arXiv:2305.05721v1 [math.PR] 9 May 2023

Quickest Real-Time Detection of Multiple Brownian Drifts

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Consider the motion of a Brownian particle in n dimensions, whose coordinate processes are standard Brownian motions with zero drift initially, and then at some random/unobservable time, exactly k of the coordinate processes get a (known) non-zero drift permanently. Given that the position of the Brownian particle is being observed in real time, the problem is to detect the time at which the k coordinate processes get the drift as accurately as possible. We solve this problem in the most uncertain scenario when the random/unobservable time is (i) exponentially distributed and (ii) independent from the initial motion without drift. The solution is expressed in terms of a stopping time that minimises the probability of a false early detection and the expected delay of a missed late detection. The elliptic case $k=1$ has been settled in [4] where the hypoelliptic case $1<k<n$ resolved in the present paper was left open (the case $k=n$ reduces to the classic case $n=1$ having a known solution). We also show that the methodology developed solves the problem in the general case where exactly k is relaxed to *any* number of the coordinate processes getting the drift. To our knowledge this is the first time that such a multi-dimensional hypoelliptic problem has been solved exactly in the literature.

1. Introduction

Imagine the motion of a Brownian particle in n dimensions, whose coordinate processes are standard Brownian motions with zero drift initially, and then at some random/unobservable time θ, exactly k of the coordinate processes get a (known) non-zero drift μ permanently. Assuming that the position of the Brownian particle is being observed in real time, the problem is to detect the time θ at which the k coordinate processes get the drift μ as accurately as possible. In the most uncertain scenario, where θ is assumed to be (i) exponentially distributed and (ii) independent from the initial motion without drift, the solution to the problem when $k=1$ has been derived in [4]. The purpose of the present paper is to derive the solution to the problem when $1<k<n$ that was left open in [4]. (Note that the case $k=n$ reduces to the classic case $n=1$ having a known solution and is therefore excluded throughout.)

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Mathematics Subject Classification 2020. Primary 60G40, 60J65, 60H30. Secondary 35H10, 45G10, 62C10.
Key words and phrases: Quickest detection, Brownian motion, optimal stopping, Hörmander's condition, hypoelliptic partial differential equation, free-boundary problem, smooth fit, nonlinear Fredholm integral equation, the change-of-variable formula with local time on surfaces.

Denoting the position of the Brownian particle in n dimensions by X, the error to be minimised over all stopping times τ of X is expressed as the the linear combination of the probability of the *false alarm* $\mathsf{P}_\pi(\tau < \theta)$ and the expected *detection delay* $\mathsf{E}_\pi(\tau - \theta)^+$ where $\pi \in [0,1]$ denotes the probability that θ has already occurred at time 0. This problem formulation of quickest detection dates back to [15] and has been extensively studied to date (see [18] and the references therein). The linear combination represents the Lagrangian and once the optimal stopping problem has been solved in that form this will also lead to the solution of the constrained problems where an upper bound is imposed on either the probability of the false alarm or the expected detection delay respectively.

Quickest detection problems related to the present problem have been studied by a number of authors and we refer to [4, Section 1] for an overview of this literature (see also Remark 9 in [4]). The initial optimal stopping problem in these quickest detection problems is equivalent to an optimal stopping problem for the posterior probability distribution ratio process Φ of θ given X. The infinitesimal generator of the Markov/diffusion process Φ (combined with X when needed) in these optimal stopping problems is of *parabolic* type. In contrast, this was no longer the case in the present problem when $k = 1$ as noted in [4], where the infinitesimal generator $I\!\!L_\Phi$ of the multi-dimensional Markov/diffusion process Φ is of *elliptic* type.

The present quickest detection problem when $1 < k < n$ was left open in [4]. We will see below that the ellipticity of $I\!\!L_\Phi$ remains valid for $k = n-1$ but breaks down when $1 < k < n-1$. We show however that $I\!\!L_\Phi$ satisfies the *Hörmander condition* (cf. [6]) in this case so that $I\!\!L_\Phi$ is *hypoelliptic*. This fact may appear somewhat surprising at first glance given that Φ is governed by a system of $N = \binom{n}{k}$ stochastic differential equations driven by n Brownian motions. For example, when $n = 10$ and $k = 5$ then $N = 252$ so that the system governing Φ consists of 252 stochastic differential equations driven by 10 Brownian motions. Nonetheless, we establish that the structure of this system when $1 < k < n-1$ is sufficiently supported to ensure *hypoellipticity* as a substitute for the broken ellipticity of k being either 1 or $n-1$. This provides regularity of the value function in the optimal stopping set. Moreover, the previous conclusions on the space operator $I\!\!L_\Phi$ extend to the backward time-space operator $-\partial_t + I\!\!L_\Phi$ as well which in turn imply that Φ is a strong Feller process. This opens avenues to regularity of the value function at the optimal stopping points and closes our exploratory analysis of the problem.

In the reminder we show that the hypoelliptic structure of the infinitesimal generator combined with the concavity of the loss functional in the optimal stopping problem is sufficiently robust to yield the solution. Finding the *exact* solution to the quickest detection problem for the observed process X in n dimensions when $1 < k < n$ is the main contribution of the present paper. We also show that the methodology developed solves the problem in the general case where exactly k is relaxed to *any* number of the coordinate processes getting the drift. To our knowledge this is the first time that such a multi-dimensional hypoelliptic problem has been solved exactly in the literature.

2. Formulation of the problem

In this section we formulate the quickest detection problem under consideration. The initial formulation of the problem will be revaluated under a change of measure in the next section.

1. We consider a Bayesian formulation of the problem where it is assumed that one observes a sample path of the standard n-dimensional Brownian motion $X = (X^1, \ldots, X^n)$, whose coordinate processes $X^1, \ldots, X^n$ are standard Brownian motions with zero drift initially, and then at some random/unobservable time θ taking value 0 with probability $\pi \in [0,1]$ and being exponentially distributed with parameter $\lambda > 0$ given that $\theta > 0$, exactly k of the coordinate processes $X^1, \ldots, X^n$ get a (known) non-zero drift μ permanently. The problem is to detect the time θ at which the k coordinate processes get the drift μ as accurately as possible (neither too early nor too late). This problem belongs to the class of quickest real-time detection problems as discussed in Section 1 above.

2. The observed process $X = (X^1, \ldots, X^n)$ satisfies the stochastic differential equations

$$dX_t^i = \mu I(i \in \beta, t \geq \theta)\, dt + dB_t^i \;\; (1 \leq i \leq n) \tag{2.1}$$

driven by a standard n-dimensional Brownian motion $B = (B^1, \ldots, B^n)$ under the probability measure P_π specified below, where the random variable β taking values in the set $C_k^n := \{(n_1, \ldots, n_k) \,|\, 1 \leq n_1 < \ldots < n_k \leq n\}$ satisfies $\mathsf{P}_\pi(\beta = (n_1, \ldots, n_k)) = p_{n_1,\ldots,n_k}$ for some $p_{n_1,\ldots,n_k} \in [0,1]$ with $\sum p_{n_1,\ldots,n_k} = 1$ given and fixed where the sum is taken over all $(n_1, \ldots, n_k)$ in C_k^n. With a slight abuse of notation, in (2.1) we write $i \in \beta$ to express the fact that i belongs to the set $\{n_1, \ldots, n_k\}$ consisting of the elements which form $\beta = (n_1, \ldots, n_k)$ in C_k^n. This means that $n_1, \ldots, n_k \in \beta$ if and only if the coordinate processes $X^{n_1}, \ldots, X^{n_k}$ get drift μ at time θ with probability $p_{n_1,\ldots,n_k}$ for $(n_1, \ldots, n_k) \in C_k^n$. With a similar abuse of notation, which will be helpful in what follows, we will arrange the elements of C_k^n in a lexicographic order and identify the i-th element of the ordered set C_k^n by its index i itself for $1 \leq i \leq N$ where we set $N := \binom{n}{k}$ to denote the total number of elements in C_k^n. Often we will write $i = (n_1, \ldots, n_k)$ or $i \in C_k^n$ to express this identification explicitly for $1 \leq i \leq N$ while at other places it will be clear from the context whether the index i belongs to the set C_k^n in this sense or the set $\{1, \ldots, n\}$ as in (2.1) above. The unobservable time θ, the unknown coordinates β, and the driving Brownian motion B are all assumed to be independent under P_π for $\pi \in [0,1]$ given and fixed.

3. Standard arguments imply that the previous setting can be realised on a probability space $(\Omega, \mathcal{F}, \mathsf{P}_\pi)$ with the probability measure P_π being decomposable as follows

$$\mathsf{P}_\pi = \sum_{i \in C_k^n} p_i \Big(\pi\, \mathsf{P}_i^0 + (1-\pi) \int_0^\infty \lambda e^{-\lambda t}\, \mathsf{P}_i^t\, dt \Big) \tag{2.2}$$

for $\pi \in [0,1]$ where P_i^t is the probability measure under which the coordinate processes $X^{n_1}, \ldots, X^{n_k}$ of the observed process X get drift μ at time $t \in [0, \infty)$ for $i = (n_1, \ldots, n_k)$ in C_k^n and $\lambda > 0$ is given and fixed. The decomposition (2.3) expresses the fact that the unobservable time θ is a non-negative random variable satisfying $\mathsf{P}_\pi(\theta = 0) = \pi$ and $\mathsf{P}_\pi(\theta > t \,|\, \theta > 0) = e^{-\lambda t}$ for $t > 0$. Thus $\mathsf{P}_i^t(X \in \cdot\,) = \mathsf{P}_\pi(X \in \cdot \,|\, \beta = i, \theta = t)$ is the probability law of the standard n-dimensional Brownian motion process X whose coordinate processes $X^{n_1}, \ldots, X^{n_k}$ get drift μ at time $t \in [0, \infty)$ for $i = (n_1, \ldots, n_k)$ in C_k^n. To remain consistent with this notation we also denote by P_i^∞ the probability measure under which the coordinate processes $X^{n_1}, \ldots, X^{n_k}$ of X get no drift μ at a finite time for $i = (n_1, \ldots, n_k)$ in C_k^n. Thus $\mathsf{P}_i^\infty(X \in \cdot\,) = \mathsf{P}_\pi(X \in \cdot \,|\, \beta = i, \theta = \infty)$ is the probability law of the standard n-dimensional

Brownian motion process for $i \in C_k^n$. Clearly the subscript i is superfluous in this case and we will often write P^∞ instead of P_i^∞ for $i \in C_k^n$. Moreover, by P_i we denote the probability measure under which the coordinate processes $X^{n_1}, \ldots, X^{n_k}$ of X get drift μ at time θ for $i = (n_1, \ldots, n_k)$ in C_k^n . From (2.2) we see that

$$\mathsf{P}_\pi = \sum_{i \in C_k^n} p_i \mathsf{P}_i \tag{2.3}$$

where $\mathsf{P}_i = \pi\, \mathsf{P}_i^0 + (1-\pi) \int_0^\infty \lambda\, e^{-\lambda t}\, \mathsf{P}_i^t\, dt$ for $i \in C_k^n$ and $\pi \in [0,1]$. Note that P_i depends on $\pi \in [0,1]$ as well but we will omit this dependence from its notation for $i \in C_k^n$.

4. Being based upon continuous observation of $X = (X^1, \ldots, X^n)$, the problem is to find a stopping time τ_* of X (i.e. a stopping time with respect to the natural filtration $\mathcal{F}_t^X = \sigma(X_s \,|\, 0 \le s \le t)$ of X for $t \ge 0$) that is 'as close as possible' to the unknown time θ . More precisely, the problem consists of computing the value function

$$V(\pi) = \inf_\tau \Big[\mathsf{P}_\pi(\tau < \theta) + c\, \mathsf{E}_\pi(\tau - \theta)^+ \Big] \tag{2.4}$$

and finding the optimal stopping time τ_* at which the infimum in (2.4) is attained for $\pi \in [0,1]$ and $c > 0$ given and fixed ˘1pt (recalling also that $p_i \in [0,1]$ with $\sum_{i \in C_k^n} p_i = 1$ are given and fixed). Note in (2.4) that $\mathsf{P}_\pi(\tau < \theta)$ is the probability of the *false alarm* and $\mathsf{E}_\pi(\tau - \theta)^+$ is the expected *detection delay* associated with a stopping time τ of X for $\pi \in [0,1]$. Recall also that the expression on the right-hand side of (2.4) is the Lagrangian associated with the constrained problems as discussed in Section 1 above.

5. To tackle the optimal stopping problem (2.4) we consider the *posterior probability distribution process* $\varPi = (\varPi_t)_{t \ge 0}$ of θ given X that is defined by

$$\varPi_t = \mathsf{P}_\pi(\theta \le t \,|\, \mathcal{F}_t^X) \tag{2.5}$$

for $t \ge 0$. Note that we have

$$\varPi_t = \sum_{i=1}^N \varPi_t^i \tag{2.6}$$

where the summands are defined as follows

$$\varPi_t^i = \mathsf{P}_\pi(\beta = i,\ \theta \le t \,|\, \mathcal{F}_t^X) \tag{2.7}$$

for $t \ge 0$ and $1 \le i \le N$ in C_k^n . The right-hand side of (2.4) can be rewritten to read

$$V(\pi) = \inf_\tau \mathsf{E}_\pi \Big(1 - \varPi_\tau + c \int_0^\tau \varPi_t\, dt \Big) \tag{2.8}$$

for $\pi \in [0,1]$.

