DYNAMICS OF TIME-PERIODIC REACTION-DIFFUSION EQUATIONS WITH FRONT-LIKE INITIAL DATA ON $\mathbb R$

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ABSTRACT. This paper is concerned with the Cauchy problem

$$\begin{cases} u_t = u_{xx} + f(t, u), & x \in \mathbb{R}, t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where f is a rather general nonlinearity that is periodic in t, and satisfies $f(\cdot,0) \equiv 0$ and that the corresponding ODE has a positive periodic solution p(t). Assuming that u_0 is front-like, that is, $u_0(x)$ is close to p(0) for $x \approx -\infty$ and close to 0 for $x \approx \infty$, we aim to determine the long-time dynamical behavior of the solution u(t,x) by using the notion of propagation terrace introduced by Ducrot, Giletti and Matano (2014). We establish the existence and uniqueness of propagating terrace for a very large class of nonlinearities, and show the convergence of the solution u(t,x) to the terrace as $t \to \infty$ under various conditions on f or u_0 . We first consider the special case where u_0 is a Heaviside type function, and prove the converge result without requiring any non-degeneracy on f. Furthermore, if u_0 is more general such that it can be trapped between two Heaviside type functions, but not necessarily monotone, we show that the convergence result remains valid under a rather mild non-degeneracy assumption on f. Lastly, in the case where f is a non-degenerate multistable nonlinearity, we show the global and exponential convergence for a much larger class of front-like initial data.

1. Introduction and main results

In this paper, we consider the following Cauchy problem

$$u_t = u_{xx} + f(t, u), \qquad x \in \mathbb{R}, \ t > 0, \tag{1.1a}$$

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.1b}$$

where the initial data $u_0 \in L^{\infty}(\mathbb{R})$ is piecewise continuous. The nonlinearity $f : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is locally Hölder continuous in $\mathbb{R} \times [0, \infty)$, and it is of class C^1 with respect to u. We assume that

$$f(t,0) = 0 \text{ for all } t \in \mathbb{R}, \tag{1.2}$$

and that f is T-periodic in t for some T > 0, that is,

$$f(t+T,u) = f(t,u) \text{ for all } t \in \mathbb{R}, u \ge 0.$$

$$(1.3)$$

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If the solution u is spatially homogeneous, then u = u(t) satisfies the following ODE, which will play an important role in the later argument:

$$\frac{du}{dt} = f(t, u), \quad t > 0. \tag{1.4}$$

We assume that (1.4) has a positive T-periodic solution p(t). Namely, p(t) is a function that satisfies

$$\begin{cases} \frac{dp}{dt} = f(t, p) & \text{for } t \in \mathbb{R}, \\ p(t) \equiv p(t+T) & \text{for } t \in \mathbb{R}. \end{cases}$$
 (1.5)

In the special case where f is independent of t, that is, when (1.1a) is autonomous, the solution p of (1.5) is nothing but a zero of f = f(u).

In the present work, we study the long-time behavior of solutions of (1.1) with "front-like" initial data. Roughly speaking, u_0 is assumed to satisfy $0 \le u_0(\cdot) \le p(0)$ and that it is close to p(0) for $x \approx -\infty$ and close 0 for $x \approx \infty$ (our actual hypotheses on u_0 will be formulated later). Our aim is to establish results that cover a large class of nonlinearities, including but not limited to the classical monostable, bistable, ignition nonlinearities.

For f belonging to one of the three classic types of nonlinearities mentioned above, it is well known that the asymptotic behavior of u(t,x) can be described by periodic traveling waves. By a **periodic traveling wave** connecting 0 and p, we mean an entire solution U(t,x) of (1.1a) satisfying that, for some $c \in \mathbb{R}$,

$$U(t+T,x+cT) \equiv U(t,x) \tag{1.6}$$

along with the asymptotics

$$\lim_{x\to\infty} U(t,x) = 0, \quad \lim_{x\to -\infty} U(t,x) = p(t) \quad \text{locally uniformly in } t\in \mathbb{R}.$$

The real number c is called the **wave speed** of U. It is easily checked that U(t,x) is a periodic traveling wave connecting 0 and p with wave speed c if and only if it has the form $U(t,x) = \widetilde{U}(t,x-ct)$, where \widetilde{U} is T-periodic in its first variable and satisfies $\widetilde{U}(t,\infty) = 0$ and $\widetilde{U}(t,-\infty) = p(t)$ uniformly in $t \in \mathbb{R}$.

It is known that, in the bistable or combustion cases, there exists a unique (up to spatial shifts) periodic traveling wave connecting 0 and p, and its speed is uniquely determined (see [1, 4, 26, 28]), while in the monostable case, there exists a continuum of admissible speeds $[c_*, \infty)$ (see [17, 27]). It was also proved in [1, 4, 25] that for bistable equations, any solution with front-like initial data converges to the periodic traveling wave. For the convergence in the monostable or combustion cases, some additional assumption on the asymptotics of $u_0(x)$ as $x \to \infty$ is needed (see [16, 27, 29]).

For general f, the situation is more complicated, even for the autonomous equation

$$u_t = u_{xx} + f(u) \quad \text{for } t > 0, \ x \in \mathbb{R}. \tag{1.7}$$

Indeed, if there are other stable zeros of f between 0 and p, a traveling wave of (1.7) connecting 0 and p may not exist (see e.g., [11]). In such a case, the asymptotic behavior of (1.7) cannot be represented by a single wave; it is represented by a combination of multiple waves, namely, a stacked family of traveling waves whose speeds may differ from one another. Such systems of traveling waves were first studied by Fife and McLeod in [11, 12] under the name "minimal decomposition". Assuming that f is a stacked composition of two bistable nonlinearities and that the zeros of f are non-degenerate, they proved in [11] that, for all front-like initial data,

The notion of propagating terrace was first introduced by Ducrot, Giletti and Matano in [9], which is concerned with a more general framework of spatially periodic equations, namely, f = f(x, u) is periodic in x (we will give the precise definition of propagating terrace later, directly in the framework of time-periodic equation (1.1a)). Under some stability assumption on the state p, it was shown in [9] that any solution with Heaviside type initial data converges to a propagating terrace. In a follow-up paper [13], further properties of propagating terrace were studied, and the convergence result was generalized. Apart from the aforementioned works for one-dimensional equations, it is known from some recent progress [7, 14, 22] that the propagating terrace is also a fundamental concept in understanding the propagation dynamics of high-dimensional equations.

In the present paper, we focus our attention on problem (1.1) with time-periodic nonlinearity f, and assume that the initial function u_0 is front-like. Our results reveal that the propagating terrace also plays a crucial role in determining the long-time behavior of solutions of (1.1) with general f. We first show the existence and uniqueness of propagating terrace under some generic condition on f and prove that any solution of (1.1) starting from Heaviside type function converges to this propagating terrace. These results do not require any non-degeneracy of f. The proof is similar to that given in [9, 13] for spatially periodic equations.

