TOPOLOGY OF CUT COMPLEXES OF GRAPHS

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Abstract. We define the *k-cut complex* of a graph G with vertex set $V(G)$ to be the simplicial complex whose facets are the complements of sets of size k in $V(G)$ inducing disconnected subgraphs of G. This generalizes the Alexander dual of a graph complex studied by Fröberg (1990), and Eagon and Reiner (1998). We describe the effect of various graph operations on the cut complex, and study its shellability, homotopy type and homology for various families of graphs, including trees, cycles, complete multipartite graphs, and the prism $K_n \times K_2$, using techniques from algebraic topology, discrete Morse theory and equivariant poset topology.

1. Introduction

This paper, a companion to [2], deals with a class of graph complexes. In recent years there has been much interest in the topology of simplicial complexes associated with graphs. A major contribution to this subject is the book [17] by Jonsson, who considers simplicial complexes defined on edge sets of graphs. Simplicial complexes defined on vertex sets of graphs include clique complexes (see, e.g., [15]), independence complexes (see, e.g., [24]), neighborhood complexes (see, e.g., [22]), and dominance complexes (see, e.g., [23]). In this paper we introduce a new family of simplicial complexes associated to the vertex set of a graph, which we call *cut complexes*. We consider only simple graphs. Our work is motivated by a famous theorem of Ralf Fröberg [13] connecting commutative algebra and graph theory through topology. We investigate our new complexes in the spirit of Fröberg's theorem, relating topological properties of the cut complex to structural properties of the graph.

For a field $\mathbb{K}$ and a finite simplicial complex Δ with vertex set $[n] = \{1, 2, \ldots, n\}$, the Stanley–Reisner ideal of Δ is the ideal I_Δ of the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials $x_{i_1} \cdots x_{i_k}$ running over the inclusion-minimal subsets $\{i_1, \ldots, i_k\}$ of $[n]$ that are NOT faces of Δ. The Stanley–Reisner ring $\mathbb{K}[\Delta]$ is the quotient of the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ by the ideal I_Δ.

For a graph G, the *clique complex* $\Delta(G)$ is the simplicial complex whose simplices are subsets of vertices of G, in which every pair of vertices is connected by an edge of G. Fröberg [13] characterized monomial ideals that have a 2-*linear resolution*, by first reducing to the case of square-free monomial ideals. The ideal I_Δ is generated by quadratic square-free monomials precisely when the simplicial complex Δ is $\Delta(G)$ for some graph G (see [12, Proposition 8]). Hence Fröberg's theorem can be stated as follows:

Theorem 1.1 (Fröberg [13], [12, p. 274])**.** *A Stanley–Reisner ideal I_Δ generated by quadratic square-free monomials has a 2-linear resolution if and only if Δ is the clique complex $\Delta(G)$ of a chordal graph G.*

Define the combinatorial *Alexander dual* of a simplicial complex Δ [9, p.188] on n vertices to be

$$\Delta^\vee \coloneqq \{F \subset [n] : [n] \setminus F \notin \Delta\}.$$

The ith homology of Δ and the $(n-i-3)$th cohomology of $\Delta^\vee$ are isomorphic by Alexander duality in the sphere $\mathbb{S}^{n-2}$.

For a graph G, write $\Delta_2(G)$ for the Alexander dual $\Delta(G)^\vee$ of the clique complex $\Delta(G)$. The facets of $\Delta_2(G)$ are the complements of independent sets of size 2 in G.

Eagon and Reiner's reformulation [12, Proposition 8] of Fröberg's theorem includes the following equivalences.

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Theorem 1.2. *The graph G is chordal $\iff \Delta_2(G)$ is shellable $\iff \Delta_2(G)$ is vertex decomposable.*

Fröberg's Theorem deals with ideals generated by monomials of degree two. Consideration of higher degree monomials leads us to the following generalization of the simplicial complex $\Delta_2(G)$. Let $k \geq 1$. Define a complex whose facets are complements of sets F of size k in G such that the induced subgraph of G on the vertex set F is disconnected; we call this the k-cut complex of G, and denote it by $\Delta_k(G)$. This complex was first introduced in [10]. A different generalization, the total k-cut complex $\Delta_k^t(G)$, is treated in [2]. The two notions coincide for $k = 2$.

The paper is organized as follows. In Sections 2 and 3, we collect basic definitions and background facts about simplicial complexes and posets. We begin Section 4 with a construction in Theorem 4.1, showing that any pure simplicial complex can be realized as the k-cut complex of some graph G, which is in fact chordal. We examine the topology of the cut complex $\Delta_k(G)$ in Sections 4 and 5, and consider the effect of properties of the graph G, and graph constructions such as join and disjoint union. For example, in analogy with Fröberg's theorem, Corollary 5.3 asserts that graph chordality implies shellability of the 3-cut complex Δ_3. This result is the best possible: we show that for any $k \geq 4$, there is a chordal graph G whose k-cut complex is not shellable, and is minimal with respect to this property. Section 6 describes the face lattice of the cut complex, from which we can deduce information about its homology and compute the Euler characteristic. We also determine completely the homotopy type of $\Delta_2(G)$ for connected triangle-free graphs G; see Theorem 6.4. In Section 7 we show that for many common families of graphs, the homotopy type of $\Delta_k(G)$ is a wedge of spheres in a single dimension. Often there is a simplicial group action on the cut complex $\Delta_k(G)$, which in turn acts on the rational homology. This homology representation is particularly interesting in the case of complete multipartite graphs; see Section 7.1.

Table 1 summarizes our results for various families of graphs.

TABLE 1. k-cut complexes for different graphs

Graph	Shellable?	Homotopy type and Betti numbers	Equivariant homology
Complete bipartite, $K_{m,n}$	Theorem 7.1: Yes if and only if $m < k$	Theorem 7.4	Theorem 7.4
Complete multipartite, $K_{m_1,\ldots,m_r}$	Theorem 7.6: Yes if and only if $m_{r-1} < k$	Theorems 7.8 & 7.9	Theorems 7.8 & 7.9
Cycle, C_n	Theorem 7.11: Yes if $k \geq 3$	Proposition 7.10 ($k = 2$), Proposition 7.13 ($k \geq 3$)	Theorem 7.15
Squared cycle, W_n	Proposition 7.27: No if $k = n - 4$	Proposition 7.22 ($k = 2$), Proposition 7.27 ($k = n - 4$)	Proposition 7.29 ($k = 2$ and $k = n - 4$)
Prism over clique, $K_n \times K_2$	Theorem 7.18: Yes if and only if $k > n$ (void complex)	Theorem 7.18 ($k \leq n$)	
Tree	Corollary 4.21: Yes for all $k \geq 2$	Proposition 7.12	
Threshold graph	Corollary 4.14: Yes for all $k \geq 2$		
Connected & triangle-free	Fröberg's theorem (Theorem 1.2) No for $k = 2$, except trees	Theorem 6.4 ($k = 2$)	

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2. Definitions

General references for simplicial complexes, shellability and related concepts are [3], [18], [29, Chapter II, Chapter III, Section 2] and [36], and [37] for graph theory. All graphs in this paper are simple (no loops and no multiple edges) and finite.

Definition 2.1. A *simplicial complex* Δ is a collection of subsets such that

$$\sigma \in \Delta \text{ and } \tau \subseteq \sigma \Rightarrow \tau \in \Delta.$$

The elements of Δ are called its *faces* or *simplices*. If the collection of subsets is empty, i.e., Δ has no faces, we call Δ the *void complex*. Otherwise Δ always contains the empty set as a face.

The *dimension of a face* σ, $\dim(\sigma)$, is one less than its cardinality; thus the dimension of the empty face is (-1), and the 0-dimensional faces are the *vertices* of Δ. A *d-face* or *d-simplex* is a face of dimension d. The maximal faces of Δ are called its *facets*, and the maximum dimension of a facet is the *dimension* $\dim(\Delta)$ of the simplicial complex Δ. The dimension of the void complex is defined to be $-\infty$ (see [18]).

A (nonvoid) simplicial complex is *pure* if all its facets have the same dimension, which is then the dimension of the complex. We write $\Delta = \langle \mathcal{F} \rangle$ to denote the simplicial complex Δ whose set of facets is $\mathcal{F}$. In this paper all simplicial complexes will be *finite*, i.e., the vertex set is finite.

We will be using the following constructions.

Definition 2.2 ([18])**.** Let Δ be a simplicial complex and σ a face of Δ.

- The *link* of σ in Δ is $\operatorname{lk}_\Delta \sigma := \{\tau \in \Delta : \sigma \cap \tau = \emptyset, \text{ and } \sigma \cup \tau \in \Delta\}$.
- The (closed) *star* of σ in Δ is $\operatorname{st}_\Delta \sigma := \{\tau \in \Delta : \sigma \cup \tau \in \Delta\}$.
- The *deletion* of σ in Δ is $\operatorname{del}_\Delta \sigma := \{\tau \in \Delta : \sigma \not\subseteq \tau\}$.

(Note that in the deletion, we are not removing proper faces of σ.)

Thus $\operatorname{lk}_\Delta(\emptyset) = \operatorname{st}_\Delta(\emptyset) = \Delta$, and $\operatorname{del}_\Delta(\emptyset)$ is the void complex.

For v a vertex of Δ, we also have the following useful facts (see [18]):

$$\Delta = \operatorname{st}_\Delta(v) \cup \operatorname{del}_\Delta(v) \quad \text{and} \quad \operatorname{lk}_\Delta(v) = \operatorname{st}_\Delta(v) \cap \operatorname{del}_\Delta(v). \tag{1}$$

Definition 2.3 ([29, Chapter III, Section 2], [3, Section 11.2])**.** An ordering $F_1, F_2, \ldots, F_t$ of the facets of a simplicial complex Δ is a *shelling* if, for every j with $1 < j \le t$,

$$\left(\bigcup_{i=1}^{j-1} \langle F_i \rangle \right) \cap \langle F_j \rangle$$

is a simplicial complex whose facets all have cardinality $|F_j| - 1$, where $\langle F_i \rangle$ is the simplex generated by the face F_i.

Equivalently, an ordering $F_1, F_2, \ldots, F_t$ of the facets of Δ is a shelling if and only if for all i, j such that $1 \le i < j \le t$, there exists $k < j$ such that

$$F_i \cap F_j \subset F_k \cap F_j \quad \text{and} \quad |F_k \cap F_j| = |F_j| - 1.$$

If the simplicial complex Δ has a shelling, it is called *shellable*.

In combinatorial topology, shellability is an important tool for determining the homotopy type of simplicial complexes, thanks to the following theorem of Björner.

Theorem 2.4 ([5, Theorem 1.3])**.** *A pure shellable simplicial complex of dimension d has the homotopy type of a wedge of spheres, all of dimension d. The complex is contractible if there are no spheres in the wedge.*

Figure 3 depicts a triangulation of the Möbius strip, a 2-dimensional surface which is homotopy equivalent to a one-dimensional circle. Thus the triangulation, a pure 2-dimensional complex, is not shellable.

Remark 2.5. By convention, the void complex is shellable. The complex whose only face is the empty set is vacuously shellable. The complex with a unique nonempty facet (i.e., a simplex) is shellable, and contractible.

Definition 2.6. Let $G = (V, E)$ be a graph on $|V| = n$ vertices. If S is a subset of the vertex set V, we write $G[S]$ to denote the *induced subgraph* of G whose vertex set is S.

For $k \geq 2$ define $D_k(G) \coloneqq \{S \subseteq V : G[S]$ is disconnected and $|S| = k\}$. We call the elements of $D_k(G)$ *disconnected k-sets* of G, and the complements of elements of $D_k(G)$ *separating $(n-k)$-sets* of G. When $k = 1$, we define $D_1(G)$ to be the empty set.

See Figure 1.

(A) $F = \{2, 3, 4\}$ (B) $G \setminus F$

FIGURE 1. Example of a separating set

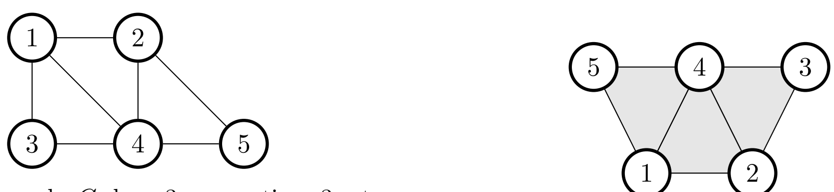
Definition 2.7. Let $G = (V, E)$ be a graph on $|V| = n$ vertices, and let $k \geq 1$. Define the *k-cut complex* of the graph G to be the simplicial complex

$$\Delta_k(G) \coloneqq \langle F \subseteq V \mid V \setminus F \in D_k(G) \rangle.$$

The facets of the cut complex $\Delta_k(G)$ are the separating sets of G of size $(n-k)$. Thus the cut complex $\Delta_k(G)$ (if not void) has dimension $n-k-1$. Equivalently, σ is a face of the cut complex $\Delta_k(G)$ if and only if its complement $V \setminus \sigma$ contains a subset S of size k such that the induced subgraph $G[S]$ is disconnected. Note the following inclusion, when $k \geq 2$:

$$\Delta_{k+1}(G) \subseteq \Delta_k(G), \tag{2}$$

and the fact that the vertices of $\Delta_k(G)$ may be a proper subset of the vertices of the graph G. See Figure 2 and Figure 6 for contrasting examples.



(A) The graph G has 3 separating 3-sets: $\{1, 2, 4\}, \{1, 4, 5\}$ and $\{2, 3, 4\}$

(B) The cut complex $\Delta_2(G) = \langle 234, 145, 124 \rangle$

FIGURE 2. 2-cut complex of graph G

Example 2.8. Let G be a graph on n vertices. We record some easy facts about cut complexes.

(1) $\Delta_k(G)$ is void if $k = 1$ or $k > n$.

(2) $\Delta_n(G)$ is $\begin{cases} \text{the void complex,} & \text{if } G \text{ is connected,} \\ \text{the } (-1)\text{-dimensional complex } \{\emptyset\}, & \text{otherwise.} \end{cases}$

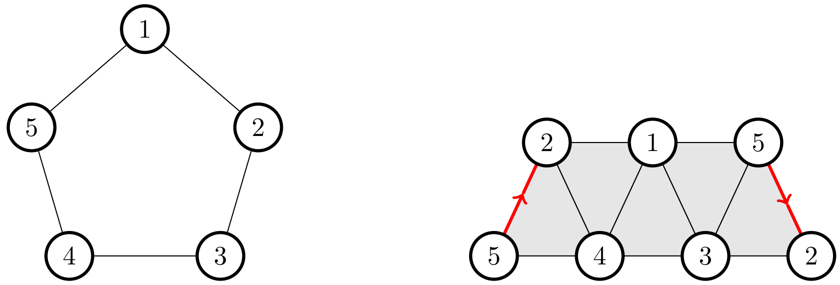
(3) $\Delta_k(G)$ is void for $k \geq n - r + 1$ if G is r-connected: at least r vertices must be removed to disconnect the graph, so $n - k \leq r - 1$.

(4) If G is the complete graph K_n, then $\Delta_k(G)$ is void for all $k \geq 1$.

(5) If G is the edgeless graph, then for $2 \leq k \leq n-1$, $\Delta_k(G)$ is the $(n-k-1)$-skeleton of an $(n-1)$-dimensional simplex.

Definition 2.9 ([37])**.** A graph is *chordal* if it contains no induced cycle of size greater than 3.

Recall Eagon–Reiner's reformulation of Fröberg's theorem from the Introduction (Theorem 1.2) that $\Delta_2(G)$ is shellable if and only if G is chordal. This is illustrated in Figures 2 and 3. Figure 2 shows a chordal graph and its shellable 2-cut complex, whereas Figure 3 is an example of a non-chordal graph, with nonshellable 2-cut complex.



(A) The cycle C_5 (B) $\Delta_2(C_5) = \langle 245, 124, 134, 135, 235 \rangle$

FIGURE 3. The 2-cut complex for C_5 is a Möbius strip

3. Background from topology

The following facts are taken from [3] and [18]. Note that in this paper we assume a nonvoid simplicial complex always contains the empty set as a face.

Definition 3.1 ([3, Section 9]). The *join* of two simplicial complexes Δ_1 and Δ_2 with disjoint vertex sets is the complex

$$\Delta_1 * \Delta_2 \coloneqq \{\sigma \cup \tau : \sigma \in \Delta_1, \tau \in \Delta_2\}.$$

Thus the join $\Delta_1 * \Delta_2$ contains Δ_1 and Δ_2 as subcomplexes.

The *cone* over Δ and the *suspension* of Δ are the complexes

$$\mathrm{cone}(\Delta) \coloneqq \Delta * \Gamma_1, \quad \mathrm{susp}(\Delta) \coloneqq \Delta * \Gamma_2 = \Delta * \{u\} \cup \Delta * \{v\},$$

where Γ_1 is the 0-dimensional simplicial complex with one vertex, and Γ_2 is the 0-dimensional complex with two vertices u, v.

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : ||x|| = 1\}$ and $\mathbb{B}^d = \{x \in \mathbb{R}^d : ||x|| \le 1\}$ denote respectively the $(d-1)$-sphere and the d-ball. Then

$$\mathbb{S}^{-1} = \emptyset, \quad \mathbb{S}^0 = \{\text{two points}\}, \quad \mathbb{B}^0 = \{\text{point}\}.$$

Furthermore, we have the following homeomorphism for spheres under the join operation:

$$\mathbb{S}^m * \mathbb{S}^n \cong \mathbb{S}^{m+n+1}. \tag{3}$$

We will consider reduced simplicial homology [14] over the integers $\mathbb{Z}$, or, for representation-theoretic purposes, over the rationals $\mathbb{Q}$, writing $\widetilde{H}_i(\Delta)$ for the ith reduced homology of the simplicial complex Δ. We record the following facts: $\widetilde{H}_i(\emptyset)$ is nonzero if and only if $i = -1$, in which case it is free of rank one. If Δ is nonvoid, $\widetilde{H}_i(\Delta)$ is nonzero only when $0 \le i \le \dim(\Delta)$, and the reduced homology $\widetilde{H}_0(\Delta)$ is free of rank one less than the number of connected components of Δ.

Theorem 3.2 ([14], [3, Eqn (9.12)]). *Let Δ_1 and Δ_2 be finite complexes. Assume at least one of $\widetilde{H}_p(\Delta_1)$, $\widetilde{H}_q(\Delta_2)$ over $\mathbb{Z}$ is torsion-free when $p + q = r - 1$. Then the reduced homology of the join $\Delta_1 * \Delta_2$ in degree r is given by*

$$\widetilde{H}_r(\Delta_1 * \Delta_2) \cong \bigoplus_{p+q=r-1} \left(\widetilde{H}_p(\Delta_1) \otimes \widetilde{H}_q(\Delta_2)\right).$$

In particular, when the appropriate homology groups are torsion-free, the Künneth Theorem confirms the well-known group isomorphism

$$\widetilde{H}_r(\mathrm{susp}(\Delta)) \cong \widetilde{H}_{r-1}(\Delta). \tag{4}$$

Also note that if Δ_1 has the homotopy type of a wedge of β_p spheres of dimension p, and Δ_2 has the homotopy type of a wedge of β_q spheres of dimension q, then the join $\Delta_1 * \Delta_2$ has the homotopy type of a wedge of $(\beta_p\beta_q)$ spheres of dimension $p+q+1$.

Definition 3.3 ([29, 30])**.** Let Δ be a finite simplicial complex of dimension d, and let

$$\beta_i \coloneqq \operatorname{rank} \widetilde{H}_i(\Delta, \mathbb{Z}) = \dim_{\mathbb{Q}} \widetilde{H}_i(\Delta, \mathbb{Q}),\ i \geq -1.$$

The β_i are the *(reduced) Betti numbers* of Δ. Let f_i be the number of i-dimensional faces of the d-dimensional complex Δ. Then $(f_0, f_1, \dots f_d)$ is the *f-vector* of Δ. The *Euler characteristic* of Δ is defined to be $\sum_{i\geq 0}(-1)^i f_i$. The *reduced Euler characteristic* $\mu(\Delta)$ of Δ is defined to be one less than the Euler characteristic:

$$\mu(\Delta) = \left(\sum_{i\geq 0} (-1)^i f_i \right) - 1.$$

Letting $f_{-1} = 1$ for the empty face, we have the reduced Euler-Poincaré formula

$$\mu(\Delta) = \sum_{i\geq -1} (-1)^i f_i = \sum_{i\geq -1} (-1)^i \beta_i.$$

3.1. **Poset Topology.** In order to determine the homotopy type of a simplicial complex Δ, and in particular the group representation on the rational homology, it is often helpful to work with the face lattice $\mathcal{L}(\Delta)$ of Δ. We recall some notions about posets and poset topology. See [3], [36] for more details.

A poset Q is bounded if it has a unique maximal element $\widehat{1}$ and a unique minimal element $\widehat{0}$. Let $\bar{Q}$ denote the proper part $Q \setminus \{\widehat{0}, \widehat{1}\}$ of a bounded poset Q. By the order complex of Q, we will always mean the simplicial complex, denoted $\Delta(\bar{Q})$, of chains in the *proper part* $\bar{Q}$ of Q. Let Q_1 and Q_2 be bounded posets. It is well known ([3], [36]) that one has the homotopy equivalence

$$\Delta(\overline{Q_1 \times Q_2}) \simeq \operatorname{susp}\left(\Delta(\bar{Q}_1) * \Delta(\bar{Q}_2)\right) \simeq \mathbb{S}^0 * \Delta(\bar{Q}_1) * \Delta(\bar{Q}_2). \tag{5}$$

In particular, if the order complex of Q_i has the homotopy type of a wedge of q_i spheres of dimension d_i, $i = 1, 2$, then the order complex of $Q_1 \times Q_2$ is a wedge of $q_1 q_2$ spheres of dimension $d_1 + d_2 + 2$.

Equation (5) generalizes inductively to an r-fold product of bounded posets Q_i, $i = 1, \dots, r$, $r \geq 2$, and we record this homotopy equivalence for later use:

$$\Delta(\overline{Q_1 \times \cdots \times Q_r}) \simeq \mathbb{S}^{r-2} * \left(\Delta(\bar{Q}_1) * \cdots * \Delta(\bar{Q}_r)\right). \tag{6}$$

Recall that the face lattice of a simplicial complex Δ is the poset of faces ordered by inclusion, with the empty face as the unique bottom element, and an artificially appended top element. This makes the face lattice $\mathcal{L}(\Delta)$ of a finite simplicial complex Δ into a bounded poset. Its proper part is the poset consisting of the nonempty faces of Δ. The order complex of the proper part of the face lattice $\mathcal{L}(\Delta)$ is the barycentric subdivision of Δ, and hence is homeomorphic to Δ, and therefore has the same homotopy type. See, e.g., [3, Section 9.3]. When G is a finite group with a simplicial action on Δ, the representation on the rational homology of Δ coincides with the representation on the homology of the face lattice.

Let B_p denote the Boolean lattice of subsets of a set with p elements, and let $P(p,k) = B_p^{\leq p-k} \cup \{\widehat{1}\}$ denote the truncated Boolean lattice, i.e., the subposet of B_p consisting of subsets with at most $p-k$ elements, with an artificially appended top element $\widehat{1}$. (This makes $P(p,k)$ a bounded poset with unique top and bottom elements.) Note that B_p is the face lattice of the boundary of a $(p-1)$-simplex, and $P(p,k)$, $0 \leq k \leq p-1$, is the face lattice of the $(p-k-1)$-skeleton of a $(p-1)$-simplex. The poset $P(p,k)$ is an example of a rank-selected subposet, and there is a large literature on the topic of rank-selection in posets and group actions [28], [36].