6. To connect the process $\varPi$ to the observed process X we define the *posterior probability distribution ratio process* $\varPhi = (\varPhi^1, \ldots, \varPhi^N)$ of θ given X by

$$\varPhi_t^i = \frac{\varPi_t^i}{\bar{\varPi}_t^i} \tag{2.9}$$

where the denominator is defined is follows

$$\bar{\Pi}_t^i = \mathsf{P}_\pi(\beta = i,\, \theta > t \,|\, \mathcal{F}_t^X) \tag{2.10}$$

for $t \ge 0$ and $1 \le i \le N$ in C_k^n . By the Girsanov theorem we see that the *likelihood ratio process* $L = (L^1, \ldots L^N)$ can be expressed as follows

$$L_t^i = \frac{d\mathsf{P}_{i,t}^0}{d\mathsf{P}_t^\infty} = \exp\Big(\mu \sum_{j=1}^k X_t^{n_j} - k\,\frac{\mu^2}{2} t\Big) \tag{2.11}$$

for $t \ge 0$ and $1 \le i \le N$ identified with $(n_1, \ldots, n_k)$ in C_k^n where $\mathsf{P}_{i,t}^0$ and P_t^∞ denote the restrictions of the measures P_i^0 and P^∞ to $\mathcal{F}_t^X$ for $t \ge 0$ and $1 \le i \le N$ in C_k^n . Moreover, using the same arguments as in [4, Section 2] we find that

$$\Phi_t^i = e^{\lambda t} L_t^i \Big(\Phi_0^i + \lambda \int_0^t \frac{ds}{e^{\lambda s} L_s^i}\Big) \tag{2.12}$$

with $\Phi_0^i = \pi/(1{-}\pi)$ for $t \ge 0$ and $1 \le i \le N$ in C_k^n . From (2.11) and (2.12) we see that the process $\Phi = (\Phi^1, \ldots \Phi^N)$ is an explicit (path-dependent) functional of the observed process $X = (X^1, \ldots, X^n)$ and hence observable (by observing a sample path of X we are also seeing a sample path of Φ both in real time).

3. Measure change

In this section we show that changing the probability measure P_π for $\pi \in [0,1]$ to P^∞ in the optimal stopping problem (2.4) or (2.8) provides crucial simplifications of the setting which make the subsequent analysis possible. This will be achieved by invoking the decomposition of P_π into P_i for $i \in C_k^n$ as stated in (2.2) above, changing each probability measure P_i to P_i^∞ , and recalling that each P_i^∞ coincides with P^∞ for $i \in C_k^n$.

1. We show that the optimal stopping problem (2.8) admits a transparent reformulation under the probability measure P^∞ in terms of the process $\Phi = (\Phi^1, \ldots, \Phi^N)$ defined in (2.9) above. Recall that Φ^i starts at $\pi/(1{-}\pi)$ and this dependence on the initial point will be indicated by a superscript to Φ^i when needed for $1 \le i \le N$ in C_k^n .

Proposition 1. *The value function* V *from* (2.8) *satisfies the identity*

$$V(\pi) = (1{-}\pi)\,\big[1 + c\hat{V}(\pi)\big] \tag{3.1}$$

where the value function $\hat{V}$ *is given by*

$$\hat{V}(\pi) = \inf_\tau \mathsf{E}^\infty\Big[\int_0^\tau e^{-\lambda t}\Big(\sum_{i=1}^N p_i \Phi_t^{i,\pi/(1-\pi)} - \frac{\lambda}{c}\Big)\,dt\Big] \tag{3.2}$$

for $\pi \in [0,1)$ *and the infimum in* (3.2) *is taken over all stopping times* τ *of* X *.*

Proof. A derivation of (3.1) and (3.2) can be reduced to one dimension where the change-of-measure identity (4.12) from [7] is applicable in exactly the same way as in the proof of

Proposition 1 in [4]. Given that the present multi-dimensional setting creates no additional difficulties we will omit details and this completes the proof. □

2. From Proposition 1 we see that the optimal stopping problem (2.4) or (2.8) is equivalent to the optimal stopping problem (3.2). From (2.11)+(2.12) using Itô's formula we find that

$$d\varPhi_t^i = \lambda(1+\varPhi_t^i)\,dt + \sum_{j=1}^{k} \mu\varPhi_t^i\,dB_t^{n_j} \quad (i=(n_1,\ldots,n_k)\in C_k^n) \tag{3.3}$$

under P^∞ with $\varPhi_0^i = \varphi_i$ in $[0,\infty)$ all being equal to $\pi/(1-\pi)$ for $1 \le i \le N$ in C_k^n and $\pi \in [0,1)$. Moreover, the system of stochastic differential equations (3.3) has a unique strong solution given by (2.11)+(2.12) above, where X equals B under P^∞, so that $\varPhi = (\varPhi^1,\ldots,\varPhi^N)$ is a *strong Markov* process (see e.g. [14, pp 158-163]). We will establish further in Section 5 below that $\varPhi$ is a strong Feller process but these arguments are more subtle at this stage. Noticing that $W_t^i := (1/\sqrt{k})\sum_{j=1}^k B_t^{n_j}$ defines a standard Brownian motion process for $t \ge 0$, we see from (3.3) that each $\varPhi^i$ is a Shiryaev diffusion process for $1 \le i \le N$ identified with $(n_1,\ldots,n_k)$ in C_k^n. Basic properties of the Shiryaev diffusion processes are reviewed in [9, Section 2]. In particular, it is known that $\varPhi^i$ is *recurrent* in $[0,\infty)$ if and only if $\lambda \le k\mu^2/2$ for $1 \le i \le N$ in C_k^n. If $\lambda > k\mu^2/2$ then $\varPhi^i$ is *transient* in $[0,\infty)$ with $\varPhi_t^i \to \infty$ almost surely under P^∞ as $t \to \infty$ for $1 \le i \le N$ in C_k^n.

3. To tackle the equivalent optimal stopping problem (3.2) for the strong Markov process $\varPhi = (\varPhi^1,\ldots,\varPhi^N)$ solving (3.3) we will enable $\varPhi = (\varPhi^1,\ldots,\varPhi^N)$ to start at any point $\varphi = (\varphi_1,\ldots,\varphi_N) \in [0,\infty)^N$ under the probability measure $\mathsf{P}_\varphi^\infty$ so that the optimal stopping problem (3.2) extends as follows

$$\hat{V}(\varphi) = \inf_\tau \mathsf{E}_\varphi^\infty\Big[\int_0^\tau e^{-\lambda t}\Big(\sum_{i=1}^N p_i\varPhi_t^i - \frac{\lambda}{c}\Big)\,dt\Big] \tag{3.4}$$

for $\varphi \in [0,\infty)^N$ with $\mathsf{P}_\varphi^\infty(\varPhi_0 = \varphi) = 1$ ˇ-1pt where the infimum is taken over all stopping times τ of $\varPhi$ and we recall that $p_i \in [0,1]$ for $1 \le i \le N$ in C_k^n with $\sum_{i=1}^N p_i = 1$. In this way we have reduced the initial quickest detection problem (2.4) or (2.8) to the optimal stopping problem (3.4) for the strong Markov process $\varPhi = (\varPhi^1,\ldots,\varPhi^N)$ solving (3.3) and being explicitly given by the Markovian flow (2.11)+(2.12) of the initial point $(\varPhi_0^1,\ldots,\varPhi_0^N) = (\varphi_1,\ldots\varphi_N) =: \varphi$ in $[0,\infty)^N$ under $\mathsf{P}_\varphi^\infty$. Note that the optimal stopping problem (3.4) is inherently/fully N-dimensional and the infinitesimal generator of $\varPhi = (\varPhi^1,\ldots,\varPhi^N)$ is of elliptic type when $k = 1$ or $k = n-1$ (because $N = n$ and the diffusion matrix in (3.3) is regular in either case) but is not of elliptic (or parabolic) type when $1 < k < n-1$ (because $N > n$ and hence the diffusion matrix in (3.3) cannot be regular). We will return to this issue in Section 5 below.

4. Mayer formulation

The optimal stopping problem (3.4) is Lagrange formulated. In this section we derive its Mayer reformulation which is helpful in the subsequent analysis.

1. From (3.3) we read that the infinitesimal generator of the strong Markov process $\varPhi = (\varPhi^1, \ldots, \varPhi^N)$ is given by

$$\mathbb{L}_{\varPhi} = \sum_{i=1}^{N} \lambda(1+\varphi_i)\,\partial_{\varphi_i} + \frac{1}{2}\sum_{i,j=1}^{N} \mu^2 \varphi_i\varphi_j (I_i, I_j)\,\partial_{\varphi_i\varphi_j} \tag{4.1}$$

for $(\varphi_1, \ldots \varphi_N) \in (0,\infty)^N$ where $I_i = (I_{i1}, \ldots, I_{in})$ with $I_{ip} = 1_i(p)$ which by definition equals 1 if p belongs to $\{n_1, \ldots, n_k\}$ for $1 \le i \le N$ identified with $(n_1, \ldots, n_k)$ in C_k^n and 0 otherwise, and similarly $I_j = (I_{j1}, \ldots, I_{jn})$ with $I_{jp} = 1_j(p)$ which by definition equals 1 if p belongs to $\{m_1, \ldots, m_k\}$ for $1 \le j \le N$ identified with $(m_1, \ldots, m_k)$ in C_k^n and 0 otherwise, while (I_i, I_j) denotes the scalar product of I_i and I_j given by $\sum_{p=1}^{n} I_{ip} I_{jp}$ for $1 \le i, j \le N$ in C_k^n. From (2.12) we see that the topological boundary $\{\,(\varphi_1, \ldots, \varphi_N) \in [0,\infty)^N \mid \varphi_i = 0 \text{ for some } 1 \le i \le N\,\}$ of the state space $[0,\infty)^N$ consists of *entrance* boundary points for $\varPhi$ (meaning that $\varPhi$ can be started at any boundary point never to return to the boundary) and clearly the differential operator $\mathbb{L}_{\varPhi}$ is of elliptic type when $k=1$ or $k=n-1$ but is not of elliptic (or parabolic) type when $1<k<n-1$ which is the case of main interest in what follows. We will determine its type when $1<k<n-1$ in Section 5 below.

2. For the Mayer reformulation of the problem (3.4) we need to look for a function $M : [0,\infty)^N \rightarrow \mathbb{R}$ solving the partial differential equation

$$\mathbb{L}_{\varPhi} M - \lambda M = L \tag{4.2}$$

on $(0,\infty)^N$ where in view of (3.4) we set

$$L(\varphi_1, \ldots, \varphi_N) = \sum_{i=1}^{N} p_i \varphi_i - \lambda/c \tag{4.3}$$

for $(\varphi_1, \ldots, \varphi_N) \in [0,\infty)^N$. Noting that the mixed derivatives in (4.1) vanish, and ignoring the constant $-\lambda/c$ on the right-hand side of (4.2) with (4.3) for now, we see that a possible attempt to solve the resulting partial differential equation is to separate the variables φ_i by considering the ordinary differential equations

$$\lambda(1+\varphi_i)\,M_i' + \frac{\mu^2}{2}\varphi_i^2\, M_i'' - \lambda M_i = \varphi_i \tag{4.4}$$

where $M_i = M_i(\varphi_i)$ is a function/solution to be found for $\varphi_i \in (0,\infty)$ with $1 \le i \le N$ in C_k^n. Note that $M_i = M_j$ for $i \neq j$ in C_k^n so that the subscript to M is superfluous but we will keep it to distinguish M_i for $1 \le i \le N$ in C_k^n from the general function M solving (4.2) to be defined shortly below. It was shown in [4, Section 4] that the sought solution to (4.4) is given by

$$M_i(\varphi_i) = \frac{2}{\mu^2}\,(1+\varphi_i) \int_0^{\varphi_i/(1+\varphi_i)} \Big(\frac{1-v}{v}\Big)^{\kappa} e^{\kappa/v} \int_0^v \frac{u^{\kappa-1}}{(1-u)^{\kappa+2}}\, e^{-\kappa/u}\, du\, dv \tag{4.5}$$

for $\varphi_i \in [0,\infty)$ and $1 \le i \le N$ in C_k^n where we set $\kappa = 2\lambda/\mu^2$. Define a function $M : [0,\infty)^N \to I\!R$ by setting

$$M(\varphi_1,\ldots,\varphi_N) = \sum_{i=1}^{N} p_i M_i(\varphi_i) + 1/c \tag{4.6}$$

for $(\varphi_1,\ldots,\varphi_N) \in [0,\infty)^N$. The arguments above then show that the function M from (4.6) solves the equation (4.2) above (notice that the final term $1/c$ yields the missing constant $-\lambda/c$ on the right-hand side of (4.2) with (4.3) as needed).

