Failure of Smooth Pasting Principle and Nonexistence of Equilibrium Stopping Rules under Time Inconsistency*

Ken Seng Tan   Wei Wei   Xun Yu Zhou †

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Abstract

This paper considers time-inconsistent stopping problems in which the inconsistency arises from a class of non-exponential discount functions called the weighted discount functions. We show that the smooth pasting principle, the main approach that is used to construct explicit solutions for the classical time-consistent optimal stopping problems, may fail under time-inconsistency. Specifically, we prove that the smooth pasting solves a time-inconsistent problem, within the intra-personal game theoretic framework with a general nonlinear cost functional and a geometric Brownian motion, if and only if certain inequalities on the model primitives are satisfied. In the special case of a real option problem, we show that the violation of these inequalities can happen even for very simple non-exponential discount functions. Moreover, we show that the real option problem actually does not admit any equilibrium whenever the smooth pasting approach fails. The negative results in this paper caution blindly extending the classical approach for time-consistent stopping problems to their time-inconsistent counterparts.

†Tan: Department of Statistics and Actuarial Science University of Waterloo Mathematics 3, 200 University Avenue West Waterloo, Ontario, Canada N2L 3G1. E-mail: kstan@uwaterloo.ca. Wei: Department of Statistics and Actuarial Science, University of Waterloo, 200 University Avenue West, Waterloo, N2L 3G1, ON, Canada. E-mail: wei.wei@uwaterloo.ca. Zhou: Department of IEOR, Columbia University, New York, NY 10027, USA. Email: xz2574@columbia.edu.

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1   Introduction

A crucial assumption imposed on classical optimal stopping models is that an agent has a constant time preference rate and hence discounts her future payoff exponentially. When this assumption is violated, an optimal stopping problem becomes generally time-inconsistent in that any optimal stopping rule obtained today may no longer be optimal from the perspective of a future date. The problem then becomes largely descriptive rather than normative because there is generally no *dynamically* optimal solution that can be used to guide the agent's decisions. Different agents may react differently to a same time-inconsistent problem, and a goal of the study is to *describe* the different behaviors. Strotz (1955) is the first to observe that non-constant time preference rates result in time-inconsistency, and to categorize three types of agents when facing such time-inconsistency. One of the types is called a "non-committed, sophisticated agent" who, at *any* given time, optimizes the underlying objective taking as constraints the stopping decisions chosen by her future selves. Such a problem has been formulated within an intra-personal game theoretic framework and the corresponding equilibria are used to describe the behaviors of this type of agents; see, for example, Phelps and Pollak (1968); Laibson (1997); O'Donoghue and Rabin (2001); Krussell and Smith (2003) and Luttmer and Mariotti (2003). An extended dynamic programming equation for continuous-time deterministic equilibrium controls is derived in Ekeland and Lazrak (2006), followed by a stochastic version in Bjork and Murgoci (2010) and application to a mean–variance portfolio model in Björk et al. (2014).

This paper studies a time-inconsistent stopping problem in continuous time within the intra-personal game framework, in which the source of time-inconsistency is the so-called weighted discount function (WDF), a very general class of non-exponential discount functions.[1] We make two main contributions. First, we demonstrate that the smooth pasting (SP) principle, which is almost the *exclusive* approach in solving classical optimal stopping problems, may fail when

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[1] The WDF, proposed in Ebert et al. (2016), is a weighted average of a set of exponential discount functions. It has been shown in Ebert et al. (2016) that it can be used to model the time preference of a group of individuals as well as that of behavioral agents, and that most commonly used non-exponential discount functions are WDFs.

time-consistency is lost. Second, for a stopping model whose time-consistent counterpart is the well-studied real option problem, we establish a condition under which no equilibrium stopping rule exists. These results are constructive and they caution blindly extending the SP principle to time-inconsistent stopping problems.

Let us now elaborate on the first contribution. Recall that the SP is used to derive (often explicit) solutions to conventional, time-consistent optimal stopping problems. It conjectures a candidate solution to the underlying Bellman equation (or variational inequalities), which is a free boundary PDE, based on the C^1 smooth pasting around the free boundary, and then checks that it solves the PDE under some standard regularity/convexity conditions on the model primitives. Finally it verifies that the first hitting time of the free boundary indeed solves the optimal stopping problem using the standard verification technique. Recently, Grenadier and Wang (2007) and Hsiaw (2013), among others, extend the application of the SP principle to solving time-inconsistent stopping problems. While the SP *happened to* work in the specific settings of these papers, it is more an exception than a rule. Indeed, in the present paper we show that, for a geometric Brownian motion with a nonlinear cost functional, while the SP always yields a candidate solution, the latter actually gives rise to an equilibrium stopping rule if and only if certainty inequalities on the model primitives are satisfied. These inequalities hold trivially for the time-consistent exponential discount case, but does not in general for its time-inconsistent non-exponential counterpart, *even if all the other parameters and assumptions (state dynamics, running cost, etc.) are identical.* Indeed, the violation of such inequalities is not rare even in very simple cases. For example, we show that in the special case of a real option problem with some WDFs including the pseudo-exponential discount function (Ekeland and Lazrak 2006; Karp 2007; Harris and Laibson 2013), the inequalities do not hold for plausible sets of parameter values of the chosen discount functions. The bottom line is that one cannot blindly apply the SP to any stopping model when time-inconsistency is present, even if the SP does work for its time-consistent counterpart.

The second contribution is on the nonexistence of an intra-personal equilibrium. For a time-

consistent stopping problem, optimal stopping rules exist when the cost functional and the underlying process satisfy some mild regularity conditions (see, e.g., Peskir and Shiryaev 2006). However, this is no longer the case for the time-inconsistent counterpart. To demonstrate this, we again take the real option problem with a WDF. For such a problem, we prove that there simply does not exist any equilibrium stopping rule whenever the aforementioned inequalities are violated and hence the SP principle fails. Our result therefore reveals that equilibrium stopping rules within the intra-personal game theoretic framework may not exist no matter what regularity conditions are imposed on the underlying models.

There are studies in the literature on time-inconsistent stopping including nonexistence results, albeit in considerably different settings especially in terms of the source of time-inconsistency and the definition of an equilibrium. Bayraktar et al. (2018) consider a stopping problem with a discrete-time Markov chain, whereas the time-inconsistency comes from the mean-variance objective functional. The Markov chain takes value in a set of finite numbers, which allows them to discuss the nonexistence of equilibrium stopping rules by enumeration. Christensen and Lindensjö (2018a) and Christensen and Lindensjö (2018b) study continuous-time stopping problems where the time-inconsistency follows from the types of payoff functions (mean-variance or endogenous habit formation). In particular, Christensen and Lindensjö (2018a) show that the candidate solution derived from the SP may not lead to an equilibrium stopping for some range of parameters. However, the definition of equilibria in these papers is entirely different from the one based on the "first-order" spike variation; the latter seems to be widely adopted by many papers (see, e.g., O'Donoghue and Rabin 2001; Bjork and Murgoci 2010; Ekeland et al. 2012; and Björk et al. 2014) including the present one.[2] Huang and Nguyen-Huu (2018) investigate a continuous-time stopping problem with non-exponential discount functions. They define an equilibrium via a fixed point of a mapping, which is essentially based on a zeroth-order condition and hence is different from our definition. Under their setting, immediately stopping is always a (trivial) equilibrium

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[2]Christensen and Lindensjö (2018a) also consider mixed strategies as opposed to the pure strategies studied in our paper and many other papers. Time-inconsistent problems using mixed strategies are interesting, and it is possible that no equilibrium may be found in the class of mixed strategies either. However, the main point of this paper is to show that a change of discounting factor from exponential to non-exponential may cause a stopping problem that has an equilibrium to one that does not, *even though both are using pure strategies.*

(so there is no issue of nonexistence), which is not the case according to our definition.