It is a well-known fact that the poset $P(p,k)$ is lexicographically shellable ([4], [6]), with Möbius number $\binom{n-1}{k-1}$, and hence its order complex is Cohen–Macaulay, and is homotopy equivalent to a wedge of $\binom{p-1}{k-1}$ spheres of dimension $p-k-1$. The homotopy type follows from the fact that the order complex of $P(p,k)$ is the barycentric subdivision of the $(p-k-1)$-skeleton of a $(p-1)$-simplex, and the latter is shellable by [7, Theorem 2.9]. See also [6, Corollary 4.4, Theorem 8.1].

Our main tools from poset topology for determining homotopy type are as follows:

Theorem 3.4 (Quillen fiber lemma [26], [18, Theorem 15.28], [36, Theorem 5.2.1])**.** *Let P and Q be bounded posets and $f : \bar{P} \to \bar{Q}$ a poset map. If for all $q \in \bar{Q}$, the order complex of the fiber $f_{\le}^{-1}(q) := \{p \in \bar{P} : f(p) \le q\}$ is contractible, then the map f induces a homotopy equivalence of order complexes $\Delta(\bar{P}) \simeq \Delta(\bar{Q})$. Furthermore, if G is a finite group of automorphisms of P and Q, and the poset map f commutes with the action of G, then the homotopy equivalence is group equivariant and hence induces a G-module isomorphism in rational homology.*

Recall (see [30]) that the reduced Euler characteristic of a simplicial complex coincides with the Möbius number of its face lattice. The following result of Baclawski will be useful.

Theorem 3.5 ([1, Lemma 4.6], [30, Lemma 3.16.4])**.** *If P is a bounded poset and Q is a subposet of P containing $\widehat{0}, \widehat{1}$, then*

$$\mu(Q) - \mu(P) = \sum_{\substack{\widehat{0} < x_1 < x_2 < \cdots < x_k < \widehat{1} \\ k \ge 1, x_i \notin Q}} (-1)^k \mu_P(\widehat{0}, x_1)\mu_P(x_1, x_2) \dots \mu_P(x_k, \widehat{1}),$$

where the sum runs over all nonempty chains with elements not in Q. Here μ_P denotes the Möbius function of the poset P.

When Q is a subposet obtained from P by removing an antichain $\mathcal{A}$, this simplifies to

$$\mu(Q) - \mu(P) = \sum_{\substack{\widehat{0} < x < \widehat{1} \\ x \in \mathcal{A}}} (-1)\mu_P(\widehat{0}, x)\mu_P(x, \widehat{1}). \tag{7}$$

Finally, given a graph G on n vertices, and k such that the cut complex $\Delta_k(G)$ is nonvoid, we have the inclusion of posets

$$\mathcal{L}(\Delta_k(G)) \subseteq P(n, k) = B_n^{\le n-k} \cup \{\widehat{1}\}. \tag{8}$$

We now immediately obtain our first nontrivial homotopy result for a cut complex.

Proposition 3.6. *Let G be the edgeless graph with n vertices. If $k \ge n$, the cut complex $\Delta_k(G)$ is void. If $2 \le k \le n-1$, then $\Delta_k(G)$ is shellable and*

$$\Delta_k(G) \simeq \bigvee_{\binom{n-1}{k-1}} \mathbb{S}^{n-k-1}.$$

Proof. If the cut complex $\Delta_k(G)$ is nonvoid, then $2 \le k \le n-1$ and Example 2.8, Part (5) tells us that it is the $(n-k-1)$-skeleton of an $(n-1)$-simplex, hence shellable by [7, Theorem 2.9]. The homotopy type and Betti number were given in the discussion above Theorem 3.4. □

In many cases, for example, when every vertex of the graph G is a vertex of the cut complex $\Delta_k(G)$, the automorphism group of G induces a simplicial action (one that sends simplices to simplices) on the cut complex $\Delta_k(G)$, and hence in turn acts on the rational homology. In subsequent sections, we will use the face lattice of the cut complex to determine this homology representation.

Finally we will make use of the following.

Proposition 3.7 ([35, Chapter 21, (21.3)], [2, Proposition 2.7])**.** *Let X be a topological space with subspaces $A, B \subseteq X$ such that $X = A \cup B$, $A \cap B \neq \emptyset$, and A, B are both closed subspaces or both open subspaces. Then the quotient map $A/(A \cap B) \to X/B$ of the inclusion $A \hookrightarrow X$ is a homeomorphism.*

Proposition 3.8 ([14, Proposition 0.17, Example 0.14])**.** *Let (X, A) be a CW pair consisting of a CW complex X and a subcomplex A.*

(1) If the subcomplex A is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence.
(2) If A is contractible in the complex X, then there is a homotopy equivalence

$$X/A \simeq X \vee \mathrm{susp}(A).$$

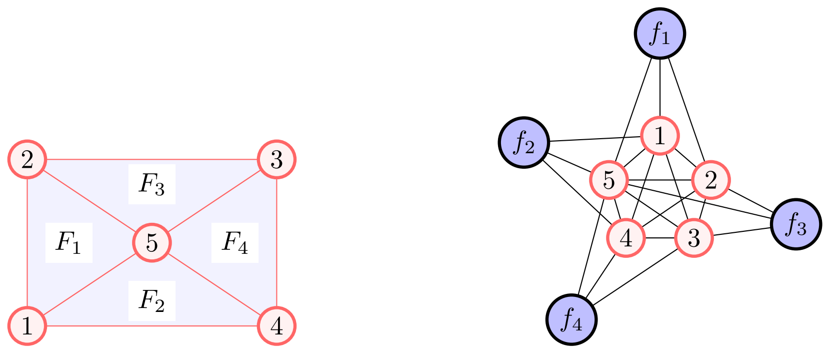
4. Constructive Theorems

In this section we consider the effect of some common graph operations on the cut complex. We begin with the following universality property.

4.1. **Any simplicial complex is a cut complex.**

Theorem 4.1 (Natalie Behague)**.** *Let Δ be any pure simplicial complex. There exists some k and some chordal graph G such that Δ is equal to the cut complex $\Delta_k(G)$.*

Proof. The idea of the construction is as follows: start with a clique whose vertices correspond to the vertex set of the complex Δ. For each facet of Δ, add a vertex that is connected to every vertex of that facet. If n is the number of vertices of Δ, t is the number of facets of Δ and d is the dimension of Δ, the resulting graph G has $(n+t)$ vertices, and our claim is that $\Delta_{n+t-(d+1)}(G) = \Delta$. Figure 4 illustrates the procedure.



(A) A pure simplicial complex Δ (B) A graph G such that $\Delta = \Delta_6(G)$

Figure 4. The construction of Theorem 4.1

Let the vertices of Δ be labelled $v_1, v_2, \ldots, v_n$ and let $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ be the set of facets of Δ. First assume $t > 1$.

Let the vertex set of G consist of $n+t$ vertices labelled $u_1, u_2, \ldots, u_n, f_1, f_2, \ldots, f_t$. We define the edge set as follows: u_iu_j is an edge for all $1 \leq i, j \leq n$, and for all $1 \leq i \leq n$, $1 \leq j \leq t$, we have that u_if_j is an edge if and only if vertex v_i is contained in facet F_j of Δ.

Let $k = n+t-(d+1)$. Consider the separating sets of G of size $d+1 = n+t-k$. Since each f_j has degree $d+1$, the neighbourhood $\Gamma(f_j)$ is a separating set of size $d+1$ for each $1 \leq j \leq t$.

In fact, these are the only separating sets of size $d+1$. Suppose S is a set not containing $\Gamma(f_j)$ for any j. Given any $x, y \notin S$, either $x = u_i$ for some i or x is adjacent to some $u_i \notin S$. Similarly, either $y = u_j$ or y is adjacent to some $u_j \notin S$. Then either $u_i = u_j$ or u_iu_j is an edge, and either way we have a path from x to y using only vertices outside the set S. Thus S cannot be separating.

The facets of the cut complex $\Delta_k(G)$ are $\Gamma(f_j) = \{u_i : v_i \in F_j\}$ for $1 \leq j \leq t$. The vertices f_j are not in any facets and so the complex $\Delta_k(G)$ has vertex set $\{u_1, u_2, \ldots, u_n\}$. Identifying u_i with v_i for each $1 \leq i \leq n$ makes it immediately clear that $\Delta_k(G) = \Delta$.

Note that if $t = 1$, then $n+t-(d+1) = 1$, and we cannot identify Δ as $\Delta_1(G)$. In this case $n = d+1$.

If $n = 1$, then Δ can be realized as the cut complex $\Delta_m(H)$ of the star graph $K_{1,m}$ for any $m \geq 2$. The center of the star constitutes the unique separating set of size 1, provided m is at least 2.

If $n = d+1 \geq 2$, then $\Delta = \{v_1, \ldots, v_n\}$ is a d-simplex, and the above construction for $t > 1$ is modified by taking the graph G to be the complete graph on n vertices, and then adding n additional vertices f_j, each of which is adjacent to each of the vertices of the K_n. Then Δ will be the cut complex $\Delta_n(G)$ (note G has $2n$ vertices), since the n vertices in the complete graph K_n constitute the only separating set of size n.

Finally, this construction always produces a *chordal* graph G. Since G contains the clique on the vertex set V of Δ as an induced subgraph, and the set of facets $\mathcal{F}$ of Δ gives an independent set of G, any cycle in G not contained in the clique on V must have at least one vertex f corresponding to a facet $F \in \mathcal{F}$, and the two adjacent vertices must be vertices $u, v \in F$, by construction of G. Thus $\{u, f, v\}$ is a 3-cycle, and G is chordal. □

The value of k given by Theorem 4.1 may be very large. In particular the theorem gives us limited information on which simplicial complexes can arise as cut complexes for a specific fixed value of k.

Remark 4.2. This theorem gives us an example of a cut complex with torsion in its homology, in contrast to the other specific cut complexes studied in this paper. Let $\Delta_{\mathbb{RP}^2}$ be the minimal triangulation of the projective plane $\mathbb{RP}^2$ with 6 vertices, 15 edges and 10 faces. The integral homology is $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}_2$, $H_i = 0$, $i \geq 2$. Theorem 4.1 constructs a graph G on $n + t = 16$ vertices such that $\Delta_{\mathbb{RP}^2} = \Delta_{13}(G)$, since $k = (n + t) - (d + 1) = 13$. Hence $\Delta_{13}(G)$ is not torsion-free.

4.2. **Facets of Cut Complexes.**

Definition 4.3. A *ridge* of a pure simplicial complex is a face of dimension one lower than the dimension of the complex.

Theorem 4.4. *Let $k \geq 2$, $G = (V, E)$ a graph. Then the facets of $\Delta_{k+1}(G)$ are precisely the ridges of $\Delta_k(G)$ that are contained in at least k facets.*

Proof. Let K be a facet of $\Delta_{k+1}(G)$, so $V \setminus K$ is a disconnected $(k+1)$-set. Let $V \setminus K = \{v_1, v_2, \ldots, v_{k+1}\}$. We write

$$K = \bigcap_{i=1}^{k+1} \big(K \cup v_i\big).$$

If $G \setminus K$ has at least three components, then for all v_i, $G \setminus (K \cup v_i)$ has at least two components. If $G \setminus K$ has two components, and each of those components has at least two vertices, then for all v_i, $G \setminus (K \cup v_i)$ has at least two components. Finally, if $G \setminus K$ has two components and one of those components has just one vertex (say v_1), then the other component has $k \geq 2$ vertices, and for all $i \geq 2$, $G \setminus (K \cup v_i)$ has at least two components. Therefore there are at least k vertices v_i such that $K \cup v_i$ is a facet of $\Delta_k(G)$.

On the other hand, suppose K is a ridge of $\Delta_k(G)$ that is contained in at least k facets. Assume K is **not** a facet of $\Delta_{k+1}(G)$; then $G \setminus K$ is connected, and so there exists a spanning tree T of $G \setminus K$. Since $k \geq 2$, this tree has at least two leaves, say, v and w. Since removing a leaf of a spanning tree does not disconnect the graph, $G \setminus (K \cup v)$ and $G \setminus (K \cup w)$ are both connected. Since $|V \setminus K| = k + 1$, there can only be at most $k - 1$ facets of $\Delta_k(G)$ that contain K, which contradicts our assumption. □

The above theorem implies that the faces of the $(k+1)$-cut complex of a graph G are completely determined by those of the k-cut complex.

4.3. **Links and Induced Subgraphs.**

Lemma 4.5. *Let $k \geq 2$, $G = (V, E)$ a graph, and $W \subseteq V$. Then $\Delta_k(G \setminus W) = \mathrm{lk}_{\Delta_k(G)} W$ if W is a face of $\Delta_k(G)$, and is void otherwise.*

Proof. First note that if W is not a face of Δ_k, then $G \setminus W$ contains no disconnected induced subgraph with k vertices, so $\Delta_k(G \setminus W)$ is void.

Consider the case of a single vertex v of $\Delta_k(G)$. Let $|V| = n$. A subset $F \subset V$ is a facet of $\Delta_k(G \setminus v)$ if and only if $v \notin F$, $|F| = n - k - 1$, and $(V \setminus v) \setminus F$ is disconnected. Since $(V \setminus v) \setminus F = V \setminus (v \cup F)$, this is equivalent to $v \notin F$, $|F \cup v| = n - k$ and $v \cup F \in \Delta_k(G)$, i.e., $F \cup v$ is a facet of $\Delta_k(G)$, and thus F is a facet of $\mathrm{lk}_{\Delta_k(G)}\{v\}$. Equivalently, $\Delta_k(G \setminus v) = \mathrm{lk}_{\Delta_k(G)}\{v\}$.

The result for an arbitrary face W now follows by repeated application. □

Proposition 4.6. *Let $k \geq 2$, $G = (V, E)$ a graph, and $W \subseteq V$. If the k-cut complex $\Delta_k(G)$ is shellable, so is $\Delta_k(G \setminus W)$. Equivalently, if $\Delta_k(G)$ is shellable for a graph G, then $\Delta_k(H)$ is shellable for every induced subgraph H of G.*

Proof. Immediate from Lemma 4.5, since the void complex is shellable, and shellability is preserved by the operation of taking links of faces; see [8, Proposition 10.14], [36, Theorem 3.1.5]. □

4.4. **Disjoint Union of Graphs.**

Definition 4.7. If G_1, G_2 are graphs, their *disjoint union* is the graph $G_1 + G_2$ having vertex set equal to the disjoint union of the vertex sets of G_1 and G_2, and edge set equal to the disjoint union of the edge sets of G_1 and G_2.

Theorem 4.8. *Let $k \geq 2$, and G_1, G_2 graphs. Then $\Delta_k(G_1 + G_2)$ is shellable if and only if $\Delta_k(G_1)$ and $\Delta_k(G_2)$ are shellable.*

Proof. $\Delta_k(G_1+G_2)$ shellable implies $\Delta_k(G_1)$ and $\Delta_k(G_2)$ are shellable as G_1 and G_2 are induced subgraphs of G_1+G_2.

Now suppose $\Delta_k(G_1)$ and $\Delta_k(G_2)$ are shellable. We construct a shelling of $\Delta_k(G_1+G_2)$. The facets of $\Delta_k(G_1+G_2)$ are the separating sets of size $|V_1|+|V_2|-k$ of the graph G_1+G_2. There are three types of facets. Type 1: $(|V_1|+|V_2|-k)$-sets containing some, but not all vertices of each of V_1 and V_2. Type 2: sets of the form $V_1 \cup A$, where A is a $(|V_2|-k)$-subset of V_2 that disconnects G_2. Type 3: sets of the form $B \cup V_2$, where B is a $(|V_1|-k)$-subset of V_1 that disconnects G_1.

We give an ordering of the facets of Type 1, followed by the facets of Type 2, followed by the facets of Type 3. First, we know that a shelling order of the facets of Type 1 exists, because the complex spanned by the facets of Type 1 is the $(|V_1|+|V_2|-k-1)$-skeleton of the join of the boundary of the simplex with vertex set V_1 with the boundary of the simplex with vertex set V_2. See [7, Theorem 2.9] for the skeleton result; the join result is due to [25, Corollary 2.9]; see also [8, Remark 10.22]. Also, we know that a shelling order of the facets of Type 2 exists, because the complex spanned by the facets of Type 2 is the $(|V_1|+|V_2|-k-1)$-skeleton of the join of the 1-facet complex consisting of the $(|V_1|-1)$-simplex with vertex set V_1 with $\Delta_k(G_2)$. Similarly, a shelling order of the facets of Type 3 exists, because the complex spanned by the facets of Type 3 is the $(|V_1|+|V_2|-k-1)$-skeleton of the join of the 1-facet complex consisting of the $(|V_2|-1)$-simplex with vertex set V_2 with $\Delta_k(G_1)$.

It remains to show that the resulting ordering of the facets of Type 1, followed by the facets of Type 2, followed by the facets of Type 3, is a shelling order. Clearly, the intersection of two facets of the same type satisfies the shelling condition. Now suppose that F_i is a facet of Type 1 and F_j is a facet of Type 2. Then $(F_i \cap F_j) \cap V_1 \subseteq F_i \cap V_1 \subseteq V_1 \setminus \{x\}$ for some $x \in V_1$, and $(F_i \cap F_j) \cap V_2 \subseteq F_j \cap V_2 \subseteq V_2 \setminus \{y, z\}$ for some $y, z \in V_2$ (since $k \geq 2$), so $F_i \cap F_j \subseteq (V_1 \setminus \{x\}) \cup (F_j \cap V_2 \cup \{y\})$, which is a $((|V_1|-1)+(|V_2|-k+1)) = (|V_1|+|V_2|-k)$-subset that is a facet of $\Delta_k(G_1+G_2)$ of Type 1. The same argument shows that the intersection of a facet of Type 1 with a facet of Type 3 is contained in a facet of Type 1. Finally, suppose F_i is a fact of Type 2 and F_j is a facet of Type 3. Then for some set $A \subset V_1$ of size $|V_1|-k$ and some set $B \subset V_2$ of size $|V_2|-k$, $F_i \cap F_j = A \cup B$ with $|A \cup B| = |V_1|+|V_2|-2k$. Choose, say, $k-1$ elements of $V_1 \setminus A$ and 1 element of $V_2 \setminus B$ to extend $A \cup B$ to a $(|V_1|+|V_2|-k)$-element subset of $V_1 \cup V_2$, not containing all of V_1 or all of V_2. This is then a facet of Type 1 that contains $F_i \cap F_j$.

Finally, we can conclude that a shelling order of Type 1 facets, followed by a shelling order of Type 2 facets, followed by a shelling order of Type 3 facets is a shelling of $\Delta_k(G_1+G_2)$. $\square$

4.5. **Join of Graphs.**

Definition 4.9. Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ on disjoint vertex sets, their *join* $G_1 * G_2$ is their disjoint union with the set of all edges between V_1 and V_2 added as well.

Theorem 4.10. *Let $k \geq 2$, and G_1, G_2 graphs. Then $\Delta_k(G_1 * G_2)$ is shellable if and only if between $\Delta_k(G_1)$ and $\Delta_k(G_2)$, one is shellable and the other is the void complex.*

Proof. Let $k \geq 2$, $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $\mathcal{F}$ the set of facets of $\Delta_k(G_1 * G_2)$, and $\mathcal{F}_1, \mathcal{F}_2$ the facets of $\Delta_k(G_1)$ and $\Delta_k(G_2)$, respectively. Furthermore let $\mathcal{F}_1^+ = \{F \cup V_2 \mid F \in \mathcal{F}_1\}$ and $\mathcal{F}_2^+ = \{F \cup V_1 \mid F \in \mathcal{F}_2\}$. We consider $D_k(G_1 * G_2)$. If S induces a disconnected subgraph of $G_1 * G_2$, then $S \subseteq V_1$ or $S \subseteq V_2$. So $D_k(G_1 * G_2) = D_k(G_1) \sqcup D_k(G_2)$, and $\mathcal{F} = \mathcal{F}_1^+ \sqcup \mathcal{F}_2^+$. Let $F_1 \in \mathcal{F}_1^+$ and $F_2 \in \mathcal{F}_2^+$; then $F_1^c \in D_k(G_1)$ so $F_1^c \subseteq V_1$, and similarly $F_2^c \subseteq V_2$. Suppose $F_1 \cap F_2 \subseteq F \in \mathcal{F}$; then $F^c \subseteq (F_1 \cap F_2)^c = F_1^c \cup F_2^c$. However F^c has size k and is contained entirely in V_1 or V_2. So $F^c = F_1^c$ or $F^c = F_2^c$, and thus $F = F_1$ or $F = F_2$; so the only facets to contain $F_1 \cap F_2$ are F_1 and F_2. However, $|F_1 \cap F_2| = |V_1 \cup V_2| - 2k = |F_1| - k < |F_1| - 1$. Thus if both $\mathcal{F}_1$ and $\mathcal{F}_2$ are non-empty, then $\Delta_k(G_1 * G_2)$ is not shellable. Assume without loss of generality that $\mathcal{F}_2 = \emptyset$; then $\mathcal{F} = \mathcal{F}_1^+$ and $\Delta_k(G_1 * G_2)$ is shellable if and only if $\Delta_k(G_1)$ is shellable. $\square$

In many ways the join of graphs behaves like the opposite of the disjoint union of graphs; see Theorem 4.8. It destroys cut complex shellability except in very specific boundary conditions. However, the following special case is worth highlighting.

Corollary 4.11. *Given a graph G, let $G * 1$ denote the join of the graph G with the graph consisting of a single vertex. Then $\Delta_k(G * 1)$ is shellable if and only if $\Delta_k(G)$ is shellable.*

Definition 4.12. A vertex v of a graph G is a *dominating* vertex if it is connected by an edge to every other vertex of G.

Thus the new vertex in $G * 1$ is a dominating vertex.

Definition 4.13. A *threshold graph* is a graph constructed from a single vertex by adding a sequence of isolated and dominating vertices.

Since the cut complex of the graph on one vertex is the void complex for all $k \geq 2$, it is trivially shellable. Hence Corollary 4.11 and Theorem 4.8 immediately give us the following corollary.

Corollary 4.14. *If G is a threshold graph, then $\Delta_k(G)$ is shellable for all $k \geq 2$.*

We now give a precise structural description of the cut complex of the join of two graphs, which will allow us to determine the homotopy type from that of the cut complexes of the individual graphs. Passing to the face lattice gives an alternative proof of the homotopy equivalence, and also shows that it is group-equivariant, a fact we will need in Section 7.1. We begin with a more general proposition. We write $\langle V \rangle$ for the simplex on the vertex set V.

Proposition 4.15. *For $i = 1, 2$ let Δ_i be a simplicial complex on a finite vertex set V_i, such that $V_1 \cap V_2 = \emptyset$. Define a new simplicial complex Δ by*

$$\Delta \coloneqq (\Delta_1 * \langle V_2 \rangle) \cup (\langle V_1 \rangle * \Delta_2).$$

(1) There is a homotopy equivalence $\Delta \simeq \operatorname{susp}(\Delta_1 * \Delta_2)$.
(2) Moreover, there is a group-equivariant poset map from the product of face lattices

$$\mathcal{L}(\Delta_1) \times \mathcal{L}(\Delta_2)$$

to the face lattice of the simplicial complex Δ, thereby inducing a group-equivariant homotopy equivalence of the respective order complexes. That is, if H_i is a group acting simplicially on Δ_i, $i = 1, 2$, then there is a group equivariant homotopy equivalence

$$\operatorname{susp}(\Delta_1 * \Delta_2) \simeq_{H_1 \times H_2} \Delta.$$

Proof. For Part (1):

For simplicity write $A = \Delta_1 * \langle V_2 \rangle$ and $B = \langle V_1 \rangle * \Delta_2$. Since $\Delta = A \cup B$, Proposition 3.7 gives a homotopy equivalence

$$\Delta / A \simeq B/(A \cap B).$$

Note that A and B are contractible. Hence by Proposition 3.8, the space on the left is homotopy equivalent to Δ, and the space on the right is homotopy equivalent to $\operatorname{susp}(A \cap B) = \operatorname{susp}(\Delta_1 * \Delta_2)$.