The main part of our paper is devoted to the study of the convergence to propagating terrace for more general front-like initial data. Although this problem has been well addressed in the autonomous case [23], the presence of time heterogeneity makes it significantly more difficult. Indeed, the proof given in [23] relies strongly on the method of phase plane analysis, while the usual ODE tools no longer work in our nonautonomous case. In our results, we develop the steepness arguments introduced in [9, 13], which allow us to handle the case where u_0 can be trapped between two Heaviside type initial functions, but not necessarily monotone (see (H2) below). We first give a precise description of the asymptotic behavior of solutions of (1.1) with such initial data, and then prove the convergence to a propagating terrace under a mild non-degeneracy condition on f. Moreover, in the case where f is a non-degenerate multistable nonlinearity, we show the global and exponential convergence for a much larger class of u_0 (see (H3) below). The proof of this result is based on a super and sub-solution method.

As announced above, the present work is concerned with the propagation dynamics of (1.1) with front-like initial data. We mention here that, for other types of initial data, such as nonnegative and compactly supported functions or more general functions with limits at $x \to \infty$

 $\pm \infty$ equal to 0, the asymptotic behavior of (1.1) has been extensively studied in the autonomous case (see e.g., [6, 7, 8, 19, 33]) and the nonautonomous case (see e.g., [5, 10, 21]).

1.1. **Propagating terrace: some definitions.** As mentioned above, the notion of propagating terrace was introduced in [9] for spatially periodic equations (see also [13] for a slightly generalized version). For our time-periodic equation (1.1a), the definition of propagating terrace can be presented as follows.

Definition 1.1. A propagating terrace connecting 0 to p is a pair of finite sequences $(p_i)_{0 \le i \le N}$ and $(U_i, c_i)_{1 \le i \le N}$ such that

• Each p_i is a nonnegative solution of (1.5) satisfying

$$p = p_0 > p_1 > \dots > p_N = 0;$$

- For each $1 \leq i \leq N$, $U_i(t, x)$ is a periodic traveling wave solution of (1.1a) connecting p_i to p_{i-1} with wave speed $c_i \in \mathbb{R}$;
- The sequence $(c_i)_{1 \le i \le N}$ satisfies $c_1 \le c_2 \le \cdots \le c_N$.

We denote such a propagating terrace by $\mathcal{T} := ((p_i)_{0 \leq i \leq N}, (U_i, c_i)_{1 \leq i \leq N})$ and call $(p_i)_{0 \leq i \leq N}$ the platforms of \mathcal{T} .

Hereinafter, by a periodic traveling wave U_i connecting p_i to p_{i-1} with wave speed c_i , we always mean that U_i is an entire solution of (1.1a) satisfying (1.6) with $c = c_i$, along with the asymptotics

$$\lim_{x \to -\infty} U(x,t) = p_{i-1}(t), \quad \lim_{x \to \infty} U(x,t) = p_i(t) \quad \text{locally uniformly in } t \in \mathbb{R}.$$

Note that, in the above definition of propagating terrace, we do not assume any sign condition on the speed c_i , $1 \le i \le N$. If $c_i < 0$ (resp. $c_i > 0$), then U_i propagates to the left (resp. right); if $c_i = 0$, then U_i is a T-periodic solution of (1.1a).

We will show below that, among all propagating terraces, only some particular terraces can be used to determine the propagation dynamics of (1.1). To explain what they are, let us first introduce the following notion:

Definition 1.2. Let $v_1(x)$ and $v_2(x)$ be two piecewise continuous functions defined on $x \in \mathbb{R}$. We say that v_1 is **steeper than** v_2 if for any $x_1, x_2 \in \mathbb{R}$ such that $v_1(x_1) = v_2(x_2)$, we have

$$v_1(x+x_1) \ge v_2(x+x_2)$$
 for $x > 0$, $v_1(x+x_1) \le v_2(x+x_2)$ for $x < 0$;

we say that v_1 is **strictly steeper than** v_2 if the above two inequalities hold strictly. Furthermore, for any two entire solutions $u_1(t,x)$ and $u_2(t,x)$ of (1.1a), we say that u_1 is steeper (resp. strictly steeper) than u_2 if for each $t \in \mathbb{R}$, $u_1(t,\cdot)$ is steeper (resp. strictly steeper) than $u_2(t,\cdot)$.

From the above definition, one easily sees that the concept of steepness is independent of the spatial positions of the two functions v_1, v_2 . More precisely, if v_1 is steeper (resp. strictly steeper) than v_2 , then for any constants $a, b \in \mathbb{R}$, $v_1(\cdot + a)$ is steeper (resp. strictly steeper) than $v_2(\cdot + b)$. In other words, the steepness property is preserved by spatial translations. It is also easily seen that v_1 and v_2 are mutually steeper than each other if and only if either $v_1 \equiv v_2$ up to a spatial translation or the ranges of v_1 and v_2 are disjoint. Moreover, if v_1 is strictly steeper than v_2 , then for any $a, b \in \mathbb{R}$, the graph of $v_1(\cdot + a)$ and that of $v_2(\cdot + b)$ intersect at most once; the converse is also true.

Additionally, it is easily seen from the strong maximum principle that for any two entire solutions u_1 , u_2 of (1.1a), if u_1 is steeper than u_2 , then either $u_1(t,x) \equiv u_2(t,x+x_0)$ for some $x_0 \in \mathbb{R}$ or u_1 is strictly steeper than u_2 . We will also show in Lemma 2.3 below that, if $u_1(0,x)$ is steeper than $u_2(0,x)$, then such steepness is preserved for any t > 0. This property will be a key tool in showing the main results of the present paper (except Theorem 1.19).

Remark 1.3. As mentioned above, the notion of steepness which we introduced in Definition 1.2 is independent of the spatial positions of the solutions $u_1(t,x)$, $u_2(t,x)$ of (1.1a). Note that this definition is different from that given in [9, 13] in which the notion of steepness is defined by using time shifts $u_1(t+t_1,x)$, $u_2(t+t_2,x)$ instead of spatial translations, thus this notion is independent of time shifts. The difference comes from the fact that the papers [9, 13] deal with equations of the form $u_t = u_{xx} + f(x,u)$ that is spatially heterogeneous but time-homogeneous (autonomous), while our equation (1.1a) is time-heterogeneous but spatially homogeneous.

We are now ready to define a special class of propagating terraces.

Definition 1.4. A propagating terrace $\mathcal{T} = ((p_i)_{0 \leq i \leq N}, (U_i, c_i)_{1 \leq i \leq N})$ is said to be **minimal** if it satisfies the following:

- For any propagating terrace $\mathcal{T}' = ((p'_i)_{0 \le i \le N'}, (U'_i, c'_i)_{1 \le i \le N'})$ connecting 0 to p, one has $\{p_i \mid 0 \le i \le N\} \subset \{p'_i \mid 0 \le i \le N'\};$
- For each $1 \le i \le N$, U_i is steeper than any other periodic traveling wave of (1.1a) connecting p_i to p_{i-1} .