The remainder of the paper is organized as follows. In Section 2, we recall the definition and some important properties of the WDF introduced by Ebert et al. (2016), formulate a general time-inconsistent stopping problem within the intra-personal game theoretic framework, and characterize the equilibrium stopping rules by a Bellman system and provide the verification theorem. In Section 3 we consider the case when the state process is a geometric Brownian motion, apply the SP principle to derive a candidate solution and establish certain equivalent conditions for the derived candidate solution to actually solve the Bellman system. Then we present a real option problem, in which the aforementioned equivalent conditions reduce to a single inequality, failing which there is simply no equilibrium at all. Finally, Section 4 concludes the paper. Appendix A contains proofs of some results.

2 The Model

2.1 Time preferences

Throughout this paper we consider weighted discount functions defined as follows.

Definition 1 (Ebert et al. 2016) *Let $h : [0, \infty) \to (0, 1]$ be strictly decreasing with $h(0) = 1$. We call h a weighted discount function (WDF) if there exists a distribution function F concentrated on $[0, \infty)$ such that*

$$h(t) = \int_0^\infty e^{-rt} dF(r). \tag{1}$$

Moreover, we call F the weighting distribution *of h.*

Many commonly used discount functions can be represented in weighted form. For example, exponential function $h(t) = e^{-rt}, r > 0$ (Samuelson 1937) and pseudo-exponential function $h(t) = \delta e^{-rt} + (1-\delta)e^{-(r+\lambda)t}, 0 < \delta < 1, r > 0, \lambda > 0$ (Ekeland and Lazrak 2006; Karp 2007) are WDFs with degenerate and binary distributions respectively. A more complicated example is the generalized

hyperbolic discount function (Loewenstein and Prelec 1992) with parameters $\gamma > 0, \beta > 0$, which can be represented as

$$h(t) = \frac{1}{(1+\gamma t)^{\frac{\beta}{\gamma}}} \equiv \int_0^\infty e^{-rt} f\left(r; \frac{\beta}{\gamma}, \gamma\right) dr \tag{2}$$

where $f(r; k, \theta) = \frac{r^{k-1} e^{-\frac{r}{\theta}}}{\theta^k \Gamma(k)}$ is the density function of the Gamma distribution with parameters k and θ, and $\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx$ the Gamma function evaluated at k. See Ebert et al. (2016) for more examples and discussions about the types of discount functions that are of weighted form.

The following result is a restatement of the well-known Bernstein's theorem in terms of WDFs, which actually provides a characterization of the latter.

Theorem 1 (Bernstein 1928) *A discount function h is a WDF if and only if it is continuous on $[0, \infty)$, infinitely differentiable on $(0, \infty)$, and satisfies $(-1)^n h^{(n)}(t) \geq 0$, for all non-negative integers n and for all $t > 0$.*

Bernstein's theorem can be used to examine if a given function is a WDF without necessarily representing it in the form of (1). For example, it follows from this theorem that the constant sensitivity discount function $h(t) = e^{-at^k}, a, k > 0$, and the constant absolute decreasing impatience discount function $h(t) = e^{e^{-ct}-1}, c > 0$, are both WDFs.

2.2 Stopping rules and equilibria

On a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, P)$ there lives a one dimensional Brownian motion W, and a family of Markov diffusion processes $X = X^x$ parameterized by the initial state $X_0 = x \in \mathbb{R}$ and governed by the following stochastic differential equation (SDE)

$$dX_t = b(X_t)dt + \sigma(X_t)W_t, \quad X_0 = x, \tag{3}$$

where b, σ are Lipschitz continuous functions, i.e., there exists an $L > 0$ such that for any $x \neq y$

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|. \tag{4}$$

We assume that $\mathbb{F}$ is the P-augmentation of the natural filtration generated by X. To avoid an uninteresting case we also assume that $|\sigma(x)| \geq c > 0 \ \forall x \in \mathbb{R}$ so that X is non-degenerate.[3]

For any fixed $x \in \mathbb{R}$, an agent monitors the process $X = X^x$ and aims to minimize the following cost functional

$$J(x; \tau) = \mathbb{E}\left[\int_0^\tau h(s)f(X_s)ds + h(\tau)g(X_\tau)\Big|X_0 = x\right] \tag{5}$$

by choosing $\tau \in \mathcal{T}$, the set of all $\mathbb{F}$-stopping times. Here h is a WDF with a weighting distribution F, g is continuous and bounded, and f is continuous with polynomial growth, i.e., there exists $m \geq 1$ and $C > 0$ such that

$$|f(x)| \leq C(|x|^m + 1). \tag{6}$$

Moreover, we assume that there exists $n \geq 1, C(r) > 0$ satisfying $\int_0^\infty C(r)dF(r) + \int_0^\infty rC(r)dF(r) < \infty$ such that

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\int_0^\tau e^{-rs}|f(X_s)|ds + e^{-r\tau}|g(X_\tau)|\Big|X_0 = x\right] \leq C(r)(|x|^n + 1), \ \ \forall r \in \text{supp}(F). \tag{7}$$

This is a (weak) assumption to ensure that the optimal value of the stopping problem is finite, and hence the problem is well-posed.

We now define stopping rules which are essentially binary feedback controls. These stopping rules induce Markovian stopping times for any given Markov process.

Definition 2 (Stopping rule) *A stopping rule is a measurable function* $u : \mathbb{R} \to \{0, 1\}$ *where*

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[3]Here we assume that the Brownian motion is one dimensional just for notational simplicity. There is no essential difficulty with a multi-dimensional Brownian motion.

0 *indicates "continue" and* 1 *indicates "stop". For any given Markov process* $X = \{X_t\}_{t \geq 0}$, *a stopping rule* u *defines a Markovian stopping time*

$$\tau_u = \inf\{t \geq 0, u(X_t) = 1\}. \tag{8}$$

Given a stopping rule u, we can define the stopping region $\mathcal{S}_u = \{x \in (0, \infty) : u(x) = 1\}$. For any $x \in \bar{\mathcal{S}}_u$, since the underlying process X is non-degenerate, a standard result (e.g., Chapter 3 of Ito and McKean Jr 1965) yields that $\mathbb{P}(\tau_u = 0 | X_0 = x) = 1$, and hence $J(x; \tau_u) = g(x)$. This means that the agent stops immediately once the process reaches at any point in $\bar{\mathcal{S}}_u$. As a result, in the setting of this paper, the continuation region is $\mathcal{C}_u = \bar{\mathcal{S}}_u^{\,c}$.

As discussed earlier the non-exponential discount function h in the cost functional (5) renders the underlying optimal stopping problem generally time-inconsistent. In this paper we consider a sophisticated and non-committed agent who is aware of the time-inconsistency but unable to control her future actions. In this case, she seeks to find the so-called equilibrium strategies within the intra-personal game theoretic framework, in which the individual is represented by different players at different dates.[4]

We now give the precise definition of an equilibrium stopping rule $\hat{u}$, which essentially entails a solution to a game in which no self at any time (or, equivalently in the current setting, at any state) is willing to deviate from $\hat{u}$.