For Part (2):

Let H_i be a group acting simplicially on Δ_i, $i = 1, 2$. Recall that the face lattice $\mathcal{L}(\Delta_i)$ has an artificially appended top element $\widehat{1}_{\mathcal{L}(\Delta_i)}$. We claim that there is a group-equivariant poset map

$$\phi : \mathcal{L}(\Delta_1) \times \mathcal{L}(\Delta_2) \to \mathcal{L}(\Delta)$$

which induces a $(H_1 \times H_2)$-homotopy equivalence of order complexes. More precisely, for faces $\sigma_i \in \mathcal{L}(\Delta_i)$, define ϕ to be the map sending

$$\begin{aligned}
&(\sigma_1, \sigma_2) \mapsto \sigma_1 \sqcup \sigma_2,\\
&(\sigma_1, \widehat{0}) \mapsto \sigma_1 \sqcup \emptyset = \sigma_1,\\
&(\sigma_1, \widehat{1}_{\mathcal{L}(\Delta_2)}) \mapsto \sigma_1 \sqcup V_2,\\
&(\widehat{0}, \sigma_2) \mapsto \emptyset \sqcup \sigma_2 = \sigma_2,\\
&(\widehat{1}_{\mathcal{L}(\Delta_1)}, \sigma_2) \mapsto V_1 \sqcup \sigma_2.
\end{aligned}$$

One checks that ϕ is a poset map, which clearly commutes with the action of $H_1 \times H_2$.

The facets of the complex Δ are of the form $F_1 \sqcup V_2$, $V_1 \sqcup F_2$, where F_i is a facet of Δ_i, $i = 1, 2$. Hence $\sigma = \alpha_1 \sqcup \alpha_2$ is a face of Δ if and only if $\alpha_i \subseteq V_i,\ i = 1, 2$ and $\alpha_i \in \Delta_i$ for at least one i.

We show that the fibers

$$\phi_{\geq}^{-1}(\alpha_1 \sqcup \alpha_2) = \Big\{(\sigma_1, \sigma_2) \neq \big(\widehat{1}_{\mathcal{L}(\Delta_1)}, \widehat{1}_{\mathcal{L}(\Delta_2)}\big) : \text{either } \sigma_i = \widehat{1}_{\mathcal{L}(\Delta_i)} \text{ or } \sigma_i \in \Delta_i \text{ and } \sigma_i \supseteq \alpha_i, i = 1, 2\Big\}$$

have a unique minimal element, and hence are contractible.

Suppose $\alpha_i \in \Delta_i$ for both $i = 1, 2$. Then clearly (α_1, α_2) is in the fiber, and is its unique minimal element.

Otherwise $\alpha_i \in \Delta_i$ for only one of $i = 1, 2$, say for $i = 2$. Since $\alpha_1 \notin \Delta_1$, if (σ_1, σ_2) is in the fiber, this forces $\sigma_1 = \widehat{1}_{\mathcal{L}(\Delta_1)}$. This is because Δ_1 is a simplicial complex, so $\alpha_1 \notin \Delta_1$ and $\sigma_1 \supseteq \alpha_1$ implies $\sigma_1 \notin \Delta_1$.

Hence $(\widehat{1}_{\mathcal{L}(\Delta_k(G_1))}, \alpha_2)$ is in the fiber, and by the above argument is its unique minimal element.

We have shown that in all cases the fiber is contractible, and hence by Theorem 3.4, ϕ induces a group-equivariant homotopy equivalence of order complexes

$$\Delta(\overline{\mathcal{L}(\Delta_1) \times \mathcal{L}(\Delta_2)}) \simeq \Delta(\overline{\mathcal{L}(\Delta)}).$$

But the right-hand side is the barycentric subdivision of Δ, and from (5), the left-hand side is homotopy equivalent to

$$\operatorname{susp}\big(\Delta(\mathcal{L}(\Delta_1)) * \Delta(\mathcal{L}(\Delta_2))\big),$$

where again the order complex $\Delta(\mathcal{L}(\Delta_i))$ of the face lattice is homeomorphic to the barycentric subdivision of Δ_i, and hence to Δ_i, $i = 1, 2$. Hence the left-hand side is homeomorphic and thus homotopy equivalent to $\operatorname{susp}(\Delta_1 * \Delta_2)$, and the claim follows. □

Theorem 4.16. *Let $k \geq 1$, let G_i be a graph with vertex set V_i, $i = 1, 2$, where $V_1 \cap V_2 = \emptyset$. Assume $\Delta_k(G_i)$ is nonvoid for at least one $i = 1, 2$.*

(1) We have the decomposition

$$\Delta_k(G_1 * G_2) = (\Delta_k(G_1) * \langle V_2 \rangle) \cup (\langle V_1 \rangle * \Delta_k(G_2)). \tag{9}$$

(2) Assume only one cut complex, say $\Delta_k(G_1)$, is void. Then $\Delta_k(G_1 * G_2)$ is contractible.

(3) Assume both cut complexes $\Delta_k(G_i)$, $i = 1, 2$, are nonvoid. Then there is a homotopy equivalence

$$\operatorname{susp}(\Delta_k(G_1) * \Delta_k(G_2)) \simeq \Delta_k(G_1 * G_2). \tag{10}$$

Moreover, there is a group-equivariant poset map from the product of face lattices

$$\mathcal{L}(\Delta_k(G_1)) \times \mathcal{L}(\Delta_k(G_2))$$

to the face lattice of the simplicial complex $\Delta_k(G_1 * G_2)$ which induces a group-equivariant homotopy equivalence of the respective order complexes. This in turn gives a group-equivariant $(H_1 \times H_2)$-homotopy equivalence

$$\operatorname{susp}(\Delta_k(G_1) * \Delta_k(G_2)) \simeq_{H_1 \times H_2} \Delta_k(G_1 * G_2),$$

where H_i is a group acting simplicially on the cut complex $\Delta_k(G_i)$ of the graph G_i.

Proof. Recall that σ is a face of the cut complex of a graph G if and only if it is contained in a facet, which by definition is the complement of a disconnected set of size k, i.e., if and only if the complement of σ contains a disconnected set of size k.

There is an edge between any vertex of G_1 and any vertex of G_2 in the join of graphs $G_1 * G_2$. Hence F is a face of $\Delta_k(G_1 * G_2)$ if and only if $F = F_1 \sqcup F_2$, $F_i \subset V_i$, where

- the complement of F_1 in V_1 contains a disconnected set of size k in G_1, or
- the complement of F_2 in V_2 contains a disconnected set of size k in G_2.

This shows that $\Delta_k(G_1 * G_2)$ satisfies the hypotheses of Proposition 4.15. Equation (9) is now immediate, as are the remaining parts of the theorem. Note that $\Delta_k(G_i) * \langle V_{3-i} \rangle$, $i = 1, 2$, is contractible since the full simplex is contractible. □

Remark 4.17. We note that exactly the same theorem holds for the *total* cut complex $\Delta_k^t(G_1 * G_2)$ of the join of two graphs, which is studied in [2].

From the Künneth Theorem we now also have:

Corollary 4.18. *Let G_i, $i = 1, 2$ be graphs and k be such that the cut complexes $\Delta_k(G_i)$, $i = 1, 2$ are nonvoid. Assume the homology of one of $\Delta_k(G_1), \Delta_k(G_2)$ is always free. Then we have the following isomorphism in homology, which is group-equivariant in rational homology:*

$$\widetilde{H}_d(\Delta_k(G_1 * G_2)) \cong \bigoplus_{p+q=d-2} \widetilde{H}_p(\Delta_k(G_1)) \otimes \widetilde{H}_q(\Delta_k(G_2)).$$

Proof. From the isomorphism of face lattices in the preceding theorem and the Künneth Theorem, Theorem 3.2, we have

$$\begin{aligned}\widetilde{H}_d(\Delta_k(G_1 * G_2)) &\cong \widetilde{H}_d(\text{susp}(\Delta_k(G_1) * \Delta_k(G_2))) \\ &= \widetilde{H}_{d-1}(\Delta_k(G_1) * \Delta_k(G_2)) \\ &\cong \bigoplus_{p+q=d-2} \widetilde{H}_p(\Delta_k(G_1)) \otimes \widetilde{H}_q(\Delta_k(G_2)),\end{aligned}$$

as claimed. □

4.6. **Wedge of Graphs.**

Definition 4.19. Given graphs G_1 and G_2, a *wedge* of G_1 and G_2, denoted $G_1 \vee G_2$, is formed by taking a vertex from G_1 and a vertex from G_2 and identifying them.

Note that a wedge of two graphs is not unique in general. See Figure 5.

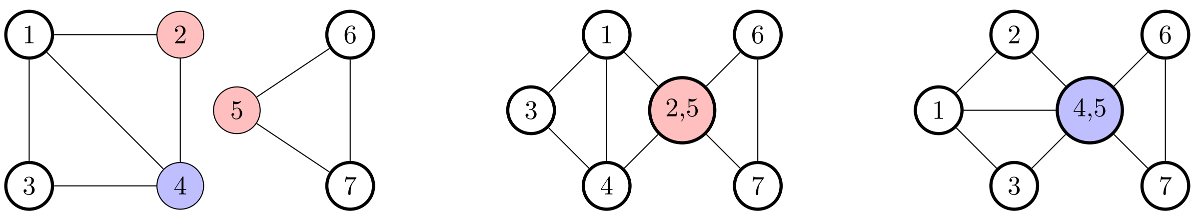


Figure 5. Example of different wedges of the same two graphs

Theorem 4.20. *Let $k \geq 2$, and let G_1, G_2 be graphs. Then $\Delta_k(G_1 \vee G_2)$ is shellable if and only if $\Delta_k(G_1)$ and $\Delta_k(G_2)$ are shellable.*

Proof. We know $\Delta_k(G_1 \vee G_2)$ being shellable implies $\Delta_k(G_1)$ and $\Delta_k(G_2)$ are shellable, since G_1 and G_2 are induced subgraphs of $G_1 \vee G_2$. Now suppose $\Delta_k(G_1)$ and $\Delta_k(G_2)$ are shellable. If $\Delta_k(G_1 \vee G_2)$ is the void complex, then it is shellable, so assume $\Delta_k(G_1 \vee G_2)$ is not the void complex. Let V_i be the vertex set of G_i and let $\{v_0\} = V_1 \cap V_2$, so that v_0 is the common wedge point. We consider the facet set $\mathcal{F}$ of $\Delta_k(G_1 \vee G_2)$; they are separating sets of size $|V_1| + |V_2| - k - 1$ of the graph $G_1 \vee G_2$. There are four types of facets:

Type 1: Sets of the form $v_0 \cup A \cup B$, where $A \subsetneq V_1 - v_0$ and $B \subsetneq V_2 - v_0$.

Type 2: Sets of the form $A \cup B$, where $\emptyset \neq A \subsetneq V_1 - v_0$ and $\emptyset \neq B \subsetneq V_2 - v_0$, where either A is a separating set for G_1 or B is a separating set for G_2, or both.

Type 3: Sets of the form $A \cup (V_2 - v_0)$, where A is a $(|V_1| - k)$-subset of V_1 whose removal disconnects G_1.

Type 4: Sets of the form $(V_1 - v_0) \cup B$, where B is a $(|V_2| - k)$-subset of V_2 whose removal disconnects G_2.

Note that if $k \geq |V_1| - 1$, then there is no Type 3 facet, and if $k \geq |V_2| - 1$, then there is no Type 4 facet.

We start the shelling with Type 1 facets. Order the Type 1 facets of $\Delta_k(G_1 \vee G_2)$ first in order of decreasing size of A, then, among facets with sets A of the same size, in lexicographic order of the sets A, and then, among facets with fixed A, in lexicographic order of the sets B. Label the facets of Type 1 in the resulting order $F_1, F_2, \ldots, F_r$.

Consider $F_i = v_0 \cup A_i \cup B_i$ and $F_j = v_0 \cup A_j \cup B_j$, with $i < j$. Here $|A_i| \geq |A_j|$. If $A_i = A_j$, then $|B_i| = |B_j|$ and B_i precedes B_j in lexicographic order. By the lexicographic shellability of the $(|B_j| - 1)$-skeleton of the simplex on $V_2 - v_0$, there exists a set $B \subsetneq V_2 - v_0$, with $|B| = |B_j|$, preceding B_j in lexicographic order such that $B_i \cap B_j \subseteq B \cap B_j$ and $|B \cap B_j| = |B_j| - 1$. Then for $F = v_0 \cup A_j \cup B$, F precedes F_j, $F_i \cap F_j \subseteq F \cap F_j$, and $|F \cap F_j| = |F_j| - 1$.

Now suppose $A_i \neq A_j$ but $|A_i| = |A_j|$. Then A_i precedes A_j in lexicographic order. By the lexicographic shellability of the $(|A_j| - 1)$-skeleton of the simplex on $V_1 - v_0$, there exists a set $A \subsetneq V_1 - v_0$, with $|A| = |A_j|$, preceding A_j in lexicographic order such that $A_i \cap A_j \subseteq A \cap A_j$ and $|A \cap A_j| = |A_j| - 1$. Then for $F = v_0 \cup A \cup B_j$, F precedes F_j, $F_i \cap F_j \subseteq F \cap F_j$, and $|F \cap F_j| = |F_j| - 1$.

Finally, suppose $|A_i| > |A_j|$. Then $|B_i| < |B_j|$. Let $x \in A_i \setminus A_j$, $y \in B_j \setminus B_i$, and $F = v_0 \cup (A_j \cup x) \cup (B_j - y)$. Then F comes before F_j and

$$F_i \cap F_j = v_0 \cup (A_i \cap A_j) \cup (B_i \cap B_j) \subseteq v_0 \cup (A_j \cup x) \cup (B_j - y) = F,$$

so $F_i \cap F_j \subseteq F \cap F_j$ and

$$|F \cap F_j| = |(v_0 \cup A_j \cup B_j) \cap (v_0 \cup (A_j \cup x) \cup (B_j - y))| = |v_0 \cup A_j \cup (B_j - y)| = |F_j| - 1.$$

So $F_1, F_2, \ldots, F_r$ is a shelling order of the Type 1 facets.

We now claim that the Type 2 facets can be added to the shelling order in any order. We will demonstrate this by showing that the intersection of a Type 2 facet and any other Type 2 or Type 1 facet is contained in a Type 1 facet that differs by only one vertex. Let F be a Type 2 facet and X be a distinct Type 1 or Type 2 facet. As $F \neq X$, there exists $v \in F \setminus X$. We claim $F' = F - v + v_0$ is a Type 1 facet. As $v_0 \notin F$, $|F'| = |F|$. As $V_1 - v_0 \nsubseteq F$, then $V_1 - v_0 \nsubseteq F - v$, so $V_1 - v_0 \nsubseteq F'$; similarly, $V_2 - v_0 \nsubseteq F'$. Then $F' = v_0 \cup A \cup B$ where $A \subsetneq V_1 - v_0$ and $B \subsetneq V_2 - v_0$. So F' is a Type 1 facet, and $|F \setminus F'| = 1$, and thus any order of Type 2 facets placed after the Type 1 facets will result in a shelling order.

Finally we add the Type 3 and Type 4 facets. As $\Delta_k(G_1)$ is shellable, the join of $\Delta_k(G_1)$ with the simplex on $V_2 - v_0$ is shellable, and we add the Type 3 facets in such a shelling order. We do the same for the Type 4 facets. Now we just need to verify that the intersection of a Type 3 or 4 facet with any facet of a different type is contained in a Type 1 facet. Without loss of generality, assume F is a Type 4 facet, and X is a facet of another type. Type 4 facets are the only facets that contain all of $V_1 - v_0$, so we know there exists $v \in (F \setminus X) \cap (V_1 - v_0)$. We now need to break into two small cases. In the first case, $v_0 \notin F$, and so we choose $F' = F - v + v_0$. Then $|F'| = |F| - 1 + 1$, and $V_1 - v_0 \nsubseteq F'$ as $v \notin F'$. Now, $V_2 - v_0 \nsubseteq F'$ as F was missing k vertices from V_2 and we only added one vertex to it. So $F' = v_0 \cup A \cup B$ where $A \subsetneq V_1 - v_0$ and $B \subsetneq V_2 - v_0$. So F' is a Type 1 facet, and $|F \setminus F'| = 1$. The second case has $v_0 \in F$; we choose any $v' \in F^c \cap V_2$, and set $F' = F - v + v'$. Then $|F'| = |F|$, and still $V_1 - v_0 \nsubseteq F'$ as $v \notin F'$. Also, $V_2 - v_0 \nsubseteq F'$ as F was missing k vertices from V_2 and we only added one vertex to it. So $F' = v_0 \cup A \cup B$ where $A \subsetneq V_1 - v_0$ and $B \subsetneq V_2 - v_0$. So F' is a Type 1 facet, and $|F \setminus F'| = 1$. This demonstrates that the intersection of any Type 4 facet with a facet of another type is contained in a Type 1 facet with only one vertex different, and very similar arguments can be used to show the same is true with Type 3 facets. This is the final step in confirming our shelling order, demonstrating that the complex $\Delta_k(G_1 \vee G_2)$ is shellable. □

As an immediate corollary we obtain the next result, which was originally proved in [10] by exhibiting an explicit shelling order.

Corollary 4.21. *If G is a tree, then $\Delta_k(G)$ is shellable for all $k \geq 2$.*

Proof. A tree on n vertices is the wedge of $n - 1$ copies of K_2. The cut complex of any complete graph is the void complex for all $k > 2$, and so is trivially shellable. Hence the wedge of these shellable graphs is shellable. For $k = 2$ the claim follows from Theorem 1.2, since trees are chordal. □

4.7. **Minimal nonshellable graphs.** Recall that Proposition 4.6 says that if G has a shellable k-cut complex, then any induced subgraph of G also has a shellable k-cut complex. In light of this result, it is natural to seek a description of the *minimal* graphs (with respect to induced inclusion) whose k-cut complex fails to be shellable. We will call such a graph a *minimal nonshellable graph* or a *minimal forbidden subgraph* for k-cut complex shellability.

Figure 7 shows some minimal nonshellable graphs for $k = 3$. The first graph in Figure 7 is in fact the $k = 3$ case of a family of graphs considered in Lemma 7.19. More generally, for arbitrary $k \geq 4$ we have the following proposition.

Proposition 4.22. *For each $k \geq 4$, there is a chordal graph G_k with $k+2$ vertices such that $\Delta_k(G_k)$ is NOT shellable. Furthermore, G_k is a minimal forbidden subgraph for k-cut complex shellability.*

Proof. The graph G_k has $k+2$ vertices, which we partition into three disjoint subsets according to the parity of k: $\{2i-1 : 1 \leq i \leq m = \lfloor k/2 \rfloor\}, \{2i : 1 \leq i \leq m = \lfloor k/2 \rfloor\}$ and $\begin{cases} \{a\} & \text{for odd } k = 2m+1, \\ \{a,b\} & \text{for even } k = 2m. \end{cases}$

See Figure 6. The edges are precisely those specified by requiring that the induced subgraph on each 4-vertex subset $\{2i-1, 2i, 2i+1, 2i+2\}$ forms a clique for all $1 \leq i \leq m = \lfloor k/2 \rfloor$, as do the induced subgraphs on the 3-vertex sets $\{a, 1, 2\}$ and $\{b, 2m-1, 2m\}$, the latter existing only if $k = 2m$ is even. Then $\Delta_k(G_k)$ has facets $\{\{2i-1, 2i\} : 1 \leq i \leq m = \lfloor k/2 \rfloor\}$ (highlighted in the figures), and is thus not shellable since $m = \lfloor k/2 \rfloor \geq 2$. Also $\Delta_{2m}(G_{2m}) = \Delta_{2m+1}(G_{2m+1})$, $m \geq 2$. Finally, for each k, G_k is minimal nonshellable: for all vertices v, $G_k \backslash v$ has only $k+1$ vertices, and $\Delta_k(G_k \backslash v)$ is either void or 0-dimensional. □

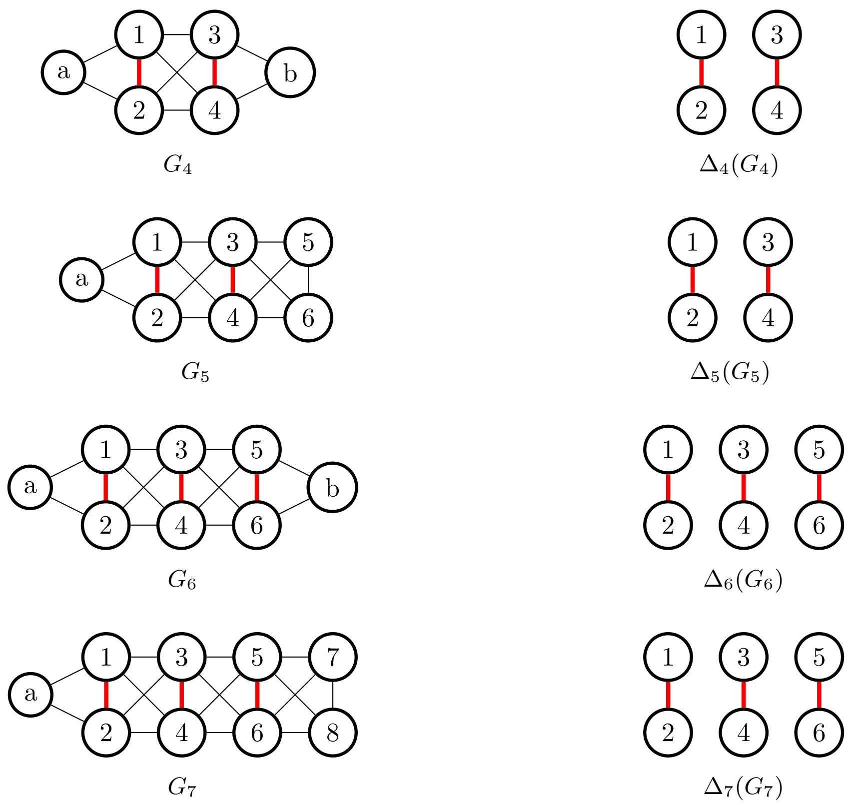


Figure 6. Graphs and cut complexes described in Proposition 4.22

We will encounter some more families of minimal nonshellable graphs in this paper: in particular, the prism over a k-clique discussed in Section 7.3 and the squared cycle on $k+4$ vertices in Section 7.4 are both minimal forbidden subgraphs for k-cut complex shellability, by Lemma 7.19 and Proposition 7.28, respectively.

5. Some theorems for the case $k = 3$

In this section we consider an operation which simultaneously generalizes disjoint unions and wedges of graphs.