Before stating our results, let us recall some basic notions on stability, which will be frequently used below. Let \mathcal{X}_{per} denote the set of all nonnegative solutions of (1.5). An element $q \in \mathcal{X}_{per}$ is said to be **stable from above** (resp. **below**) with respect to the initial-value problem

$$\frac{dh}{dt} = f(t,h) \quad \text{for } t > 0, \quad h(0) = h_0 \in \mathbb{R}, \tag{1.8}$$

if it is stable under nonnegative (resp. nonpositive) perturbations of the initial values around $h_0 = q(0)$. Otherwise q is called **unstable from above** (resp. **below**). An element $q \in \mathcal{X}_{per}$ is said to be **isolated from above** (resp. **below**) if there exists no sequence of other solutions of (1.5) converging to q from above (resp. **below**). Moreover, $q \in \mathcal{X}_{per}$ is said to be **linearly stable** (resp. **linearly unstable**) if $\int_0^T \partial_u f(t, q(t)) dt < 0$ (resp. > 0); it is said to be degenerate, if it is neither linearly stable nor linearly unstable.

1.2. Existence and uniqueness of minimal propagating terrace. We now proceed to the statements of our main results. We begin with the uniqueness of minimal propagating terrace. The uniqueness is meant here up to spatial shifts, that is, given two propagating terraces $((p_i)_i, (U_i, c_i)_i)$ and $((p_i')_i, (U_i', c_i')_i)$, we say that they are equal up to spatial shifts if $p_i \equiv p_i'$ and $c_i = c_i'$ for every i, and $U_i(t, x + a_i) = U_i'(t, x)$ for some constants a_i , $i = 1, \dots, N$. The uniqueness result actually follows immediately from the definition. For the convenience of later discussions, we state it precisely as follows:

Proposition 1.5. If there exists a propagating terrace $\mathcal{T} = ((p_i)_i, (U_i, c_i)_i)$ that is minimal in the sense of Definition 1.4, then it is unique up to spatial shifts.

Proof. According to Definition 1.4, all minimal propagating terraces should share the same platforms. It is also easily seen that, given two adjacent platforms, the steepest periodic traveling wave connecting them should be unique up to spatial shifts. \Box

We now discuss the existence of minimal propagating terrace. Note that, in our definition of propagating terrace, only finitely many platforms can appear. Actually, if we only assume (1.2), (1.3) and the existence of positive solution p of (1.5), there can exist a minimal propagating terrace with infinitely many platforms in some pathological cases (see [23] for the existence of such terraces in the autonomous case). To exclude the possibility of such pathological cases, we impose the following assumption, which is satisfied by virtually all the important examples of f.

Assumption 1.6. There exists a **decomposition** between 0 and p, that is, there exists a finite sequence of solutions $(q_i)_{0 \le i \le M}$ of (1.5) such that $q_0 = p > q_1 > \cdots > q_M = 0$, and that for each $1 \le m \le M$, there exists a periodic traveling wave V_m connecting q_m to q_{m-1} .

The term decomposition was introduced by Fife and McLeod [11] for the case f = f(u), and the above notion of decomposition can be viewed as a generalization of their notion. Note that a similar generalization has been given in [13] for spatially periodic equations. Unlike the definition of propagating terrace (see Definition 1.1), a decomposition does not require the speeds of the traveling waves to be ordered. Thus, a decomposition is a much weaker concept than a propagating terrace. However, we will show in Theorem 1.8 below that existence of a decomposition is enough to guarantee existence of a terrace. Before stating our result, let us first give some simple sufficient conditions for the existence of a decomposition.

Proposition 1.7. Assume that either of the following conditions holds:

- (a) There are finitely many solutions of (1.5) between 0 and p;
- (b) f = f(u) is independent of t, and the function $F(u) := \int_0^u f(s)ds$ has only finitely many global maximizers in [0,p], all of which are isolated zeros of f in [0,p].

Then there exists a decomposition between 0 and p.

It is clear that solutions of (1.5) are all ordered. Thus, under condition (a), the solutions between 0 and p are numbered, say $q_0 = p > q_1 > q_2 > \cdots > q_n = 0$, and obviously they are isolated. Furthermore, equation (1.1a) restricted to the region between any adjacent q_i and q_{i+1} has a monostable structure. It then follows from the work [17] on time-periodic monostable semiflows that there exists a periodic traveling wave connecting q_{i+1} to q_i . This immediately implies that $(q_i)_{0 \le i \le n}$ is a decomposition between 0 and p. Note that finiteness of the number of solutions of (1.5) between 0 and p is by no means necessary, since it is known that periodic traveling wave exists for a time-periodic combustion nonlinearity (see e.g., [28]). The condition (a) can be relaxed to include such nonlinearities or even a stacked composition of finitely many such nonlinearities.

Under condition (b), the existence of a decomposition follows from [23, Theorem 1.2], which actually shows the existence of a propagating terrace directly from (b). It is also known from [23] that, in the autonomous case, the following condition implies (b):

(b)' There exists a solution of (1.7) with compactly supported initial function that converges to p from below as $t \to \infty$ locally uniformly on \mathbb{R} .

Indeed, (b)' holds if and only if u = p is the unique global maximizer of the function F in [0, p] and it is an isolated zero of f. In the more general case where f = f(x, u) is allowed to depend

on x periodically, under a similar condition to (b), it was shown in [9] that there exists a minimal terrace consisting of traveling waves with positive speeds. After the completion of the present work, we learned that similar existence result was proved for time-periodic equation (1.1) in [32]. Here, inspired by [13], we show the existence of a minimal terrace under the more general Assumption 1.6. Our theorem is stated as follows:

Theorem 1.8. Let Assumption 1.6 holds. Then there exists a unique (up to spatial shifts) minimal propagating terrace $((p_i)_{0 \le i \le N}, (U_i, c_i)_{1 \le i \le N})$ connecting 0 to p. Moreover, it satisfies

- (i) For any $1 \le i \le N$, if $c_i > 0$, then p_{i-1} is isolated from below and stable from below; if $c_i < 0$, then p_i is isolated from above and stable from above; if $c_i = 0$, then p_{i-1} is stable from below and p_i is stable from above.
- (ii) All p_i and U_i are steeper than any other entire solutions of (1.1a) between 0 and p.

Apart from the existence of a minimal propagating terrace, the above theorem also provides some information about what kind of solutions of (1.5) can possibly be selected as platforms of the terrace. For example, statement (i) implies in particular that for each $1 \le i \le N-1$, p_i is either stable from below or stable from above. This property will be used in handling a case of non-degenerate multistable nonlinearity (see Assumption 1.18 below and its followed discussion). In addition, the steepness of p_i in statement (ii) also implies that p_i is contained in any decomposition between 0 and p.

1.3. Convergence to minimal propagating terrace. In this subsection, we assume the existence of a minimal propagating terrace, and establish results on the convergence of solutions (1.1) to the minimal terrace.

In order to formulate our convergence theorems, let us introduce a notion of limit sets. Given a bounded solution u(t, x) of (1.1), we call w(t, x) an Ω -limit solution of u if

$$u(t+k_jT,x+x_j) \to w(t,x) \text{ as } j \to \infty$$
 (1.9)

for some subsequence of positive integers $k_j \to \infty$ (as $j \to \infty$) and some sequence $(x_j) \subset \mathbb{R}$. Here the convergence is understood in the topology of $L^\infty_{loc}(\mathbb{R}^2)$. By parabolic estimates, this convergence also takes place in the $C^1(\mathbb{R}^2)$ topology. Clearly, w(t,x) is an entire solution of (1.1a). Denote by $\Omega(u)$ the set of all Ω -limit solutions. It is easily checked that if w is an element of $\Omega(u)$, then so is $w(\cdot + kT, \cdot + z)$ for any $k \in \mathbb{Z}$ and $z \in \mathbb{R}$. In Section 2.2 below, we will summarize more basic properties of $\Omega(u)$.