Definition 3 (Equilibrium stopping rule) *The stopping rule* $\hat{u}$ *is an equilibrium stopping rule if*

$$\limsup_{\epsilon \to 0+} \frac{J(x; \tau_{\hat{u}}) - J(x; \tau^{\epsilon, a})}{\epsilon} \leq 0, \quad \forall x \in \mathbb{R}, \ \forall a \in \{0, 1\}, \tag{9}$$

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[4]Given the infiniteness of the time horizon, the stationarity of the process X as well the time-homogeneity of the running objective function f, each self at any given time t faces exactly the same decision problem as the others, which only depends on the current state $X_t = x$, but not on time t directly. We can thus identify self t by the current state $X_t = x$. That is why we need to consider only stationary stopping rules u, which are functions of the state variable x only. For details on this convention, see, e.g. Grenadier and Wang (2007); Ekeland et al. (2012); Harris and Laibson (2013) and, in particular, Section 3.2 of Ebert et al. (2016).

where

$$\tau^{\epsilon,a} = \begin{cases} \inf\{t \geq \epsilon, \hat{u}(X_t) = 1\} & \text{if } a = 0, \\ \\ 0 & \text{if } a = 1 \end{cases} \tag{10}$$

with $\{X_t\}_{t \geq 0}$ *being the solution to (3).*

This definition of an equilibrium is consistent with the majority of definition for time-inconsistent control problems in the literature (see, e.g., Bjork and Murgoci 2010; Ekeland et al. 2012; and Björk et al. 2014) when a stopping rule is interpreted as a binary control. Indeed, $\tau^{\epsilon,a}$ is a stopping time that might be different from $\tau_{\hat{u}}$ only in the very small initial time interval $[0, \epsilon)$; hence it is a "pertubation" of the latter.

2.3 Equilibrium characterization

The following result, Theorem 2, formally establishes the Bellman system and provides the verification theorem for verifying equilibrium stoppings.

Theorem 2 (Equilibrium characterization) *Consider the cost functional (22) with WDF* $h(t) = \int_0^\infty e^{-rt} dF(r)$, *a stopping rule* $\hat{u}$, *an underlying process* X *defined by (3), functions* $w(x; r) = \mathbb{E}\left[\int_0^{\tau_{\hat{u}}} e^{-rt} f(X_t) dt + e^{-r\tau_{\hat{u}}} g(X_{\tau_{\hat{u}}}) \Big| X_0 = x\right]$ *and* $V(x) = \int_0^\infty w(x; r) dF(r)$. *Suppose that* w *is continuous in* x *and* V *is continuously differentiable with its first-order derivative being absolutely continuous. If* $(V, w, \hat{u})$ *solves*

$$\min\left\{\frac{1}{2}\sigma^2(x)V_{xx}(x) + b(x)V_x(x) + f(x) - \int_0^\infty rw(x; r) dF(r), g(x) - V(x)\right\} = 0, \ x \in \mathbb{R}, \tag{11}$$

$$\hat{u}(x) = \begin{cases} 1 & \text{if } V(x) = g(x), \\ \\ 0 & \text{otherwise,} \end{cases} \quad x \in \mathbb{R}, \tag{12}$$

then $\hat{u}$ *is an equilibrium stopping rule and* V *is the value function of the problem, i.e.,* $V(x) = J(x; \tau_{\hat{u}}) \ \forall x \in \mathbb{R}$.

A proof to the above proposition is relegated to the appendix.[5]

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3 Failure of SP and Nonexistence of Equilibrium

In the classical literature on (time-consistent) stopping, optimal solutions are often obtained by the SP, because the candidate solution obtained from the SP must solve the Bellman system (and hence the optimal stopping problem) under some mild conditions, such as the smoothness and convexity/concavity of the cost functions. In economics terms, the SP principle amounts to the matching of the marginal cost at the stopped state; hence some economists apply the SP principle without even explicitly introducing the Bellman system. However, as we will show in this section, the SP approach in the presence of time-inconsistency may *not* yield a solution to the Bellman system (and therefore *not* to the stopping problem within the game theoretic framework), no matter how smooth and convex/concave the cost functions might be.

3.1 A time-consistent benchmark

Let us start with a time-consistent optimal stopping problem which we use as a benchmark for comparison purpose and outline the way to use the SP principle in constructing explicit solutions. Consider the following classical optimal stopping problem

$$\inf_{\tau \in \mathcal{T}} \mathbb{E}\left[\int_0^\tau e^{-rs} f(X_s)ds + e^{-r\tau}K \Big| X_0 = x\right], \tag{13}$$

where the underlying process X is a geometric Brownian motion

$$dX_t = bX_t dt + \sigma X_t dW_t, \ x > 0, \tag{14}$$

and $\mathcal{T}$ is the set of all stopping times with respect to $\mathbb{F}$.[6]

In what follows we assume that the running cost f is continuously differentiable, increasing and concave. Moreover, to rule out the "trivial cases" where either immediately stopping or never stopping is optimal for this time consistent benchmark, we assume that $f(0) < rK, b < r$,

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[6]In this formulation the final cost is assumed to be a constant lump sum K without loss of generality. In fact, by properly modifying the running cost, we are able to reduce the stopping problem with a final cost function g to one with a final cost being any given constant $K > 0$. To see this, applying Ito's formula to $e^{-rt}(g(X_t) - K)$,

and $\lim_{x\to\infty} f_x(x)x = \infty$.

Define $L(x; r) = \mathbb{E}[\int_0^\infty e^{-rs} f(X_s)ds | X_0 = x]$. Noting that X is a geometric Brownian motion, we have after straightforward manipulations

$$L(x; r) = \int_0^\infty \int_0^\infty f(yx)e^{-rs} G(y, s) dy ds, \tag{15}$$

where $G(y, s) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma y \sqrt{s}} e^{-\frac{(\ln y - (b - \frac{1}{2}\sigma^2)s)^2}{2\sigma^2 s}}$. To ensure L and L_x are well defined, we further assume that f has linear growth and $f_x(0+) < \infty$.

We now characterize the optimal stopping rule as follows.

Proposition 1 *There exists $x_B > 0$ such that the stopping rule $u_B(x) = \mathbf{1}_{x \geq x_B}(x)$ solves optimal stopping problem (13). Moreover, x_B is the unique solution of the following algebraic equation in y:*

$$\alpha(r)[K - L(y; r)] + L_x(y; r)y = 0 \tag{16}$$

where

$$\alpha(r) = \frac{-(b - \frac{1}{2}\sigma^2) + \sqrt{(b - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}. \tag{17}$$

The key to proving this theorem is to make us of the SP; see Appendix A.2.

we get

$$\mathbb{E}\left[\int_0^\tau e^{-rs} f(X_s)ds + e^{-r\tau} g(X_\tau) \Big| X_0 = x\right]$$

$$= \mathbb{E}\left[\int_0^\tau e^{-rs} f(X_s)ds + e^{-r\tau} K + e^{-r\tau}(g(X_\tau) - K) \Big| X_0 = x\right]$$

$$= \mathbb{E}\left\{\int_0^\tau e^{-rs} f(X_s)ds + e^{-r\tau} K + \int_0^\tau e^{-rs}\left[\frac{1}{2}\sigma^2 x^2 g_{xx}(X_s) + bxg_x(X_s) - r(g(X_s) - K)\right]ds \Big| X_0 = x\right\}.$$

Letting $\tilde{f}(x) := f(x) + \frac{1}{2}\sigma^2 x^2 g_{xx}(x) + bxg_x(x) - r(g(x) - K)$, the cost functional now becomes the one in problem (13) with running cost $\tilde{f}$.

3.2 Equivalent conditions under time-inconsistency

We now consider exactly the same stopping problem as the above time-consistent benchmark except that the exponential discount function is replaced by a WDF, namely, the cost functional is changed to

$$J(x;\tau) = \mathbb{E}\left[\int_0^\tau h(s)f(X_s)ds + h(\tau)K \Big| X_0 = x\right], \tag{18}$$

where h is a WDF with a weighting distribution F.