Theorem 5.1 (Generalized Wedge)**.** *Let $G = (V, E)$, $A \cup B = V$ such that if $e \in E$, e is between two vertices in A or two vertices in B, and $\Delta_3(G[A \cap B])$ is the void complex. Then $\Delta_3(G)$ is shellable if and only if $\Delta_3(G[A])$ and $\Delta_3(G[B])$ are shellable.*

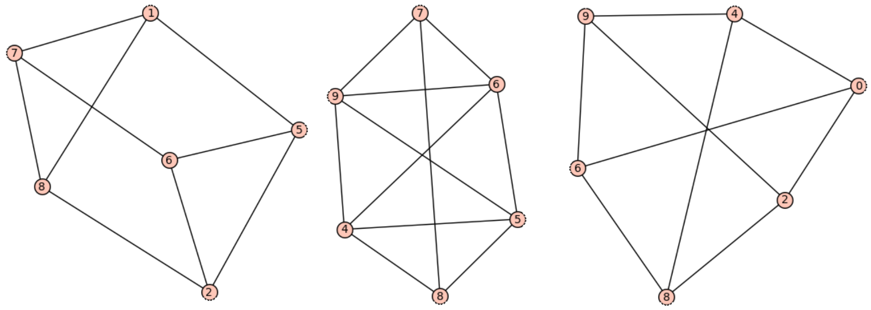


FIGURE 7. Some minimal nonshellable graphs for $k = 3$

Proof. We know that G having a shellable cut complex implies its induced subgraphs also have shellable cut complexes. So we assume $\Delta_3(G[A])$ and $\Delta_3(G[B])$ are shellable. We will now construct a shelling order for $\Delta_3(G)$ on its set of facets $\mathcal{F}$.

Let $S = A \cap B$. Note that the cases $|S| = 0, 1$ were settled in Theorem 4.8 and Theorem 4.20, respectively.

Because $k = 3$, it is more convenient to classify our facets in terms of their complements, the disconnected 3-sets. Let $\mathcal{F}_A = \{F \in \mathcal{F} \mid F^c \subseteq A\}$ be the set of facets whose complement lies entirely in A, and $\mathcal{F}_B = \{F \in \mathcal{F} \mid F^c \subseteq B\}$ be facets whose complement lies in B. The intersection $\mathcal{F}_A \cap \mathcal{F}_B$ is empty because $\Delta_3(G[A \cap B])$ is the void complex, so no disconnected 3-sets lie in $S = A \cap B$. For the remaining facets, we characterize them by the intersection of their complement with S, $\mathcal{F}_i = \{F \in \mathcal{F} \setminus (\mathcal{F}_A \cup \mathcal{F}_B) \mid |F^c \cap S| = i\}$. We know S cannot contain a disconnected set of size 3, so $\mathcal{F}_3 = \emptyset$, and if a set of three elements has 2 elements in S, then it is completely contained in either A or B, so $\mathcal{F}_2 = \emptyset$. So only $\mathcal{F}_0$ and $\mathcal{F}_1$ are potentially nonempty. This gives a complete classification of facets $\mathcal{F} = \mathcal{F}_0 \sqcup \mathcal{F}_1 \sqcup \mathcal{F}_A \sqcup \mathcal{F}_B$.

Our shelling order construction starts by ordering the facets in $\mathcal{F}_0$, the facets whose complement shares no elements with S, or, equivalently, the facets of $\Delta_3(G)$ that contain S. This subcomplex of $\Delta_3(G)$ is precisely $\Delta_3(G[V \setminus S]) * \langle S \rangle$ as in Proposition 4.6, since $\Delta_3(G[V \setminus S])$ is the link of $\langle S \rangle$, but $\Delta_3(G[V \setminus S]) = \Delta_3(G[A \setminus S] + G[B \setminus S])$, and so is shellable from Theorem 4.8. This provides us with a shelling order for $\mathcal{F}_0$, call it S_0. (We note in passing that the complex $\langle \mathcal{F}_0 \rangle$ is contractible.)

Now we append an order of the facets in $\mathcal{F}_1$; in this case any order will do. If $F \in \mathcal{F}_1$ and $X \in \mathcal{F}_0 \cup \mathcal{F}_1$, then either $|F \cap X| = |F| - 1$ (in which case $X = F'$ suffices for the shellability criterion) or there exists $x \in F \setminus (X \cup S)$ as X^c has at most one element in S. In that case, let $y \in F^c \cap S$, and consider $F' = F - x + y$. Then $F \cap X \subseteq F'$, and $|F \cap F'| = |F| - 1$. Additionally, $F' \in \mathcal{F}_0$ as $F'^c \cap S = \emptyset$, and $F' \nsubseteq A$, $F' \nsubseteq B$, so any order of $\mathcal{F}_1$ will do. Choose some order and call it S_1.

The facets in $\mathcal{F}_A$ are the facets of $\Delta_3(G[A])$ with the elements of $B \setminus A$ added, and so can inherit the shelling order of $\Delta_3(G[A])$: call it S_A. A similar argument holds for S_B. We claim that the order of S_0 followed by S_1 followed by S_A followed by S_B is a shelling order. We have already demonstrated that S_0 followed by S_1 fulfills the shelling criterion, and S_A and S_B work internally. To complete the proof we will demonstrate that the intersection of facets of $\mathcal{F}_B$ with facets outside of $\mathcal{F}_B$ are always contained in facets in $\mathcal{F}_0$ or $\mathcal{F}_1$ that fulfill the shelling criterion, and a similar argument will hold for the facets of $\mathcal{F}_A$.

Let $F \in \mathcal{F}_B$, and $X \in \mathcal{F} \setminus \mathcal{F}_B$. Then there exists $x \in X^c \cap B^c$. We will choose $F' = F - x + y$ so that $F \cap X \subset F'$ and $|F \cap F'| = |F| - 1$, but we need to decide which y to add. If $|F^c \cap S| = 0$, then any $y \in F^c$ will do, as then $F' \in \mathcal{F}_0$; if $|F^c \cap S| = 1$, add the $y \in F^c \cap S$, and again $F' \in \mathcal{F}_0$. If $|F^c \cap S| = 2$, then as $|F^c| = 3$ and $G[F^c]$ is not connected, the element of F^c not in S cannot be connected to both elements in S. Choose the $y \in F^c \cap S$ such that the remaining two elements of F^c do not share an edge. Then $F' = F - x + y \in \mathcal{F}_1$, as the element of F' contained in $B \setminus A$ cannot share an edge with x as $x \in A \setminus B$, and also does not share an edge with the element in S, so it is an isolated vertex in $G[F'^c]$ and so $F' \in \mathcal{F}_1$. In each case $|F \cap F'| = |F| - 1$, $F \cap X \subseteq F'$, and F' comes before F in the shelling order as $F' \in \mathcal{F}_0 \cup \mathcal{F}_1$. A similar argument holds for every facet in $\mathcal{F}_A$ and every facet not in $\mathcal{F}_A$. So every pair of facets in the shelling order meets the shelling criterion, and $S_0 + S_1 + S_A + S_B$ is a shelling order for $\Delta_3(G)$. □

The construction of Proposition 4.22 shows that the conclusion of Theorem 5.1 is false for $k \geq 4$.

Corollary 5.2. *If G contains a separating set S such that $\Delta_3(S)$ is the void complex, then G is not a minimal forbidden subgraph for 3-cut complex shellability.*

Proof. Let $G = (V, E)$, $V = A \sqcup S \sqcup B$, where S separates A and B, both non-empty. If $\Delta_3(G)$ is not shellable, either $\Delta_3(G[A \cup S])$ or $\Delta_3(G[B \cup S])$ is not shellable, so G is not a minimal forbidden subgraph. □

The following is in contrast to Proposition 4.22.

Corollary 5.3. *If G is chordal, then $\Delta_3(G)$ is shellable.*

Proof. Suppose G is chordal, but $\Delta_3(G)$ is not shellable. Let H be a minimal induced subgraph of G with a nonshellable 3-cut complex. H is chordal as it is an induced subgraph of G, so it has a vertex v whose neighborhood $N(v) = S$ is a clique. Either H is the complete graph, in which case $\Delta_3(H)$ is the void complex and therefore shellable, or S is a separating set such that $\Delta_3(S)$ is the void complex, so H is shellable. In either case we have a contradiction, so $\Delta_3(G)$ is in fact shellable. □

Corollary 5.4. *Every minimal forbidden subgraph for 3-cut complex shellability is at least 3-connected.*

Proof. If G has a separating set S such that $|S| < 3$, then $\Delta_3(S)$ is trivially the void complex, so G is not a minimal forbidden subgraph for 3-cut complex shellability. □

Corollary 5.5. *Suppose G is 3-connected, every vertex of G has a neighbor of degree 3, and $\Delta_3(G)$ is not shellable. Then G is a minimal forbidden subgraph for 3-cut complex shellability.*

Proof. Assume the hypothesis. Then every proper induced subgraph H of G has a vertex of degree at most 2. So H has a component that is either K_i for $1 \le i \le 3$ or is not 3-connected. So H is not a minimal forbidden subgraph for 3-cut complex shellability. So G is a minimal forbidden subgraph for 3-cut complex shellability. □

6. The face lattice of the k-cut complex

In this section we investigate the face lattice of the cut complex. Note that a simplicial complex Δ is shellable if and only if its face lattice admits a recursive coatom ordering [6, Theorem 4.3, Corollary 4.4]. Under certain favourable conditions (see Theorem 6.1 below), the face lattice admits a particularly simple description which allows us to conclude that homology is torsion-free and occurs in at most two dimensions, and also gives us a way to compute the Betti numbers of the cut complex.

The reduced Euler characteristic of a simplicial complex Δ is the Möbius number $\mu(\mathcal{L}(\Delta))$ of its face lattice $\mathcal{L}(\Delta)$ [30]. We exploit the fact that the face lattice of the cut complex is a subposet of the truncated Boolean lattice, and then use poset topology techniques to determine the Möbius number, using Theorem 3.5. Recall that the truncated Boolean lattice, by definition, has an artificially appended top element.

For a subset A of the vertex set $V(G)$ of a graph G, we say A is a *connected set* if the induced subgraph $G[A]$ is connected. Recall from Section 3 that $P(n,k)$ denotes the truncated Boolean lattice $B_n^{\le n-k}$, and let

$$\mathcal{Z}_k(G) := \{A^c : |A| = k, A \text{ is a connected subset of } V(G)\} \subseteq P(n,k),$$

where A^c denotes the complement of A in the vertex set $V(G)$. Thus $\mathcal{Z}_k(G)$ is the set of those $(n-k)$-element subsets of $[n]$ that are not facets of the cut complex, and the number of facets of $\Delta_k(G)$ is $\binom{n}{k} - |\mathcal{Z}_k(G)|$. Clearly the face lattice of $\Delta_k(G)$ is a subposet of $P(n,k) \setminus \mathcal{Z}_k(G)$.

Theorem 6.1. *Let G be a graph with vertex set $V(G)$ of size n, and let $k \ge 2$. Then the face lattice of $\Delta_k(G)$ coincides with $P(n,k) \setminus \mathcal{Z}_k(G)$ if and only if the $(n-k-1)$-dimensional complex $\Delta_k(G)$ contains a complete $(n-k-2)$-skeleton, that is, if and only if either of the following equivalent conditions holds:*

(11)
for every subset X of a set $A^c \in \mathcal{Z}_k(G)$ with $|X| = n-k-1$, X^c contains a disconnected set of size k;

(12) *if $A^c \in \mathcal{Z}_k(G)$ and $x \notin A$, there is a $y \in A$ such that $(A \setminus \{y\}) \cup \{x\}$ is disconnected.*

If condition (11) holds, the reduced Euler characteristic of $\Delta_k(G)$ is given by

$$(-1)^{n-k-1}\mu(\Delta_k(G)) = \binom{n-1}{k-1} - |\mathcal{Z}_k(G)| = |\{F : F \text{ is a facet of } \Delta_k(G)\}| - \binom{n-1}{k}.$$

Furthermore, in this case the nonzero homology of $\Delta_k(G)$ is torsion-free and occurs in at most two dimensions, $n-k-1$ and $n-k-2$.

If condition (11) holds and $\Delta_k(G)$ is shellable, then it is homotopy equivalent to

$$\begin{cases} \text{a point, if } \mu(\Delta_k(G)) = 0, \text{ i.e., if the number of facets of } \Delta_k(G) \text{ is } \binom{n-1}{k}, \\ \text{a wedge of } \left(\binom{n-1}{k-1} - |\mathcal{Z}_k(G)|\right) \text{ spheres in dimension } n-k-1, \text{ otherwise.} \end{cases}$$

Proof. The equivalence of the two conditions, and the fact that $\Delta_k(G)$ contains a complete $(n-k-2)$-skeleton, follow from the definitions. Either condition ensures that any faces contained in the removed facets $\mathcal{Z}_k(G)$ are already faces of $\Delta_k(G)$, since X^c contains a disconnected set of size k if and only if X is a subset of a facet of $\Delta_k(G)$.

For the remaining statements, we apply Theorem 3.5 to $P = P(n,k)$ and $Q = P \setminus \mathcal{Z}_k(G)$; note that $\mathcal{A} = \mathcal{Z}_k(G)$ is an antichain.

Also note that the intervals $(\widehat{0}, A^c)$ are Boolean intervals of $P(n,k)$ for all $A^c \in \mathcal{Z}_k(G)$, where A ranges over all connected k-subsets of $V(G)$. We obtain

$$\mu(P(n,k) \setminus \mathcal{Z}_k(G)) = \mu(P(n,k)) - \sum_{A^c \in \mathcal{Z}_k(G)} \mu(\widehat{0}, A^c)\mu(A^c, \widehat{1}) = \mu(P(n,k)) - (-1)^{n-k}(-1)|\mathcal{Z}_k(G)|.$$

Since $\mu(P(n,k)) = (-1)^{n-k-1}\binom{n-1}{k-1}$, the result follows.

The second equality follows from the discussion preceding the theorem.

The last statement about the homology of $\Delta_k(G)$ is a consequence of [34, Theorem 2.1, Theorem 2.5], which applies because we are deleting an antichain from the Cohen–Macaulay poset $B_n^{\leq n-k}$.

Finally, note that a nonzero reduced Euler characteristic gives the dimension of the unique nonvanishing homology, if the nonzero homology is concentrated in a single degree, as is the case for a shellable complex. If the Euler characteristic is zero, the shellable complex must be contractible. □

Proposition 6.2. *Let G be a graph on n vertices, let $k \geq 2$, and assume that G contains no cycles of length less than or equal to $(k+1)$. Then $\Delta_k(G)$ contains a complete $(n-k-2)$-skeleton.*

Proof. If G has no cycles of length less than or equal to $(k+1)$, any $(k+1)$-subset of the vertex set induces a forest, and thus contains a disconnected set of size k. By Definition 2.7, all subsets of size $(n-k-1)$ are faces of the cut complex. □

For instance, consider the disjoint union $C_n + C_m$ of two cycle graphs C_n, C_m with $n, m \geq 4$ and the cut complex $\Delta_2(C_n + C_m)$. There are no triangles, and there are $(n+m)$ connected subsets of size 2, so the preceding results give the reduced Euler characteristic $\mu(\Delta_2(C_n + C_m)) = (-1)^{n+m}$. Sage code (for $9 \geq m \geq n \geq 4$) gives homology of rank 1 in the top dimension and rank 2 in the dimension one below the top, which is consistent with the Euler characteristic.

The converse of Proposition 6.2 is false, as shown by Theorem 4.1.

Since trees are acyclic, Corollary 4.21 together with Theorem 6.1 and Proposition 6.2 immediately give:

Corollary 6.3. *If T is a tree, then $\Delta_2(T)$ is contractible.*

If the graph G has $(k+1)$-cycles, the conditions in Theorem 6.1 can be modified to obtain a precise description of the face lattice of $\Delta_k(G)$, from which we again obtain a formula for the reduced Euler characteristic. This is described in a forthcoming paper.

6.1. **A theorem about Δ_2 for connected triangle-free graphs.** Theorem 6.1 asserts that if G has n vertices and e edges and $\Delta_2(G)$ contains a complete $(n-4)$-skeleton, then $\Delta_2(G)$ has torsion-free homology, with Euler characteristic $(-1)^{n-4}(e-n+1)$, and the homology is nonzero in at most the top two dimensions. In this section we refine this result, noting the following special case of Proposition 6.2: if G is triangle-free, $\Delta_2(G)$ contains a complete codimension 1 skeleton.

Theorem 6.4. *Let G be a graph on n vertices with e edges, and assume G is connected and triangle-free. If G is not a tree, then $\Delta_2(G)$ is a wedge of $e-n+1$ spheres in dimension $n-4$, that is, in codimension 1.*

We will prove this below.

First, we refer the reader to [17] for the necessary background on discrete Morse matchings. The particular case of element matchings is described in [11]; a brief summary appears in [2, Appendix]. We begin by showing that there is a Morse matching on $\Delta_2(T)$ for any tree T. If G is a triangle-free connected graph, we can then choose a spanning tree T of G, and restrict the Morse matching of $\Delta_2(T)$ to $\Delta_2(G)$.

Lemma 6.5. *Let T be any tree on n vertices. Then $\Delta_2(T)$ admits a perfect Morse matching.*

Proof. We construct an element matching for $\Delta_2(T)$. First choose a root for T, labelled 1, and then label T recursively with labels $1, 2, 3, \ldots$, so that $x < y$ if x is the parent of y in T. We use this recursive labelling as the order for our element matching. By Proposition 6.2, as T is triangle-free, the $(n-3)$-dimensional cut complex $\Delta_2(T)$ has a complete $(n-4)$-skeleton.

For each vertex a, denote by $\mathcal{M}_a$ the set of element matchings

$$\{(\sigma, \sigma \cup a) : a \notin \sigma, \text{ and } \sigma,\ \sigma \cup a \text{ are both faces of } \Delta_2(G) \text{ that have not been matched by any } \mathcal{M}_b,\ b < a\},$$

and consider the sequence $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots$.

The unmatched faces after $\mathcal{M}_1$ are precisely the faces σ not containing 1 such that $\sigma \cup 1$ is not a face; since $\Delta_2(T)$ has a complete $(n-4)$-skeleton, $\sigma \cup 1$ must have dimension greater than $n-4$, and hence σ is either a facet or a face of dimension $(n-4)$.

Suppose σ is a face of dimension $(n-4)$, so that its (set) complement is $\sigma^c = \{1, x, y\}$. Since $\sigma \cup 1$ is NOT a face, xy must be an edge. Also σ must be contained in some facet, and G is triangle-free, so exactly one of $1x$, $1y$ is not an edge, and exactly one of $\sigma \cup x$, $\sigma \cup y$ is a facet. Assume without loss of generality that x is the parent of y; thus $1x$ is the edge, and the matching $\mathcal{M}_x$ matches the pair $(\sigma, \sigma \cup x)$.

This shows that all $(n-4)$-dimensional faces are matched.

Now suppose τ is an unmatched facet after $\mathcal{M}_1$, so that its complement $\tau^c = \{x, y\}$ where $x \neq 1, y \neq 1$, and xy is not an edge. But then x, y have a common ancestor $a \neq 1$, and $\{xya\}^c$ will be matched to τ^c by the element matching $\mathcal{M}_a$.

Since a sequence of element matchings is acyclic (see [2, Appendix], [11]), we have exhibited a Morse matching for $\Delta_2(T)$. In fact we have shown that all faces are matched, and hence there are no critical cells, recovering the result of Corollary 6.3 that $\Delta_2(T)$ is contractible. □

Proof of Theorem 6.4. Let G be a connected, triangle-free graph. If G is chordal, it must be a tree, and hence $\Delta_2(G)$ is shellable and contractible, and the theorem is verified, since the number of edges is one less than the number of vertices.

Now assume G is nonchordal, so that G is not a tree. Let T be a spanning tree of G. Then $\Delta_2(G) \subset \Delta_2(T)$, $\Delta_2(G) \neq \Delta_2(T)$. We will show that there is a Morse matching for $\Delta_2(G)$ with $e - n + 1$ critical cells in codimension 1, one below the top.

By the preceding lemma, the cut complex $\Delta_2(T)$ has a Morse matching $\mathcal{M}$ with no unmatched cells. The restriction $\mathcal{M}|_{\Delta_2(G)}$ to the subcomplex $\Delta_2(G)$ remains acyclic, because there are fewer cells, and hence we have a Morse matching for $\Delta_2(G)$. Unmatched faces in $\mathcal{M}|_{\Delta_2(G)}$ must be of the form $\sigma \setminus \{b\}$, where σ is a face in $\Delta_2(T) \setminus \Delta_2(G)$ and $(\sigma \setminus \{b\}, \sigma)$ was a matched pair under $\mathcal{M}$ in $\Delta_2(T)$. Since G is triangle-free, $\Delta_2(G)$ contains a complete skeleton of codimension 1, so $\sigma \setminus \{b\} \in \Delta_2(G)$ and $\sigma \in \Delta_2(T) \setminus \Delta_2(G)$ must be a face in the top dimension.

Hence in the restriction $\mathcal{M}|_{\Delta_2(G)}$, there is exactly one critical cell of codimension 1 corresponding to each facet of $\Delta_2(T) \setminus \Delta_2(G)$. Each of the $e - (n-1)$ edges not in the spanning tree yields exactly one such facet, giving $e - (n-1)$ critical cells, where e is the number of edges of G, and n is the number of vertices. It follows that $\Delta_2(G)$ is homotopy equivalent to a wedge of $(e - n + 1)$ spheres of dimension $(n-4)$. □

Theorem 6.4 is corroborated by the nonchordal triangle-free families studied in [2], namely complete bipartite graphs, cycles, and grid graphs. In addition, we have the following examples. Recall that the Kneser graph $K(m, r)$ [37] has one vertex for each r-subset of $[m]$, with an edge between subsets if and only if they are disjoint.

Corollary 6.6. *For each of the following graphs G, $\Delta_2(G)$ is a wedge of spheres in codimension 1.*

(1) *Let G be the Kneser graph $K(m, r)$, $m < 3r$. Then $\Delta_2(G)$ is a wedge of $\frac{1}{2}\binom{m}{r}\binom{m-r}{r} - \binom{m}{r} + 1$ spheres in dimension $\binom{m}{r} - 4$.*

(2) *In particular, for the Petersen graph* $G = K(5,2)$, $\Delta_2(G)$ *is a wedge of* 6 *spheres in dimension 6.*
(3) *Let* G *be the Cartesian product* $H_1 \times H_2$ *of two triangle-free connected graphs* H_1 *and* H_2, *with* $n_i = |V(H_i)| \geq 2$ *and* $m_i = |E(H_i)|$. *Then* $\Delta_2(G)$ *is a wedge of* $n_1m_2 + n_2m_1 - n_1n_2 + 1$ *spheres in dimension* $n_1 + n_2 - 4$.

Proof. The Kneser graph is connected and, by the pigeonhole principle, it is triangle-free if and only if $m < 3r$. It has $\binom{m}{r}$ vertices and $\frac{1}{2}\binom{m}{r}\binom{m-r}{r}$ edges, so the result follows.

One can check that Cartesian products preserve connectedness and the triangle-free property. □

Remark 6.7. The hypotheses in Theorem 6.4 are necessary. If G is not triangle-free, the conclusion is false, the counterexample being multipartite graphs with three or more parts. (These contain a 4-cycle without a chord as an induced subgraph.) The smallest counterexample is the multipartite graph with $r \geq 4$ parts each with 2 vertices: we will prove in Proposition 7.7 that $\Delta_2(G)$ has dimension $(2r-3)$, but is homotopy equivalent to one sphere $\mathbb{S}^{r-2}$ in dimension $(r-2)$.

Likewise, if G is triangle-free but not connected, the conclusion of Theorem 6.4 is again false. Sage computations show that the disjoint union $\Delta_2(C_m + C_n)$ has homology of rank 1 in the top dimension and rank 2 in codimension 1, for $m \geq n \geq 4$. Again, we already know homology is torsion-free and concentrated in the top two dimensions.

6.2. **Applications to computing Betti numbers of cut complexes.** In this subsection we show how to use Theorem 6.1 to compute the reduced Euler characteristic for cut complexes for some families of graphs. This in turn determines the Betti number if the homotopy type is a wedge of spheres in a single dimension, as is the case for the families studied in the next section.

