed uniquely into disjoint cycles (cycles that have no common indices). For example, when $n = 4$, the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ can be represented by a single 4-cycle (1234); while the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ is the composition of two transpositions (2-cycles): (13)(24). Given a permutation σ of X_n and an integer $i, 1 \leq i \leq n$, the *orbit* of i is formed under the cyclic group generated by σ . So for $\sigma = (1234)$, the orbit of 2 is $\{\sigma^i(2) : i \in \mathbb{N}\} = \{2, \sigma(2), \sigma^2(2), \sigma^3(2)\} = \{1, 2, 3, 4\}$; and for $\sigma = (13)(24)$, the orbit of 2 is $\{\sigma^i(2) : i \in \mathbb{N}\} = \{2, \sigma(2)\} = \{2, 4\}$. The *symmetric group* S_n is defined as the group of permutations on X_n , with its group operation being the composition of bijections.

Appendix B. Basics of Representation Theory.

Representation theory is a branch of algebra which studies structure theory by representing elements in an algebraic object, such as a group, a module, or an algebra, using linear transformations of vector spaces. In this appendix, we will review some basic concepts and results in the representation theory of Lie algebras that are used in this paper. Detailed discussions of Lie representation theory can be found in [9, 17].

To study the algebraic structure of a Lie algebra, let us introduce some related definitions.

DEFINITION B.1.

- A Lie algebra \mathfrak{g} is said to be abelian if

$$[\mathfrak{g}, \mathfrak{g}] := \text{span} \{[X, Y] : X, Y \in \mathfrak{g}\} = 0.$$

- A subspace \mathfrak{h} of \mathfrak{g} is a Lie subalgebra of \mathfrak{g} if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. In other words, \mathfrak{h} is a Lie algebra itself w.r.t. $[\cdot, \cdot]$.
- A Lie subalgebra $\mathfrak{h} \leq \mathfrak{g}$ is an ideal in \mathfrak{g} if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.
- The Lie algebra \mathfrak{g} is said to be simple if it is nonabelian and has no proper nonzero ideals, and semisimple if it has no nonzero abelian ideals.

It can be shown that every semisimple Lie algebra \mathfrak{g} can be decomposed into a direct sum of simple Lie algebras which are ideals in \mathfrak{g} . Moreover, this decomposition is unique, and the only ideals of \mathfrak{g} are the direct sums of some of these simple Lie algebras. For example, each special orthogonal Lie algebra $\mathfrak{so}(n) = \{\Omega \in \mathbb{R}^{n \times n} : \Omega + \Omega^\top = 0\}$, as we use extensively in this paper, is simple except for $n = 4$, while $\mathfrak{so}(4)$ is semisimple but not simple: as shown in [Example 4.4](#), $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

The study of algebraic structures of semisimple Lie algebras plays a central role in representation theory. One of the most dominant results is the Cartan decomposition that traces back to the work of Élie Cartan and Wilhelm Killing in the 1880s, which generalizes the notion of singular value decomposition for matrices. Given a semisimple Lie algebra \mathfrak{g} , its *Cartan subalgebra* \mathfrak{h} is a maximal abelian subalgebra of \mathfrak{g} such that ad_H is diagonalizable for all $H \in \mathfrak{h}$, where $\text{ad}_X Y = [X, Y]$ for all $X, Y \in \mathfrak{g}$. Moreover, the dimension of \mathfrak{h} is called the *rank* of \mathfrak{g} . Let \mathfrak{h}^* denote the dual space of \mathfrak{h} , i.e., the space of linear functionals on \mathfrak{h} , then a nonzero element $\alpha \in \mathfrak{h}^*$ is called a *root* of \mathfrak{g} if there exists some $X \in \mathfrak{g}$ such that $\text{ad}_H X = \alpha(H)X$ for all $H \in \mathfrak{h}$, and $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : \text{ad}_H X = \alpha(H)X, \forall H \in \mathfrak{h}\}$ is a vector space called the *root space* of \mathfrak{g} , which can be shown to be one-dimensional. Let R denote the set of roots of \mathfrak{g} , then R is finite and spans \mathfrak{h}^* . With the above notations, the *root space decomposition*, which generalizes the classical *Cartan decomposition*, is defined as

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right).$$

A major tool to study the properties of R is the Weyl group, which is defined as follows: Let $\alpha \in R$ be a root and $s_\alpha : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ denote the reflection about the hyperplane in \mathfrak{h}^* orthogonal to α , i.e., $s_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$ for all $\beta \in \mathfrak{h}^*$, where $\langle \cdot, \cdot \rangle$ is an inner product on \mathfrak{h} , then the *Weyl group* W of R is the subgroup of the orthogonal group $O(\mathfrak{h}^*)$ of \mathfrak{h}^* generated by all s_α for $\alpha \in R$. It can be shown that W is a finite group and hence a subgroup of a symmetric group by Cayley's theorem.

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