As in the case of exponential discounting, we need to impose the following regularity conditions on the parameters of the problem:

$$b < r, \ \forall r \in \text{supp}(F); \ \text{and} \ \max\left\{\int_0^\infty \frac{1}{r-b}dF(r), \int_0^\infty \frac{1}{r}dF(r), \int_0^\infty rdF(r)\right\} < \infty.$$

These conditions either hold automatically or reduce to the respective counterparts when the discount function degenerates into the exponential one. On the other hand, they hold valid with many genuine WDFs, including the generalized hyperbolic discount function (2) when $\gamma < \beta$ and the pseudo-exponential discount function.

We now attempt to use the SP principle to solve the Bellman system in Theorem 2 with the cost functional (18). We start by conjecturing that the equilibrium stopping region is $[x_*, \infty)$ for some $x_* > 0$. (As in the time-consistent case, x_* is called the triggering boundary or the stopping threshold.)

It follows from the Feynman–Kac formula that w in the Bellman system is given by

$$w(x;r) = \begin{cases} (K - L(x_*;r))\left(\frac{x}{x_*}\right)^{\alpha(r)} + L(x;r), & x < x_*, \\ K, & x \geq x_*, \end{cases}$$

where $L(x; r)$ is defined by (15) and $\alpha(r)$ by (17). Recall we have defined V and $\hat{u}$ by

$$V(x) = \int_0^\infty w(x; r) dF(r) = \begin{cases} \int_0^\infty ((K - L(x_*; r))(\frac{x}{x_*})^{\alpha(r)} + L(x; r)) dF(r), & x < x_*, \\ K, & x \geq x_* \end{cases}$$

and

$$\hat{u}(x) = \begin{cases} 0 & x < x_*, \\ 1 & \text{otherwise.} \end{cases}$$

The SP applied to V (*not* to w) yields $V_x(x_*) = 0$, implying that x_* is the solution to the following algebraic equation in y

$$\int_0^\infty [\alpha(r)(K - L(y; r)) + L_x(y; r)y] \, dF(r) = 0. \tag{19}$$

Clearly, this equation is a generalization of its time-consistent counterpart, (16). The following proposition stipulates that it has a unique solution.

Proposition 2 *Equation (19) admits a unique solution in* $(0, \infty)$.

 Proof. Following the same lines of proof of Proposition 1 (Appendix A.2), we have that $Q(x) := \int_0^\infty (\alpha(r)(K - L(x; r)) + L_x(x; r)x) dF(r)$ is strictly decreasing in $x > 0$, with $Q(0) > 0$ and $Q(\infty) < 0$. This completes the proof. ∎

 Proposition 2 indicates that following the conventional SP line of argument does indeed give rise to a *candidate* solution to the Bellman system, even under time inconsistency. We may be tempted to claim, as taken for granted in the time-consistent case, that this candidate solution solves the Bellman system in Theorem 2 and hence the corresponding stopping rule $\hat{u}$ solves the equilibrium stoppping problem. Unfortunately, this is not always the case, as shown in the following result.

Theorem 3 *Assume that* $\alpha(r)[\alpha(r) - 1][K - L(x_*; r)]$ *is increasing in* $r \in \text{supp}(F)$, *and let* x_* *be the unique solution to (19). Then the triplet* $(V, w, \hat{u})$ *solves the Bellman system in Theorem 2*

and in particular û is an equilibrium stopping rule if and only if

$$f(x_*) \geq \int_0^\infty r dF(r) K, \tag{20}$$

and

$$\int_0^\infty \alpha(r)[\alpha(r) - 1][K - L(x_*; r)]dF(r) + \int_0^\infty x_*^2 L_{xx}(x_*; r)dF(r) \leq 0. \tag{21}$$

As a proof is lengthy, we defer it to Appendix A.3.

The above theorem presents characterizing conditions (on the model primitives) for the SP to work for stopping problems with general WDFs. These conditions are satisfied automatically in the classical time-consistent case, but not in the time-inconsistent case in general. We will demonstrate this with a classical real option problem in the next subsection.

3.3 A real option problem: failure of SP and nonexistence of equilibrium

In this subsection we consider a special case of the model studied in the previous subsection, which is a time-inconsistent counterpart of the well-studied (time-consistent) problem of real options. Such a problem can be used to model, among others, when to start a new project or to abandon an ongoing project; see Dixit (1993) for a systematic account on the classical real options theory.

The problem is to minimize

$$\mathbb{E}\left[\int_0^\tau h(s)X_s ds + h(\tau)K \Big| X_0 = x\right] \tag{22}$$

by choosing $\tau \in \mathcal{T}$, where $X = \{X_t\}_{t \geq 0}$ is governed by[7]

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[7]Here we assume that the geometric Brownian motion is driftless without loss of generality.

$$dX_t = \sigma X_t dW_t. \tag{23}$$

We now apply Theorem 3 to this problem, and see what the equivalent conditions (20) and (21) boil down to.

First of all,

$$L(x;r) = \mathbb{E}\Big[\int_0^\infty e^{-rt} X_t dt \Big| X_0 = x\Big] = \frac{x}{r}.$$

Hence

$$w(x;r) = \begin{cases} \left(K - \frac{x_*}{r}\right)\left(\frac{x}{x_*}\right)^{\alpha(r)} + \frac{x}{r}, & x < x_*, \\ K, & x \geq x_*, \end{cases}$$

$$V(x) = \begin{cases} \int_0^\infty \left(K - \frac{x_*}{r}\right)\left(\frac{x}{x_*}\right)^{\alpha(r)} dF(r) + \int_0^\infty \frac{x}{r} dF(r), & x < x_*, \\ K, & x \geq x_*, \end{cases}$$

and

$$\hat{u}(x) = \begin{cases} 0 & x < x_*, \\ 1 & \text{otherwise}, \end{cases}$$

where

$$\alpha(r) = \frac{\frac{1}{2}\sigma^2 + \sqrt{\frac{1}{4}\sigma^4 + 2\sigma^2 r}}{\sigma^2}. \tag{24}$$

Moreover, it follows from (19) that x_* is the solution to the following equation in y:

$$\int_0^\infty \left(K - \frac{y}{r}\right)\alpha(r)dF(r) + \int_0^\infty \frac{y}{r}dF(r) = 0.$$

Thus

$$x_* = \frac{\int_0^\infty \alpha(r)dF(r)}{\int_0^\infty \frac{\alpha(r)-1}{r}dF(r)}K. \tag{25}$$

Next, it is easy to verify that $\alpha(r)[\alpha(r)-1][K-L(x_*;r)] = \frac{2}{\sigma^2}(Kr-x_*)$; hence it is an increasing function in $r \geq 0$. Moreover, substituting the explicit representation of x_* in (25) into (20) and (21) we find that the latter two inequalities are both identical to the following single inequality

$$\int_0^\infty \alpha(r)dF(r) \geq \int_0^\infty rdF(r) \int_0^\infty \frac{\alpha(r)-1}{r}dF(r). \tag{26}$$

We have proved the following

Proposition 3 *The triplet $(V, w, \hat{u})$ solves the Bellman system of the real option problem if and only if (26) holds.*

Inequality (26) is a critical condition on the model primitives we must verify before we can be sure that the solution constructed through the SP is indeed an equilibrium solution to the time-inconsistent real option problem. It is immediate to see that the strict inequality of (26) is satisfied *trivially* when the distribution function F is degenerate corresponding to the classical time-consistent case with an exponential discount function. In this case, x_* defined by (25) coincides with the stopping threshold derived in Subsection 3.1. This reconciles with the time-consistent setting.