Proposition 6.8. *Fix* $k \geq 2$ *and let* G_i *be a graph on* n_i *vertices,* $1 \leq i \leq r$, *such that for all* i, *the* k*-cut complex* $\Delta_k(G_i)$ *satisfies condition* (11) *of Theorem 6.1. Let* $N = \sum_{i=1}^{r} n_i$. *Then the disjoint union* $G_1 + \cdots + G_r$ *also satisfies the condition, and has Euler characteristic equal to*

$$(-1)^{N-k-1}\left(\binom{N-1}{k-1} - \sum_{i=1}^{r} |\mathcal{Z}_k(G_i)|\right).$$

If, in addition, $\Delta_k(G_1 + \cdots + G_r)$ *is nonvoid and shellable, its Betti number is given by* $(-1)^{N-k-1}$ *times the above formula.*

In the special case when each $\Delta_k(G_i)$ *is nonvoid and shellable with Betti number* β_i, $\Delta_k(G_1 + \cdots + G_r)$ *is shellable with Betti number given by*

$$\binom{N-1}{k-1} - \sum_{i=1}^{r}\binom{n_i - 1}{k-1} + \sum_{i=1}^{r}\beta_i.$$

Proof. This is clear from Theorem 6.1, since A is a connected subset of the disjoint union if and only if it is a connected subset of some G_i. In particular the last two statements are a consequence of the shellability condition, by the final statement of Theorem 6.1. Note that $|\mathcal{Z}_k(G_i)| = 0$ if $n_i < k$. □

For instance, for paths P_{n_i}, $1 \leq i \leq r$, where each P_{n_i} has n_i vertices, we can use Proposition 7.12 in the next section to compute that the rank of the unique nonvanishing homology group of $P_{n_1} + \cdots + P_{n_r}$ is given by the explicit formula $\binom{N-1}{k-1} - \sum_{i \in \{1,\ldots,r\}, n_i \geq k}(n_i - k + 1)$.

Next we consider the wedge of two graphs, as defined in Section 4.6. Examples are the wedge of two trees (also a tree), two paths end-to-end (also a path), two cycles (a figure 8), a cycle and a tree, a cycle and a path (balloon).

Proposition 6.9. *Let* G_i *be a graph on* n_i *vertices,* $i = 1, 2$. *Let* $k \geq 2$, $k \leq n_1 + n_2 - 3$. *Let* $G_1 \vee_{w_0} G_2$ *be the wedge of the two graphs at some vertex* w_0. *If* $G_i, i = 1, 2$ *satisfy the condition* (11), *then so does* $G_1 \vee_{w_0} G_2$. *Furthermore, its Euler characteristic* $\mu(G_1 \vee_{w_0} G_2)$ *satisfies*

$$(-1)^{n_1+n_2-k-2}\mu(G_1 \vee_{w_0} G_2) = \binom{n_1 + n_2 - 2}{k-1} - |\mathcal{Z}_k(G_1)|\delta_{k \leq n_1} - |\mathcal{Z}_k(G_2)|\delta_{k \leq n_2} - \bar{\xi}_k(G_1, G_2, w_0),$$

where δ_A *is the Kronecker delta, equal to 1 or 0 according as the statement* A *is true or false, and*

$$\bar{\xi}_k(G_1, G_2, w_0) = |\{(A_1, A_2) \in V(G_1) \times V(G_2) : w_0 \in A_i,\ A_i^c \in \mathcal{Z}_{r_i}(G_i),\ r_1 + r_2 = k+1,\ A_i \setminus \{w_0\} \neq \emptyset\}|.$$

In particular, if $\Delta_k(G_i)$ is shellable for $i = 1, 2$, then so is $\Delta_k(G_1 \vee_{w_0} G_2)$, with Betti number given by

$$(-1)^{n_1+n_2-k-2}\mu(G_1 \vee_{w_0} G_2) = \binom{n_1+n_2-2}{k-1} - \xi_k(G_1, G_2, w_0),$$

where

$$\xi_k(G_1, G_2, w_0) = |\{(A_1, A_2) \in V(G_1) \times V(G_2) : w_0 \in A_i,\ A_i^c \in \mathcal{Z}_{r_i}(G_i),\ r_1 + r_2 = k+1\}|.$$

Proof. Denote the vertex set of G_i by $V(G_i), i = 1, 2$. We assume $V(G_1) \cap V(G_2) = \{w_0\}$, the wedge point. The connected subsets of the wedge $G_1 \vee_{w_0} G_2$ fall into three categories (some of which may be empty):

(1) Connected subsets A_1 of $V(G_1)$ of size k such that $A_1 \cap V(G_2) \subseteq \{w_0\}$, provided $k \leq n_1$; thus $A_1^c \in \mathcal{Z}_k(G_1) \subset \mathcal{Z}_k(G_1 \vee_{w_0} G_2)$.
(2) Connected subsets A_2 of $V(G_2)$ of size k such that $A_2 \cap V(G_1) \subseteq \{w_0\}$, provided $k \leq n_2$; thus $A_2^c \in \mathcal{Z}_k(G_2) \subset \mathcal{Z}_k(G_1 \vee_{w_0} G_2)$.
(3) Connected sets in $G_1 \vee_{w_0} G_2$ of size k, of the form $A_1 \cup A_2$ where for $i = 1, 2$ we have $w_0 \in A_i \subset V(G_i)$ and $A_i \setminus \{w_0\} \neq \emptyset$; thus $A_i \cap V(G_{3-i}) = \{w_0\}$ and $|A_1 \cup A_2| = k = |A_1| + |A_2| - 1$. Note that $(A_1 \cup A_2)^c \in \mathcal{Z}_k(G_1 \vee_{w_0} G_2)$.

Clearly these account for all the connected subsets of size k of $G_1 \vee_{w_0} G_2$. We claim that condition (12) is satisfied for all three cases. For Case (3) it is clear that removing w_0 from $A_1 \cup A_2$ results in two disconnected components, and hence makes $((A_1 \cup A_2) \setminus \{w_0\}) \cup \{x\}$ disconnected for any $x \notin A_1 \cup A_2$.

For Case (1), let $X = A_1 \cup \{x\}$, $x \notin A_1$. If $x \in V(G_1)$ we are done since G_1 satisfies condition (12).

Otherwise $x \in V(G_2) \setminus V(G_1)$. If $w_0 \notin A_1$, then $A_1 \cup \{x\}$ is itself disconnected and we are done. If $w_0 \in A_1$, then clearly $(A_1 \setminus \{w_0\}) \cup \{x\}$ results in a disconnected set of size k. Case (2) follows similarly.

Since the number of vertices of the wedge is $n_1 + n_2 - 1$, the dimension of the cut complex $\Delta_k(G_1 \vee_{w_0} G_2)$ is $n_1 + n_2 - 2 - k$ and the expression for the Euler characteristic follows from Theorem 6.1. ☐

For instance, the wedge of two paths P_{n_1} and P_{n_2} wedged at a leaf is the path $P_{n_1+n_2-1}$. Assume $k \leq \min(n_1 - 2, n_2 - 2)$; one can check that in this case $\xi_k(P_{n_1}, P_{n_2}, w_0) = k - 2$, and hence the formula of Proposition 6.9 agrees with Proposition 7.12 in the next section.

We record the following computation that will be needed to write down a formula for the Betti numbers for wedges of cycles and paths.

Lemma 6.10. *Assume $k \geq 1$.*

(1) Let $n \geq 1$. Then $|\mathcal{Z}_k(P_n)| = n - k + 1$ if $k \leq n$ and $|\mathcal{Z}_k(P_n)|$ is zero otherwise.
(2) Let $n \geq 3$. Then $|\mathcal{Z}_k(C_n)| = n$ if $k < n$, $|\mathcal{Z}_n(C_n)| = 1$, and $|\mathcal{Z}_k(C_n)|$ is zero otherwise.
(3) Let $\xi_n(a)$ denote the number of connected subsets of size a in the cycle C_n passing through a fixed vertex w_0. Then $\xi_n(a) = a$ if $a < n$, $\xi_n(n) = 1$, and $\xi_n(a)$ is zero if $a > n$.
(4) Let $n_1 \geq 3$, $n_2 \geq 1$. Let w_0 be a leaf of P_{n_2} and let G be the wedge $G = C_{n_1} \vee_{w_0} P_{n_2}$. Then

$$\bar{\xi}_k(G) = \sum_{a=\max(2,k+1-n_2)}^{\min(n_1,k-1)} \xi_n(a) \tag{13}$$

Proof. Only Equation (13) requires comment. By definition of $\bar{\xi}_k(G)$, we are counting connected subsets A of size k, passing through w_0, consisting of $a \geq 2$ vertices in C_{n_1} through w_0 and $k - a + 1 \geq 2$ vertices in P_{n_2}, also passing through w_0. Thus $a \leq n_1$ and $k - a + 1 \leq n_2$. Since the set A is uniquely determined by its intersection with C_{n_1}, the expression follows. ☐

Now define $\xi_k(C_{n_1}, P_{n_2}, w_0)$ to be the sum

$$\xi_k(C_{n_1}, P_{n_2}, w_0) := \sum_{a=\max(1,k+1-n_2)}^{\min(n_1,k)} \xi_n(a).$$

Proposition 6.11. *Let $G = C_{n_1} \vee_{w_0} P_{n_2}$ where C_n is the cycle graph on n vertices and P_n is the path on n vertices, and w_0 is a leaf of the path. Then G has $n_1 + n_2 - 1$ vertices. If $k = n_1 + n_2 - 2$, $\Delta_k(G)$ is a* *single point $\{w_0\}$. Assume $2 \leq k \leq n_1+n_2-3$. If $k \neq n_1-1$, then G satisfies condition* (12) *and $\Delta_k(G)$ has reduced Euler characteristic $\mu(\Delta_k(G))$ where*

$$(-1)^{n_1+n_2-1-k}\mu(\Delta_k(G)) = \binom{n_1+n_2-2}{k-1} - |\mathcal{Z}_k(C_{n_1})| - (n_2-k+1)\delta_{k\leq n_2} - \bar{\xi}_k(C_{n_1}, P_{n_2}, w_0)$$
$$= \binom{n_1+n_2-2}{k-1} - \xi_k(C_{n_1}, P_{n_2}, w_0).$$

If $k \geq 3$, $k \neq n_1-1$, , $\Delta_k(G)$ is shellable, and has the homotopy type of a wedge of $(-1)^{n_1+n_2-1-k}\mu(\Delta_k(G))$ spheres in dimension n_1+n_2-2-k.

If $k=2$, $k \neq n_1-1$, we obtain $(-1)^{n_1+n_2-3}\mu(\Delta_k(G)) = -1$. Nonzero homology can only be in the top two dimensions.

Proof. First we observe that the k-cut complex of the cycle graph C_n satisfies the condition (12) provided $k \leq n-1$. If $k = n-1$, the condition fails because all subsets of C_n of size $n-1$ are connected.

By definition, $\bar{\xi}_k(C_{n_1}, P_{n_2}, w_0)$ counts connected k-subsets passing through the wedge point w_0, and these consist of a path with $k = a+1 \geq 2$ vertices in P_{n_2} starting at w_0, attached to a path in C_{n_1} through w_0 with $a \geq 2$ vertices. The precise counts are provided in Lemma 6.10.

When $3 \leq k$, we know from Proposition 6.2 that $\Delta_k(C_{n_1})$ and $\Delta_k(P_{n_2})$ satisfy condition (12), and hence so does $\Delta_k(G)$, by Proposition 6.9. The Betti number follows from the general expression for the reduced Euler characteristic. □

A completely analogous argument gives us the following for a wedge of two cycles.

Proposition 6.12. *Consider the wedge $G = C_{n_1} \vee_{w_0} C_{n_2}$ where C_n is the cycle graph on n vertices, and w_0 is a common vertex of both C_{n_1} and C_{n_2}. Then G has n_1+n_2-1 vertices. If $k = n_1+n_2-1$, $\Delta_k(G)$ is a point $\{w_0\}$. Assume $2 \leq k \leq n_1+n_2-3$. If $k \neq n_i-1, i=1,2$, then G satisfies condition* (12) *and $\Delta_k(G)$ has reduced Euler characteristic $\mu(\Delta_k(G))$ where*

$$(-1)^{n_1+n_2-1-k}\mu(\Delta_k(G)) = \binom{n_1+n_2-2}{k-1} - |\mathcal{Z}_k(C_{n_1})| - |\mathcal{Z}_k(C_{n_2})| - \bar{\xi}_k(C_{n_1}, C_{n_2}, w_0)$$
$$= \binom{n_1+n_2-2}{k-1} - \xi_k(C_{n_1}, C_{n_2}, w_0),$$

where

$$\xi_k(C_{n_1}, C_{n_2}, w_0) = \sum_{a=\max(1,1+k+1-n_2)}^{\min(n_1,k)} \xi_{n_1}(a)\xi_{n_2}(k+1-a).$$

If $k \geq 3$, $\Delta_k(G)$ is shellable, and has the homotopy type of a wedge of $(-1)^{n_1+n_2-1-k}\mu(\Delta_k(G))$ spheres in dimension n_1+n_2-2-k.

If $k=2$ we obtain $(-1)^{n_1+n_2-3}\mu(\Delta_k(G)) = -1$. Nonzero homology can only be in the top two dimensions.

Proof. We invoke Proposition 6.9 again, noting that the caveat $k \neq n_i-1$, $i=1,2$, still applies. It remains to observe that

$$\xi_k(C_{n_1}, C_{n_2}, w_0) = \sum_{a=\max(1,1+k+1-n_2)}^{\min(n_1,k)} \xi_{n_1}(a)\xi_{n_2}(k+1-a),$$

but this follows since we are counting connected subsets of size k passing through the wedge point w_0; these separate into two parts, one in each cycle, intersecting in w_0. Again Lemma 6.10 provides the precise counts. □

7. Families of graphs

7.1. **Complete Multipartite Graphs.** In this section we determine the homotopy type of the cut complex $\Delta_k(G)$ for various families of graphs G. With the exception of squared cycle graphs, the families we consider have the property that their homology is concentrated in a single dimension.

Let E_n denote the edgeless graph on n vertices. The complete bipartite graph $K_{m,n}$ is the join of two edgeless graphs E_m and E_n. In this section we will first determine completely the equivariant homotopy type for cut complexes of $K_{m,n}$, and then proceed to do the same for multipartite graphs. Note that the automorphism group of a complete multipartite graph acts simplicially on the cut complexes.

Let $G = K_{m,n}$ be the complete bipartite graph with $1 \le m \le n$. We point out for clarity that if we label the vertices with the sets $[m]$ and $m + [n] := \{m+1, \dots, m+n\}$, so that the edges are (i, j) for $1 \le i \le m$ and $m+1 \le j \le m+n$, then for each $k \ge 2$, the nonvoid cut complex $\Delta_k(G)$ has facets F of size $(m+n-k)$ where F must contain the vertex subset $[m]$ or the vertex subset $m + [n]$.

Theorem 7.1 was proved in [2, Theorem 3.3] by different methods. Here we deduce it from the structure theorems of Section 4. From Theorem 4.10, Theorem 4.16 and Proposition 3.6, we have the following, since $K_{m,n} = E_m * E_n$:

Theorem 7.1. *Let $1 \le m \le n$ and $2 \le k$.*

(1) $\Delta_k(K_{m,n})$ is shellable if and only if $m < k$. Furthermore, if $m < k \le n$, then $\Delta_k(K_{m,n})$ is contractible, and if $k > n$, the cut complex is void and hence shellable.

(2) If $k \le m \le n$, the $(m+n-k-1)$-dimensional complex $\Delta_k(K_{m,n})$ is homotopy equivalent to a wedge of $\binom{m-1}{k-1}\binom{n-1}{k-1}$ spheres of dimension $m+n-2k$.

Proof. From Theorem 4.10, $\Delta_k(K_{m,n})$ is shellable if and only if one of $\Delta_k(E_m)$, $\Delta_k(E_n)$ is shellable and the other is void. Since $1 \le m \le n$, the first statement follows from Proposition 3.6. The contractibility when $m < k \le n$ follows from Part (3) of Theorem 4.16.

If $k > n$ both $\Delta_k(E_m)$, $\Delta_k(E_n)$ are void and again the result follows from Theorem 4.10.

Now let $k \le m \le n$. From Proposition 3.6 and Equation (10) of Theorem 4.16, the cut complex is homotopy equivalent to

$$\operatorname{susp}(\Delta_k(E_m) * \Delta_k(E_n)) \simeq \bigvee_{\binom{m-1}{k-1}\binom{n-1}{k-1}} \mathbb{S}^0 * (\mathbb{S}^{m-k-1} * \mathbb{S}^{n-k-1}) \simeq \bigvee_{\binom{m-1}{k-1}\binom{n-1}{k-1}} \mathbb{S}^{m+n-2k}. \qquad \square$$

Thus bipartite graphs are an obstruction to shellability: if a graph H contains a bipartite graph $K_{m,n}$ such that $k = m = n$ or $k \le m < n$ as an induced subgraph, then $\Delta_k(H)$ is not shellable.

In order to describe the homology representation for the case $k \le m \le n$, we will need a well-known result of Solomon, as well as a result about group actions on products of posets.

Theorem 7.2 ([27], [28])**.** *The $\mathfrak{S}_p$-representation on the unique nonvanishing homology module of the order complex of $P(p, k)$ is the irreducible V_λ indexed by the integer partition $\lambda = (k, 1^{p-k})$.*

Note that a special case of this is the classical result that $\mathfrak{S}_p$ acts on the homology of the boundary of a $(p-1)$-simplex like the sign.

Theorem 7.3 ([32, Proposition 2.3])**.** *Let Q be any poset. Assume the nonzero reduced homology of Q is concentrated in dimension d. Then the r-fold product of Q has nonzero reduced homology concentrated in dimension $dr + 2r - 2$, and the symmetric group $\mathfrak{S}_r$ acts on this unique nonvanishing homology module like $(\mathrm{sgn}_{\mathfrak{S}_r})^d$, the dth power of the sign representation of $\mathfrak{S}_r$.*

By Theorem 7.2, the representation of $\mathfrak{S}_n$ on the unique nonvanishing reduced homology $\widetilde{H}_{n-k-1}(\Delta_k(E_n))$ is the irreducible $V_{(k,1^{n-k})}$, for $2 \le k \le n$.

Denote by $\mathfrak{S}_m[\mathfrak{S}_n]$ the wreath product group of $\mathfrak{S}_m$ with $\mathfrak{S}_n$, with $\mathfrak{S}_m$ acting on m copies of $\mathfrak{S}_n$, so that its order is $m!(n!)^m$. Let V, W be $\mathfrak{S}_m$- and $\mathfrak{S}_n$-modules, respectively. The wreath product module $V[W]$ is the tensor product of vector spaces $W^{\otimes m} \otimes V$, equipped with a canonical action of $\mathfrak{S}_m[\mathfrak{S}_n]$. See [16, Section 4.3].

We compute the equivariant homotopy type of the cut complex $\Delta_k(K_{m,n}) = \Delta_k(E_m * E_n)$ of the complete bipartite graph in the case $k \le m$ using Theorem 4.16. In fact that theorem immediately gives:

Theorem 7.4. *Let $G = K_{m,n}$ be the complete bipartite graph with $1 \le m \le n$. Let $2 \le k \le m$. Recall that $P(p, k)$ is the face lattice of the $(p-k-1)$-skeleton of a $(p-1)$-simplex. Then there is a poset map from the face lattice $\mathcal{L}(\Delta_k(G))$ of the cut complex to the product of posets $P(m, k) \times P(n, k)$, which induces a group-equivariant homotopy equivalence between the respective order complexes:*

$$\Delta(\overline{P(m,k) \times P(n,k)}) \simeq \operatorname{susp}(\Delta(\overline{P(m,k)}) * \Delta(\overline{P(n,k)})) \simeq \Delta_k(G).$$

Hence the $(m+n-k-1)$-dimensional cut complex $\Delta_k(G)$ has the homotopy type of a wedge of $\binom{m-1}{k-1}\binom{n-1}{k-1}$ spheres in dimension $m+n-2k$. In particular, it is not shellable.

If $m<n$, the automorphism group is $\mathfrak{S}_m \times \mathfrak{S}_n$, and the representation on the homology is the irreducible $V_{(k,1^{m-k})} \otimes V_{(k,1^{n-k})}$, indexed by the pair of integer partitions $((k,1^{m-k}),(k,1^{n-k}))$.

If $m=n$, the automorphism group is the wreath product $\mathfrak{S}_2[\mathfrak{S}_n]$, and the representation on the homology of the cut complex is $\Phi_{\mathfrak{S}_2}[V_{(k,1^{n-k})}]$, where the one-dimensional representation $\Phi_{\mathfrak{S}_2}$ equals sgn^{n-k-1}, and sgn is the sign representation of $\mathfrak{S}_2$.

Proof. The only statement that needs to be verified is the homology representation, since the homotopy type was determined in Theorem 7.1. (Alternatively, it follows immediately from Theorem 4.16 and Equation 5.) The homology representation is clear for the automorphism group $\mathfrak{S}_m \times \mathfrak{S}_n$, because the poset map induces an equivariant isomorphism in homology.

Note that when $k=m<n$, $\Delta_m(E_m)$ is the empty complex, and $\Delta_k(G) \sim \mathrm{susp}\,\Delta(\overline{P(n,k)})$.

When $m=n$, the full automorphism group is the wreath product group $\mathfrak{S}_2[\mathfrak{S}_n]$ of order $2(n!)^2$ (with $\mathfrak{S}_2$ acting on two copies of $\mathfrak{S}_n$), and the action of $\mathfrak{S}_2$ is given as stated, by Theorem 7.3. □

The equivariant homotopy and homology representation determined in Theorem 7.4 bear a striking resemblance to that of a complex studied by Linusson, Shareshian and Welker [20]. In that paper, the authors consider the following simplicial complex of graphs. For positive integers k,m,n, they define $B_k(m,n)$ to be the simplicial complex consisting of (edge-sets of) all bipartite graphs, with bipartite vertex set $[m]\cup[n]$, that do not contain a matching of size k. Clearly we may assume $k \le m \le n$. Their result is the following.

Theorem 7.5 ([21, Theorem 1.4])**.** *The complex $B_k(m,n)$ has the homotopy type of a wedge of $\binom{m-1}{k-1}\binom{n-1}{k-1}$ spheres of dimension $2k-3$.*

They then make the conjecture [21, Conjecture 1.15] that (implicitly, for $k<m<n$) the representation of the group $\mathfrak{S}_m \times \mathfrak{S}_n$ on the unique nonvanishing homology is, in our notation, $(\mathrm{sgn}_{\mathfrak{S}_m} \otimes V_{(k,1^{m-k})}) \otimes (\mathrm{sgn}_{\mathfrak{S}_n} \otimes V_{(k,1^{n-k})})$. It would be interesting to know if there is a topological connection between their complex and ours.

We turn now to multipartite graphs, where the results are similar. Let G be the complete multipartite graph $K_{m_1,\dots,m_r}$, $r\ge 3$. We may assume $m_1 \le \cdots \le m_{r-1} \le m_r$. Then $G = E_{m_1} * \cdots * E_{m_r}$, and by associativity, $G = G_1 * G_2$ where $G_1 = K_{m_1,\dots,m_t}$ and $G_2 = K_{m_{t+1},\dots,m_r}$ for any t, $1 \le t \le r-1$.