3. Having defined the function M in (4.6) we can now describe the Mayer reformulation of the optimal stopping problem (3.4) as follows.

Proposition 2. *The value function* $\hat{V}$ *from* (3.4) *can be expressed as*

$$\hat{V}(\varphi) = \inf_\tau \mathsf{E}_\varphi^\infty \big[e^{-\lambda\tau} M(\Phi_\tau^1,\ldots,\Phi_\tau^N) \big] - M(\varphi) \tag{4.7}$$

for $\varphi \in [0,\infty)^N$ *where the infimum is taken over all (bounded) stopping times* τ *of* $\Phi = (\Phi^1,\ldots,\Phi^N)$ *and the function* M *is given by* (4.6) *using* (4.5) *above.*

Proof. By Itô's formula using (3.3) we get

$$e^{-\lambda t} M(\Phi_t) = M(\varphi) + \int_0^t e^{-\lambda s} \big(I\!L_\Phi M - \lambda M \big)(\Phi_s)\, ds + N_t \tag{4.8}$$

for $\varphi \in [0,\infty)^N$ ˜1pt where $N_t = \sum_{i \in C_k^n} \sum_{j=1}^k \int_0^t e^{-\lambda s} M_{\varphi_i}(\Phi_s) \mu \Phi_s^i \, dB_s^{n_j}$ is a continuous local martingale for $t \ge 0$. Making use of a localisation sequence of stopping times for this local martingale if needed, applying the optional sampling theorem and recalling that M solves (4.2), we find by taking $\mathsf{E}_\varphi^\infty$ on both sides in (4.8) that

$$\mathsf{E}_\varphi^\infty \big[e^{-\lambda\tau} M(\Phi_\tau^1,\ldots,\Phi_\tau^N) \big] = M(\varphi) + \mathsf{E}_\varphi^\infty \Big[\int_0^\tau e^{-\lambda t} L(\Phi_t)\, dt \Big] \tag{4.9}$$

for all $\varphi \in [0,\infty)^N$ and all (bounded) stopping times τ of Φ. From (3.4) and (4.9) using (4.3) we see that (4.7) holds as claimed and the proof is complete. □

4. From Proposition 2 we see that the optimal stopping problem (3.4) is equivalent to the optimal stopping problem defined by

$$\check{V}(\varphi) = \inf_\tau \mathsf{E}_\varphi^\infty \big[e^{-\lambda\tau} M(\Phi_\tau^1,\ldots,\Phi_\tau^N) \big] \tag{4.10}$$

for $\varphi \in [0,\infty)^N$ where the infimum is taken over all (bounded) stopping times τ of $\Phi = (\Phi^1,\ldots,\Phi^N)$ and the function M is given by (4.6) using (4.5) above. The optimal stopping problem (4.10) is Mayer formulated. From (4.7) and (4.10) we see that

$$\hat{V}(\varphi) = \check{V}(\varphi) - M(\varphi) \tag{4.11}$$

for $\varphi \in [0,\infty)^N$. The Mayer reformulation (4.10) has certain advantages that will be exploited in the subsequent analysis of the optimal stopping problem (3.4) below.

5. Hypoellipticity

In this section we show that the differential operator $\mathbb{L}_\Phi$ from (4.1) satisfies the *Hörmander condition* (cf. [6]) and therefore is *hypoelliptic*. This will provide a needed regularity of the value function $\hat{V}$ from (3.4) in the continuation set of the optimal stopping problem. We also show that the previous conclusions on the space operator $\mathbb{L}_\Phi$ extend to the backward time-space operator $-\partial_t+\mathbb{L}_\Phi$ which in turn imply that Φ is a *strong Feller* process. This will provide a needed regularity of $\hat{V}$ at the optimal stopping boundary (between the continuation set and the stopping set). The regularity of $\hat{V}$ in the continuation set and at the optimal stopping boundary will be discussed in Sections 6 and 7 below.

1. *Hörmander's condition.* To recall what it means that $\mathbb{L}_\Phi$ satisfies the Hörmander condition, we will connect to Part 6 of Section 4 in [11] from where one reads that $\mathbb{L}_\Phi$ can be rewritten as the 'sum of squares' as follows

(5.1) $$\mathbb{L}_\Phi = \sum_{i=1}^N \mu_i \partial_{\varphi_i} + \frac{1}{2}\sum_{i,j=1}^N (\sigma\sigma^t)_{ij}\partial_{\varphi_i\varphi_j} = D_0 + \sum_{i=1}^N D_i^2$$

where D_i is a first-order differential operator given by

(5.2) $$D_i = \sum_{j=1}^N \beta_{ij}\partial_{\varphi_j}$$

for $0 \le i \le N$ with the coefficients β_{ij} expressed explicitly as ˇ6pt

(5.3) $$\beta_{0j} = \mu_j - \frac{1}{2}\sum_{k,l=1}^N \sigma_{lk}\,\partial_{\varphi_l}\sigma_{jk} \;\; (1 \le j \le N)$$

(5.4) $$\beta_{ij} = \tfrac{1}{\sqrt{2}}\,\sigma_{ji} \;\; (1 \le i \le N)\; (1 \le j \le N)\,.$$

Comparing (4.1) with (5.1) we see that the drift vector $\mu = (\mu_1, \ldots, \mu_N)$ and the diffusion matrix $\sigma = (\sigma_1; \ldots ; \sigma_N)$ of Φ (the latter written as a sequence of its rows) are given by

(5.5) $$\mu_i(\varphi) = \lambda(1+\varphi_i)$$

(5.6) $$\sigma_i(\varphi) = \mu\varphi_i I_i$$

for $\varphi = (\varphi_1, \ldots, \varphi_N) \in [0,\infty)^N$ and $1 \le i \le N$ where $I_i = (I_{i1}, \ldots, I_{in}, 0, \ldots, 0) \in \{0,1\}^N$ with $I_{ip} = 1_i(p)$ which by definition equals 1 if p belongs to $\{n_1, \ldots, n_k\}$ for $1 \le i \le N$ identified with $(n_1, \ldots, n_k)$ in C_k^n and 0 otherwise (note that we have extended the range of I_i appearing in the text following (4.1) above by adding $N-n$ zeros to make σ a square $N \times N$ matrix). Identifying D_i with $(\beta_{i1}, \ldots, \beta_{iN})$ we see that each D_i may be viewed as a function from $[0,\infty)^N$ (or its subset) to $\mathbb{R}^N$ defined by $D_i(\varphi) = (\beta_{i1}(\varphi), \ldots, \beta_{iN}(\varphi))$ for $\varphi \in [0,\infty)^N$ and $0 \le i \le N$. The *Lie bracket* of D_i and D_j understood as differential operators is defined by

(5.7) $$[D_i, D_j] = D_iD_j - D_jD_i$$

for $0 \le i,j \le N$. The smallest vector space in $I\!R^N$ that (i) contains all $D_0, D_1, \ldots, D_N$ understood as vectors in $I\!R^N$ and (ii) is closed under the Lie bracket operation (5.7) is referred to as the *Lie algebra* generated by $D_0, D_1, \ldots, D_N$ and is denoted by $Lie(D_0, D_1, \ldots, D_N)$. In other words $Lie(D_0, D_1, \ldots, D_N) = \text{span}\{D_i, [D_i, D_j], [[D_i, D_j], D_k], \ldots \mid 0 \le i, j, k, \ldots \le N\}$. Note that $Lie(D_0, D_1, \ldots, D_N)$ may be viewed as a function from $[0,\infty)^N$ into the family of linear subspaces of $I\!R^N$ whose (algebraic) dimensions could also be strictly smaller than N . The Hörmander condition states that

$$\dim Lie(D_0, D_1, \ldots, D_N) = N \tag{5.8}$$

on $[0,\infty)^N$ (or its subset). If (5.8) is satisfied then $I\!L_\Phi$ is hypoelliptic on $[0,\infty)^N$ (or its subset) by the Hörmander theorem (cf. [6, Theorem 1.1]).

Proposition 3. *The differential operator* $I\!L_\Phi$ *from* (4.1) *satisfies the Hörmander condition* (5.8)*. Consequently* $I\!L_\Phi$ *is hypoelliptic.*

Proof. We need to show that (5.8) holds. For this, we find by direct calculation using (5.3) with (5.5)+(5.6) that

$$\beta_{0j} = a\varphi_j + \lambda \;\; (1 \le j \le N) \tag{5.9}$$

where we set $a := \lambda - k\mu^2/2$. By (5.4) and (5.6) we see that

$$\begin{aligned} \beta_{ij} &= \frac{\mu}{\sqrt{2}}\varphi_j 1_j(i) \;\; \text{if } 1 \le i \le n \\ &= \quad 0 \qquad\quad \text{if } n < i \le N \end{aligned} \tag{5.10}$$

for $1 \le j \le N$ where $1_j(i)$ by definition equals 1 if i belongs to $\{m_1, \ldots, m_k\}$ for $1 \le j \le N$ identified with $(m_1, \ldots, m_k)$ in C_k^n and 0 otherwise. Inserting the right-hand sides of (5.9) and (5.10) into (5.2) with i & j swapped we find that

$$\begin{aligned} D_0 = \sum_{i=1}^N (a\varphi_i + \lambda)\partial_{\varphi_i} &\sim \sum_{i=1}^N (\varphi_i + b)\partial_{\varphi_i} \;\; \text{if } a \ne 0 \\ &\sim \sum_{i=1}^N \partial_{\varphi_i} \;\; \text{if } a = 0 \end{aligned} \tag{5.11}$$

$$D_j = \sum_{i=1}^N \frac{\mu}{\sqrt{2}}\varphi_i I_{ij}\partial_{\varphi_i} \sim \sum_{i=1}^N \varphi_i I_{ij}\partial_{\varphi_i} \;\; (1 \le j \le N) \tag{5.12}$$

where in (5.11) we set $b := \lambda/a$ and $L \sim R$ by definition means that R is a constant multiple of L (making R equivalent to L when searching for the Lie algebra generated by a set containing L) and I_{ij} in (5.12) equals 1 if j belongs to $\{n_1, \ldots, n_k\}$ for $1 \le i \le N$ identified with $(n_1, \ldots n_k)$ in C_k^n and 0 otherwise.

Calculating (iterated) Lie brackets of $D_0, D_1, \ldots, D_N$ from (5.11) and (5.12) one arrives at the following recipe for verifying the Hörmander condition (5.8). For this, let any $1 \le j \le N$ with $j = (n_1, \ldots, n_k) \in C_k^n$ be given and fixed. Using (5.11) and (5.12) we find by direct calculation that

$$[D_0, D_{n_1}] \sim \sum_{i=1}^N b I_{in_1}\partial_{\varphi_i} \sim \sum_{i=1}^N I_{in_1}\partial_{\varphi_i} \;\; \text{if } a \ne 0 \tag{5.13}$$

$$= \sum_{i=1}^{N} I_{in_1} \partial_{\varphi_i} \;\text{ if }\; a = 0$$

for any $1 \le n_1 \le n$ and thus the specified one in forming j as well. Using (5.12) and (5.13)we find by direct calculation that

$$[[D_0, D_{n_1}], D_{n_2}] \sim \sum_{i=1}^{N} I_{in_1} I_{in_2} \partial_{\varphi_i} \tag{5.14}$$

for any $1 \le n_1 < n_2 \le n$ and thus the specified ones in forming j as well. Continuing this calculation by induction we find that

$$[[[D_0, D_{n_1}], D_{n_2}], \ldots, D_{n_k}] \sim \sum_{i=1}^{N} I_{in_1} I_{in_2} \ldots I_{in_k} \partial_{\varphi_i} \tag{5.15}$$

for any $1 \le n_1 < n_2 < \ldots < n_k \le n$ and thus the specified ones in forming j as well. By definition of I recalled following (5.12) above we see that

$$\begin{aligned} I_{in_1} I_{in_2} \ldots I_{in_k} &= 1 \;\text{ if }\; i = j \\ &= 0 \;\text{ if }\; i \ne j\,. \end{aligned} \tag{5.16}$$

Combining (5.15) and (5.16) we find that

$$[[[D_0, D_{n_1}], D_{n_2}], \ldots, D_{n_k}] \sim \partial_{\varphi_j}\,. \tag{5.17}$$

Since $1 \le j \le N$ was arbitrary this shows that the iterated Lie brackets (5.17) span the entire $\mathbb{R}^N$ so that the Hörmander condition (5.8) is satisfied as claimed. The final claim follows by the Hörmander theorem as recalled following (5.8) above. This completes the proof. □