The condition (26) may hold for some non-exponential discount functions. Consider a generalized hyperbolic discount function

$$h(t) = \frac{1}{(1+\gamma t)^{\frac{\beta}{\gamma}}} \equiv \int_0^\infty e^{-rt} \frac{r^{\frac{\beta}{\gamma}-1}e^{-\frac{r}{\gamma}}}{\gamma^{\frac{\beta}{\gamma}}\Gamma(\frac{\beta}{\gamma})}dr, \quad \gamma > 0, \beta > 0.$$

We assume that $\gamma < \beta \leq \frac{\sigma^2}{2}$. Noting that $\alpha(r) - 1 = -\frac{1}{2} + \frac{\sqrt{\frac{1}{4}\sigma^4 + 2\sigma^2 r}}{\sigma^2}$ is a concave function in r, we

have

$$\alpha(r) - 1 \le (\alpha(r) - 1)'|_{r=0} r + \alpha(0) - 1 = \frac{2}{\sigma^2} r.$$

Moreover, it is easy to see that

$$\int_0^\infty r dF(r) = \beta \text{ and } \alpha(r) \ge 1.$$

Therefore,

$$\int_0^\infty \frac{\alpha(r) - 1}{r} dF(r) \int_0^\infty r dF(r) \le \beta \frac{2}{\sigma^2} \le 1 \le \int_0^\infty \alpha(r) dF(r)$$

which is (26). So, in this case the SP works and the stopping threshold x_* is given by

$$x_* = \frac{\displaystyle\int_0^\infty \alpha(r) \frac{r^{\frac{\beta}{\gamma}-1} e^{-\frac{r}{\gamma}}}{\gamma^{\frac{\beta}{\gamma}} \Gamma(\frac{\beta}{\gamma})} dr}{\displaystyle\int_0^\infty \frac{\alpha(r) - 1}{r} \frac{r^{\frac{\beta}{\gamma}-1} e^{-\frac{r}{\gamma}}}{\gamma^{\frac{\beta}{\gamma}} \Gamma(\frac{\beta}{\gamma})} dr} K.$$

However, it is also possible that (26) fails, which is the case even with the simplest class of non-exponential WDFs – the pseudo-exponential discount functions. To see this, let $h(t) = \delta e^{-rt} + (1 - \delta) e^{-(r+\lambda)t}, 0 < \delta < 1, r > 0, \lambda > 0$. It is straightforward to obtain that

$$\int_0^\infty \alpha(r) dF(r) = \delta \alpha(r) + (1 - \delta) \alpha(r + \lambda)$$

and

$$\int_0^\infty r dF(r) \int_0^\infty \frac{\alpha(r) - 1}{r} dF(r) > (1 - \delta)(r + \lambda) \delta \left(\frac{\alpha(r) - 1}{r} \right).$$

Since $(1 - \delta)(r + \lambda)\delta(\frac{\alpha(r)-1}{r})$ grows faster than $\delta\alpha(r) + (1 - \delta)\alpha(r + \lambda)$ when λ becomes large, we conclude that (26) is violated when r, δ are fixed and λ is sufficiently large.

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What we have discussed so far shows that the solution constructed through the SP does not solve the time-inconsistent real option problem whenever inequality (26) fails. A natural question in this case is whether there might exist equilibrium solutions that cannot be obtained by the SP or even by the Bellman system. The answer is resoundingly negative.

Proposition 4 *For the real option problem (22)–(23), if (26) does not hold, then no equilibrium stopping rule exists.*

Proof. We prove by contradiction. Suppose $\hat{u}$ is an equilibrium stopping rule. We first note that $\mathcal{C}_{\hat{u}} \equiv \{x > 0 : \hat{u}(x) = 0\} \neq (0, \infty)$; otherwise $\hat{u} \equiv 0$, leading to $J(x; \tau_{\hat{u}}) = \int_0^\infty L(x; r)dF(r) = \int_0^\infty \frac{x}{r}dF(r)$, and hence $J(x; \tau_{\hat{u}}) \to \infty$ as $x \to \infty$ contradicting Lemma 2 in Appendix A.2.

Define $x_* = \inf\{x : x \in \bar{\mathcal{S}}_{\hat{u}}\}$. It follows from Lemma 3 in Appendix A.2 that $x_* \in (0, \infty)$. A standard argument then leads to

$$J(x; \tau_{\hat{u}}) = \int_0^\infty \left(K - \frac{x_*}{r}\right)\left(\frac{x}{x_*}\right)^{\alpha(r)} dF(r) + \int_0^\infty \frac{x}{r}dF(r), \quad x \in (0, x_*].$$

Because $J(x; \tau_{\hat{u}}) \leq K$ and $J(x_*; \tau_{\hat{u}}) = K$, we have $J_x(x_*-; \tau_{\hat{u}}) \geq 0$, i.e.,

$$\int_0^\infty \left(K - \frac{x_*}{r}\right)\alpha(r)\frac{1}{x_*}dF(r) + \int_0^\infty \frac{1}{r}dF(r) \geq 0,$$

which in turn gives

$$x_* \leq \frac{\int_0^\infty \alpha(r)dF(r)}{\int_0^\infty \frac{\alpha(r)-1}{r}dF(r)}K.$$

Combining with the failure of condition (26), we derive

$$x_* < \int_0^\infty rdF(r),$$

which contradicts Lemma 3. This completes the proof. ∎

The above is a stronger result. It suggests that for the problem to have any equilibrium

stopping rule at all (not necessarily the one obtainable by the SP principle), condition (26) *must* hold. So, when it comes to a time-inconsistent stopping problem with non-exponential discounting, it is highly likely that no equilibrium stopping rule exists, even if the SP principle does generate a "solution", or even if the time-consistent counterpart (in which everything else is identical except the discount function) is indeed solvable by the SP. Applying these conclusions to the pseudo-exponential discount functions discussed above, we deduce that there is no equilibrium stopping when λ is sufficiently large.

Having said this, a logical conclusion from Propositions 3 and 4 is that when equilibria do exist, one of them must be a solution generated by the SP. In general if there exists an equilibrium then there may be multiple ones; see, for example, Krussell and Smith (2003) and Ekeland and Pirvu (2008) for multiple equilibria in time-inconsistent control problems. In this case, the SP can only generate *one* of them, but not necessarily *all* of them. (This statement is true even for a classical time-consistent stopping problem.) So, after all, the SP is still a useful, proper method for time-inconsistent problems; we can simply apply it to generate a candidate solution. If the solution is an equilibrium (which we must verify), then we have found one (but not necessarily other equilibria); if it is not an equilibrium, then we know there is no equilibrium at all.

4   Conclusions

While the SP principle has been widely used to study time-inconsistent stopping problems, our results indicate the risk of using this principle on such problems. We have shown that the SP principle solves the time-inconsistent problem if and only if certain inequalities are satisfied.

By a simple model of the classical real option problem, we have found that these inequalities may be violated even for simple and commonly used non-exponential discount functions. When the SP principle fails, we have shown the intra-personal equilibrium does not exist. The nonexistence result and the failure of the SP principle suggest that it is imperative that the techniques for conventional optimal stopping problems be used more carefully when extended to solving

time-inconsistent stopping problems.

A   Appendix: Proofs

A.1   Proof of Theorem 2

For the stopping time $\tau^{\epsilon,a}$, if $a = 1$, then $J(x; \tau^{\epsilon,a}) = g(x)$. The Bellman equation (11) implies that $g(x) \geq V(x) \equiv J(x; \tau_{\hat{u}})$. This yields (9).