Remark 1.9. Note that, if $x_j \equiv 0$ in (1.9), then $\Omega(u)$ coincides with the set of ω -limit solutions defined in [5]. The latter can capture the asymptotic behavior of u(t,x) around each fixed point $x \in \mathbb{R}$ but cannot capture the profile of fronts that propagate at non-zero speeds. The above notion of $\Omega(u)$, on the other hand, can capture the profile of propagating fronts of any speeds. This multi-speed observation is particularly important for our study, since, as we will see later, multiple fronts with different speeds may coexist in a solution.

We are now ready to state our convergence theorems. Let us begin with a special case where the initial data are of the Heaviside type, that is,

(H1) There is some $a \in \mathbb{R}$ such that

$$u_0(x) = p(0)H(a-x)$$

where H denotes the Heaviside function defined by H(x) = 0 if x < 0 and H(x) = 1 if $x \ge 0$.

Theorem 1.10. Assume that $((p_i)_{0 \le i \le N}, (U_i, c_i)_{1 \le i \le N})$ is a minimal propagating terrace of (1.1a) connecting 0 to p. Let u(t, x) be the solution of (1.1) with u_0 satisfying (H1). Then

$$\Omega(u) = \{ U_i(\cdot, \cdot + \xi) : \xi \in \mathbb{R}, \ 1 \le i \le N \} \cup \{ p_i : 0 \le i \le N \}.$$
 (1.10)

Furthermore, there are $C^1([0,\infty))$ functions $\eta_1(t), \dots, \eta_N(t)$ such that the following statements hold:

- (i) $\eta_i(t) = o(t)$ as $t \to \infty$ for $i = 1, \dots, N$;
- (ii) $\eta_{i+1}(t) \eta_i(t) \to \infty$ as $t \to \infty$ whenever $i \in \{1, \dots, N-1\}$ satisfies $c_i = c_{i+1}$;
- (iii) The following convergence holds:

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}} \left| u(t, x) - \left(\sum_{i=1}^{N} U_i(t, x - \eta_i(t)) - \sum_{i=1}^{N} p_i(t) \right) \right| = 0.$$
 (1.11)

Clearly, (1.10) follows immediately from statement (iii). Note that, since c_i is the wave speed of U_i , statements (i) and (iii) imply that the *i*-th front of u has the asymptotic speed c_i . In the special case where $c_i = 0$, the *i*-th front of u moves with asymptotically vanishing speed. A convergence result similar to the above theorem has been established in [9, 13] for spatially periodic problem (i.e., f = f(x, u) is periodic in x), but the speed c_i was required to be non-zero.

We remark that the functions $(\eta_i(t))_{1 \leq i \leq N}$ may not be convergent or even bounded. Actually, by statement (ii), if $c_i = c_{i+1}$ for some i, then at least either of $\eta_i(t)$ and $\eta_{i+1}(t)$ is unbounded. Besides, even in the simple case where the terrace consists of a single wave, it is known that, for autonomous KPP equations, the corresponding drift function $\eta(t)$ grows logarithmically as $t \to \infty$ (see [3, 15]). On the other hand, if $N \geq 2$ and if the speeds $(c_i)_{1 \leq i \leq N}$ are all different, we will show in Theorem 1.19 below that $(\eta_i(t))_{1 \leq i \leq N}$ are convergent provided that f is a non-degenerate multistable nonlinearity.

Theorem 1.10 is stated under very general assumptions on f (only the standing hypotheses and the existence of a minimal propagating terrace), but the initial data are rather special. In the next two theorems, we relax the assumption on u_0 as follows:

(H2) $u_0(x)$ is piecewise continuous, and there are two constants $a_- < a_+$ such that

$$p(0)H(a_{-}-x) \leq u_0(x) \leq p(0)H(a_{+}-x)$$
 for $x \in \mathbb{R}$,

where H is the Heaviside function introduced in (H1).

Note that any u_0 satisfying (H2) is not necessarily monotone. By the comparison principle, the solution u(t,x) of (1.1) with such an initial function is bounded from above and below by two solutions starting from Heaviside type functions for all $t \geq 0$, each of which converges to the minimal terrace as shown in Theorem 1.10. In the case of the autonomous equation (1.7), it follows immediately from [23, Theorem 1.2] that u(t,x) converges to the minimal terrace. However, in the present time-periodic problem, the technique used in [23] does not apply, and it is much harder to prove the convergence. At the moment, all we can show without any extra condition on f is the following:

Theorem 1.11. Assume that $((p_i)_{0 \le i \le N}, (U_i, c_i)_{1 \le i \le N})$ is a minimal propagating terrace of (1.1a) connecting 0 to p. Let u(t,x) be the solution of (1.1) with u_0 satisfying (H2). Then

$$\{p_i\}_{0 \le i \le N} \subset \Omega(u).$$

Furthermore, for every $w \in \Omega(u)$, w is either spatially constant or strictly decreasing in $x \in \mathbb{R}$, and one of the following cases holds:

- (a) $w(t,x) \equiv p_i(t)$ for some $0 \le i \le N$;
- (b) w(t,x) is a periodic traveling wave of (1.1a) connecting p_i to p_{i-1} with wave speed c_i for some $1 \le i \le N$;
- (c) There are two periodic traveling waves V_{\pm} of (1.1a) connecting p_i to p_{i-1} and sharing the same wave speed c_i for some $1 \le i \le N$ such that

$$w(t,x) - V_{\pm}(t,x) \to 0$$
 as $t \to \pm \infty$ uniformly in $x \in \mathbb{R}$,

and that V_{+} is either strictly steeper or strictly less steep than V_{-} .

The above theorem immediately implies that $\{p_i\}_{0 \le i \le N}$ are the only spatially homogeneous functions in $\Omega(u)$. Moreover, since the set $\{w(t,\cdot): t \in \mathbb{R}, w \in \Omega(u)\}$ is compact and connected in $L^{\infty}_{loc}(\mathbb{R})$ (see Section 2.2 below for more details), we have the following corollary:

Corollary 1.12. Let the assumptions of Theorem 1.11 hold. Then for each $1 \le i \le N$, $\Omega(u)$ contains at least one periodic traveling wave connecting p_i to p_{i-1} with speed c_i . Furthermore, if U_i is the unique (up to spatial shifts) such periodic traveling wave, then case (c) of Theorem 1.11 does not occur; hence, (1.10) holds.