Theorem 7.6. *Let $m_1 \le \cdots \le m_{r-1} \le m_r, r \ge 3$. Let G be the complete multipartite graph $K_{m_1,\dots,m_r}$. Then*

(1) If $m_r < k$, then $\Delta_k(G)$ is void and hence shellable.
(2) If $m_{r-1} < k \le m_r$, then $\Delta_k(G)$ is contractible and shellable.
(3) If $m_1 < k \le m_{r-1}$, then $\Delta_k(G)$ is contractible and not shellable.
(4) If $k \le m_1$, then $\Delta_k(G)$ is not shellable.

Proof. Item (1) is clear.

So assume $k \le m_r$, and consider Item (2). Take $G_1 = K_{m_1,\dots,m_{r-2}}$ and $G_2 = K_{m_{r-1},m_r}$, so $G = G_1 * G_2$. If $k \le m_{r-1}$, Theorem 7.1 tells us that the cut complex $\Delta_k(G_2)$ of the bipartite graph is not shellable, and hence Theorem 4.10 shows that $\Delta_k(G)$ is not shellable. On the other hand, if $m_{r-1} < k \le m_r$, then $\Delta_k(G_1)$ is void and $\Delta_k(G_2)$ is shellable and not void, so by Theorem 4.16 $\Delta_k(G)$ is contractible and shellable.

Next consider Item (3). If $m_t < k \le m_{t+1}$ for $1 \le t \le r-2$, now write $G = G_1 * G_2$ with $G_1 = K_{m_1,\dots,m_t}$ and $G_2 = K_{m_{t+1},\dots,m_r}$. Since $\Delta_k(G_1)$ is void, and $\Delta_k(G_2)$ is not void, Part (2) of Theorem 4.16 says that $\Delta_k(G)$ is contractible (but by above not shellable).

Finally, for Item (4), if $k \le m_1$, let $G_1 = K_{m_1,m_2}$ and $G_2 = K_{m_3,\dots,m_r}$, so that $G = G_1 * G_2$. Then Part (1) of Theorem 7.1 says that $\Delta_k(G_1)$ is not shellable, and hence Theorem 4.10 shows that $\Delta_k(G)$ is not shellable. □

For complete bipartite graphs, Theorem 7.4 asserts that $\Delta_n(K_{n,n}) \simeq \mathbb{S}^0$. The next result generalizes this to complete multipartite graphs.

Proposition 7.7. *Let $G = K_{\underbrace{n,n,\ldots,n}_{r}}$ be the complete multipartite graph whose vertex set is a disjoint union of r sets of size n, $r \geq 2$. Then the $(nr-n-1)$-dimensional cut complex $\Delta_n(G)$ is homotopy equivalent to a single sphere $\mathbb{S}^{r-2}$ in dimension $r-2$. The automorphism group is the wreath product $\mathfrak{S}_r[\mathfrak{S}_n]$, with $\mathfrak{S}_r$ permuting the r copies of $\mathfrak{S}_n$. The one-dimensional representation afforded by the unique nonzero homology module in degree $r-2$ is*

$$\mathrm{sgn}_{\mathfrak{S}_r}[1_{\mathfrak{S}_n}].$$

Proof. We have $G = E_n * \cdots * E_n$, and $\Delta_n(E_n) = \{\emptyset\} \simeq \mathbb{S}^{-1}$. Iterating Theorem 4.16 gives the equivariant isomorphism of face lattices

$$\mathcal{L}(\Delta_n(G)) \cong \underbrace{\mathcal{L}(\Delta_n(E_n)) \times \cdots \times \mathcal{L}(\Delta_n(E_n))}_{r};$$

using the fact that $\mathbb{S}^a * \mathbb{S}^b = \mathbb{S}^{a+b+1}$, this (with Equation 6) gives the homotopy equivalence

$$\Delta_n(G) \simeq \underbrace{\mathbb{S}^0 * \cdots * \mathbb{S}^0}_{r-1} * \underbrace{\Delta_n(E_n) * \cdots * \Delta_n(E_n)}_{r} \simeq \mathbb{S}^{r-2} * \mathbb{S}^{-1} \simeq \mathbb{S}^{r-2}.$$

The equivariant isomorphism of face lattices makes the homology of $\Delta_n(G)$ isomorphic to the r-fold tensor product of $\widetilde{H}_{-1}(\Delta_n(E_n))$. Clearly $\mathfrak{S}_n$ acts trivially on each factor, and $\mathfrak{S}_r$ acts like $(\mathrm{sgn})^{-1} = \mathrm{sgn}$ by Theorem 7.3. This finishes the proof. □

For the remainder of this section, we record the following notational conventions, which will be helpful in describing the homology representations. Assume $M = \{m_1, \ldots, m_r\}$ is the multiset consisting of r_i indices equal to i. The automorphism group of the complete multipartite graph $G = K_{m_1,\ldots,m_r}$ is the group $\mathfrak{G}_M$, where $\mathfrak{G}_M$ is the product of wreath products $\mathfrak{G}_M = \times_i \mathfrak{S}_{r_i}[\mathfrak{S}_i]$.

The next case is a straightforward generalization of the bipartite case, Theorem 7.4. Recall that $P(n,m)$ is the truncated Boolean lattice $B_n^{\leq n-m} \cup \{\widehat{1}\}$, and it is the face lattice of the $(n-m-1)$-skeleton of an $(n-1)$-simplex. From Part (1) of Theorem 4.16 and Theorem 7.3, we obtain:

Theorem 7.8. *Let $2 \leq k < m_1 \leq m_2 \leq \cdots \leq m_r$, and let G be the complete multipartite graph $G = K_{m_1,\cdots,m_r}$. There is a group-equivariant homotopy equivalence*

$$\Delta_k(G) \simeq \Delta(\overline{P(m_1,k) \times \cdots \times P(m_r,k)}).$$

Hence the $((\sum_{i=1}^r m_i) - k - 1)$-dimensional cut complex $\Delta_k(G)$ has the homotopy type of a wedge of $\prod_{i=1}^r \binom{m_i-1}{k-1}$ spheres in dimension $\sum_{i=1}^r (m_i - k) + (r-2)$.

The homology representation of the automorphism group $\mathfrak{G}_M$ is then the tensor product of wreath product representations

$$\bigotimes_{i \geq k+1} \Phi_{\mathfrak{S}_{r_i}}[V_{(k,1^{i-k})}],$$

where $\Phi_{\mathfrak{S}_{r_i}}$ is the one-dimensional representation of $\mathfrak{S}_{r_i}$ equal to $(\mathrm{sgn}\,)^{i-k-1}$.

We end this section with the last case of multipartite graphs that needs to be examined. If $m_1 = \ldots = m_t = m$ and $m < m_{t+1} \leq \cdots \leq m_{t+r}$, for convenience we write $K_{m^t, m_{t+1},\ldots,m_{t+r}}$ for the complete multipartite graph $K_{\underbrace{m,\ldots,m}_{t},m_{t+1},\ldots,m_{t+r}}$. Once again the theorem follows immediately from Part (1) of Theorem 4.16 and Theorem 7.3, invoking in addition Proposition 7.7 and Theorem 7.8. The case $r = 1$ is listed separately only for clarity.

Theorem 7.9. *Let $m = m_1 = \cdots = m_t < m_{t+1} \leq \cdots \leq m_{t+r}$, where $t \geq 1$.*

(1) Let $r \geq 2$. Then we have the equivariant homotopy equivalence of cut complexes

$$\Delta_m(K_{m^t, m_{t+1},\ldots,m_{t+r}}) \simeq \mathrm{susp}\,(\,\Delta_m(K_{m^t}) * \Delta_m(K_{m_{t+1},\ldots,m_{t+r}})\,)$$

which in turn is homotopy equivalent to $\mathbb{S}^{t-1} * \Delta_m(K_{m_{t+1},\ldots,m_{t+r}})$. Let $N = \sum_{i=1}^r m_{t+i}$. The $((t-1)m + N - 1)$-dimensional cut complex $\Delta_m(K_{m_{t+1},\ldots,m_{t+r}})$ is homotopy equivalent to a wedge *of* $\prod_{i=1}^{r}\binom{m_i-1}{m-1}$ *spheres in dimension* $t+N-rm+(r-2)$*. The homology representation of the automorphism group* $\mathfrak{S}_t[\mathfrak{S}_m]\times\mathfrak{G}_M$*, where* $\mathfrak{G}_M$ *is the automorphism group of* $K_{m_{t+1},\ldots,m_{t+r}}$*, is*

$$V_{(1^t)}[V_{(m)}]\otimes\bigotimes_{i\geq m+1}\Phi_{\mathfrak{S}_{\mathrm{r_i}}}[V_{(m,1^{i-m})}].$$

(2) Let $r=1$*. Then we have the equivariant homotopy equivalence of complexes*

$$\Delta_m(K_{m^t,m_{t+1}})\simeq\operatorname{susp}(\Delta_m(K_{m^t})*\Delta(P(m_{t+1},m))),$$

which in turn is homotopy equivalent to $\mathbb{S}^{t-1}*\Delta(P(m_{t+1},m))$*. The order complex* $\Delta(P(m_{t+1},m))$ *is the barycentric subdivision of the* $(m_{t+1}-m-1)$*-skeleton of an* $(m_{t+1}-1)$*-dimensional simplex. The* $((t-1)m+m_{t+1}-1)$*-dimensional cut complex* $\Delta_m(K_{m^t,m_{t+1}})$ *is homotopy equivalent to a wedge of* $\binom{m_{t+1}-1}{m-1}$ *spheres in dimension* $t+m_{t+1}-m-1$*. The homology representation of the automorphism group* $\mathfrak{S}_t[\mathfrak{S}_m]\times\mathfrak{S}_{m_{t+1}}$ *is*

$$V_{(1^t)}[V_{(m)}]\otimes V_{(m,1^{m_{t+1}-m})}.$$

7.2. **Cycle Graphs.** For the cycle graph C_n, we may assume $n\geq 4$ and $2\leq k\leq n-2$. If $n=4$, the cut complex $\Delta_2(C_4)$ is 1-dimensional with two facets $\{1,3\}$ and $\{2,4\}$, and is homotopy equivalent to the 0-sphere $\mathbb{S}^0$; it is thus NOT shellable. In fact $\Delta_2(C_4)=\Delta_2(K_{2,2})$.

Recall from Figure 3 that the 2-dimensional cut complex $\Delta_2(C_5)$ is in fact a Möbius strip, and hence it is homotopy equivalent to the one-sphere $\mathbb{S}^1$. This holds more generally for the $(n-3)$-dimensional cut complex $\Delta_2(C_n)$: it was shown in [2] that:

Proposition 7.10 ([2, Theorem 3.9])**.** *Let* C_n *be a cycle graph,* $n\geq 5$*. Then the* $(n-3)$*-dimensional cut complex* $\Delta_2(C_n)$ *is homotopy equivalent to the sphere* $\mathbb{S}^{n-4}$*.*

For $k\geq 3$, we have the following:

Theorem 7.11 (Dane Miyata)**.** *For all* $n-1\geq k\geq 3$*, the* k*-cut complex of the cycle graph* C_n *on* n *vertices,* $\Delta\coloneqq\Delta_k(C_n)$*, is shellable.*

Proof. When $k=n-1$, $\Delta_k(G)$ is void and thus trivially shellable, so assume $k\leq n-2$.

Fix $k>2$ and n, and let $s=n-k$. Order the vertices of C_n by going clockwise around the cycle, starting at some initial vertex and, for simplicity, identify the vertex set of C_n with $[n]\coloneqq\{1,\ldots,n\}$ with the usual order. The facets of Δ are exactly the size s separating sets of C_n. We will identify each facet of Δ with the unique increasing sequence of its vertices. We claim that the lexicographic ordering on the facets of Δ gives a shelling order for Δ.

Let A and B be facets of Δ such that $A<B$ in the lexicographic order. Suppose A is given by the sequence $a_1,\ldots,a_s$ of vertices of C_n and B is given by the sequence $b_1,\ldots,b_s$ of vertices of C_n. Since $A<B$, there is a unique index i, such that $a_i\lneq b_i$, and $a_j=b_j$ for all $j<i$. Furthermore, there is a minimum index ℓ such that $b_\ell\notin A$. Let $C=(B\cup\{a_i\})\setminus\{b_\ell\}$. In other words, C is given by the sequence

$$b_1,\ldots,b_{i-1},a_i,b_i,\ldots,\widehat{b_\ell},\ldots b_s.$$

(Here the hat means we omit that element from the sequence). Intuitively, C is obtained from B by taking the smallest element of B that is not in A and replacing it with the smallest element of A that is not in B. Now there are two cases; either C gives a separating set or it does not.

In the case where C gives a separating set, C is a facet of Δ. Notice $C<B$ in the lexicographic order, since the ith term in C is a_i which is less than b_i. Furthermore, $C\cap B=B\setminus\{b_\ell\}$ and $b_\ell\notin A\cap B$, which means $A\cap B\subseteq B\cap C$ and $|B\cap C|=s-1$. Thus, in this case, the shelling order condition is satisfied.

In the case where C is not a separating set, it must either be an interval, or the complement of an interval. If C is an interval, we conclude that $b_\ell=b_s$, meaning b_ℓ is the largest element of B. Otherwise $b_s>b_\ell>a_i$, and because C contains a_i and b_s but not b_ℓ, it would not be an interval. Since $b_\ell=b_s$ is the smallest element of $B\setminus A$, we conclude $b_1,\ldots,b_{s-1}$ are all in A. In our construction of C, we took B and replaced $b_\ell=b_s$ with a_i and so all elements of C are in A, which means $C=A$. This is a contradiction because A is a separating set but C is not. Therefore, C is not an interval, so it must be the complement of an interval, namely,

$$C=[n]\setminus[p,q]$$

where $[p,q]$ is some interval of size k. In this case, we can immediately deduce the following from the construction of C:

$$B \cap [p,q] = \{b_\ell\} \qquad \text{and} \qquad p > a_i.$$

Since A is a facet, it is distinct from C so there exists some $a' \in A$ such that $a' \in [p,q]$. We know that $a' \notin B$ since it cannot be equal to b_ℓ and $B \cap [p,q] = \{b_\ell\}$. Thus, B is missing at least two elements of A, namely a_i and a'. We know $|B| = |A|$ so B must contain an element b' distinct from b_ℓ such that $b' \notin A$. We defined b_ℓ to be the minimum element of $B \setminus A$ so $b' > b_\ell$. This implies $b' > q$ as $B \cap [p,q] = \{b_\ell\}$. In particular, we have shown that b' is greater than all elements of $[n] \setminus B$ since $[n] \setminus B = a_i \cup [p,q] \setminus b_\ell$.

Note that $B = [1, a_i - 1] \cup [a_i + 1, p - 1] \cup \{b_\ell\} \cup [q+1, n]$ and so $[n] \setminus B = \{a_i\} \cup [p, b_\ell - 1] \cup [b_\ell + 1, q]$.

Now recall that $k \geq 3$, which means $[n] \setminus B$ has cardinality at least 3. Let x, y, z be three distinct elements of $[n] \setminus B$ and without loss of generality assume $x < y < z$. Now we have four cases, $b_\ell < x$, $x < b_\ell < y$, $y < b_\ell < z$, and $b_\ell > z$.

Case 1. ($b_\ell < x$) In this case, let $D = \big(B \cup \{y\}\big) \setminus \{b'\}$.
Notice D must necessarily be a separating set since $b_\ell < x < y < b'$ and $b_\ell, y \in D$, but $x, b' \notin D$.

Case 2. ($x < b_\ell < y$) In this case, let $D = \big(B \cup \{z\}\big) \setminus \{b'\}$.
Notice D must necessarily be a separating set since $b_\ell < y < z < b'$ and $b_\ell, z \in D$, but $y, b' \notin D$.

Case 3. ($y < b_\ell < z$) In this case, let $D = \big(B \cup \{x\}\big) \setminus \{b'\}$.
Notice D must necessarily be a separating set since $x < y < b_\ell < b'$ and $b_\ell, x \in D$, but $y, b' \notin D$.

Case 4. ($z < b_\ell$) In this case, let $D = \big(B \cup \{y\}\big) \setminus \{b'\}$.
Notice D must necessarily be a separating set since $y < z < b_\ell < b'$ and $b_\ell, y \in D$, but $z, b' \notin D$.

In each case, we see that D is a facet of Δ and since it is constructed by taking B and replacing b' with a smaller element, we have that $D < B$ in the lexicographic order. Furthermore, the only element of $B \setminus D$ is b' so $|B \cap D| = s - 1$. Finally, as $b' \notin A$, we see that $B \cap A \subseteq B \cap D$ and so the shelling order condition is satisfied. □

The results of Section 6 can be applied to compute Betti numbers for cut complexes of trees and cycles. These belong to families of graphs whose cut complexes satisfy the condition (11) of Theorem 6.1. Recall from that theorem that for a forest $\mathcal{F}_n$ with n vertices, $|\mathcal{Z}_k(\mathcal{F}_n)|$ is the number of subgraphs of $\mathcal{F}_n$ that are trees.

In this case $|\mathcal{Z}_k(\mathcal{F}_n)| = n - c$, where c is the number of connected components of the forest $\mathcal{F}_n$. Also if the tree is a path P_n, then $|\mathcal{Z}_k(P_n)| = (n - k + 1)$.

Proposition 7.12. *Let $\mathcal{F}_n$ be a forest on n vertices. Then $\Delta_k(\mathcal{F}_n)$ is homotopy equivalent to a wedge of $\left(\binom{n-1}{k-1} - |\mathcal{Z}_k(\mathcal{F}_n)|\right)$ spheres in dimension $n-k-1$ if this number is nonzero, and contractible otherwise. In particular, if $\mathcal{F}_n$ is a tree T_n on n vertices, we have*

$$\Delta_k(T_n) \simeq \begin{cases} \{\text{a point}\}, & k = 2, \\ \bigvee_{\binom{n-1}{k-1} - |\mathcal{Z}_k(T_n)|} \mathbb{S}^{n-k-1}, & k \geq 3. \end{cases}$$

Proof. It is immediate from Proposition 6.2 that condition (11) in Part (3) of Theorem 6.1 is satisfied, since forests are acyclic. Since forests are shellable by Corollary 4.21 and Theorem 4.8, the result follows. □

Observe that when $k = 2$, $|\mathcal{Z}_k(\mathcal{F}_n)|$ is just the number of edges in the forest, namely, $n - c$ if the forest has c connected components, and the Betti number is thus $c - 1$. In the special case when the forest is a single tree, this makes the reduced Euler characteristic zero, and so the 2-cut complex of a tree is contractible, recovering the result of Corollary 6.3.

For the cycle graph C_n, we may assume $n \geq 4$ and $2 \leq k \leq n - 2$. If $n = 4$, the one-dimensional cut complex $\Delta_2(C_4)$ has two facets $\{1, 3\}$ and $\{2, 4\}$, and is homotopy equivalent to the 0-sphere $\mathbb{S}^0$; it is thus not shellable.

The cut complex $\Delta_k(C_n)$ has dimension $n - k - 1$. When $k = 2$, we already know that the cut complex $\Delta_2(C_n)$ has the homotopy type of one sphere in codimension 1, i.e., in dimension $n - 4$.

Proposition 7.13. *Let C_n be the cycle graph on $n \geq 5$ vertices. For $n-2 \geq k \geq 3$, the shellable cut complex $\Delta_k(C_n)$ is homotopy equivalent to a wedge of $\binom{n-1}{k-1} - n$ spheres in (top) dimension $n-k-1$.*

Proof. Proposition 6.2 shows that condition (11) in Part (3) of Theorem 6.1 is satisfied, since C_n has no cycles of length less than n. The trees on k vertices may be described as the intervals $[i, i+k-1], 1 \leq i \leq n$, but now with arithmetic modulo n, so that there are n such trees in all, i.e., $|\mathcal{Z}_k(C_n)| = n$.

Since $\Delta_k(C_n)$ is shellable for $k \geq 3$ by Theorem 7.11, the statement follows. For the cut complex $\Delta_2(C_n)$, we already know that it has the homotopy type of one sphere in dimension $n-4$, by Proposition 7.10; this is confirmed by the value (-1) of the reduced Euler characteristic $\mu(\Delta_2(C_n))$. □

Corollary 7.14. *The subposet $P(n,k) \backslash \mathcal{Z}_k(C_n)$, where $\mathcal{Z}_k(C_n)$ is the antichain of n complements of trees on k vertices $[i, i+k-1]$ (modulo n), $1 \leq i \leq n$, as above, is homotopy equivalent to a wedge of $\binom{n-1}{k-1} - n$ spheres in the top dimension $n-k-1$ if $k \geq 3$, and to a single sphere in dimension one less than the top if $k = 2$.*

The Euler characteristic computation of Proposition 7.13 can be made equivariant. The cycle graph C_n and its cut complexes are invariant under the action of the cyclic subgroup $\mathfrak{C}_n$ of the symmetric group $\mathfrak{S}_n$, generated by the n-cycle $(1, 2, \ldots, n)$. We have the following result.

Theorem 7.15. *The cyclic group $\mathfrak{C}_n$ of order n acts on the unique nonvanishing homology of the cut complex $\Delta_k(C_n)$, $2 \leq k \leq n-2$, as follows:*

If $k = 2$, the action of $\mathfrak{C}_n$ is the 1-dimensional module given by

$$\begin{cases} \text{the trivial representation}, & n \text{ odd}, \\ \text{the sign representation}, & n \text{ even}. \end{cases}$$

If $k \geq 3$, the action of $\mathfrak{C}_n$ is given by $V_{(k,1^{n-k})}\downarrow_{\mathfrak{C}_n}^{\mathfrak{S}_n} - \mathrm{Reg}_{\mathfrak{C}_n}$, where $V_{(k,1^{n-k})}$ is the $\mathfrak{S}_n$-irreducible indexed by the partition $(k, 1^{n-k})$, the down arrow indicates restriction to the subgroup $\mathfrak{C}_n$, and $\mathrm{Reg}_{\mathfrak{C}_n}$ is the regular representation of $\mathfrak{C}_n$.

Proof. Recall from Theorem7.2 that the homology module of the subposet $P(n,k)$ affords the representation $V_{(k,1^{n-k})}$ of $\mathfrak{S}_n$. In [33], tools were developed to compute the group action on the Lefschetz module of the order complex of a poset, in particular when an antichain is deleted. Applying [33, Theorem 1.10] to Corollary 7.14 and the antichain $\mathcal{Z}_k(C_n)$ consisting of complements of trees of size k, we obtain the following $\mathfrak{C}_n$-equivariant versions of Equation (7):

$$\widetilde{H}_{n-4}(\Delta_2(C_n)) = \bigoplus_{\substack{\hat{0}<x<\hat{1} \\ x \in \mathcal{Z}_k(C_n)}} \widetilde{H}(\hat{0}, x) \otimes \widetilde{H}(x, \hat{1}) - V_{(2,1^{n-2})}\downarrow_{\mathfrak{C}_n}^{\mathfrak{S}_n}, \tag{14}$$

and for $k \geq 3$,

$$\widetilde{H}_{n-k-1}(\Delta_k(C_n)) = V_{(k,1^{n-k})}\downarrow_{\mathfrak{C}_n}^{\mathfrak{S}_n} - \bigoplus_{\substack{\hat{0}<x<\hat{1} \\ x \in \mathcal{Z}_k(C_n)}} \widetilde{H}(\hat{0}, x) \otimes \widetilde{H}(x, \hat{1}), \ k \geq 3. \tag{15}$$

We have omitted the homology degrees in the right-hand side since the intervals involved are Boolean lattices, and it is clear where the unique nonvanishing homology occurs.