2. *Parabolic Hörmander's condition.* To recall what it means that $-\partial_t + \mathbb{L}_\Phi$ satisfies the (parabolic) Hörmander condition, we will connect to Part 3 of Section 5 in [11] from where one reads that $-\partial_t + \mathbb{L}_\Phi$ can be rewritten as the 'sum of squares' as follows

$$-\partial_t + \mathbb{L}_\Phi = -\partial_{\varphi_0} + \sum_{i=1}^{N} \mu_i \partial_{\varphi_i} + \frac{1}{2} \sum_{i,j=1}^{N} (\sigma\sigma^t)_{ij} \partial_{\varphi_i \varphi_j} = \bar{D}_0 + \sum_{i=1}^{N} \bar{D}_i^2 \tag{5.18}$$

where $\bar{D}_i$ is a first-order differential operator given by

$$\bar{D}_i = \sum_{j=0}^{N} \beta_{ij} \partial_{\varphi_j} \tag{5.19}$$

for $0 \le i \le N$ with the coefficients β_{ij} expressed explicitly as

$$\beta_{00} = -1 \;\;\&\;\; \beta_{i0} = 0 \;\; (1 \le i \le N) \tag{5.20}$$

in addition to (5.3) and (5.4) above. Viewing $\bar{D}_i$ as functions from $[0, \infty)^{N+1}$ (or its subset) to $\mathbb{R}^{N+1}$ this amounts to setting

$$\bar{D}_0 = (-1, \beta_{01}, \ldots \beta_{0N}) \;\;\&\;\; \bar{D}_i = (0, \beta_{i1}, \ldots, \beta_{iN}) \;\; (1 \le i \le N)\,. \tag{5.21}$$

Note that $Lie(\bar{D}_0, \bar{D}_1, \ldots, \bar{D}_N)$ can be viewed as a function from $[0,\infty)^{N+1}$ into the family of linear subspaces of $I\!R^{N+1}$ whose (algebraic) dimensions could also be strictly smaller than $N+1$. The (parabolic) Hörmander condition states that

$$\dim Lie(\bar{D}_0, \bar{D}_1, \ldots, \bar{D}_N) = N+1 \tag{5.22}$$

on $[0,\infty)^{N+1}$ (or its subset). If (5.22) is satisfied then $-\partial_t + I\!L_\Phi$ is hypoelliptic on $[0,\infty)^{N+1}$ (or its subset) by the Hörmander theorem (cf. [6, Theorem 1.1]).

Proposition 4. *The backward time-space differential operator* $-\partial_t + I\!L_\Phi$ *of* Φ *satisfies the parabolic Hörmander condition* (5.22). *Consequently* Φ *is a strong Feller process.*

Proof. 1. We need to show that (5.22) holds. For this, note from (5.12) that with a slight abuse of notation we have

$$\bar{D}_0 = (-1, D_0) \;\;\&\;\; \bar{D}_i = (0, D_i) \;\; (1 \le i \le N) \tag{5.23}$$

where D_0 and D_i are defined in (5.2) with (5.3)+(5.4) above followed by the more explicit expressions in (5.11)+(5.12) above. Since the first coordinates -1 and 0 in (5.23) are constants, and $\partial_{\varphi_0} = \partial_t$ plays no role because μ and σ are time independent, we see that calculating the (iterated) Lie brackets of $\bar{D}_0, \bar{D}_1, \ldots, \bar{D}_N$ reduces to calculating the (iterated) Lie brackets of $D_0, D_1, \ldots, D_N$. More specifically, letting any $1 \le i \le N$ with $i = (n_1, \ldots, n_k) \in C_k^n$ be given and fixed, and proceeding in the same way as in the proof of Proposition 3, we find by (5.17) above that

$$[[[\bar{D}_0, \bar{D}_{n_1}], \bar{D}_{n_2}], \ldots, \bar{D}_{n_k}] \sim [[[D_0, D_{n_1}], D_{n_2}], \ldots, D_{n_k}] \sim \partial_{\varphi_i}\,. \tag{5.24}$$

Viewed as a function from $[0,\infty)^{N+1}$ to $I\!R^{N+1}$ we see that the differential operator (5.24) is identified with $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in I\!R^{N+1}$ where the number 1 is placed on the $i+1$ coordinate. Enlarging (5.24) by

$$\bar{D}_0 = (-1, D_0) = -\partial_{\varphi_0} + \sum_{i=1}^{N} (a\varphi_i + \lambda)\partial_{\varphi_i} \tag{5.25}$$

which viewed as a function from $[0,\infty)^{N+1}$ to $I\!R^{N+1}$ is identified with $e_0 = (-1, a\varphi_1 + \lambda, \ldots, a\varphi_N + \lambda)$ for $(\varphi_1, \ldots, \varphi_N) \in [0,\infty)^N$, we see that the vectors e_i for $0 \le i \le N$ are linearly independent in $I\!R^{N+1}$ and clearly span the entire $I\!R^{N+1}$ so that the parabolic Hörmander condition (5.22) is satisfied as claimed.

2. To see that Φ is a strong Feller process, let any bounded measurable function $f : [0,\infty)^N \to I\!R$ be given and fixed. Then by Corollary 9 in [11] we know that

$$\partial_t \mathsf{P}_t f \overset{w}{=} I\!L_\Phi \mathsf{P}_t f \quad \text{in } (0,\infty) \times [0,\infty)^N \tag{5.26}$$

where $\mathsf{P}_t f(\varphi) = \mathsf{E}_\varphi^\infty f(\Phi_t)$ for $t \ge 0$ and $\varphi \in [0,\infty)^N$. Since $-\partial_t + I\!L_\Phi$ satisfies the Hörmander condition (5.22) we know that $-\partial_t + I\!L_\Phi$ is hypoelliptic by the Hörmander theorem as recalled following (5.22) above. It follows therefore that the weak solution $(t,\varphi) \mapsto \mathsf{P}_t f(\varphi)$ in (5.26) is C^∞ on $(0,\infty) \times [0,\infty)^N$. In particular, the mapping $\varphi \mapsto \mathsf{P}_t f(\varphi)$ is C^∞ and therefore continuous as well on $[0,\infty)^N$ for every $t > 0$ given and fixed. This shows that Φ is a strong Feller process as claimed and the proof is complete. □

6. Properties of the optimal stopping boundary

In this section we establish the existence of an optimal stopping time in the problem (3.4) and derive basic properties of the optimal stopping boundary.

1. Looking at (3.4) we may conclude that the (candidate) continuation and stopping sets in this problem need to be defined as follows

$$C = \{ \varphi \in [0,\infty)^N \mid \hat{V}(\varphi) < 0 \} \tag{6.1}$$

$$D = \{ \varphi \in [0,\infty)^N \mid \hat{V}(\varphi) = 0 \} \tag{6.2}$$

respectively. Since the integral in the optimal stopping problem (3.4) is uniformly bounded from below by $-1/c$, it follows by [16, Theorem 6(2), p. 137] that the first entry time of the process Φ into the closed set D defined by

$$\tau_D = \inf \{ t \ge 0 \mid \Phi_t \in D \} \tag{6.3}$$

is optimal in (4.3). Within this general existence result one formally allows that the optimal stopping time τ_D takes the value ∞ as well, however, we will now show that this is not the case in the present problem.

Proposition 5. *We have*

$$\mathsf{P}^\infty_\varphi(\tau_D < \infty) = 1 \tag{6.4}$$

for all $\varphi \in [0,\infty)^N$.

Proof. Since τ_D is optimal in (3.4) it follows from the result and proof of Proposition 1 above that τ_D is optimal in (2.8) as well. The result of Lemma 1 in [7] identifies the Radon-Nikodym derivative corresponding to the measure change from P_π to P^∞ to be

$$\frac{d\mathsf{P}_{\pi,\tau}}{d\mathsf{P}^\infty_\tau} = e^{-\lambda\tau}\,\frac{1-\pi}{1-\Pi_\tau} \tag{6.5}$$

for all stopping times τ of X and all $\pi \in [0,1)$, where P^∞_τ and $\mathsf{P}_{\pi,\tau}$ denote the restrictions of measures P^∞ and P_π to $\mathcal{F}^X_\tau$ for $\pi \in [0,1)$ respectively. Using (6.5) we recognise $e^{\lambda t}(1-\Pi_t)$ as a constant multiple of the Radon-Nikodym derivative $d\mathsf{P}^\infty_t/d\mathsf{P}_{\pi,t}$ and hence the process is a martingale under P_π for $t \ge 0$ whenever $\pi \in [0,1)$ is given and fixed. Moreover, from the fact that the probability measures P_π and P^∞ are singular, we can conclude that

$$e^{\lambda t}(1-\Pi_t) \to 0 \tag{6.6}$$

and hence $\Pi_t \to 1$ both almost surely under P_π as $t \to \infty$ for $\pi \in [0,1)$ (cf. Theorem 2 in [17, p. 527]). Using the latter fact on the right-hand side of (2.8) with τ_D in place of τ we see that $\mathsf{P}_\pi(\tau_D < \infty) = 1$ since otherwise $1-\pi \ge V(\pi) = \infty$ for $\pi \in [0,1)$ which is a contradiction. Since the set $\{\tau_D < \infty\}$ belongs to $\mathcal{F}^X_{\tau_D}$ and by (6.5) above the probability measures P_π and P^∞ restricted to $\mathcal{F}^X_{\tau_D}$ are equivalent (i.e. $\mathsf{P}_\pi(F) = 0$ if and only if $\mathsf{P}^\infty(F) = 0$ for $F \in \mathcal{F}^X_{\tau_D}$) for $\pi \in [0,1)$, it follows that $\mathsf{P}^\infty_\varphi(\tau_D < \infty) = 1$ for all $\varphi \in [0,\infty)^N$ as claimed. $\square$

2. The topological boundary between the sets C and D in $[0,\infty)^N$ is referred to as the *optimal stopping boundary* in the problem (3.4). We will denote it by ∂C although we could

also use ∂D without altering its meaning. To derive an upper bound on the size of ∂C, recall that the optimal stopping boundary/point φ_* in the one-dimensional problem (3.4) (i.e. when $N=1$) can be characterised as a unique solution to

$$\frac{e^{\kappa(1+\varphi_*)/\varphi_*}}{\varphi_*^\kappa} \int_0^{\varphi_*/(1+\varphi_*)} \frac{u^{\kappa-1}}{(1-u)^{\kappa+2}}\, e^{-\kappa/u}\, du = \frac{\mu^2}{2c} \tag{6.7}$$

on $(\lambda/c,\infty)$ where we set $\kappa = 2\lambda/\mu^2$ (cf. [4, Section 5] & [12, Section 22]). Note from (6.7) that we have $\varphi_* = \varphi_*(\lambda/\mu^2, \lambda/c)$ and set

$$\varphi_i^* := \varphi_*\big(\tfrac{\lambda}{k\mu^2}, \tfrac{\lambda}{p_i c}\big) \tag{6.8}$$

for $\lambda > 0$, $\mu \in I\!\!R$, $c>0$ and $p_i \in [0,1]$ for $1 \le i \le N$ with $\sum_{i=1}^N p_i = 1$. Recall that $\varphi_i^* \in (\lambda/(p_i c),\infty)$ for $1 \le i \le N$. We can now expose basic properties of the the value function and the continuation/stopping set in the problem (3.4) as follows.

Proposition 6.

(6.9) *The value function* $\hat{V}$ *is concave and continuous on* $[0,\infty)^N$.

(6.10) *If* $\varphi_1 \le \psi_1, \ldots, \varphi_N \le \psi_N$ *then* $\hat{V}(\varphi_1,\ldots,\varphi_N) \le \hat{V}(\psi_1,\ldots\psi_N)$.

(6.11) *If* $(\varphi_1,\ldots,\varphi_N) \in D$ *and* $\psi_1 \ge \varphi_1, \ldots, \psi_N \ge \varphi_N$ *then* $(\psi_1,\ldots\psi_N) \in D$.

(6.12) *The stopping set* D *is convex and the polytope* $\{\, (\varphi_1,\ldots\varphi_N) \in [0,\infty)^N \mid \sum_{i=1}^N \varphi_i/\varphi_i^* - 1 \ge 0 \,\}$ *is contained in* D.

(6.13) *The simplex* $\{\, (\varphi_1,\ldots,\varphi_N) \in [0,\infty)^N \mid \sum_{i=1}^N p_i\varphi_i - \lambda/c < 0 \,\}$ *is contained in the continuation set* C.

Proof. (6.9): Combining the fact that the Markovian flow (2.12) is linear as a function of its initial point with the fact that the integral in (3.4) is a linear function of its argument, and using that the infimum of a convex combination is larger than the convex combination of the infima, we find that $\hat{V}$ is concave on $[0,\infty)^N$ as claimed. Hence we can also conclude that $\hat{V}$ is continuous on the open set $(0,\infty)^N$. To see that $\hat{V}$ is continuous at the boundary points of $[0,\infty)^N$ we may recall the well-known (and easily verified) fact that the concave function $\hat{V}$ is lower semicontinuous on the closed and convex set $[0,\infty)^N$. Moreover, recalling that (2.12) defines a Markovian functional of the initial point $\Phi_0^i := \varphi_i$ in $[0,\infty)$ of the process Φ^i for $1 \le i \le N$, we see that the expectation in (4.10) defines a continuous function of the initial point $\varphi = (\varphi_1,\ldots,\varphi_N)$ of the process $\Phi = (\Phi^1,\ldots,\Phi^N)$ for every (bounded) stopping time τ of Φ given and fixed. Taking the infimum over all (bounded) stopping times τ of Φ we can thus conclude from (4.11) that the value function $\hat{V}$ is upper semicontinuous on $[0,\infty)^N$. Being also lower semicontinuous it follows that $\hat{V}$ is continuous on $[0,\infty)^N$ as claimed.