As mentioned earlier, in the autonomous case, it is known from [23, Theorem 1.2] that (1.10) holds for any solution of (1.7) with (H2)-type initial function. In fact, this result can also be derived directly from the above corollary, as a simple ODE argument can prove that, up to spatial shifts, U_i is the unique traveling wave connecting p_i to p_{i-1} with speed c_i . Our next theorem shows that, the same is true for the time-periodic equation (1.1) under an additional assumption on f. Recall that \mathcal{X}_{per} denotes the set of all nonnegative solutions of (1.5). Our assumption is stated as follows:

Assumption 1.13. Each element $q \in \mathcal{X}_{per}$ between 0 and p satisfies the following:

(i) If q > 0 is stable from below, then there exist a real number $\sigma_0 > 0$ and a T-periodic function g(t) such that

$$\int_0^T g(t)dt \le 0, \quad and \quad \partial_u f(t,u) \le g(t) \quad for \ all \ u \in (q(t) - \sigma_0, q(t)], \ t \in \mathbb{R};$$
 (1.12)

(ii) If q < p is stable from above, then there exist a real number $\sigma_0 > 0$ and a T-periodic function g(t) such that

$$\int_0^T g(t)dt \le 0, \quad and \quad \partial_u f(t,u) \le g(t) \quad for \ all \ u \in [q(t), q(t) + \sigma_0), \ t \in \mathbb{R}.$$
 (1.13)

By an easy comparison argument applied to (1.8), one can check that a simple sufficient condition for our Assumption 1.13 to hold is that:

Assumption 1.14. Each element $q \in \mathcal{X}_{per}$ between 0 and p which is stable from above or from below is linearly stable.

It should be noted that Assumption 1.13 is weaker than Assumption 1.14. A simple example is that $f(t,u) = b(t)\bar{f}(u)$, where b(t) is a positive T-periodic $C(\mathbb{R})$ function, $\bar{f}(u)$ is an autonomous combustion nonlinearity satisfying $\bar{f}(u) = 0$ for $u \in [0,\theta] \cup \{p\}$, $\bar{f}(u) > 0$ for $u \in (\theta,p)$ and $\bar{f}'(u) \leq 0$ for u close to p. Clearly, such a nonlinearity satisfies Assumption

1.13, but not Assumption 1.14. In fact, Assumption 1.13 allows infinitely many elements of \mathcal{X}_{per} between 0 and p, while there can only be finitely many such elements if Assumption 1.14 is satisfied. In the latter case, all the elements of \mathcal{X}_{per} between 0 and p are isolated, since any element of \mathcal{X}_{per} that is unstable both from above and from below is isolated by definition, and any linearly stable element is also isolated as it is asymptotic stable.

Notice that the above two assumptions do not require anything on the elements of \mathcal{X}_{per} that are unstable both from above and from below, therefore those elements can be degenerate.

Under Assumption 1.13, we have the following uniqueness result.

Proposition 1.15. Let Assumption 1.13 hold and let q_1 , q_2 be elements of \mathcal{X}_{per} satisfying $0 \le q_1 < q_2 \le p$. Let V_1 and V_2 be two periodic traveling waves connecting q_1 to q_2 with wave speeds c_1 and c_2 , respectively. Assume that V_1 is steeper than V_2 . Then there holds $c_1 \le c_2$. Furthermore, if $c_1 = c_2$, then V_1 is equal to V_2 up to a spatial shift.

Combining Theorem 1.11 and Proposition 1.15, we easily obtain that (1.10) holds for solutions of (1.1) with (H2)-type initial data. Furthermore, we have the following theorem.

Theorem 1.16. Let Assumption 1.13 hold. Assume that there exists a minimal propagating terrace $((p_i)_{0 \le i \le N}, (U_i, c_i)_{1 \le i \le N})$ connecting 0 to p. Then all the conclusions of Theorem 1.10 hold for solutions of (1.1) with (H2)-type initial data.

The following result is an easy consequence of Proposition 1.7 and Theorems 1.8, 1.16.

Corollary 1.17. Let Assumption 1.14 hold. Then there exists a unique (up to spatial shifts) minimal propagating terrace $((p_i)_{0 \leq i \leq N}, (U_i, c_i)_{1 \leq i \leq N})$ connecting 0 to p. Furthermore, all the conclusions of Theorem 1.10 hold for solutions of (1.1) with (H2)-type initial data.

1.4. Convergence in a non-degenerate multistable case. Our last main result is concerned with the asymptotic behavior of solutions of (1.1) where f is of multistable type in the following sense:

Assumption 1.18. The elements 0 and p of \mathcal{X}_{per} are linearly stable, and any other element between 0 and p which is stable from above or from below is linearly stable.

Under the above assumption, it is clear that all the elements of \mathcal{X}_{per} between 0 and p are isolated, and hence, there are only finitely many such elements. It then follows immediately from Proposition 1.7 and Theorem 1.8 that a minimal propagating terrace $((p_i)_{0 \le i \le N}, (U_i, c_i)_{1 \le i \le N})$ exists. Furthermore, in view of statement (i) of Theorem 1.8, for each $0 \le i \le N$, p_i is linearly stable. In other words, for each $1 \le i \le N$, U_i is a periodic traveling wave connecting two linearly stable solutions of (1.5).

Clearly, Assumption 1.18 is stronger than Assumption 1.13, therefore Theorem 1.16 immediately implies that the minimal terrace attracts solutions with initial data satisfying (H2). But in this subsection, we will show that this holds for a larger class of initial data. To formulate our hypotheses on u_0 , we denote I_+ and I_- by the intervals of attraction with respect to the equation (1.8) of the periodic solutions p(t) and 0, respectively. Namely, the set I_+ (resp. I_-) consists of element $h_0 \in \mathbb{R}$ such that the solution of (1.8) with initial value h_0 converges to p(t) (resp. 0) as $t \to \infty$. Since we have assumed that p and 0 are linearly stable, it is easily checked that I_+ and I_- are open intervals containing p(0) and 0, respectively. Our hypotheses on u_0 are stated as follows:

(H3) $u_0(x)$ is bounded and piecewise continuous, and it satisfies

$$\liminf_{x \to -\infty} u_0(x) \in I_+, \quad \sup_{x \in \mathbb{R}} u_0(x) \in I_+, \tag{1.14}$$

$$\lim_{x \to -\infty} \inf u_0(x) \in I_+, \quad \sup_{x \in \mathbb{R}} u_0(x) \in I_+,
\lim_{x \to \infty} \sup u_0(x) \in I_-, \quad \inf_{x \in \mathbb{R}} u_0(x) \in I_-. \tag{1.15}$$

Theorem 1.19. Let Assumption 1.18 hold. Assume further that $\partial_u f(t,u)$ is locally Lipschitz continuous in u uniformly for t. Let u(t,x) be the solution of (1.1) with u_0 satisfying (H3). Then there exists a unique (up to spatial shifts) minimal propagating terrace $((p_i)_{0 \le i \le N}, (U_i, c_i)_{1 \le i \le N})$ connecting 0 to p. Furthermore, the following statements hold:

- (i) There are $C^1([0,\infty))$ functions $\eta_1(t), \dots, \eta_N(t)$ such that statements (i)-(iii) of Theorem 1.10 hold for u(t,x);
- (ii) If the speeds satisfy $c_1 < c_2 < \cdots < c_N$, then for each $i = 1, \dots, N$, there is some constant $\bar{\eta}_i \in \mathbb{R}$ such that

$$\lim_{t\to\infty}\eta_i(t)=\bar{\eta}_i,$$

and there are constants $\nu > 0$, C > 0 such that

$$\left| u(t,x) - \left(\sum_{i=1}^{N} U_i(t,x - \bar{\eta}_i) - \sum_{i=1}^{N} p_i(t) \right) \right| \le C e^{-\nu t} \text{ for all } t \ge 0, x \in \mathbb{R}.$$

We remark that Theorem 1.16 and Theorem 1.19 (i) treat different cases, and the methods used to prove them are different. On the one hand, Theorem 1.16 allows nonlinearities to be degenerate, but requires more restrictions on initial data. The proof relies heavily on zeronumber arguments. On the other hand, a key step in the proof of Theorem 1.19 (i) is to show that, up to some error terms with exponential decay, the solution u(t,x) can be bounded from above and from below by solutions with Heaviside type initial data for all large times (see Lemma 5.2 below). The non-degeneracy of $(p_i)_{1 \le i \le N}$ plays an important role in this step.