To see that these decompositions are indeed group-equivariant, we observe that the n elements of the antichain $\mathcal{Z}_k(C_n)$ are transitively permuted by the cyclic group $\mathfrak{C}_n$. This is also true of the one-dimensional homology modules $\widetilde{H}(\hat{0}, x), x \in \mathcal{Z}_k(C_n)$. Since the stabilizer of an $x \in \mathcal{Z}_k(C_n)$ is clearly the trivial group, the cyclic group transitively permutes the summands $\widetilde{H}(\hat{0}, x) \otimes \widetilde{H}(x, \hat{1})$, and hence acts like the regular representation on the direct sum $\oplus_{\substack{\hat{0}<x<\hat{1} \\ x \in \mathcal{Z}_k(C_n)}} \widetilde{H}(\hat{0}, x) \otimes \widetilde{H}(x, \hat{1})$. Note also that the homology of $(x, \hat{1})$ is the one-dimensional trivial module for all $x \in \mathcal{Z}_k(C_n)$. Hence Equation (15) becomes

$$\widetilde{H}_{n-k-1}(\Delta_2(C_n)) = V_{(k,1^{n-k})}\downarrow_{\mathfrak{C}_n}^{\mathfrak{S}_n} - \mathrm{Reg}_{\mathfrak{C}_n},$$

as claimed.

In the case $k = 2$, Equation (14) becomes $\widetilde{H}_{n-4}(\Delta_2(C_n)) = \mathrm{Reg}_{\mathfrak{C}_n} - V_{(2,1^{n-2})}\downarrow_{\mathfrak{C}_n}^{\mathfrak{S}_n}$.

We can determine the restriction of the $\mathfrak{S}_n$-irreducible $V_{(2,1^{n-2})}$ to $\mathfrak{C}_n$ precisely as follows.

The permutation action of the cyclic group $\mathfrak{C}_n$ on the set $[n]$ is the restriction of the natural action of $\mathfrak{S}_n$ on $[n]$. It is well known that the latter decomposes into two irreducible components, the trivial representation and the *reflection* representation, see e.g., [31, Example 7.18.8]: $V_{(n)} \oplus V_{(n-1,1)}$.

Since $V_{(n)}$ is the one-dimensional trivial representation of $\mathfrak{S}_n$, we obtain

$$V_{(n-1,1)}\downarrow_{\mathfrak{C}_n}^{\mathfrak{S}_n} = \mathrm{Reg}_{\mathfrak{C}_n} - 1_{\mathfrak{C}_n}.$$

Now observe that $V_{(2,1^{n-2})}$ is the sign representation of $\mathfrak{S}_n$ tensored with $V_{(n-1,1)}$. It is easy to see that the sign representation restricted to $\mathfrak{C}_n$ is the trivial representation $1_{\mathfrak{C}_n}$ if n is odd, and the sign representation otherwise. The regular representation is invariant with respect to tensoring with the sign, and the result follows. $\square$

7.3. **Prism over a clique.**

Definition 7.16. The *prism over a clique* is the graph G_n with vertex set $\{1^+, \ldots, n^+, 1^-, \ldots, n^-\}$ for a given integer n, and an edge between i^+ and j^+, between i^- and j^-, and between i^+ and i^-, for every $i, j \in \{1, \ldots, n\}$. Note that this is the Cartesian product, $G_n = K_n \times K_2$.

Remark 7.17. If $n < k$, then $\Delta_k(G_n)$ is the void complex, since there are no separating sets of size $2n - k$, which is less than n. So $\Delta_k(G_n)$ is shellable.

We determine the homotopy type of $\Delta_k(G_n)$ when $n \geq k$ precisely in Theorem 7.18 below. The case $k = 2$ was also proved in [2].

Theorem 7.18. *Let $n \geq k \geq 2$. The $(2n - k - 1)$-dimensional cut complex $\Delta_k(G_n)$ has homotopy type*

$$\Delta_k(G_n) \simeq \bigvee_{\binom{n-1}{k-1}} \mathbb{S}^{2n-k-2},$$

one lower than full dimension. Thus for $n \geq k$, $\Delta_k(G_n)$ is not shellable.

Proof. We prove this theorem by using discrete Morse theory, precisely by constructing a sequence of element matchings [2, Appendix, Theorems 7.2, 7.5]. Let $V = \{1^+, \ldots, n^+, 1^-, \ldots, n^-\}$ be the vertex set, $V^+ = \{1^+, \ldots, n^+\}$ and $V^- = \{1^-, \ldots, n^-\}$. For an arbitrary $X \subset V$ let $X^+ = X \cap V^+$, and $X^- = X \cap V^-$. We observe that if $X \subset V^+$, $X \subset V^-$, or if there is an i such that $\{i^+, i^-\} \subset X$, then X is connected. Pairs $\{i^+, i^-\}$ will be called *columns*. Thus the complement of a disconnected set (in particular, a facet of $\Delta_k(G_n)$) must contain at least one element of each column. A set of vertices X contains a disconnected set of size m if and only if there are m columns indexed by $i_1 < i_2 < \cdots < i_m$ such that for each $j \in \{1, \ldots, m\}$, $X \cap \{i_j^+, i_j^-\} \neq \emptyset$, $X \cap \{i_1^+, \ldots, i_m^+\} \neq \emptyset$, and $X \cap \{i_1^-, \ldots, i_m^-\} \neq \emptyset$. We will denote by $d(X)$ the maximal cardinality of a disconnected set contained in X.

First, we perform an element matching $\mathcal{M}_{1^+}$ using vertex 1^+. Faces $\sigma \in \Delta_k(G_n)$ that remain unmatched satisfy $1^+ \notin \sigma$ and $\sigma \cup \{1^+\} \notin \Delta_k(G_n)$. Then we perform an element matching $\mathcal{M}_{1^-}$ using vertex 1^-. There are two possible types of unmatched faces after the sequence $\mathcal{M}_{1^+}$ followed by $\mathcal{M}_{1^-}$:

(1) Faces $\sigma \in \Delta_k(G_n)$ that satisfy:

$$1^+, 1^- \notin \sigma, \ \sigma \cup \{1^+\} \notin \Delta_k(G_n), \text{ and } \sigma \cup \{1^-\} \notin \Delta_k(G_n).$$

(2) Faces $\sigma \cup \{1^-\} \in \Delta_k(G_n)$ that satisfy:

$$1^+, 1^- \notin \sigma, \ \sigma \cup \{1^+\} \in \Delta_k(G_n), \text{ and } \sigma \cup \{1^+, 1^-\} \notin \Delta_k(G_n).$$

To see that there are actually no faces of type (1), note that every face σ is in a facet, and every facet has to contain either 1^+ or 1^-. So either $\sigma \cup \{1^+\}$ or $\sigma \cup \{1^-\}$ is in $\Delta_k(G_n)$. Therefore, there are only unmatched faces of type (2). Denote by K the set of all unmatched faces after the sequence of matchings $\mathcal{M}_{1^+}$ followed by $\mathcal{M}_{1^-}$.

Let $\sigma \cup \{1^-\}$ be an arbitrary face in K. Conditions from (2) imply that $d(V \setminus (\sigma \cup \{1^+\})) \geq k$, $d(V \setminus (\sigma \cup \{1^-\})) \geq k$, while $d(V \setminus (\sigma \cup \{1^+, 1^-\})) \leq k - 1$. Consequently,

$$d(V \setminus (\sigma \cup \{1^+\})) = k = d(V \setminus (\sigma \cup \{1^-\})), \text{ and } d(V \setminus (\sigma \cup \{1^+, 1^-\})) = k - 1.$$

Because of its repeated use, we write

$$X_\sigma = V \setminus (\sigma \cup \{1^+, 1^-\}).$$

We conclude that the unmatched faces are all those $\sigma \cup \{1^-\}$ which satisfy the condition that there are $k-1$ indices $i_1 < i_2 < \cdots < i_{k-1}$ ($i_1 \geq 2$) such that:

(a) $X_\sigma \subset \{i_1^+, i_1^-\} \cup \cdots \cup \{i_{k-1}^+, i_{k-1}^-\}$;
(b) For each $j \in \{1, \ldots, k-1\}$, $X_\sigma \cap \{i_j^+, i_j^-\} \neq \emptyset$;
(c) $X_\sigma \cap \{i_1^+, \ldots, i_{k-1}^+\} \neq \emptyset$;
(d) $X_\sigma \cap \{i_1^-, \ldots, i_{k-1}^-\} \neq \emptyset$.

The set $\{i_1, i_2, \ldots, i_{k-1}\}$ of indices will be called the *support* of X_σ, and the individual i_j the *supporting indices*.

Further, we perform a sequence of element matchings $\mathcal{M}_{2^+}, \ldots, \mathcal{M}_{n^+}$, where each $\mathcal{M}_{i^+}$ denotes the element matching using vertex i^+. We claim that the set of unmatched faces after all these element matchings will be

$$\mathcal{C} = \{V \setminus \{1^+, i_1^+, i_1^-, i_2^-, \ldots, i_{k-1}^-\} : \{i_1, \ldots, i_{k-1}\} \subset \{2, 3, \ldots, n\} \text{ and } i_1 < i_2 < \ldots < i_{k-1}\},$$

i.e., the set of all $\sigma \cup \{1^-\}$ satisfying $X_\sigma = \{i_1^+, i_1^-, i_2^-, \ldots, i_{k-1}^-\}$ for an arbitrary subset $\{i_1, \ldots, i_{k-1}\} \subset \{2, 3, \ldots, n\}$ and order $i_1 < i_2 < \ldots < i_{k-1}$. See Figure 8 for an example of such a matched face, and Figure 9 for an example of an unmatched face.

Let us explain which pairs are made by these element matchings. For each face $\sigma \cup \{1^-\} \in K$, consider the set of supporting indices for X_σ: $i_1 < i_2 < \cdots < i_{k-1}$ ($i_1 \geq 2$). Consider the smallest $j \in \{1, \ldots, k-1\}$ such that $i_j^- \in X_\sigma$ (it exists because of condition (d)). We have two possibilities:

(P1) If $i_j^+ \in \sigma$, we claim that $\sigma \cup \{1^-\}$ will be matched with $(\sigma \setminus \{i_j^+\}) \cup \{1^-\}$ in the element matching $\mathcal{M}_{i_j^+}$. One can easily check that if $\sigma \cup \{1^-\} \in K$, then $(\sigma \setminus \{i_j^+\}) \cup \{1^-\} \in K$ as well. Also, j is the smallest index for $(\sigma \setminus \{i_j^+\}) \cup \{1^-\} \in K$ such that $i_j^- \in X_{\sigma \setminus \{i_j^+\}}$.

(P2) If $i_j^+ \notin \sigma$, and if $(\sigma \cup \{i_j^+\}) \cup \{1^-\} \in K$, we claim that $\sigma \cup \{1^-\}$ will be matched with $(\sigma \cup \{i_j^+\}) \cup \{1^-\}$ in the element matching $\mathcal{M}_{i_j^+}$. Again, the smallest negative entry j is the same for σ and $\sigma \cup \{i_j^+\}$.

Consider an arbitrary σ for which $(\sigma \cup \{i_j^+\}) \cup \{1^-\} \notin K$. Then X_σ satisfies conditions (a)–(d), while $X_{\sigma \cup \{i_j^+\}}$ does not satisfy all of them. The only condition which can be violated is condition (c), i.e., $X_{\sigma \cup \{i_j^+\}} \cap \{i_1^+, \cdots, i_{k-1}^+\} = \emptyset$. From this relation, and from (a)–(d) for X_σ, we conclude that $X_\sigma = \{i_j^+\} \cup \{i_1^-, i_2^-, \cdots, i_{k-1}^-\}$. Further, since j is the smallest negative entry in X_σ, then $j = 1$, and $X_\sigma = \{i_1^+\} \cup \{i_1^-, i_2^-, \cdots, i_{k-1}^-\}$. This means that $\sigma \cup \{1^-\}$ belongs precisely to $\mathcal{C}$. Therefore we have proved that all faces in $K \setminus \mathcal{C}$ are divided into "pairs" of type

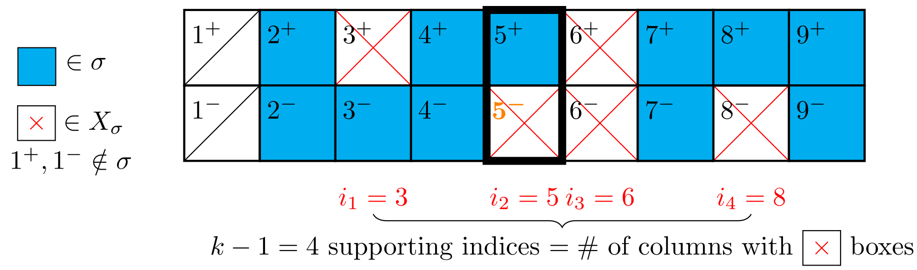
$$\sigma \cup \{1^-\} \longleftrightarrow (\sigma \cup \{i_j^+\}) \cup \{1^-\}$$

by considering that smallest negative entry j.

It remains to prove that for each $i \geq 2$, the element matching $\mathcal{M}_{i^+}$ makes exactly the pairs in $K \setminus \mathcal{C}$ (based on the minimal negative entry), as stated in (P1) and (P2), while it does not match any other face, and it does not match any of the faces from $\mathcal{C}$. We prove this by induction on i.

First we prove the base case $i = 2$ (one can observe that the base case is just a simplified version of the induction step, but we include it for completeness). If $\sigma \cup \{1^-\}$ is an arbitrary face in $K \setminus \mathcal{C}$ such that $2^- \in X_\sigma$, then $\sigma \cup \{1^-\}$ is paired in $\mathcal{M}_{2^+}$ as explained in (P1) and (P2). Next, let $\sigma \cup \{1^-\} \in \mathcal{C}$, and let $X_\sigma = \{i_1^+\} \cup \{i_1^-, i_2^-, \cdots, i_{k-1}^-\}$. If $2^+ \notin \sigma$ (i.e.,$i_1 = 2$), then 2^+ cannot be added to $\sigma \cup \{1^-\}$ because $X_{\sigma \cup \{2^+\}}$ would not satisfy condition (c). If $2^+ \in \sigma$ (i.e.,$i_1 \geq 3$), then $\sigma \cup \{1^-\}$ cannot be paired by using 2^+ because $d(X_{\sigma \setminus \{2^+\}}) = k$, contradicting the condition of exactly $k-1$ supporting indices. So we have proved that $\mathcal{M}_{2^+}$ does not match any of the faces in $\mathcal{C}$. It remains to be proved that $\mathcal{M}_{2^+}$ does not make any other pairs. Consider an arbitrary $\sigma \cup \{1^-\} \in K \setminus \mathcal{C}$, with supporting indices $i_1 < i_2 < \cdots < i_{k-1}$ for X_σ, and let j be the smallest index such that $i_j^- \in X_\sigma$. Now $i_j > 2$, so $2^- \in \sigma$ ($2^- \notin X_\sigma$). We claim that the corresponding addition/removal of 2^+ would change the cardinality of the maximal disconnected set in the complement (which has to be $k-1$). Indeed, if $2^+ \notin \sigma$, then $(\sigma \cup \{2^+\}) \cup \{1^-\} \notin K$ because

$$\sigma = \{2^+, 2^-, 3^-, 4^+, 4^-, 5^+, 7^+, 7^-, 8^+, 9^+, 9^-\} \notin \mathcal{C}.$$



$$X_\sigma = \{3^+, 5^-, 6^+, 6^-, 8^-\}$$

$j = 2$, $i_2^- = 5^-$ – smallest negative in X_σ

$5^+ \in \sigma$, hence $\sigma \cup \{1^-\}$ will be matched with $\big(\sigma \backslash \{5^+\}\big) \cup \{1^-\}$ in $\mathcal{M}_{5^+}$

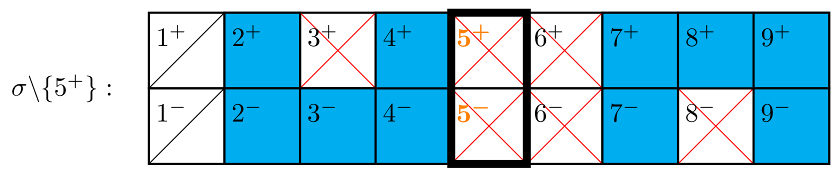
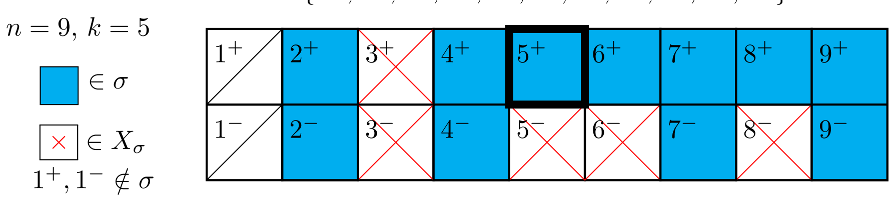


FIGURE 8. An example of a matched pair $(\sigma \cup \{1^-\},\ \sigma \setminus \{5^+\} \cup \{1^-\})$

$$\sigma = \{2^+, 2^-, 4^+, 4^-, 5^+, 6^+, 7^+, 7^-, 8^+, 9^+, 9^-\} \in \mathcal{C}.$$



$\sigma \cup \{1^-\}$ cannot be matched in $\mathcal{M}_{3^+}$, because $\sigma \cup \{1^+, 1^-\} \cup \{3^+\}$ is not a face by condition (c).
$\sigma \cup \{1^-\}$ is not matched in $\mathcal{M}_{5^+}$ with the following face τ, because τ is matched with $\tau \cup \{3^+\}$.

$$\tau = \{1^-, 2^+, 2^-, 4^+, 4^-, 6^+, 7^+, 7^-, 8^+, 9^+, 9^-\}.$$

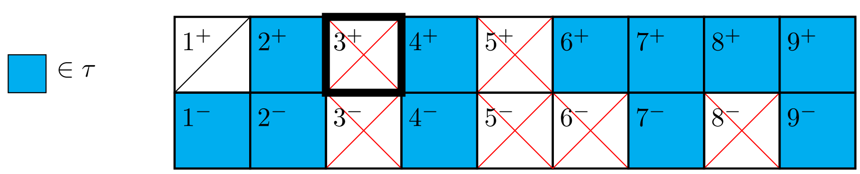


FIGURE 9. An unmatched face $\sigma \cup \{1^-\}$ and an example showing why it is unmatched.

$d(X_{\sigma \cup \{2^+\}}) < k - 1$. Otherwise, if $2^+ \in \sigma$, $(\sigma \setminus \{2^+\}) \cup \{1^-\} \notin K$ because $d(X_{\sigma \setminus \{2^+\}}) = k$. This confirms that $\mathcal{M}_{2^+}$ does not make any other pairs, and finishes the proof for the base case.

Now assume that the statement holds for all $2, \dots, i-1$ $(i \geq 3)$, and we prove it for the element matching $\mathcal{M}_{i^+}$ using i^+.

First, consider an arbitrary $\sigma \cup \{1^-\} \in K$, with the supporting indices for X_σ: $i_1 < i_2 < \cdots < i_{k-1}$ $(i_1 \geq 2)$, for which the smallest $j \in \{1, \dots, k-1\}$ such that $i_j^- \in X_\sigma$ satisfies $i_j = i$. By the induction hypothesis, $\sigma \cup \{1^-\}$ has not been matched yet, and its pair defined in (P1)/(P2) has not been matched yet. Therefore this pair is formed precisely in $\mathcal{M}_{i^+}$.

The next fact we need to prove is that $\mathcal{M}_{i^+}$ does not match any of the faces from $\mathcal{C}$. Consider an arbitrary $\sigma \cup \{1^-\} \in \mathcal{C}$, $X_\sigma = \{i_1^+, i_1^-, i_2^-, \ldots, i_{k-1}^-\}$, $2 \leq i_1 < i_2 < \ldots < i_{k-1}$. Obviously, no vertex i^+ can be added to $\sigma \cup \{1^-\}$, because then $X_{\sigma \cup \{i^+\}}$ would not satisfy condition (c). The only possibility is that $\sigma \cup \{1^-\}$ is matched with $(\sigma \setminus \{i^+\}) \cup \{1^-\}$, where $i_1 < i \leq i_{k-1}$. However, by induction hypothesis, the face $(\sigma \setminus \{i^+\}) \cup \{1^-\}$ was matched with $((\sigma \setminus \{i^+\}) \cup \{i_1^+\}) \cup \{1^-\}$ when we performed the element matching $\mathcal{M}_{i_1^+}$. Therefore, $\mathcal{M}_{i^+}$ does not match any face from $\mathcal{C}$.

Finally, we claim that $\mathcal{M}_{i^+}$ does not match any other face besides the faces described in (P1) and (P2). Again, consider a face $\sigma \cup \{1^-\} \in K \setminus \mathcal{C}$, where X_σ has supporting indices $i_1 < i_2 < \cdots < i_{k-1}$, and the smallest $j \in \{1, \ldots, k-1\}$ such that $i_j^- \in X_\sigma$. If $i_j < i$ then this face was already matched by the induction hypothesis, so we can assume that $i_j > i$ (for $i_j = i$ we already know how this face is paired). There are two possibilities: either i is a supporting index for X_σ, or not. If $i \in \{i_1, \ldots, i_{j-1}\}$, then $i^- \notin X_\sigma$, i.e., $i^- \in \sigma$, so $i^+ \notin \sigma$ from condition (b). The only option would be to match $\sigma \cup \{1^-\}$ with $(\sigma \cup \{i^+\}) \cup \{1^-\}$, but this is not possible because then $X_{\sigma \cup \{i^+\}}$ would not contain a disconnected $(k-1)$-set (because $X_{\sigma \cup \{i^+\}} \cap \{i^+, i^-\} = \emptyset$). The second possibility for i is that $i \notin \{i_1, \ldots, i_{j-1}\}$, i.e., i is not a supporting index for X_σ. Then $i^+, i^- \in \sigma$, so the only option would be to match $\sigma \cup \{1^-\}$ with $(\sigma \setminus \{i^+\}) \cup \{1^-\}$. But the addition of i^+ to the complement would increase the size of a maximal disconnected set, i.e., it would imply $d(X_{\sigma \setminus \{i^+\}}) = k$, which is not possible.

By induction, we have proved that after the sequence $\mathcal{M}_{1^+}, \mathcal{M}_{1^-}$, and then $\mathcal{M}_{2^+}, \mathcal{M}_{3^+}, \ldots \mathcal{M}_{n^+}$, the unmatched faces are exactly the faces in $\mathcal{C}$. There are exactly $\binom{n-1}{k-1}$ faces in $\mathcal{C}$, and each of them contains exactly $2n-k-1$ vertices. By [2, Appendix, Theorems 7.5], a sequence of element matchings is an acyclic matching of the face poset, so we conclude that the complex $\Delta_k(G_n)$ is homotopy equivalent to a CW-complex with $\binom{n-1}{k-1}$ cells of dimension $2n-k-2$ and one additional 0-cell [2, Appendix, Theorems 7.2]. Consequently, $\Delta_k(G_n) \simeq \bigvee_{\binom{n-1}{k-1}} \mathbb{S}^{2n-k-2}$. ☐

The description in the proof shows that $\Delta_k(G_k)$ is isomorphic to the boundary of a k-dimensional crosspolytope with two opposite facets removed.

Lemma 7.19. *The prism over a clique, G_k, is a minimal forbidden subgraph for k-cut complex shellability.*

Proof. We must examine what happens to the cut complex when a vertex of G_k is deleted. By symmetry, we may assume this vertex is k^+.