If $a = 0$, then

$$
\begin{aligned}
J(x; \tau^{\epsilon,a}) &= \mathbb{E}\left[\int_0^\epsilon h(s)f(X_s)ds \Big| X_0 = x\right] + \mathbb{E}\left[\int_\epsilon^{\tau^{\epsilon,a}}(h(s) - h(s - \epsilon))f(X_s)ds \Big| X_0 = x\right] \\
&\quad + \mathbb{E}[(h(\tau^{\epsilon,a}) - h(\tau^{\epsilon,a} - \epsilon))g(X_{\tau^{\epsilon,a}})|X_t = x] + \mathbb{E}[V(X_\epsilon)|X_0 = x] \\
&= \mathbb{E}\left[\int_0^\epsilon h(s)f(X_s)ds \Big| X_0 = x\right] + \mathbb{E}\left[\int_\epsilon^{\tau^{\epsilon,a}}\int_0^\infty e^{-r(s-\epsilon)}(e^{-\epsilon r} - 1)dF(r)f(X_s)ds \Big| X_0 = x\right] \\
&\quad + \mathbb{E}\left[\int_0^\infty e^{-r(\tau^{\epsilon,a}-\epsilon)}(e^{-\epsilon r} - 1)dF(r)g(X_{\tau^{\epsilon,a}}) \Big| X_0 = x\right] + \mathbb{E}[V(X_\epsilon)|X_0 = x] \\
&= \mathbb{E}\left[\int_0^\epsilon h(s)f(X_s)ds \Big| X_0 = x\right] + \mathbb{E}\left[\int_0^\infty (e^{-\epsilon r} - 1)w(X_\epsilon; r)dF(r) \Big| X_0 = x\right] \\
&\quad + \mathbb{E}[V(X_\epsilon)|X_0 = x]. \tag{27}
\end{aligned}
$$

Define $\tau_n = \inf\{s \geq 0 : \sigma(X_s)V_x(X_s) > n\} \wedge \epsilon$. Then it follows from Ito's formula that

$$
\mathbb{E}[V(X_{\tau_n})|X_0 = x] = \mathbb{E}\left[\int_0^{\tau_n}(\frac{1}{2}\sigma^2(X_s)V_{xx}(X_s) + b(X_s)V_x(X_s))ds \Big| X_0 = x\right] + V(x).
$$

By (11), we conclude

$$
\begin{aligned}
\mathbb{E}[V(X_{\tau_n})|X_0 = x] &= \mathbb{E}\left[\int_0^{\tau_n}(\frac{1}{2}\sigma^2(X_s)V_{xx}(X_s) + b(X_s)V_x(X_s))ds \Big| X_0 = x\right] + V(x) \\
&\geq \mathbb{E}\left[\int_0^{\tau_n}(-f(X_s) + \int_0^\infty rw(X_s; r)dF(r))ds \Big| X_0 = x\right] + V(x).
\end{aligned}
$$

Note that conditions (6) and (7) ensure that $-f(x) + \int_0^\infty rw(x; r)dF(r)$ has polynomial growth,

i.e., there exist $C > 0, m \geq 1$ such that

$$\left| -f(x) + \int_0^\infty rw(x;r)dF(r) \right| \leq C(|x|^m + 1),$$

which leads to

$$\sup_{0 \leq t \leq \epsilon} \left| -f(X_s) + \int_0^\infty rw(X_s;r)dF(r) \right| \leq C(\sup_{0 \leq t \leq \epsilon} |X_t|^m + 1).$$

Moreover, under condition (4), it follows from standard SDE theory (see, for example, Chapter 1 of Yong and Zhou (1999)) that equation (3) admits a unique strong solution X satisfying

$$\mathbb{E}[\sup_{0 \leq t \leq \epsilon} |X_t|^m | X_0 = x] \leq K_\epsilon(|x|^m + 1)$$

with $K_\epsilon > 0$.

Then letting $n \to \infty$, we conclude by the dominated convergence theorem that

$$\mathbb{E}\left[V(X_\epsilon)|X_0 = x\right] \geq \mathbb{E}\left[\int_0^\epsilon (-f(X_s) + \int_0^\infty rw(X_s;r)dF(r))ds \middle| X_0 = x\right] + V(x).$$

Consequently,

$$\liminf_{\epsilon \to 0+} \frac{J(x;\tau^{\epsilon,a}) - J(x;\tau_{\hat{u}})}{\epsilon}$$

$$\geq \liminf_{\epsilon \to 0+} \mathbb{E}\left[\int_0^\epsilon h(s)f(X_s)ds \middle| X_0 = x\right] + \mathbb{E}\left[\int_0^\infty (e^{-\epsilon r} - 1)w(X_\epsilon;r)dF(r) \middle| X_0 = x\right]$$

$$+ \liminf_{\epsilon \to 0+} \frac{1}{\epsilon}\mathbb{E}\left[\int_0^\epsilon \int_0^\infty (rw(X_t;r)dF(r) - f(X_t))dt \middle| X_0 = x\right].$$

The continuity of f and w along with the polynomial growth conditions (6) and (7) allow the use of the dominated convergence theorem, which yields

$$\liminf_{\epsilon \to 0+} \frac{J(x;\tau^{\epsilon,a}) - J(x;\tau_{\hat{u}})}{\epsilon} \geq 0.$$

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This completes the proof.

A.2 Proof of Proposition 1

Let V^B be the value function of the optimal stopping problem. It follows from the standard argument (see, for example, Chapter 6 of Krylov 2008) that V^B is continuously differentiable and its first-order derivative is absolutely continuous. Moreover, V^B solves the following Bellman equation

$$\min \left\{ \frac{1}{2}\sigma^2 x^2 V_{xx}^B(x) + bx V_x^B(x) + f(x) - r V^B(x), K - V^B(x) \right\} = 0. \tag{28}$$

Define the continuation region $\mathcal{C}^B = \{x > 0 : V^B(x) < K\}$ and the stopping region $\mathcal{S}^B = \{x > 0 : V^B(x) = K\}$.

We claim that $\mathcal{S}^B \neq (0, \infty)$. If not, then $V^B \equiv K$. Thus $\frac{1}{2}\sigma^2 x^2 V_{xx}^B(x) + bx V_x^B(x) + f(x) - r V^B(x) < 0$ whenever $x \in \{x > 0 : f(x) - rK < 0\}$. However, since $f(0) < rK$, the continuity of f implies $\{x > 0 : f(x) - rK < 0\} \neq \emptyset$. This contradicts the Bellman equation (28).

We now show that $\mathcal{C}^b \neq (0, \infty)$. If it is false, then we have $V^B(x) = L(x; r)$, with L defined by (15). Since f is increasing and bounded from below by 0, we have

$$V^B(\infty) \equiv \lim_{x \to \infty} V^B(x) = \int_0^\infty \int_0^\infty \lim_{x \to \infty} f(yx) e^{-rs} G(y, s) dy ds.$$

The concavity of f yields $f(x) \geq x f_x(x) + f(0)$. It then follows from $\lim_{x \to \infty} x f_x(x) = \infty$ that $\lim_{x \to \infty} f(x) = \infty$, which yields that $V^B(\infty) = \infty$. This contradicts the fact that $V^B(x) \leq K$.