Theorem 1.19 (ii) implies that, if the speeds $(c_i)_{1 \leq i \leq N}$ are all different, then the functions $(\eta_i(t))_{1 \le i \le N}$ are convergent, and the solution u(t,x) converges to the minimal terrace with an exponential rate. In the autonomous case, similar convergence results have been proved in [11, 23] for scalar equation (1.7), and in [24] for cooperative systems. Our proof is based on ideas in these work, but new techniques are needed to overcome considerable difficulties arising from time heterogeneity.

In the special case N=1 (the terrace consists of a single wave), Theorem 1.19 (ii) covers earlier stability results of periodic traveling waves for bistable equations [1, 4], and extends their results to the case where there may exist more than one intermediate solution of (1.5) between 0 and p. It should be pointed out that, when $N \geq 2$, the assumption on the mutual distinctness of $(c_i)_{1 \leq i \leq N}$ is necessary. Otherwise, if $c_i = c_{i+1}$ for some i, then by Theorem 1.10 (i), $\eta_{i+1}(t) - \eta_i(t) \to \infty$ as $t \to \infty$, and hence, at least one of $\eta_{i+1}(t)$ and $\eta_i(t)$ cannot be convergent.

Outline of the paper. In Section 2, we will present some preliminaries. We will recall some basic properties of zero-number arguments and the Ω -limit set, and show a lemma on the stability of certain solutions of (1.5). Section 3 is concerned with the asymptotic behavior of solutions of (1.1) with Heaviside type initial functions, and the proof of Theorems 1.8 and 1.10. In Section 4, we will study the propagation dynamics of (1.1) with u_0 satisfying (H2), and prove Theorems 1.11 and 1.16. Section 5 is devoted to the proof of Theorem 1.19.

2. Preliminaries

2.1. **Zero-number properties.** In this subsection, we recall some basic properties of zero-number arguments (also known as intersection comparison arguments). They will be key ingredients of our proofs in Sections 3-4.

Let $\mathcal{Z}(w)$ denote the number of sign changes of a real-valued function w(x) defined on \mathbb{R} , namely, the supremum over all $k \in \mathbb{N}$ such that there exist real numbers $x_1 < x_2 < \cdots < x_{k+1}$ with

$$w(x_i) \cdot w(x_{i+1}) < 0 \text{ for all } i = 1, 2, \dots, k.$$

We set $\mathcal{Z}(w) = -1$ if $w \equiv 0$. Clearly, if w is a smooth function having only simple zeros on \mathbb{R} , then $\mathcal{Z}(w)$ coincides with the number of zeros of w.

The following intersection-comparison principle holds (see [2, 6, 9]).

Lemma 2.1. Let $w \not\equiv 0$ be a solution of the equation

$$w_t = w_{xx} + c(t, x)w \quad \text{for } t \in (t_1, t_2), \ x \in \mathbb{R},$$
 (2.1)

where the coefficient function c is bounded. Then the following statements hold:

- (i) For each $t \in (t_1, t_2)$, all zeros of $w(t, \cdot)$ are isolated;
- (ii) $t \mapsto \mathcal{Z}(w(t,\cdot))$ is a nonincreasing function with values in $\mathbb{N} \cup \{0\} \cup \{\infty\}$;
- (iii) If $w(t^*, x^*) = w_x(t^*, x^*) = 0$ for some $t^* \in (t_1, t_2), x^* \in \mathbb{R}$, then $\mathcal{Z}(w(t, \cdot)) > \mathcal{Z}(w(s, \cdot))$ for all $t \in (t_1, t^*), s \in (t^*, t_2)$

$$\mathcal{Z}(w(t,\cdot)) > \mathcal{Z}(w(s,\cdot))$$
 for all $t \in (t_1,t_1)$, s

whenever $\mathcal{Z}(w(s,\cdot)) < \infty$.

One can check that \mathcal{Z} is semi-continuous with respect to pointwise convergence, that is, the pointwise convergence $w_n(x) \to w(x)$ implies

$$w \equiv 0 \quad \text{or} \quad \mathcal{Z}(w) \le \liminf_{n \to \infty} \mathcal{Z}(w_n).$$
 (2.2)

This semi-continuity immediately implies the following lemma.

Lemma 2.2. Let $(w_n)_{n\in\mathbb{N}}: \mathbb{R} \to \mathbb{R}$ be a sequence of functions converging to $w: \mathbb{R} \to \mathbb{R}$ pointwise on \mathbb{R} . If for each $n \in \mathbb{N}$, w_n is steeper than $v: \mathbb{R} \to \mathbb{R}$, then w is steeper than v.

The following lemma is a consequence of the application of Lemma 2.1 when at most one intersection occurs and the fact the shape of the intersection remains the same as long as it exists.

Lemma 2.3. Let u_1 and u_2 be two bounded solutions of (1.1a). Assume that $u_1(0,x)$ is piecewise continuous and bounded, $u_2(0,x)$ is continuous and bounded, and that $u_1(0,x)$ is steeper than $u_2(0,x)$. Then for any t > 0, the function $x \mapsto u_1(t,x)$ is steeper than $x \mapsto u_2(t,x)$. Furthermore, either $u_1 \equiv u_2$ up to a spatial shift or for any t > 0 and $z \in \mathbb{R}$, the function $x \mapsto u_1(t,x) - u_2(t,x+z)$ has at most one (simple) zero.

Proof. The proof follows from that of [9, Lemma 2.4 and Corollary 2.5] with obvious modifications, therefore we omit the details.

Remark 2.4. We emphasize that, in the above two lemmas, the steepness of functions is independent of spatial translations. In view of this, Lemma 2.3 implies that if $u_1(0,x)$ is steeper than $u_2(0,x)$, then for any t > 0, the curves (not necessarily simple) $\{(u_1(t,x), \partial_x u_1(t,x)) : x \in \mathbb{R}\}$ and $\{(u_2(t,x), \partial_x u_2(t,x)) : x \in \mathbb{R}\}$ do not intersect unless they are equal. This property

is indeed a key point in showing the convergence theorems in the autonomous case [23], where the above curves are called spatial trajectories of solutions of (1.7).

We also recall the following lemma which will be used repeatedly in proving Theorem 1.11. The proof can be found in [6].