(6.10): This is a direct consequence of the fact that the Markovian flow (2.12) is increasing as a function of its initial point being used in (3.4) above.

(6.11): By (6.10) we have $\hat{V}(\varphi_1,\ldots,\varphi_N) \le \hat{V}(\psi_1,\ldots,\psi_N) \le 0$ so that $(\varphi_1,\ldots,\varphi_N) \in D$ i.e. $\hat{V}(\varphi_1,\ldots,\varphi_N) = 0$ implies that $\hat{V}(\psi_1,\ldots,\psi_N) = 0$ i.e. $(\psi_1,\ldots,\psi_N) \in D$ as claimed.

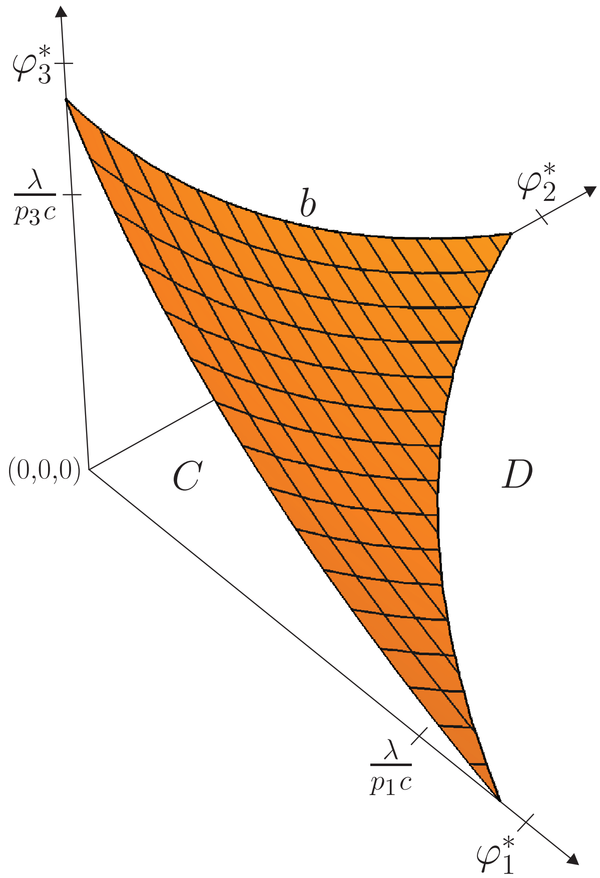

Figure 1. The optimal stopping surface b in the problem (3.4) when $n=3$ and $k=2$ with $\mu=\lambda=c=1$ and $p_{1,2}=p_{1,3}=p_{2,3}=1/3$.

(6.12): To see that D is convex, take any φ and ψ from D and note by (6.9) that $0 \geq \hat{V}(\alpha\varphi+(1-\alpha)\psi) \geq \alpha\hat{V}(\varphi)+(1-\alpha)\hat{V}(\psi)=0$ so that $\hat{V}(\alpha\varphi+(1-\alpha)\psi)=0$ i.e. $\alpha\varphi+(1-\alpha)\psi \in D$ for every $\alpha \in [0,1]$ as claimed. To see that the polytope is contained in D , note that pulling p_j in front of the infimum in (3.4) with any $1 \leq j \leq N$ given and fixed shows that the point $(0,\dots,0,\varphi_j^*,0,\dots,0)$ belongs to D because φ_j^* as defined in (6.8) above (with j in place of i) is ˇ1pt an optimal stopping point in the one-dimensional problem obtained by removing the non-negative term $\sum_{i=1,i\neq j}^N (p_i/p_j)\,\Phi_t^i$ from the integral with respect to time in (3.4) with p_j in front of ˇ1pt the infimum (note that the appearance of k in (6.8) follows from the fact that $(1/\sqrt{k})\sum_{j=1}^k B_t^{n_j}$ is a standard Brownian motion for $t \geq 0$). It follows therefore by (6.11) that the set $\{(0,\dots,0,\varphi_j,0,\dots,0) \mid \varphi_j \geq \varphi_j^*\}$ is contained in D for every $1 \leq j \leq N$. But then the entire polytope is contained in D due to its convexity.

(6.13): Taking any point φ from the (open) simplex and replacing τ in (3.4) by the first exit time of Φ from a sufficiently small ball around φ that is strictly contained in the simplex, we see that the integrand in (3.4) remains strictly negative so that $\hat{V}$ takes a strictly negative value at φ itself, showing that φ belongs to the continuation set C as claimed. □

3. From the results of Proposition 6 we see that the stopping set in the problem (3.4) can

be described as follows

$$D = \{ (\varphi_1, \ldots, \varphi_N) \in [0,\infty)^N \mid \varphi_N \ge b(\varphi_1, \ldots, \varphi_{N-1}) \} \tag{6.14}$$

where $b : [0,\infty)^{N-1} \to [0,\infty)$ is a convex, continuous, decreasing function (in the sense that $b(\varphi_1, \ldots, \varphi_{N-1}) \ge b(\psi_1, \ldots, \psi_{N-1})$ whenever $\varphi_1 \le \psi_1, \ldots, \varphi_{N-1} \le \psi_{N-1}$) satisfying

$$-\sum_{i=1}^{N-1} \frac{p_i}{p_N}\, \varphi_i + \frac{\lambda}{p_N c} \le b(\varphi_1, \ldots, \varphi_{N-1}) \le -\sum_{i=1}^{N-1} \frac{\varphi_N^*}{\varphi_i^*}\, \varphi_i + \varphi_N^* \tag{6.15}$$

for $\varphi_1 \in [0, \lambda/(p_1 c)], \ldots, \varphi_{N-1} \in [0, \lambda/(p_{N-1} c)]$ and $\varphi_1 \in [0, \varphi_1^*], \ldots, \varphi_{N-1} \in [0, \varphi_{N-1}^*]$ in the first and second inequality respectively (see Figure 1). Note that the optimal stopping boundary in the problem (3.4) can be described as follows

$$\partial C = \{ (\varphi_1, \ldots, \varphi_N) \in [0,\infty)^N \mid \varphi_N = b(\varphi_1, \ldots, \varphi_{N-1}) \}\, . \tag{6.16}$$

We address the question of characterising/determining b in the remaining two sections. To this end we conclude this section by establishing a key regularity result of ∂C for D.

4. Recall that a point $\varphi \in [0,\infty)^N$ is said to be *probabilistically regular* for D if

$$\mathsf{P}_\varphi^\infty(\sigma_D = 0) = 1 \tag{6.17}$$

where σ_D is the first hitting time of $\varPhi$ to D defined by $\sigma_D = \inf\{\, t > 0 \mid \varPhi_t \in D \,\}$ (see Sections 2 and 3 in [2] for fuller details). If every point at ∂C is probabilistically regular for D we say that ∂C is probabilistically regular for D. We now show that this is the case in the optimal stopping problem (3.4).

Proposition 7. *The optimal stopping boundary ∂C is probabilistically regular for the stopping set D in the problem* (3.4).

Proof. Let any point $\varphi = (\varphi_1, \ldots, \varphi_N)$ at ∂C be given and fixed. We need to show that (6.17) holds. For this, note that

$$\begin{aligned} \mathsf{P}_\varphi^\infty\big(\sigma_D = 0\big) &= \mathsf{P}_\varphi^\infty\big(\cap_{n=1}^\infty \cup_{t\in(0,1/n)} \{\varPhi_t \in D\}\big) = \lim_{n\to\infty} \mathsf{P}_\varphi^\infty\big(\cup_{t\in(0,1/n)} \{\varPhi_t \in D\}\big) \\ &\ge \lim_{n\to\infty} \sup_{t\in(0,1/n)} \mathsf{P}_\varphi^\infty(\varPhi_t \in D) = \limsup_{t\downarrow 0} \mathsf{P}_\varphi^\infty(\varPhi_t \in D) \\ &\ge \limsup_{t\downarrow 0} \mathsf{P}_\varphi^\infty\big(\varPhi_t^1 \ge \varphi_1, \ldots, \varPhi_t^N \ge \varphi_N\big) \\ &\ge \limsup_{t\downarrow 0} \mathsf{P}^\infty\big(L_t^1 \ge 1, \ldots, L_t^N \ge 1\big) \\ &= \limsup_{t\downarrow 0} \mathsf{P}^\infty\Big(\textstyle\sum_{j=1}^k X_t^{n_j^1} \ge k\, \frac{\mu}{2} t, \ldots, \sum_{j=1}^k X_t^{n_j^N} \ge k\, \frac{\mu}{2} t\Big) \\ &\ge \limsup_{t\downarrow 0} \mathsf{P}^\infty\big(X_t^1 \ge \tfrac{\mu}{2} t, \ldots, X_t^n \ge \tfrac{\mu}{2} t\big) \\ &= \limsup_{t\downarrow 0} \mathsf{P}\big(B_t^1 \ge \tfrac{\mu}{2} t, \ldots, B_t^n \ge \tfrac{\mu}{2} t\big) \\ &= \limsup_{t\downarrow 0} \big(\mathsf{P}\big(B_t^1 \ge \tfrac{\mu}{2} t\big)\big)^n = \lim_{t\downarrow 0} \big(\mathsf{P}\big(B_1^1 \ge \tfrac{\mu}{2} \sqrt{t}\big)\big)^n \end{aligned} \tag{6.18}$$

$$= \big(\mathsf{P}(B_1^1 > 0)\big)^n = 1/2^n > 0$$

where in the second inequality we use that $[\varphi_1,\infty)\times\ldots\times[\varphi_N,\infty)\subseteq D$ by (6.11) above, in the third inequality we use (2.12) above, and in the fourth equality we use (2.11) above (upon assuming that $\mu > 0$ without loss of generality). From (6.18) we see that $\mathsf{P}_\varphi^\infty\big(\sigma_D = 0\big) > 0$. As the latter probability can only be either zero or one by the Blumenthal 0-1 law (cf. [1, p. 30]), it follows that (6.17) holds as claimed. □

Remark 8. Note that if we replace all inequalities under the probability measures in (6.18) by strict inequalities, then the same proof shows that ∂C is probabilistically regular for the interior of the stopping set D in the problem (3.4). Although this stronger probabilistic regularity plays no role in the present setting because the process $\varPhi$ is strong Feller by Proposition 4 above, this observation may be useful in the settings where the process $\varPhi$ is only known to be strong Markov (see [2] for fuller details).

7. Free-boundary problem

In this section we derive a free-boundary problem that stands in one-to-one correspondence with the optimal stopping problem (3.4). Using the results derived in the previous sections we show that the value function $\hat{V}$ from (3.4) and the optimal stopping boundary b from (6.16) solve the free-boundary problem. This establishes the existence of a solution to the free-boundary problem. Its uniqueness in a natural class of functions will follow from a more general uniqueness result that will be established in Section 8 below. This will also yield an explicit integral representation of the value function $\hat{V}$ expressed in terms of the optimal stopping boundary b.

1. Consider the optimal stopping problem (3.4) where the Markov process $\varPhi = (\varPhi^1,\ldots,\varPhi^N)$ solves the system of stochastic differential equations (3.3) driven by a standard n-dimensional Brownian motion $B = (B^1,\ldots,B^n)$ under the probability measure P^∞. Recall that the infinitesimal generator of $\varPhi$ is the second-order hypoelliptic differential operator $I\!\!L_\varPhi$ given in (4.1) above (cf. Proposition 3). Looking at (3.4) and relying on other properties of $\hat{V}$ and b derived above, we are naturally led to formulate the following free-boundary problem for finding $\hat{V}$ and b:

(7.1) $$I\!\!L_\varPhi \hat{V} - \lambda \hat{V} = -L \quad \text{in } C$$

(7.2) $$\hat{V}(\varphi) = 0 \quad \text{for } \varphi \in D \quad \text{(instantaneous stopping)}$$

(7.3) $$\hat{V}_{\varphi_i}(\varphi) = 0 \quad \text{for } \varphi \in \partial C \text{ and } i = 1,\ldots,N \quad \text{(smooth fit)}$$

where L is defined in (4.3) above, C is the (continuation) set from (6.1) above, D is the (stopping) set from (6.2)+(6.14) above, and ∂C is the (optimal stopping) boundary between the sets C and D from (6.16) above.