Next, since X is a geometric Brownian motion and f is increasing, it is clear that V is increasing too. Now, we derive the value of the triggering boundary, x_B, via the SP principle. Specifically, it follows from (28) that

$$V^b(x) = (K - L(x_B; r))(\frac{x}{x_B})^{\alpha(r)} + L(x; r), \quad x < x_B$$

$$V^b(x) = K, x \geq x_B,$$

where $\alpha(r)$ is defined by (17). Then the SP implies that $V_x^B(x_B) = 0$ which after some calculations yields that x_B is the solution of the equation (16).

To prove the unique existence of the solution of (16), define $Q(x) := \alpha(r)(K - L(x; r)) + L_x(x; r)x$. Then $Q_x(x) = (-\alpha(r) + 1)L_x(x; r) + L_{xx}(x; r)x$. As L is strictly increasing and concave and $\alpha(r) > 1$, we deduce that Q is strictly decreasing. It remains to show $Q(0) > 0$ and $Q(\infty) < 0$. It is easy to see that $Q(0) = \alpha(r)(K - L(0; r)) = \alpha(r)(K - \frac{f(0)}{r}) > 0$ and $Q(x) = \alpha(r)(K - L(0; r) - \int_0^x L_x(s; r)ds) + L_x(x; r)x$. Since L is concave, we have $\int_0^x L_x(s; r)ds \geq xL_x(x; r)$. Thus $Q(x) \leq \alpha(r)(K - L(0; r)) + (-\alpha(r) + 1)xL_x(x; r)$. Recalling that $\lim_{x \to \infty} xL_x(x; r) = \infty$ and $\alpha(r) > 1$, we have $Q(\infty) = -\infty$. This completes the proof.

A.3 Proof of Theorem 3

We need to present a series of lemmas before giving a proof of Theorem 3.

Lemma 1 *Given a stopping rule u and a discount rate $r > 0$, the function $E(x; \tau_u, r) := \mathbb{E}[\int_0^{\tau_u} e^{-rt}f(X_t)dt + e^{-r\tau_u}K|X_0 = x]$ is continuous in $x \in (0, \infty)$.*

Proof. We prove the right continuity of $E(\cdot; \tau_u, r)$ at a given $x_0 > 0$; the left continuity can be discussed in the same way.

If there exists $\delta > 0$ such that $(x_0, x_0 + \delta) \in \mathcal{S}_u$, then the right continuity of $E(\cdot; \tau_u, r)$ at x_0 is obtained immediately. If there exists $\delta > 0$ such that $(x_0, x_0 + \delta) \in \mathcal{C}_u$, then it follows from the Feynman-Kac formula that $E(\cdot; \tau_u, r)$ is the solution to the differential equation $\frac{1}{2}\sigma^2 x^2 E_{xx} + bxE_x - rE + f = 0$ on $(x_0, x_0 + \delta)$. This in particular implies that $E(\cdot; \tau_u, r) \in C^2((x_0, x_0 + \delta)) \cap C([x_0, x_0 + \delta])$ due to the regularity of f and the coefficients of the differential equations; hence the right continuity of $E(\cdot; \tau_u, r)$ at x_0.

Otherwise, we first assume that $f(x_0) \geq rK$ and consider the set $\mathcal{C}_u \cap (x_0, \infty)$. Since it is an open set, we have $\mathcal{C}_u \cap (x_0, \infty) = \cup_{n \geq 1}(a_n, b_n)$, where $a_n, b_n \in \bar{\mathcal{S}}_u, \forall n \geq 1$. It is then easy to see that x_0 is an accumulation point of $\{a_n\}_{n \geq 1}$ and hence $x_0 \in \bar{\mathcal{S}}_u$.

Define $I(x) := E(x; \tau_u, r) - K$ for $x \in (a_n, b_n)$. It is easy to see that I solves the following

differential equation

$$\frac{1}{2}\sigma^2 x^2 I_{xx}(x) + bx I_x(x) - rI(x) + f(x) - rK = 0. \tag{29}$$

with the boundary conditions

$$I(a_n) = I(b_n) = 0.$$

Consider an auxiliary function H that solves the following differential equation

$$\frac{1}{2}\sigma^2 x^2 H_{xx}(x) + bx H_x(x) - rH(x) + f(x) - rK = 0,$$

with the boundary conditions

$$H(x_0) = H(b_1) = 0.$$

Since $f(x) > rK$ on (x_0, ∞), the comparison principle shows that $H(x) \geq 0, \forall x \in [x_0, b_1]$. Applying the comparison principle again on any $(a_n, b_n) \cap (x_0, b_1), \forall n \in \mathbb{N}^+$, we have $0 \leq I(x) \leq H(x)$. Noting that $H(x) \to H(x_0) = 0$ as $x \to x_0+$, we conclude that $I(\cdot)$ is right continuous at x_0 and so is $E(\cdot; \tau_u, r)$.

For the case $f(x_0) < rK$, a similar argument applies. Indeed, consider an auxiliary function H_1 satisfying the differential equation (29) on $(x_0, f^{-1}(rK))$ with the boundary condition $H_1(x_0) = H_1(f^{-1}(rK)) = 0$. The comparison principle yields that $H_1(x) \leq I(x) \leq 0$ on $(a_n, b_n) \cap (x_0, f^{-1}(rK)), \forall n \in \mathbb{N}^+$. The right continuity of $I(\cdot)$ and $E(\cdot; \tau_u, r)$ then follows immediately. ∎

Lemma 2 *If $\hat{u}$ is an equilibrium stopping rule, then $J(x; \tau_{\hat{u}}) \leq K \ \forall x \in (0, \infty)$.*

Proof. If there exists $x_0 \in (0, \infty)$ such that $J(x_0; \tau_{\hat{u}}) > K$, then we have

$$\limsup_{\epsilon \to 0} \frac{J(x_0; \tau_{\hat{u}}) - J(x_0; \tau^{\epsilon,1})}{\epsilon} = \infty,$$

where $\hat{u}^{\epsilon,1}$ is given by (10). This contradicts the definition of an equilibrium stopping rule. ∎

Lemma 3 *If $\hat{u}$ is an equilibrium stopping rule, then $\{x > 0 : f(x) < \int_0^\infty r dF(r) K\} \subset \mathcal{C}_{\hat{u}}$.*

Proof. Suppose that there exists $x \in \{x > 0 : f(x) < \int_0^\infty r dF(r) K\} \cap \bar{\mathcal{S}}_{\hat{u}}$, then it follows from Lemma 2 that $\mathbb{E}[J(X_t; \tau_{\hat{u}})|X_0 = x] \leq K$. Consider the stopping time $\tau^{\epsilon,0}$. Equation (27) and the fact that $J(x; \tau_{\hat{u}}) = K$ give

$$\frac{J(x; \tau^{\epsilon,0}) - J(x; \tau_{\hat{u}})}{\epsilon} \leq \frac{1}{\epsilon} \mathbb{E}\left[\int_0^\epsilon h(s) f(X_s) ds \Big| X_0 = x\right] + \mathbb{E}\left[\int_0^\infty \left(\frac{e^{-\epsilon r} - 1}{\epsilon}\right) w(X_\epsilon; r) dF(r) \Big| X_0 = x\right].$$

As $w(\cdot, r)$ is continuous (Lemma 1) and $w(x; r) = K$, we have

$$\liminf_{\epsilon \to 0} \frac{J(x; \tau^{\epsilon,0}) - J(x; \tau_{\hat{u}})}{\epsilon} \leq f(x) - \int_0^\infty r K dF(r) < 0.$$

This contradicts the definition of an equilibrium stopping rule. ∎

We now turn to the proof of Theorem 3. We begin with the sufficiency. To this end it suffices to show that $V(x) \leq K, x \in (0, x_*)$ and $f(x) - \int_0^\infty r dF(r) K \geq 0, x \in (x_*, \infty)$.