- **Lemma 2.5.** Let $w_n(t,x)$ be a sequence of functions converging to w(t,x) in $C^1((t_1,t_2) \times I)$, where I is an open finite interval in \mathbb{R} . Assume that for each $t \in (t_1,t_2)$ and $n \in \mathbb{N}$, the function $x \mapsto w_n(t,x)$ has only simple zeros in I, and that w(t,x) satisfies an equation of the form (2.1) on $(t_1,t_2) \times I$. Then for every $t \in (t_1,t_2)$, either $w \equiv 0$ on I or w(t,x) has only simple zeros on I.
- 2.2. Basic properties of the set of Ω -limit solutions. Recall that for any bounded solution u(t,x) of (1.1), $\Omega(u)$ denotes the set of Ω -limit solutions defined in Section 1.3. Obviously, our definition of $\Omega(u)$ is different from the standard notion of ω -limit set. In this subsection, we summarize some basic properties of $\Omega(u)$.

First, since $u(t,\cdot)$ is uniformly bounded in $L^{\infty}(\mathbb{R})$ for t>0, by the regularity assumption on f and the standard parabolic estimates, $u(\cdot,\cdot)$ is bounded in $C^{1,2}([1,\infty)\times\mathbb{R})$. This immediately implies that $\Omega(u)$ is a nonempty compact subset of $L^{\infty}_{loc}(\mathbb{R}^2)$.

Next, we show that the following set

$$\bar{\Omega}(u) := \{ w(t, \cdot) : t \in \mathbb{R}, w \in \Omega(u) \}$$

coincides with the set of all limit points of the trajectory $\{u(t,\cdot):t>0\}$ with arbitrary spatial translations, that is,

$$\bar{\Omega}(u) = \Omega^*(u), \tag{2.3}$$

where

$$\Omega^*(u) := \big\{ \phi: \, u(t_j, \cdot + x_j) \to \phi(\cdot) \text{ for some sequences of real numbers } t_j \to \infty \text{ and } x_j \in \mathbb{R} \big\}.$$

Here the convergence is with respect to the topology of $L^{\infty}_{loc}(\mathbb{R})$. Indeed, the relation $\bar{\Omega}(u) \subset \Omega^*(u)$ is easily seen, so it suffices to prove $\bar{\Omega}(u) \supset \Omega^*(u)$. Choose any $\phi \in \Omega^*(u)$ and let $t_j \to \infty$ and $x_j \in \mathbb{R}$ be such that $u(t_j, \cdot + x_j) \to \phi(\cdot)$ in $L^{\infty}_{loc}(\mathbb{R})$. For each $j \in \mathbb{N}$, let us write $t_j = t'_j + \tau_j$ with $t'_j \in T\mathbb{N}$ and $\tau_j \in [0, T)$. By choosing a subsequence if necessary, we can assume that the following limits exist:

$$u(t+t_j',x+x_j) \to w(t,x)$$
 in $L_{loc}^{\infty}(\mathbb{R}^2), \quad \tau_j \to \tau_{\infty} \in [0,T].$

Clearly, w belongs to $\Omega(\underline{u})$. Therefore, $\phi(\cdot) = w(\tau_{\infty}, \cdot) \in \bar{\Omega}(u)$, which proves (2.3).

Finally, we note that $\bar{\Omega}(u)$ is a nonempty, compact and connected subset of $L^{\infty}_{loc}(\mathbb{R})$. Thanks to (2.3), this follows directly from the fact that $\Omega^*(u)$ is nonempty, compact and connected in $L^{\infty}_{loc}(\mathbb{R})$ (the proof of such a property is standard in the theory of dynamical systems; at least the same argument that is known for autonomous equations applies to our time-periodic equations without any changes).

2.3. Stability of solutions of (1.5) connected by traveling wave. In this subsection, given a periodic traveling wave U connecting two solutions q_{\pm} of (1.5), we investigate the link between the stability of q_{\pm} and the sign of the speed of U. The lemma stated below will be used frequently in later sections, and it is also of independent interest in its own.

Lemma 2.6. Let $q_- < q_+$ be two solutions of (1.5). Assume that there is a periodic traveling wave U(t,x) of (1.1a) connecting q_- to q_+ with speed $c_0 \in \mathbb{R}$. Then the following statements hold:

- (i) If $c_0 > 0$, then q_+ is stable from below and isolated from below;
- (ii) If $c_0 < 0$, then q_- is stable from above and isolated from above;
- (iii) If $c_0 = 0$, then q_+ is stable from below and q_- is stable from above.

Proof. Let us first show that if $c_0 \ge 0$, then q_+ is stable from below. Assume by contradiction that q_+ is unstable from below. It then follows directly from [5, Proposition 3.5] that there exists R > 0 sufficiently large such that the following problem

$$\begin{cases} \varphi_t = \varphi_{xx} + f(t, \varphi), & \text{for } t \in \mathbb{R}, -R < x < R, \\ \varphi(t+T, x) = \varphi(t, x), & \text{for } t \in \mathbb{R}, -R \le x \le R, \\ q_-(t) < \varphi(t, x) < q_+(t), & \text{for } t \in \mathbb{R}, -R < x < R, \\ \varphi(t, \pm R) = q_+(t), & \text{for } t \in \mathbb{R}, \end{cases}$$

has a classical solution $\varphi(t,x)$ satisfying

$$\partial_x \varphi(t,x) < 0 \text{ for } t \in \mathbb{R}, x \in [-R,0) \text{ and } \partial_x \varphi(t,x) > 0 \text{ for } t \in \mathbb{R}, x \in (0,R].$$

Notice that the traveling wave U(t,x) satisfies the following asymptotics

$$\lim_{x \to \infty} U(t, x) = q_{-}(t), \quad \lim_{x \to -\infty} U(t, x) = q_{+}(t) \text{ locally uniformly in } t \in \mathbb{R}.$$
 (2.4)

It is easily checked from the above that, at t = 0, there is some $x_0 \in \mathbb{R}$ such that $U(0, x + x_0)$ and $\varphi(0, x)$ intersects at some $\xi_0 \in (-R, 0]$, and that

$$U(0, x + x_0) \le \varphi(0, x)$$
 for $x \in [-R, R]$.

Clearly, $U(t, x + x_0) < \varphi(t, x)$ for $t \ge 0$, $x = \pm R$. Then by the strong maximum principle, we have

$$U(t, x + x_0) < \varphi(t, x) \text{ for } t > 0, -R \le x \le R.$$
 (2.5)

If $c_0 = 0$, then U(t, x) is T-periodic in t. Since $\varphi(t, x)$ is also T-periodic, we have $U(T, x_0 + \xi_0) = \varphi(T, \xi_0)$, which is a contradiction with (2.5). In the case $c_0 > 0$, since

$$U(t+kT,x) = U(t,x-c_0kT) \text{ for each } k \in \mathbb{Z},$$
(2.6)

it follows from (2.4) that U(t+kT,x) converges to $p_+(t)$ as $k \to \infty$ locally uniformly in $t \in \mathbb{R}$ and $x \in \mathbb{R}$. In particular, we have $\lim_{k\to\infty} U(kT,x_0) = p_+(0)$. This also contradicts (2.5), as $\varphi(kT,0) = \varphi(0,0) < p_+(0)$ for each $k \in \mathbb{Z}$. Thus, q_+ is stable from below if $c_0 \ge 0$.

In the case $c_0 \leq 0$, one can proceed similarly as above to prove that q_- is stable from above, and the details are omitted.