2. To formulate the existence and uniqueness result for the free-boundary problem (7.1)-(7.3), we let $\mathcal{C}$ denote the class of functions (U,a) such that

(7.4) U belongs to $C^2(C_a)\cap C^1(\bar{C}_a)$ and is continuous & bounded on $[0,\infty)^N$

(7.5) a is continuous & decreasing on $[0,\infty)^{N-1}$ (in the sense that $a(\varphi_1,\ldots,\varphi_{N-1})$

$\geq a(\psi_1, \ldots, \psi_{N-1})$ whenever $\varphi_1 \leq \psi_1, \ldots, \varphi_{N-1} \leq \psi_{N-1})$ and satisfies $\sum_{i=1}^{N-1} p_i\varphi_i + p_N a(\varphi_1, \ldots, \varphi_{N-1}) - \lambda/c \geq 0$ for $(\varphi_1, \ldots, \varphi_{N-1}) \in [0, \infty)^{N-1}$

where we set $C_a = \{ (\varphi_1, \ldots, \varphi_N) \in [0, \infty)^N \mid \varphi_N < a(\varphi_1, \ldots, \varphi_{N-1}) \}$ and $\bar{C}_a = \{ (\varphi_1, \ldots, \varphi_N) \in [0, \infty)^N \mid \varphi_N \leq a(\varphi_1, \ldots, \varphi_{N-1})$ with $(\varphi_1, \ldots, \varphi_{N-1})$ belonging to the closure of the set $\{ (\psi_1, \ldots, \psi_{N-1}) \mid a(\psi_1, \ldots, \psi_{N-1}) > 0 \}$ in $[0, \infty)^{N-1} \}$.

Theorem 9. *The free-boundary problem* (7.1)-(7.3) *has a unique solution* $(\hat{V}, b)$ *in the class* $\mathcal{C}$ *where* $\hat{V}$ *is given in* (3.4) *and* b *is given in* (6.16) *above.*

Proof. We first show that the pair $(\hat{V}, b)$ belongs to the class $\mathcal{C}$ and solves the free-boundary problem (7.1)-(7.3). For this, note that the optimal stopping problem (3.4) is Lagrange formulated and recall that the infimum in (3.4) is attained at τ_D from (6.3) above. It follows therefore by the result of Corollary 6 in [11] that $\hat{V}$ from (3.4) is a weak solution to the equation (7.1) in the sense of Schwartz distributions. By the result of Proposition 3 above we know that the differential operator $\mathbb{L}_\Phi$ is hypoelliptic and hence $\mathbb{L}_\Phi - \lambda I$ is hypoelliptic too. It follows therefore that the weak solution $\hat{V}$ to (7.1) belongs to C^∞ on C (cf. [11, Corollary 8]). This shows that $\hat{V}$ belongs to $C^2(C)$ and satisfies (7.1) as claimed. Moreover, from (6.9) we know that $\hat{V}$ is continuous on $[0, \infty)^N$ and from (3.4) we readily find that

$$-\frac{1}{c} \leq \hat{V}(\varphi) \leq 0 \tag{7.6}$$

for all $\varphi \in [0, \infty)^N$. Furthermore, recall from Proposition 4 above that the process $\Phi = (\Phi^1, \ldots, \Phi^N)$ is strong Feller while by Proposition 7 we know that ∂C is probabilistically regular for D. Finally, from (2.12) we see that the process Φ can be realised as a continuously differentiable stochastic flow of its initial point so that the integrability conditions of Theorem 8 in [2] are satisfied. Recalling that $\hat{V}$ satisfies (7.2), and applying the result of that theorem, we can conclude that

$$\hat{V} \text{ is continuously differentiable on } [0, \infty)^N. \tag{7.7}$$

In particular, this shows that (7.3) holds as well as that $\hat{V}$ belongs to $C^1(\bar{C})$ as required in (7.4) above. The fact that b satisfies (7.5) was established in (6.14)-(6.15) above. This shows that $(\hat{V}, b)$ belongs to $\mathcal{C}$ and solves (7.1)-(7.3) as claimed. To derive uniqueness of the solution we will first see in the next section that any solution (U, a) to (7.1)-(7.3) from the class $\mathcal{C}$ admits an explicit integral representation for U expressed in terms of a, which in turn solves a nonlinear Fredholm integral equation, and we will see that this equation cannot have other solutions satisfying the required properties. From these facts we can conclude that the free-boundary problem (7.1)-(7.3) cannot have other solutions in the class $\mathcal{C}$ as claimed. This completes the proof. □

8. Nonlinear integral equations

In this section we show that the optimal stopping boundary b from (6.16) can be characterised as the unique solution to a nonlinear Fredholm integral equation. This also yields an explicit integral representation of the value function $\hat{V}$ from (3.4) expressed in terms of the optimal stopping boundary b. As a consequence of the existence and uniqueness result for the the nonlinear Fredholm integral equation we also obtain uniqueness of the solution to the free-boundary problem (7.1)-(7.3) as explained in the proof of Theorem 9 above. Finally, collecting the results derived throughout the paper we conclude our exposition by disclosing the solution to the initial problem.

1. Let $p = p(t; \varphi_1, \ldots, \varphi_N, \psi_1, \ldots, \psi_N)$ denote the transition probability density function of the Markov process $\Phi = (\Phi^1, \ldots, \Phi^N)$ in the sense that

$$\mathsf{P}^{\infty}_{\varphi_1,\ldots,\varphi_N}(\Phi_t \in A) = \int \cdots \int_A p(t; \varphi_1, \ldots, \varphi_N, \psi_1, \ldots, \psi_N)\, d\psi_1 \ldots d\psi_N \tag{8.1}$$

for any measurable $A \subseteq [0,\infty)^N$ with $(\varphi_1, \ldots, \varphi_N) \in [0,\infty)^N$ and $t \geq 0$ given and fixed. Having p we can evaluate the expression of interest in the theorem below as follows

$$\begin{aligned} K_b(t; \varphi_1, \ldots, \varphi_N) &:= \mathsf{E}^{\infty}_{\varphi_1,\ldots,\varphi_N}\big[L(\Phi^1_t, \ldots, \Phi^N_t)\, I\big(\Phi^N_t < b(\Phi^1_t, \ldots, \Phi^{N-1}_t)\big)\big] \\ &= \int \cdots \int_{\{\psi_N < b(\psi_1,\ldots,\psi_{N-1})\}} L(\psi_1, \ldots, \psi_N)\, p(t; \varphi_1, \ldots, \varphi_N, \psi_1, \ldots, \psi_N)\, d\psi_1 \ldots d\psi_N \end{aligned} \tag{8.2}$$

for $t \geq 0$ and $(\varphi_1, \ldots, \varphi_N) \in [0,\infty)^N$ where L is defined in (4.3) above.

Theorem 10 (Existence and uniqueness). *The optimal stopping boundary b in (3.4) can be characterised as the unique solution to the nonlinear Fredholm integral equation*

$$\int_0^\infty e^{-\lambda t} K_b(t; \varphi_1, \ldots, \varphi_{N-1}, b(\varphi_1, \ldots, \varphi_{N-1}))\, dt = 0 \tag{8.3}$$

in the class of continuous & decreasing (convex) functions b on $[0,\infty)^{N-1}$ satisfying $\sum_{i=1}^{N-1} p_i \varphi_i + p_N b(\varphi_1, \ldots, \varphi_{N-1}) - \lambda/c \geq 0$ for $(\varphi_1, \ldots, \varphi_{N-1}) \in [0,\infty)^{N-1}$. The value function $\hat{V}$ in (3.4) admits the following representation

$$\hat{V}(\varphi_1, \ldots, \varphi_N) = \int_0^\infty e^{-\lambda t} K_b(t; \varphi_1, \ldots, \varphi_N)\, dt \tag{8.4}$$

for $(\varphi_1, \ldots, \varphi_N) \in [0,\infty)^N$. The optimal stopping time in (3.4) is given by

$$\tau_b = \inf\{\, t \geq 0 \mid \Phi^N_t \geq b(\Phi^1_t, \ldots, \Phi^{N-1}_t)\,\} \tag{8.5}$$

under $\mathsf{P}^{\infty}_{\varphi_1,\ldots,\varphi_N}$ with $(\varphi_1, \ldots, \varphi_N) \in [0,\infty)^N$ given and fixed.

Proof. (i) *Existence.* We first show that the optimal stopping boundary b in (3.4) solves (8.3). Recalling that b satisfies the properties stated following (6.14) above, this will establish the existence of a solution to (8.3) in the specified class of functions.

For this, to gain control over the (individual) second partial derivatives $\hat{V}_{\varphi_i \varphi_j}$ close to the optimal stopping boundary within C for $1 \leq i,j \leq N$ (see [5] for general results of this kind in the elliptic case), consider the sets $C_n := \{\, \varphi \in [0,\infty)^N \mid \hat{V}(\varphi) < -1/n \,\}$ and $D_n := \{\, \varphi \in [0,\infty)^N \mid \hat{V}(\varphi) \geq -1/n \,\}$ for $n \geq 1$ (large). Note that $C_n \uparrow C$ and $D_n \downarrow D$ as $n \uparrow \infty$. Moreover, using the same arguments as for the sets C and D above, we find that the set D_n is convex, and the boundary $b_n = b_n(\varphi_1, \ldots, \varphi_{N-1})$ between C_n and D_n is a convex, continuous, decreasing function of $(\varphi_1, \ldots, \varphi_{N-1})$ in $[0,\infty)^{N-1}$. This also shows that

$b_n \uparrow b$ uniformly on $[0,\infty)^{N-1}$ as $n \to \infty$ (where the functions b_n and b take zero value at $(\varphi_1,\ldots,\varphi_{N-1})$ by definition whenever $(\varphi_1,\ldots,\varphi_{N-1},0)$ belongs to D_n and D respectively for $n \geq 1$). Approximate the value function $\hat{V}$ in (3.4) by functions $\hat{V}^n$ defined as $\hat{V}$ on C_n and $-1/n$ on D_n for $n \geq 1$. Note that $\hat{V}^n \uparrow \hat{V}$ uniformly on $[0,\infty)^N$ as $n \to \infty$. Moreover, letting $n \geq 1$ be given and fixed in the sequel, clearly $\hat{V}^n$ is a continuous function on $[0,\infty)^N$ and $\hat{V}^n$ restricted to C_n and D_n belongs to $C^2(\bar{C}_n)$ and $C^2(\bar{D}_n)$ respectively. Finally, since b_n is convex we know that $b_n(\Phi^1,\ldots,\Phi^{N-1})$ is a continuous semimartingale. This shows that the change-of-variable formula with local time on surfaces [10, Theorem 2.1] is applicable to $\hat{V}^n$ composed with $\Phi=(\Phi^1,\ldots,\Phi^N)$ and using (7.1) this gives

$$
\begin{aligned}
(8.6)\qquad e^{-\lambda t}\hat{V}^n(\Phi_t) &= \hat{V}^n(\Phi_0) + \int_0^t e^{-\lambda s}\big(I\!\!L_\Phi \hat{V}^n - \lambda \hat{V}^n\big)(\Phi_s)\,ds \\
&\quad + \sum_{i=1}^N \sum_{j=1}^k \int_0^t e^{-\lambda s}\hat{V}^n_{\varphi_i}(\Phi_s)\,\mu\Phi^i_s\,dB^{n_j(i)}_s - \int_0^t e^{-\lambda s}\hat{V}^n_{\varphi_N}(\Phi_s-)\,d\ell^{b_n}_s(\Phi) \\
&= \hat{V}^n(\Phi_0) - \int_0^t e^{-\lambda s} L(\Phi_s)\, I(\Phi_s \in C_n)\,ds + M^n_t - \int_0^t e^{-\lambda s}\hat{V}_{\varphi_N}(\Phi_s)\,d\ell^{b_n}_s(\Phi)
\end{aligned}
$$

where $M^n_t = \sum_{i=1}^N \sum_{j=1}^k \int_0^t e^{-\lambda s}\hat{V}_{\varphi_i}(\Phi_s)\,\mu\Phi^i_s\, I(\Phi_s \in C_n)\,dB^{n_j(i)}_s$ is a continuous martingale for $t \geq 0$ and $\ell^{b_n}(\Phi)$ is the local time of Φ on the curve b_n given by

$$
\begin{aligned}
(8.7)\qquad \ell^{b_n}_t(\Phi) = \mathsf{P}\text{-}\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t & I(-\varepsilon < \Phi^N_s - b_n(\Phi^1_s,\ldots,\Phi^{N-1}_s) < \varepsilon) \\
& d\langle \Phi^N - b_n(\Phi^1,\ldots,\Phi^{N-1}), \Phi^N - b_n(\Phi^1,\ldots,\Phi^{N-1})\rangle_s
\end{aligned}
$$

for $t \geq 0$. To gain control over the final term in (8.6), note that the Itô-Tanaka formula (cf. [13, pp 222-223]) yields