We first show that $V_{xx} \leq 0, x \in (0, x_*)$. By simple algebra, we have

$$V_{xx}(x) = \int_0^\infty \alpha(r)(\alpha(r) - 1)(K - L(x; r))(\frac{x}{x_*})^{\alpha(r)} \frac{1}{x^2} dF(r) + \int_0^\infty L_{xx}(x; r) dF(r).$$

As L is concave, we only need to prove $\int_0^\infty \alpha(r)(\alpha(r) - 1)(K - L(x; r))(\frac{x}{x_*})^{\alpha(r)} dF(r) \leq 0$. It is easy to see that $(\frac{x}{x_*})^{\alpha(r)}$ is decreasing in r given that $\alpha(r)$ is increasing in r and $x < x_*$. Then the

rearrangement inequality (e.g., Chapter 10 of Hardy et al. 1952; Lehmann et al. 1966) yields[8]

$$\int_0^\infty \alpha(r)(\alpha(r) - 1)(K - L(x;r))(\frac{x}{x_*})^{\alpha(r)}dF(r)$$

$$\leq \int_0^\infty \alpha(r)(\alpha(r) - 1)(K - L(x;r))dF(r)\int_0^\infty (\frac{x}{x_*})^{\alpha(r)}dF(r). \tag{30}$$

Therefore it follows from (21) that $V_{xx}(x) \leq 0$, $x \in (0, x_*)$. Now, $V_x(x_*) = 0$. Thus $V_x(x) \geq 0$ and consequently $V(x) \leq K \ \forall x \in (0, x_*)$, due to $V(x_*) = K$.

Next, the inequality $f(x) - \int_0^\infty rdF(r)K \ \forall x \in (x_*, \infty)$ follows from f being increasing along with inequality (20). This completes the proof of the sufficiency.

We now turn to the necessity part. As (20) is an immediate corollary of Lemma 3, we only need to prove (21). Suppose (21) does not hold. Then by a simple calculation, we have

$$V_{xx}(x_*-) = \int_0^\infty \alpha(r)(\alpha(r) - 1)(K - L(x_*;r))\frac{1}{x_*^2}dF(r) + \int_0^\infty L_{xx}(x_*;r)dF(r) > 0.$$

However, $V_x(x_*) = 0$, implying that there exists $x_1 \in (0, x_*)$ such that $V_x(x) < 0$ on $x \in (x_1, x_*)$. Then it follows from $V(x_*) = K$ that $V(x) > K$ when $x \in (x_1, x_*)$, which contradicts Lemma 2.

References

BAYRAKTAR, E., J. ZHANG, AND Z. ZHOU (2018): "Time consistent stopping for the mean-standard deviation problem—The discrete time case," *Available at SSRN 3128866*.

BERNSTEIN, S. (1928): "Sur les fonctions absolument monotones," *Acta Mathematica*, 52, 1–66.

BJORK, T. AND A. MURGOCI (2010): "A general theory of Markovian time inconsistent stochastic control problems," *Available at SSRN 1694759*.

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[8]Inequality (30) can be read as

$$\text{cov}(X, Y) \leq 0,$$

with $X = \alpha(R)(\alpha(R) - 1)(K - L(x;R))(\frac{x}{x_*})^{\alpha(R)}$ and $Y = (\frac{x}{x_*})^{\alpha(R)}$, where R is a random variable with distribution function F. Because of the monotonicity of X, Y in R, X and Y are anti-comonotonic. Then inequality (30) follows from the fact that the covirance of two anti-comonotonic random variables is non-positive.

BJÖRK, T., A. MURGOCI, AND X. ZHOU (2014): "Mean-variance portfolio optimization with state dependent risk aversion," *Mathematical Finance*, 24, 1–24.

CHRISTENSEN, S. AND K. LINDENSJÖ (2018b): "On finding equilibrium stopping times for time-inconsistent Markovian problems," *SIAM Journal on Control and Optimization*, 56, 4228–4255.

CHRISTENSEN, S. AND K. LINDENSJÖ (2018a): "On time-inconsistent stopping problems and mixed strategy stopping times," *Available at arXiv 1804.07018*.

DIXIT, A. K. (1993): *The art of smooth pasting*, vol. 2, Routledge, London and New York.

EBERT, S., W. WEI, AND X. ZHOU (2016): "Weighted discounting–On group diversity, time-inconsistency, and consequences for investment," *Available at SSRN 2840240*.

EKELAND, I. AND A. LAZRAK (2006): "Being serious about non-commitment: Subgame perfect equilibrium in continuous time," *Available at arXiv 0604264*.

EKELAND, I., O. MBODJI, AND T. A. PIRVU (2012): "Time-consistent portfolio management," *SIAM Journal on Financial Mathematics*, 3, 1–32.

EKELAND, I. AND T. PIRVU (2008): "Investment and consumption without commitment," *Mathematics and Financial Economics*, 2, 57–86.

GRENADIER, S. AND N. WANG (2007): "Investment under uncertainty and time inconsistent preferences," *Journal of Financial Economics*, 84, 2–39.

HARDY, G. H., J. E. LITTLEWOOD, AND G. PÓLYA (1952): *Inequalities*, Cambridge university press, Cambridge.

HARRIS, C. AND D. LAIBSON (2013): "Instantaneous gratification," *Quarterly Journal of Economics*, 128, 205–248.

HSIAW, A. (2013): "Goal-setting and self-control," *Journal of Economic Theory*, 148, 601–626.

HUANG, Y.-J. AND A. NGUYEN-HUU (2018): "Time-consistent stopping under decreasing impatience," *Finance and Stochastics*, 22, 69–95.

ITO, K. AND P. MCKEAN JR (1965): *Diffusion processes and their sample paths*, Springer, Berlin Heidelberg.

KARP, L. (2007): "Non-constant discounting in continuous time," *Journal of Economic Theory*, 132, 557–568.

KRUSSELL, P. AND A. SMITH (2003): "Consumption-savings decision with quasi-geometric discounting," *Econometrica*, 71, 365–375.

KRYLOV, N. V. (2008): *Controlled diffusion processes*, vol. 14, Springer, Berlin Heidelberg.

LAIBSON, D. (1997): "Golden eggs and hyperbolic discounting," *Quarterly Journal of Economics*, 112, 443–378.

LEHMANN, E. L. ET AL. (1966): "Some concepts of dependence," *The Annals of Mathematical Statistics*, 37, 1137–1153.

LOEWENSTEIN, G. AND D. PRELEC (1992): "Anomalies in intertemporal choice: Evidence and an interpretation," *The Quarterly Journal of Economics*, 573–597.

LUTTMER, E. AND T. MARIOTTI (2003): "Subjective discounting in an exchange economy," *Journal of Political Economy*, 11, 959–989.

O'DONOGHUE, T. AND M. RABIN (2001): "Choice and procrastination," *Quarterly Journal of Economics*, 116, 112–160.

PESKIR, G. AND A. SHIRYAEV (2006): *Optimal stopping and free-boundary problems*, Birkhäuser, Basel.

PHELPS, E. AND R. POLLAK (1968): "On second-best national saving and game-equilibrium growth," *Review of Economic Studies*, 35, 185–199.

SAMUELSON, P. (1937): "A note on measurement of utility," *Review of Economic Studies*, 4, 155–161.

STROTZ, R. (1955): "Myopia and inconsistency in dynamic utility maximization," *Review of Economic Studies*, 23, 165–180.

YONG, J. AND X. Y. ZHOU (1999): *Stochastic controls: Hamiltonian systems and HJB equations*, vol. 43, Springer, New York.