According to Lemma 4.5, we have $\Delta_k(G_k \setminus \{k^+\}) = \mathrm{lk}_{\Delta_k(G_k)}(k^+)$. The link of k^+ in the k-dimensional crosspolytope is the $(k-1)$-dimensional crosspolytope on vertex set $\{1^+, \ldots, (k-1)^+, 1^-, \ldots, (k-1)^-\}$. Next, consider what happens when the facets $\{1^+, \ldots, k^+\}$ and $\{1^-, \ldots, k^-\}$ are deleted from the k-dimensional crosspolytope: the vertex k^+ does not appear in the second of these facets, and deleting the first of them results in the facet $\{1^+, \ldots, (k-1)^+\}$ being removed from the link of k^+. Therefore, $\Delta_k(G_k \setminus \{k^+\})$ is a $(k-1)$-dimensional crosspolytope with a single facet removed.

Every polytope is shellable, so in particular the $(k-1)$-dimensional crosspolytope is shellable. By symmetry, there is a shelling order in which the facet $\{1^+, \ldots, (k-1)^+\}$ appears last, so removing this facet from the shelling order gives us a shelling order for the crosspolytope without this facet. Thus $\Delta_k(G_k \setminus \{k^+\})$ is shellable, so G_k is a minimal forbidden subgraph for k-cut complex shellability. ☐

7.4. **Squared Cycle Graphs.**

Definition 7.20. The *squared cycle graph* W_n is the graph with vertex set $[n]$, and edge-set $\{(i, i+1 \mod n), (i, i+2 \mod n)\}$, $i = 1, \ldots, n$.

Clearly, W_n contains the cycle graph C_n. If $n \leq 5$, W_n is the complete graph K_n. For $n \geq 6$ and $n > k+3$, the cut complex $\Delta_k(W_n)$ has dimension $n-k-1$.

Proposition 7.21. *For $n \leq k+3$, $\Delta_k(W_n)$ is void (there are no faces) and therefore shellable.*

Proof. Clearly $\Delta_k(W_n) = \emptyset$ if $n \leq k+1$. If $n = k+2$, a separating 2-set must be of the form $\{1, j\}$, $j \notin \{2, 3, k+1, k+2\}$, so $4 \leq j \leq k$. Since there is an edge between $j-1$ and $j+1$, and paths from 2 to $j-1$, and from $j+1$ to k, this is impossible.

Let $n = k+3$. Consider the set $S = \{1 < i < j\}$. If $j < k+3$ and $i < j-1$, then $i-1, i-2, \dots, 2, k+3, \dots, j+1, j-1, \dots, i+1$ is a path in $W_n \setminus S$. If $j = k+3$ and $2 < i < k+2$, then $2, 3, \dots i-1, i+1, \dots, k+2$ is a path in $W_n \setminus S$. So W_n does not have a separating set of size 3. ☐

Proposition 7.22 ([2, Theorem 3.11])**.** *The $(n-3)$-dimensional cut complex $\Delta_2(W_n)$ has the homotopy type of $\mathbb{S}^{n-4}$, one sphere in dimension one lower than the top, for all $n \geq 7$. If $n = 6$, $\Delta_2(W_6)$ is homotopy equivalent to $\mathbb{S}^1$. Hence for $n \geq 6$, the cut complex $\Delta_2(W_n)$ is not shellable.*

Sage computations suggest the following conjectures:

Conjecture 7.23. For $k \geq 3$, the 4-dimensional cut complex $\Delta_k(W_{k+5})$ has the homotopy type of $\mathbb{S}^3 \vee \bigvee_{\beta(k)} \mathbb{S}^4$ (a wedge of spheres in dimensions 3 and 4) for positive integers $\beta(k)$, and is therefore not shellable. Its nonzero homology is $\widetilde{H}_3 = \mathbb{Z}, \widetilde{H}_4 = \mathbb{Z}^{\beta(k)}$.

Conjecture 7.24. For $15 \geq k \geq 3$, the number of 4-spheres $\beta(k)$ in $\Delta_k(W_{k+5})$ is given by the formula

$$\beta(k) = \frac{(k-3)(k-2)(k+5)}{6}.$$

These numbers match OEIS sequence A006503.

Conjecture 7.25. For $k \geq 3$ the cut complex $\Delta_k(W_n)$ is shellable for $n \geq k+6$ (supported by Sage for $n \leq 13$ and $k \leq 5$). For $k = 3$ and $n \geq 9$, the Betti numbers are $\binom{n-4}{2} - 9 = \{1, 6, 12, 19, 27, \dots\}$. This is OEIS A051936.

Sage computations also suggest that for $k \geq 3$, the 3-dimensional cut complex $\Delta_k(W_{k+4})$ has the homotopy type of $\mathbb{S}^1$ and is therefore not shellable. The following results enable us to prove this. Let $n = k+4$, and label the vertices of $W_{k+4} = W_n$ with the indices $1, \dots, n$, with arithmetic done modulo n.

Lemma 7.26. *The facets of the 3-dimensional cut complex $\Delta_k(W_{k+4})$, for $k \geq 2$, are the sets of the form*

$$S = \{i, i+1, j, j+1\}$$

with i and j chosen so that $i \neq j$ and so that $i+2$ and $j+2$ are not elements of this set. In other words, S consists of two pairs of consecutive vertices, $\{i, i+1\}$ and $\{j, j+1\}$, with a gap of at least one vertex between the pairs in both directions.

Proof. First, suppose S is a set of this form. Then $W_n \setminus S$ has two components, specifically $\{i+2, i+3, \dots, j-1\}$ and $\{j+2, \dots, i-1\}$: there are no edges between these components, since the gaps between these two sets are size 2. So S is a facet of $\Delta_k(W_n)$.

Conversely, suppose S is any facet, so S has size 4 and $W_n \setminus S$ is disconnected into two non-empty subgraphs U and V with no edges between them. Let u and v be vertices of U and V, respectively. Consider the list of vertices $u, u+1, \dots, v-1, v$, read cyclically. Suppose x is the first vertex in this list that is in V. Then both $x-2$ and $x-1$ cannot be in V, and they cannot be in U because $(x-2, x)$ and $(x-1, x)$ are edges in W_n but by construction there are no edges between U and V; therefore, both $x-2$ and $x-1$ must be elements of S. Similarly, if we consider the list $v, v+1, \dots, u-1, u$ and take y to be the first element of U in this list, then $y-2$ and $y-1$ must be elements of S. But since S has size 4, S must be exactly $\{x-2, x-1, y-2, y-1\}$. Since $(x-2)+2 = x$ and $(y-2)+2 = y$ are not elements of S, as they are elements of V and U respectively by construction, we conclude that S is a set of the desired form. ☐

A routine count shows that the number of facets of $\Delta_k(W_{k+4})$ is $(k+4)(k-1)/2$.

Proposition 7.27. *The complex $\Delta_k(W_{k+4})$ is homotopy equivalent to the circle $\mathbb{S}^1$. Consequently the 3-dimensional cut complex $\Delta_k(W_{k+4})$ is not shellable.*

Proof. In view of Proposition 7.22, we need only consider the case $k \geq 3$.

Given a facet $F = \{i, i+1, j, j+1\}$ in $\Delta_k(W_{k+4})$, define its *smallest gap size* to be the number

$$\min\left(\#\{i+2, i+3, \dots, j-1\}, \#\{j+2, \dots, i-1\}\right) = \min\left(j-i-2 \bmod n,\ i-j-2 \bmod n\right)$$

with arithmetic modulo n. Now, for $g = 1, \dots, \lfloor k/2 \rfloor$, define X_g to be the complex generated by all facets of $\Delta_k(W_{k+4})$ with smallest gap size at least g. Note that all facets of $\Delta_k(W_{k+4})$ have smallest gap size at least 1, so X_1 is $\Delta_k(W_{k+4})$ itself.

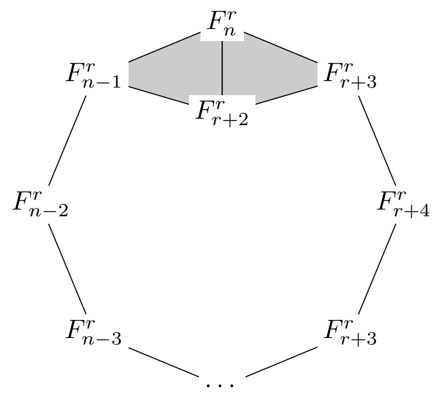


FIGURE 10. The nerve of the facets $F^r_{r+2}, \dots, F^r_n$ in X_r in the odd case of the proof of Proposition 7.27

Now we claim that X_g is always homotopy equivalent to X_{g+1}, for $g < \lfloor k/2 \rfloor$. For this we use the method of collapsible complexes, see [3, Section 11] and [19]. The difference between the two complexes X_g and X_{g+1} is the set of facets with smallest gap size exactly g, i.e., the facets $F^g_i = \{i, i+1, i+g+2, i+g+3\}$. Recall [19, Chapter 9] that a free face in a simplicial complex is one that is contained in a unique facet. The face $\{i+1, i+g+2\}$ is a free face in X_g: the only facets of $\Delta_k(W_{k+4})$ that contain it are

$$\{i, i+1, i+g+2, i+g+3\} = F^g_i,$$
$$\{i+1, i+2, i+g+2, i+g+3\},$$
$$\{i, i+1, i+g+1, i+g+2\},$$
$$\{i+1, i+2, i+g+1, i+g+2\}$$

and all of these facets but F^g_i have a smallest gap size less than g, so are not present in X_g. Therefore we can remove $\{i+1, i+g+2\}$ from X_g by a sequence of elementary collapses without changing the homotopy type. The remaining faces of F^g_i are $\{i, i+g+1, i+g+3\}$ and $\{i, i+1, i+g+3\}$ and their subfaces, but these two faces are contained respectively in the facets $\{i-1, i, i+g+2, i+g+3\}$ and $\{i, i+1, i+g+3, i+g+4\}$, which each have gap size $g+1$. Therefore we can remove all facets F^g_i with smallest gap size g from X_g, in any order, without changing the homotopy type, producing X_{g+1}.

Thus $\Delta_k(W_{k+4}) = X_1$ is homotopy equivalent to $X_{\lfloor k/2 \rfloor}$. We consider the cases where k is even or odd separately.

If $k = 2r$ is even, then $X_{\lfloor k/2 \rfloor} = X_r$ consists of the facets $F^r_i = \{i, i+1, i+r+2, i+r+3\}$ for $i = 1, \dots, r+2$, where the gaps between the pairs of vertices are size r in both directions. (Note that $F^r_{r+2+i} = \{i+r+2, i+r+3, 2r+4+i, 2r+5+i\} = F^r_i$, since $2r+4 = k+4 = n$ and arithmetic takes place modulo n.) Two facets F^r_i and F^r_j intersect each other if and only if i and j are consecutive modulo $r+2$; therefore, the nerve of this set of facets is the cycle graph C_{r+2}. By the nerve theorem [14, Corollary 4G.3], X_r is homotopy equivalent to the nerve of its facets; thus $\Delta_k(W_{k+4})$ is homotopy equivalent to a circle when k is even.

When $k = 2r+1$ is odd, then $X_{\lfloor k/2 \rfloor} = X_r$ consists of the facets $F^r_i = \{i, i+1, i+r+2, i+r+3\}$ for $i = 1, \dots, n$, where the two gaps between the pairs of vertices are size r and $r+1$.

We claim we can continue using free faces and elementary collapses to remove the facets F^r_i where $i = 1, \dots, r+1$. The face $\{i+1, i+r+2\}$ is a free face in F^r_i, since all other facets of $\Delta_k(W_{k+4})$ that contain it have a smallest gap size less than r. Its elementary collapse leaves behind the faces $\{i, i+1, i+r+3\}$ and $\{i, i+r+2, i+r+3\}$, which are shared respectively with the facets $\{i, i+1, i+r+3, i+r+4\} = F^r_{i+r+3}$ and $\{i-1, i, i+r+2, i+r+3\} = F^r_{i+r+2}$, which are not in the list of facets we are removing.

This leaves the facets F^r_i for $i = r+2, \dots, n$. This time, the nerve of these facets is the complex shown in Figure 10, which is homotopy equivalent to a circle. Thus $\Delta_k(W_{k+4})$ is homotopy equivalent to a circle for all k. □

Proposition 7.28. *Fix $k \geq 3$ and let G be a proper induced subgraph of W_{k+4}. Then $\Delta_k(G)$ is shellable. Hence W_{k+4} is a minimal nonshellable graph for k-cut complex shellability.*

Proof. It suffices to prove the case where G is W_{k+4} minus a single vertex. In this case, G may be described as the graph with vertex set $\{a, b, 1, 2, \dots, k+1\}$, and edge set

$$\big\{\{a,b\}\{a,1\},\{a,2\},\{b,k+1\},\{b,k\}\big\} \cup \big\{\{i,i+1\}\big\}_{i=1}^{k} \cup \big\{\{j,j+2\}\big\}_{j=1}^{k-1}.$$

Figure 11 shows G when $k = 4$. Since G has $k+3$ vertices, the facets of $\Delta_k(G)$ are the separating sets of size 3. Thus, from Lemma 7.26, we conclude that the facets of $\Delta_k(G)$ are exactly the sets

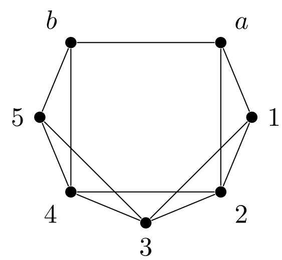
$$\{a,2,3\},\dots,\{a,k,k+1\},\{b,2,3\},\dots,\{b,k-1,k\},\{b,1,2\}.$$

We claim that the order of the facets as listed above gives a shelling order for $\Delta_k(G)$. To see this, let $F_1, \dots, F_{2k-2}$ be the facets of $\Delta_{k+3}(G)$ listed in the order as above. Then we need to show that for each $i = 2, \dots, 2k-2$, the set

$$S_i = \{F : F \subseteq F_i,\ F \not\subseteq F_j \text{ for all } j < i\}$$

has a unique minimal element with respect to inclusion. Indeed, for $i \in \{2, \dots, k-1\}$ (when $a \in F_i$), we see that the unique minimal element of S_i is $\{i+1\}$. When $i = k$, $F_i = \{b,2,3\}$ and we see that the unique minimal element of S_i is $\{b\}$. For $i \in \{k+1, 2k-1\}$ (when $b \in F_i$ but $2 \notin F_i$), we see that the unique minimal element of S_i is $\{b, i+1\}$. Finally, when $i = 2k-2$, we have that the unique minimal element of S_i is $\{1\}$. □

Figure 11. The graph W_8 minus a vertex.



The cyclic group $\mathfrak{C}_n$ acts as a group of automorphisms of the squared cycle W_n, and hence on the homology of its cut complex $\Delta_k(W_n)$. We have the following:

Proposition 7.29. *Let $k \geq 2$. The one-dimensional homology module $\widetilde{H}_1(\Delta_k(W_{k+4})$ affords the trivial representation of the cyclic group $\mathfrak{C}_{k+4}$.*

If $n \geq 7$, the cyclic group $\mathfrak{C}_n$ acts on the one-dimensional homology module $\widetilde{H}_{n-4}(\Delta_2(W_n))$ like

$$\begin{cases} \text{the trivial representation,} & \text{if } n \text{ is odd,} \\ \text{the sign representation,} & \text{if } n \text{ is even.} \end{cases}$$

Proof. We apply the Hopf trace formula [14] to the face lattice of the cut complex. In the present context, the precise fact that we need is as follows [32]. Suppose the nonzero homology of a bounded poset P is concentrated in a single degree $r-2$, and suppose g is an automorphism of P. Let P^g denote the subposet of P consisting of all elements fixed by g. Then one has the formula [32, p. 282, Eqn. (1.2)]

$$\mu(P^g) = (-1)^r \operatorname{tr}(g, \widetilde{H}_{r-2}(P)).$$

Now take P to be the face lattice of $\Delta_k(W_{k+4})$. From Proposition 7.27 the homology is concentrated in degree 1, and has vector space dimension one. In order to determine this one-dimensional representation of $\mathfrak{C}_{k+4}$, it suffices to compute the trace of the $(k+4)$-cycle $g = (1, 2, \dots, k+4)$ which generates $\mathfrak{C}_{k+4}$. But the fixed point subposet P^g is clearly the trivial poset consisting of $\{\widehat{0}, \widehat{1}\}$, and thus $\mu(P^g) = -1 = (-1)^3 \operatorname{tr}(g, \widetilde{H}_1(P))$.

Hence $\operatorname{tr}(g, \widetilde{H}_1(P)) = 1$, confirming that the action of the cyclic group $\mathfrak{C}_{k+4}$ on the homology is trivial.

Now let $n \geq 7$. Proposition 7.22 tells us that the nonzero homology of $\Delta_2(W_n)$ occurs only in degree $n-4$, and has dimension 1 as a vector space. By the Hopf trace formula above applied to the face lattice P of $\Delta_2(W_n)$, for the n-cycle $g = (1, 2, \ldots, n) \in \mathfrak{C}_n$, we have

$$\mu(P^g) = (-1)^{n-2}\mathrm{tr}(g, \widetilde{H}_{n-4}(P)).$$

Again it is easy to see that $P^g = \{\widehat{0}, \widehat{1}\}$. This time we obtain $\mathrm{tr}(g, \widetilde{H}_{n-4}(P)) = (-1)^{n-1}$. The claim follows. □

8. Conclusion and Further Directions

In this paper we introduced a new graph complex, the k-cut complex for $k > 2$, which generalizes the (2-cut) complex in the Eagon–Reiner proof of Fröberg's Theorem. We investigated how shellability and homotopy type of the k-cut complex are affected by the following graph operations: induced subgraphs, disjoint union, joins and wedges. We were able to extend one direction of Fröberg's result for chordal graphs and the 2-cut complex to the 3-cut complex. Our results for $k = 3$ are best possible: we showed that for any $k \geq 4$, there are examples of chordal graphs for which $\Delta_k(G)$ is not shellable. We also studied the face lattice of the cut complex, giving a formula for the reduced Euler characteristic for a broad family of graphs. We completely determined the homotopy type of the 2-cut complex in the case of connected triangle-free graphs.

The families of graphs we considered include trees, complete multipartite graphs, cycles, prisms over cliques, and squared cycles. In all cases except the latter, we determined completely the homotopy type of the k-cut complex. Our tools encompassed a broad range: shellability, poset topology and discrete Morse theory.

We continue the investigation of k-cut complexes in a subsequent paper, where we apply these methods to the families of grid graphs, and squared paths. We also undertake a more detailed study of how the k-cut complex behaves under the disjoint union operation of graphs, including an analysis of the face vectors and h-vectors.

References

[1] Kenneth Baclawski. Cohen-Macaulay connectivity and geometric lattices. *European J. Combin.*, 3(4):293–305, 1982.

[2] Margaret Bayer, Mark Denker, Marija Jelić Milutinović, Rowan Rowlands, Sheila Sundaram, and Lei Xue. Total cut complexes of graphs. *Discrete and Computational Geometry*, 2024. to appear.

[3] A. Björner. Topological methods. In *Handbook of Combinatorics, Vol. 1, 2*, pages 1819–1872. Elsevier Sci. B. V., Amsterdam, 1995.

[4] Anders Björner. Shellable and Cohen-Macaulay partially ordered sets. *Trans. Amer. Math. Soc.*, 260(1):159–183, 1980.

[5] Anders Björner. Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings. *Adv. in Math.*, 52(3):173–212, 1984.

[6] Anders Björner and Michelle Wachs. On lexicographically shellable posets. *Trans. Amer. Math. Soc.*, 277(1):323–341, 1983.

[7] Anders Björner and Michelle L. Wachs. Shellable nonpure complexes and posets. I. *Trans. Amer. Math. Soc.*, 348(4):1299–1327, 1996.

[8] Anders Björner and Michelle L. Wachs. Shellable nonpure complexes and posets. II. *Trans. Amer. Math. Soc.*, 349(10):3945–3975, 1997.

[9] Winfried Bruns and Jürgen Herzog. Semigroup rings and simplicial complexes. *J. Pure Appl. Algebra*, 122(3):185–208, 1997.

[10] Mark Denker. Extension of Fröberg's theorem to other graph ideals. *Formal Report for Completion of Masters Degree, Oklahoma State University*, 2018.

[11] Priyavrat Deshpande and Anurag Singh. Higher independence complexes of graphs and their homotopy type. *J. Ramanujan Math. Society*, 36(1), 2021.

[12] John A. Eagon and Victor Reiner. Resolutions of Stanley-Reisner rings and Alexander duality. *J. Pure Appl. Algebra*, 130(3):265–275, 1998.

[13] Ralf Fröberg. On Stanley-Reisner rings. In *Topics in algebra, Part 2 (Warsaw, 1988)*, volume 26 of *Banach Center Publ.*, pages 57–70. PWN, Warsaw, 1990.

[14] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[15] Aleksandr Vadimovich Ivashchenko. Contractible transformations do not change the homology groups of graphs. *Discrete Math.*, 126(1-3):159–170, 1994.

[16] Gordon James and Adalbert Kerber. *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications.* Addison-Wesley Publishing Co., Reading, Mass., 1981. With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson.

[17] Jakob Jonsson. *Simplicial complexes of graphs*, volume 1928 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008.

[18] Dmitry Kozlov. *Combinatorial algebraic topology*, volume 21 of *Algorithms and Computation in Mathematics*. Springer, Berlin, 2008.

[19] Dmitry N. Kozlov. *Organized collapse: an introduction to discrete Morse theory*, volume 207 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, [2020] ©2020.

[20] Svante Linusson and John Shareshian. Complexes of t-colorable graphs. *SIAM J. Discrete Math.*, 16(3):371–389, 2003.

[21] Svante Linusson, John Shareshian, and Volkmar Welker. Complexes of graphs with bounded matching size. *J. Algebraic Combin.*, 27(3):331–349, 2008.

[22] László Lovász. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25(3):319–324, 1978.

[23] Takahiro Matsushita. Dominance complexes and vertex cover numbers of graphs. *J. Appl. Comput. Topol.*, 7(2):363–368, 2023.

[24] Roy Meshulam. Domination numbers and homology. *J. Combin. Theory Ser. A*, 102(2):321–330, 2003.

[25] J. Scott Provan and Louis J. Billera. Decompositions of simplicial complexes related to diameters of convex polyhedra. *Math. Oper. Res.*, 5(4):576–594, 1980.

[26] Daniel Quillen. Homotopy properties of the poset of nontrivial p-subgroups of a group. *Adv. in Math.*, 28(2):101–128, 1978.

[27] Louis Solomon. A decomposition of the group algebra of a finite Coxeter group. *J. Algebra*, 9:220–239, 1968.

[28] Richard P. Stanley. Some aspects of groups acting on finite posets. *J. Combin. Theory Ser. A*, 32(2):132–161, 1982.

[29] Richard P. Stanley. *Combinatorics and commutative algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 1996.

[30] Richard P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[31] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[32] Sheila Sundaram. Applications of the Hopf trace formula to computing homology representations. In *Jerusalem Combinatorics '93*, volume 178 of *Contemp. Math.*, pages 277–309. Amer. Math. Soc., Providence, RI, 1994.

[33] Sheila Sundaram. The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice. *Adv. Math.*, 104(2):225–296, 1994.

[34] Sheila Sundaram. Homotopy of non-modular partitions and the Whitehouse module. *J. Algebraic Combin.*, 9(3):251–269, 1999.

[35] O. Ya. Viro, O. A. Ivanov, N. Yu. Netsvetaev, and V. M. Kharlamov. *Elementary topology*. American Mathematical Society, Providence, RI, 2008. Problem textbook.

[36] Michelle L. Wachs. Poset topology: tools and applications. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 497–615. Amer. Math. Soc., Providence, RI, 2007.

[37] Douglas B. West. *Introduction to graph theory*. Prentice Hall, Inc., Upper Saddle River, NJ, 1996.

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