$$
\begin{aligned}
(8.8)\qquad \big(b_n(\Phi^1_t,\ldots,\Phi^{N-1}_t) - \Phi^N_t\big)^+ &= \big(b_n(\Phi^1_0,\ldots,\Phi^{N-1}_0) - \Phi^N_0\big)^+ \\
&\quad + \int_0^t I(b_n(\Phi^1_s,\ldots,\Phi^{N-1}_s) - \Phi^N_s > 0)\, d(b_n(\Phi^1,\ldots,\Phi^{N-1}) - \Phi^N)_s + \frac{1}{2}\ell^{b_n}_t(\Phi) \\
&= \big(b_n(\Phi^1_0,\ldots,\Phi^{N-1}_0) - \Phi^N_0\big)^+ + \int_0^t I(b_n(\Phi^1_s,\ldots,\Phi^{N-1}_s) - \Phi^N_s > 0) \\
&\quad \Big(\sum_{i=1}^{N-1}\frac{\partial b_n}{\partial \varphi_i}(\Phi^1_s,\ldots\Phi^{N-1}_s)\,d\Phi^i_s + \frac{1}{2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{\partial^2 b_n}{\partial\varphi_i\partial\varphi_j}(\Phi^1_s,\ldots\Phi^{N-1}_s)\,d\langle\Phi^i,\Phi^j\rangle_s - d\Phi^N_s\Big) \\
&\quad + \frac{1}{2}\ell^{b_n}_t(\Phi)
\end{aligned}
$$

for $t \geq 0$ where we use that b_n is C^2 by the implicit function theorem since the smooth fit fails at b_n due to its suboptimality in the problem (3.4). Since b_n is convex we know that the Hessian matrix of b_n is non-negative definite and recalling from (3.3)+(4.1) that $d\langle\Phi^i,\Phi^j\rangle_s = \mu^2\Phi^i_s\Phi^j_s(I_i,I_j)\,ds$ for $1 \leq i,j \leq N-1$ we see that the integral associated with the double sum in (8.8) is non-negative. It follows therefore from (8.8) using (3.3) above that

$$
(8.9)\qquad \frac{1}{2}\ell^{b_n}_t(\Phi) \leq \big(b_n(\Phi^1_t,\ldots,\Phi^{N-1}_t) - \Phi^N_t\big)^+
$$

$$
\begin{aligned}
&- \sum_{i=1}^{N-1} \int_0^t I(b_n(\Phi_s^1,\ldots,\Phi_s^{N-1}) - \Phi_s^N > 0)\, \frac{\partial b_n}{\partial \varphi_i}(\Phi_s^1,\ldots,\Phi_s^{N-1})\, \lambda(1+\Phi_s^i)\, ds \\
&+ \int_0^t I(b_n(\Phi_s^1,\ldots,\Phi_s^{N-1}) - \Phi_s^N > 0)\, \lambda(1+\Phi_s^N)\, ds + N_t^n
\end{aligned}
$$

where ˘2pt $N_t^n = -\sum_{i=1}^{N-1} \sum_{j=1}^k \int_0^t I(b_n(\Phi_s^1,\ldots,\Phi_s^{N-1}) - \Phi_s^N > 0)(\partial b_n/\partial \varphi_i)(\Phi_s^1,\ldots,\Phi_s^{N-1})\, \mu \Phi_s^i dB_s^{n_j(i)}$ $+ \sum_{j=1}^k \int_0^t I(b_n(\Phi_s^1,\ldots,\Phi_s^{N-1}) - \Phi_s^N > 0)\, \mu \Phi_s^N dB_s^{n_j(N)}$ is a continuous local martingale for $t \ge 0$. Let $(\tau_m)_{m \ge 1}$ be a localising sequence of stopping times for N^n, define the stopping time

$$
\sigma_m = \inf \{\, t \ge 0 \mid (\Phi_t^1,\ldots,\Phi_t^{N-1}) \notin (\tfrac{1}{m}, \infty)^{N-1} \,\} \tag{8.10}
$$

and set $\rho_m := \tau_m \wedge \sigma_m$ for $m \ge 1$. Fixing $\varphi \in [0,\infty)^N$ we then find from (8.9) that

$$
\begin{aligned}
\frac{1}{2} \mathsf{E}_\varphi^\infty \big[\ell_{t \wedge \rho_m}^{b_n}(\Phi) \big] \le \varphi_N^* - \sum_{i=1}^{N-1} \frac{\partial b_n}{\partial \varphi_i}(\tfrac{1}{m},\ldots,\tfrac{1}{m}) \int_0^t \lambda\big(1+\mathsf{E}_\varphi^\infty(\Phi_s^i)\big)\, ds \\
+ \int_0^t \lambda\big(1+\mathsf{E}_\varphi^\infty(\Phi_s^N)\big)\, ds \le K_m(t)
\end{aligned} \tag{8.11}
$$

for $t \ge 0$ and $m \ge 1$ where the positive constant $K_m(t)$ does not depend on $n \ge 1$ because each b_n is convex and $b_n \uparrow b$ on $[0,1/m]^{N-1}$ as $n \to \infty$ so that $(\partial b_n/\partial \varphi_i)(1/m,\ldots,1/m)$ must stay bounded from below over $n \ge 1$ for $1 \le i \le N-1$ if b_n is to stay below b on $[0,1/m]^{N-1}$ for all $n \ge 1$. In addition, by (7.7) we know that $\hat{V}_{\varphi_N}$ is continuous on $\bar{C}$ and hence uniformly continuous too because $\bar{C}$ is a compact set. ˘1pt It follows therefore that $0 \le \hat{V}_{\varphi_N}(\varphi_1,\ldots,\varphi_{N-1}, b_n(\varphi_1,\ldots,\varphi_{N-1})) \le \varepsilon$ for all $(\varphi_1,\ldots,\varphi_{N-1}) \in [0,\infty)^{N-1}$ and all $n \ge n_\varepsilon$ with $n_\varepsilon \ge 1$ large enough depending on the given and fixed $\varepsilon > 0$. Combining this fact with (8.11), upon replacing t with $t \wedge \rho_m$ in the final integral of (8.6) and taking $\mathsf{E}_\varphi^\infty$ of the resulting expression, we see that

$$
0 \le \mathsf{E}_\varphi^\infty \bigg[\int_0^{t \wedge \rho_m} e^{-\lambda s} \hat{V}_{\varphi_N}(\Phi_s)\, d\ell_s^{b_n}(\Phi) \bigg] \le 2\varepsilon K_t(m) \tag{8.12}
$$

for all $n \ge n_\varepsilon$ with $t \ge 0$ and $m \ge 1$ given and fixed. This shows that the expectation in (8.12) tends to zero as n tends to infinity for every $t \ge 0$ and $m \ge 1$ given and fixed. Using this fact in (8.6) upon replacing t with $t \wedge \rho_m$, taking $\mathsf{E}_\varphi^\infty$ on both sides, and letting n tend to infinity, we find by the monotone convergence theorem upon recalling (7.6) that

$$
\hat{V}(\varphi) = \mathsf{E}_\varphi^\infty \big[e^{-\lambda(t \wedge \rho_m)} \hat{V}(\Phi_{t \wedge \rho_m}) \big] + \mathsf{E}_\varphi^\infty \bigg[\int_0^{t \wedge \rho_m} e^{-\lambda s} L(\Phi_s) I(\Phi_s \in C)\, ds \bigg] \tag{8.13}
$$

for all $t \ge 0$ and all $m \ge 1$. Letting $m \to \infty$ and using that $\rho_m \to \infty$ because 0 is an entrance boundary point for each Φ^i with $1 \le i \le N-1$, we see from (8.13) upon recalling (7.6) and using the dominated convergence theorem that

$$
\hat{V}(\varphi) = \mathsf{E}_\varphi^\infty \big[e^{-\lambda t} \hat{V}(\Phi_t) \big] + \mathsf{E}_\varphi^\infty \bigg[\int_0^t e^{-\lambda s} L(\Phi_s) I(\Phi_s \in C)\, ds \bigg] \tag{8.14}
$$

for all $t \geq 0$. Finally, letting $t \to \infty$ in (8.14) and using the dominated and monotone convergence theorems upon recalling (7.6), we find that

$$\hat{V}(\varphi) = \mathsf{E}_\varphi^\infty \Big[\int_0^\infty e^{-\lambda s} L(\Phi_s) I(\Phi_s \in C)\, ds \Big] \tag{8.15}$$

for all $\varphi \in [0,\infty)^N$. Recalling (6.14) and (8.2) above we see that this establishes the representation (8.4) as claimed. Moreover, the fact that τ_b from (8.5) is optimal in (3.4) follows by (6.14) above. Finally, inserting $\varphi_N = b(\varphi_1,\ldots,\varphi_{N-1})$ in (8.4) and using that $\hat{V}(\varphi_1,\ldots,\varphi_{N-1}, b(\varphi_1,\ldots,\varphi_{N-1})) = 0$, we see that b solves (8.3) as claimed.

(ii) *Uniqueness.* To show that b is a unique solution to the equation (8.3) in the specified class of functions, one can adopt the four-step procedure from the proof of uniqueness given in [3, Theorem 4.1] extending and further refining the original uniqueness arguments from [8, Theorem 3.1]. Given that the present setting creates no additional difficulties we will omit further details of this verification and this completes the proof. □

2. The nonlinear Fredholm integral equation (8.3) can be used to find the optimal stopping boundary b numerically (using Picard iteration). Inserting this b into (8.4) via (8.2) we also obtain a closed form expression for the value function $\hat{V}$. Collecting the results derived throughout the paper we now disclose the solution to the initial problem.

Corollary 11. *The value function in the initial problem* (2.4) *is given by*

$$V(\pi) = (1-\pi) \Big[1 + c\hat{V}\Big(\frac{\pi}{1-\pi},\ldots,\frac{\pi}{1-\pi}\Big)\Big] \tag{8.16}$$

for $\pi \in [0,1]$ *where the function* $\hat{V}$ *is given by* (8.4) *above. The optimal stopping time in the initial problem* (2.4) *is given by*

$$\begin{aligned} \tau_* = \inf \Big\{ t \geq 0 \;\big|\; & e^{\mu \sum_{j=1}^k X_t^{n_j(N)} + (\lambda - k\frac{\mu^2}{2})t} \Big(\tfrac{\pi}{1-\pi} + \lambda \int_0^t e^{-\mu \sum_{j=1}^k X_s^{n_j(N)} - (\lambda - k\frac{\mu^2}{2})s}\, ds \Big) \\ & \geq b\Big(e^{\mu \sum_{j=1}^k X_t^{n_j(1)} + (\lambda - k\frac{\mu^2}{2})t} \Big(\tfrac{\pi}{1-\pi} + \lambda \int_0^t e^{-\mu \sum_{j=1}^k X_s^{n_j(1)} - (\lambda - k\frac{\mu^2}{2})s}\, ds \Big), \ldots, \\ & e^{\mu \sum_{j=1}^k X_t^{n_j(N-1)} + (\lambda - k\frac{\mu^2}{2})t} \Big(\tfrac{\pi}{1-\pi} + \lambda \int_0^t e^{-\mu \sum_{j=1}^k X_s^{n_j(N-1)} - (\lambda - k\frac{\mu^2}{2})s}\, ds \Big) \Big) \Big\} \end{aligned} \tag{8.17}$$

where b *is a unique solution to* (8.3) *above (see Figure 1).*

Proof. The identity (8.16) was established in (3.1) above. The explicit form of the optimal stopping time (8.17) follows from (8.5) in Theorem 10 combined with (2.11)+(2.12) above. The final claim on b was derived in Theorem 10 above. This completes the proof. □

9. General case

In the general case of the quickest detection problem (2.4) we no longer insist that exactly k of the coordinate processes $X^1,\ldots,X^n$ get a (known) non-zero drift μ but instead allow that *any* number of them get such a drift with prescribed probabilities. In this section we show that the methodology developed in the previous sections to solve the problem for exactly k

coordinate processes can be used to solve the problem in the general case for *any* number of coordinate processes. This extension of the solution will also enable us to reduce the dimension of the problem to its minimal value which is of fundamental importance in real applications.

1. The key issue in extending the solution from exactly k to any number of the coordinate processes is whether the hypoelliptic structure established in Section 5 above using the Hörmander theorem remains preserved. For this, we first note that this is the case if we mechanically increase the problem dimension to a size which however could be alarmingly high. For example, returning to the case when $n = 10$ and $k = 5$ as discussed in Section 1 above, and allowing any (one or several) tagged 4 coordinate processes to get the drift as well, we would increase the problem dimension from $N_1 = \binom{10}{5} = 252$ to $N_2 = \binom{10}{5} + \binom{10}{4} = 462$. Clearly, the larger the number of the tagged 4 coordinate processes involved, the more justified increase of the problem dimension would be, and vice versa. This raises the question of establishing a minimal dimension of the problem given all the tagged coordinate processes that can get such a drift with prescribed probabilities. On closer inspection of the previous arguments we then note that the Hörmander condition, and hence the hypoelliptic structure established in Section 5 above as well, remain valid in the case of a minimal dimension of the problem. This makes the results derived in the previous sections applicable in the general case where the number of the tagged coordinate processes is firstly enlarged to an arbitrary value and then trimmed down to the value of a minimal problem dimension.

2. To describe the solution to the quickest detection problem (2.4) in the general case with a minimal problem dimension, we will return to the beginning of Section 2 and replace the number $1 \le k \le n$ by the numbers $1 \le k_1 < \cdots < k_m \le n$ with $m \le n$. This means that any k_l of the coordinate processes $X^1, \ldots, X^n$ get a (known) non-zero drift μ at time θ for $1 \le l \le m$ instead of exactly k of them. Setting $C^n_{k_1,\ldots,k_m} := \cup_{l=1}^m C^n_{k_l}$ we see that that the random variable β in (2.1) taking values in the set $C^n_{k_1,\ldots,k_m}$ satisfies $\mathsf{P}_\pi(\beta = (n_1, \ldots, n_{k_l})) = p_{n_1,\ldots,n_{k_l}}$ ˘1pt for some $p_{n_1,\ldots,n_{k_l}} \in [0,1]$ with $\sum_{l=1}^m \sum p_{n_1,\ldots,n_{k_l}} = 1$ given and fixed where the second sum is taken over all $(n_1, \ldots, n_{k_l}) \in C^n_{k_l}$. It is important in this setting that at least one among $p_{n_1,\ldots,n_{k_l}}$ when $(n_1, \ldots, n_{k_l})$ runs through $C^n_{k_l}$ is assumed to be strictly positive for every $1 \le l \le m$ given and fixed. If this would not be the case for some $1 \le l \le m$ then C_{k_l} could be omitted from the setting. As before, with a slight abuse of notation, in (2.1) we write $i \in \beta$ to express the fact that i belongs to the set $\{n_1, \ldots, n_{k_l}\}$ consisting of the elements which form $\beta = (n_1, \ldots, n_{k_l})$ in $C^n_{k_l}$ for $1 \le l \le m$. This means that $n_1, \ldots, n_{k_l} \in \beta$ if and only if the coordinate processes $X^{n_1}, \ldots, X^{n_{k_l}}$ get drift μ at time θ with probability $p_{n_1,\ldots,n_{k_l}}$ for $(n_1, \ldots, n_{k_l}) \in C^n_{k_l}$ with $1 \le l \le m$. With a similar abuse of notation, which will be helpful in what follows as before, we will first arrange the elements of $C^n_{k_1,\ldots,k_m}$ in a lexicographic order starting first with $C^n_{k_1}$ and moving forward until we reach $C^n_{k_m}$, then remove all $(n_1, \ldots, n_{k_l})$ from the ordered set $C^n_{k_1,\ldots,k_m}$ for which $p_{n_1,\ldots,n_{k_l}} = 0$ when $1 \le l \le m$ and denote ˘3pt the remaining (ordered) set by $\hat{C}^n_{k_1,\ldots,k_m}$, and finally identify the i-th element of the ordered set $\hat{C}^n_{k_1,\ldots,k_m}$ by its index i itself for $1 \le i \le N$ where we let N denote the total number of elements in $\hat{C}^n_{k_1,\ldots,k_m}$. ˘3pt Thus as before we write $i = (n_1, \ldots, n_{k_l}) \in \hat{C}^n_{k_1,\ldots,k_m}$ for $1 \le l \le m$ to express this identification explicitly for $1 \le i \le N$. Note that

$$\hat{C}^n_{k_1,\ldots,k_m} = \{\, (n_1, \ldots, n_{k_l}) \in C^n_{k_l} \mid p_{n_1,\ldots,n_{k_l}} > 0 \;\text{ for }\; 1 \le l \le m \,\} \tag{9.1}$$

and ˘1pt hence $N \le \binom{n}{k_1} + \ldots + \binom{n}{k_m}$ with equality being attained if $p_{n_1,\ldots,n_{k_l}} > 0$ for all

$(n_1, \ldots, n_{k_l}) \ \in C^n_{k_1,\ldots,k_m}$ with $1 \le l \le m$.

3. The rest of the analysis in Sections 2-4 above can then be carried out in exactly the same way. The only difference is notational in that the index set C^n_k needs to be firstly enlarged and then trimmed down to the index set $\hat{C}^n_{k_1,\ldots,k_m}$ as explained above. In particular, from the enlarged/trimmed identities (2.11)+(2.12) using Itô's formula we find that the enlarged/trimmed system of stochastic differential equations (3.3) reads as follows

$$d\Phi^i_t = \lambda(1+\Phi^i_t)\,dt + \sum_{j=1}^{k_l} \mu \Phi^i_t\, dB^{n_j}_t \quad (i=(n_1,\ldots,n_{k_l})\in\hat{C}^n_{k_1,\ldots,k_m} \text{ with } 1 \le l \le m) \tag{9.2}$$

under P^∞ with $\Phi^i_0 = \varphi_i$ in $[0,\infty)$ all being equal to $\pi/(1-\pi)$ for $1 \le i \le N$ in $\hat{C}^n_{k_1,\ldots,k_m}$ and $\pi \in [0,1)$. Similarly, we see that the enlarged/trimmed optimal stopping problem (3.4) remains unchanged and reads

$$\hat{V}(\varphi) = \inf_\tau \mathsf{E}^\infty_\varphi \Big[\int_0^\tau e^{-\lambda t} \Big(\sum_{i=1}^N p_i \Phi^i_t - \frac{\lambda}{c} \Big)\, dt \Big] \tag{9.3}$$

for $\varphi \in [0,\infty)^N$ with $\mathsf{P}^\infty_\varphi(\Phi_0 = \varphi) = 1$ ˇ-1pt where the infimum is taken over all stopping times τ of Φ and we recall that $p_i \in (0,1]$ for $1 \le i \le N$ in $\hat{C}^n_{k_1,\ldots,k_m}$ with $\sum_{i=1}^N p_i = 1$.

4. This raises the question whether the hypoelliptic structure established using the Hörmander theorem in Section 5 above is preserved in the enlarged/trimmed system (9.2) above. We now show that the answer is affirmative.

Corollary 12. *The results of Proposition 3 and Proposition 4 above remain valid for the enlarged/trimmed system* (9.2) *above.*

Proof. We show that the Hörmander condition (5.8) is satisfied using backward induction over $k_1 < \ldots < k_m$. Setting $k = k_m$ and connecting to the conclusion (5.17) in the proof of Proposition 3 above, we see that $\partial_{\varphi_j} \in Lie(D_0, D_1, \ldots, D_N)$ for $N_{m-1}+1 \le j \le N_m$ where N_l denotes the number of elements from $\hat{C}^n_{k_1,\ldots,k_m}$ that belong to $C^n_{k_l}$ for $1 \le l \le m$. Note that for that conclusion we did not use that N_l must be equal to its general bound $\binom{n}{k_l}$ but could also be any strictly smaller number as well for $1 \le l \le m$. Recall also that the lexicographic order of $\hat{C}^n_{k_1,\ldots,k_m}$ is without loss of generality assumed to start with $C^n_{k_1}$ and end with $C^n_{k_m}$. The induction step can therefore be realised by setting $k = k_{m-1}$. Then the same arguments as in (5.13)-(5.15) show that

$$[[[D_0, D_{n_1}], D_{n_2}], \ldots, D_{n_{k_{m-1}}}] \sim \sum_{i=1}^N I_{in_1} I_{in_2} \ldots I_{in_{k_{m-1}}} \partial_{\varphi_i} \tag{9.4}$$

for $j = (n_1, \ldots, n_{k_{m-1}}) \in \hat{C}^n_{k_1,\ldots,k_m}$ given and fixed. This reveals a crucial difference in comparison with (5.16) above because this time we have

$$\begin{aligned} I_{in_1} I_{in_2} \ldots I_{in_{k_{m-1}}} &= 1 \ \text{ if } \ i = j \ \text{ or } \ i \in \hat{C}^n_{k_m}(j) \\ &= 0 \ \text{ otherwise} \end{aligned} \tag{9.5}$$

where $\hat{C}^n_{k_m}(j)$ denotes the set of all $i \in C^n_{k_m}$ in $\hat{C}^n_{k_1,\ldots,k_m}$ which include j in the sense that $n_1, \ldots, n_{k_{m-1}}$ belong to i understood as the set of its elements. It follows therefore from (9.4) and (9.5) that we have

$$[[[D_0, D_{n_1}], D_{n_2}], \ldots, D_{n_{k_{m-1}}}] \sim \partial_{\varphi_j} + \sum_{i \in \hat{C}^n_{k_m}(j)} \partial_{\varphi_i} \, . \tag{9.6}$$

Recalling however that $\partial_{\varphi_i} \in Lie(D_0, D_1, \ldots, D_N)$ for every $i \in \hat{C}^n_{k_m}(j)$ as established in the previous step, we see from (9.6) that $\partial_{\varphi_j} \in Lie(D_0, D_1, \ldots, D_N)$ for $N_{m-2}+1 \le j \le N_{m-1}$. Continuing this procedure by backward induction until reaching $k = k_1$ we find that $\partial_{\varphi_j} \in Lie(D_0, D_1, \ldots, D_N)$ for all $1 \le j \le N$ so that the Hörmander condition (5.8) is satisfied as claimed. The remaining arguments are identical to those presented in the proofs of Proposition 3 and Proposition 4 above. This completes the proof. □

5. The rest of the analysis in Sections 6-8 above can then be carried out in exactly the same way with the index set C^n_k replaced by the enlarged/trimmed index set $\hat{C}^n_{k_1,\ldots,k_m}$. This yields the solution to the initial problem when the number of coordinate processes getting a (known) non-zero drift μ with prescribed (non-zero) probabilities is no longer fixed to be k exactly but could be *any* number instead.

Corollary 13. *The results of Theorem 10 and Corollary 11 remain valid in the general case described by* (9.1)-(9.3) *above.*

Proof. Having established the hypoelliptic structure of (9.1)-(9.3) in Corollary 12 above we can carry out the proofs of Theorem 10 and Corollary 11 in exactly the same way. Note that this also applies to all other results in Sections 6-8 (as well as to those in Sections 2-4). This completes the proof. □

Acknowledgements. The authors gratefully acknowledge support from the United States Army Research Office Grant ARO-YIP-71636-MA. The third-named author is indebted to Xinfu Chen and David Saunders for the insightful discussions on hypoellipticity in relation to [7] and related equations during a splendid stay in Beijing, China (June/July 2017).

References

[1] Blumenthal, R. M. *and* Getoor, R. K. (1968). *Markov Processes and Potential Theory*. Academic Press.

[2] De Angelis, T. *and* Peskir, G. (2020). Global C^1 regularity of the value function in optimal stopping problems. *Ann. Appl. Probab.* 30 (1007–1031).

[3] Du Toit, J. *and* Peskir, G. (2009). Selling a stock at the ultimate maximum. *Ann. Appl. Probab.* 19 (983–1014).

[4] Ernst, P. A. *and* Peskir, G. (2022). Quickest real-time detection of a Brownian coordinate drift. *Ann. Appl. Probab.* 32 (2652–2670).

[5] Gilbarg, D. *and* Trudinger, N. S. (2001). *Elliptic Partial Differential Equations of the Second Order*. Springer.

[6] Hörmander, L. (1967). Hypoelliptic second order differential equations. *Acta Math.* 119 (147–171).

[7] Johnson, P. *and* Peskir, G. (2017). Quickest detection problems for Bessel processes. *Ann. Appl. Probab.* 27 (1003–1056).

[8] Peskir, G. (2005). On the American option problem. *Math. Finance* 15 (169–181).

[9] Peskir, G. (2006). On the fundamental solution of the Kolmogorov-Shiryaev equation. *The Shiryaev Festschrift* (Metabief, 2005), Springer (535–546).

[10] Peskir, G. (2007). A change-of-variable formula with local time on surfaces. *Sém. de Probab.* XL, *Lecture Notes in Math.* 1899, Springer (69–96).

[11] Peskir, G. (2022). Weak solutions in the sense of Schwartz to Dynkin's characteristic operator equation. *Research Report* No. 1, *Probab. Statist. Group Manchester* (20 pp).

[12] Peskir, G. *and* Shiryaev, A. N. (2006). *Optimal Stopping and Free-Boundary Problems.* Lectures in Mathematics, ETH Zürich, Birkhäuser.

[13] Revuz, D. *and* Yor, M. (1999). *Continuous Martingales and Brownian Motion.* Springer-Verlag.

[14] Rogers, L. C. G. *and* Williams, D. (2000). *Diffusions, Markov Processes and Martingales: Itô Calculus* (Vol 2). Cambridge University Press.

[15] Shiryaev, A. N. (1961). The problem of the most rapid detection of a disturbance in a stationary process. *Soviet Math. Dokl.* 2 (795–799).

[16] Shiryaev, A. N. (1978). *Optimal Stopping Rules.* Springer-Verlag.

[17] Shiryaev, A. N. (1996). *Probability.* Springer.

[18] Shiryaev, A. N. (2010). Quickest detection problems: Fifty years later. *Sequential Anal.* 29 (345–385).