Let us now show that q_+ is isolated from below in the case $c_0 > 0$. Assume by contradiction that q_+ is an accumulation solution of (1.5) from below, that is, there exists some sequence $(q_j)_{j \in \mathbb{N}}$ of solutions of (1.5) such that $q_j \to q_+$ as $j \to \infty$ and $q_j < q_+$ for each $j \in \mathbb{N}$. It is then easily seen that

$$\int_0^T \partial_u f(t, q_+(t)) dt = 0,$$

and thus the following problem

$$\begin{cases}
\phi_t - \partial_u f(t, q_+(t))\phi = 0 & \text{for } t \in \mathbb{R}, \\
\phi(t+T) = \phi(t) & \text{for } t \in \mathbb{R}, \quad \phi(0) = 1,
\end{cases}$$
(2.7)

has a unique positive solution $\phi \in C^1(\mathbb{R})$.

To find a contradiction, we construct a super-solution of (1.1a) as follows. Let $c \in (0, c_0)$, $\lambda \in (0, c)$ be two constants and v(t, x) be a function defined by

$$v(t,x) := \min \left\{ q_+(t), e^{-\lambda(x-ct)}\phi(t) + q_j(t) \right\} \text{ for } t \in \mathbb{R}, x \in \mathbb{R},$$

where ϕ is given by (2.7), and $j \in \mathbb{N}$ is to be determined later. Let $t \mapsto y(t)$ be the function satisfying

$$v(t, y(t)) = q_+(t)$$
 for $t \in \mathbb{R}$, $v(t, x) < q_+(t)$ for $x > y(t), t \in \mathbb{R}$.

Clearly, $y(t)/t \to c$ as $t \to \infty$. It is also straightforward to compute on the set $D := \{(t, x) \in [0, \infty) \times \mathbb{R} : x \geq y(t)\}$ that

$$v_t - v_{xx} - f(t, v) = e^{-\lambda(x - ct)} (\phi_t + \lambda c\phi - \lambda^2 \phi) + f(t, q_j) - f(t, v)$$

= $e^{-\lambda(x - ct)} \left(\lambda c\phi - \lambda^2 \phi + \partial_u f(t, q_+(t)) - \partial_u f(t, q_j + \theta e^{-\lambda(x - ct)} \phi) \right),$

for some $\theta = \theta(t, x) \in [0, 1]$. Since $0 < \lambda < c$, and since

$$q_j(t) + \theta e^{-\lambda(x-ct)}\phi(t) \to q_+(t)$$
 as $j \to \infty$ uniformly in D.

it then follows that for any j sufficiently large

$$v_t - v_{xx} - f(t, v) > 0$$
 in D.

Thus, v is a super-solution of (1.1a) over D.

Let $x_1 > 0$ be a sufficiently large number such that

$$U(0, x + x_1) < v(0, x)$$
 for $x > y(0)$.

Then by the comparison principle, we have

$$U(t, x + x_1) < v(t, x)$$
 for $t > 0, x > v(t)$.

This implies in particular that there exists some $\sigma > 0$ such that

$$U(t, y(t) + 1 + x_1) \le v(t, y(t) + 1) \le q_+(t) - \sigma \text{ for } t \ge 0.$$
 (2.8)

On the other hand, since $y(t)/t \to c$ as $t \to \infty$ and $c < c_0$, it follows that $y(t) - c_0 t \to -\infty$ as $t \to \infty$. Combining this with (2.4) and (2.6), we obtain

$$U(kT, y(kT) + 1 + x_1) = U(0, y(kT) - c_0kT + 1 + x_1) \rightarrow q_+(0)$$
 as $k \rightarrow \infty$.

This is a contradiction with (2.8). Therefore, q_{+} is isolated from below if $c_0 > 0$.

In the case $c_0 < 0$, one can argue analogously to conclude that $q_-(t)$ is isolated from above. The proof of Lemma 2.6 is thus complete.

3. Existence of minimal terrace and convergence with Heaviside type initial data

In this section, we show the existence of minimal propagating terrace (i.e., Theorem 1.8), and the convergence to a minimal terrace when the initial data are of Heaviside type (i.e., Theorem 1.10). Recall that by a Heaviside type initial function, we mean a function u_0 has the form $u_0(x) = p(0)H(a-x)$ for some $a \in \mathbb{R}$. Hereinafter, we denote by $\widehat{u}(t,x)$ the solution of (1.1) with such an initial function. In some parts of later sections, we will write $\widehat{u}(t,x;a)$ instead of $\widehat{u}(t,x)$ to stress the dependence on a.

As mentioned earlier, the proof is inspired from the papers [9, 13] devoted to spatially periodic equations, but the method has to be adapted here to the time-periodic framework. Let us first state a key lemma.

Lemma 3.1. Any Ω -limit solution of the solution $\widehat{u}(t,x)$ is steeper than any other entire solution of (1.1a) lying between 0 and p.

Proof. Let w(t,x) be an Ω -limit solution of $\widehat{u}(t,x)$. Then there exist a sequence of positive integers $(k_j)_{j\in\mathbb{N}}$ $(k_j\to\infty)$ as $j\to\infty$ and a sequence of real numbers $(x_j)_{j\in\mathbb{N}}$ such that

$$\widehat{u}(t+k_jT,x+x_j)\to w(t,x)$$
 as $j\to\infty$ in $C^1(\mathbb{R}^2)$.

For any entire solution v(t,x) of (1.1a) between 0 and p, it is easily seen that for each $j \in \mathbb{N}$, the function $\widehat{u}(0,x+x_j)$ is steeper than $v(-k_jT,x)$ in the sense of Definition 1.2. By using Lemma 2.3, for any $t > -k_jT$, $\widehat{u}(t+k_jT,x+x_j)$ is steeper than v(t,x). Then by Lemma 2.2, passing to the limit as $j \to \infty$, we obtain that w(t,x) is steeper than v(t,x), where $t \in \mathbb{R}$ is arbitrary. This ends the proof of Lemma 3.1.

In Subsection 3.1, we will use the above lemma to show the convergence of $\widehat{u}(t,x)$ to a unique limit function around a given level set, and further prove that the limit function is either a solution of (1.5) or a periodic traveling wave connecting two solutions of (1.5). Once we obtain this convergence property, we will be able to construct a minimal propagating terrace by an iterative argument, and complete the proof of Theorem 1.8 (see Subsection 3.2). The assumption that there exists a decomposition between 0 and p will be used to ensure that the iteration process ends in a finite number of steps. In Subsection 3.3, we will give the proof of Theorem 1.10, which also relies on the convergence property established in Subsection 3.1.

3.1. Convergence around a given level set. Let $\widehat{u}(t,x)$ be the solution of (1.1) with a given Heaviside type initial function. It is clear that $0 \le \widehat{u}(t,x) \le p(t)$ for $t \ge 0$, $x \in \mathbb{R}$, and for each t > 0, $\widehat{u}(t,x)$ is decreasing in $x \in \mathbb{R}$, and satisfies

$$\lim_{x \to \infty} \widehat{u}(t, x) = 0 \quad \text{and} \quad \lim_{x \to -\infty} \widehat{u}(t, x) = p(t).$$

This implies in particular that for each $k \in \mathbb{N}$, there exists a unique $a_k \in \mathbb{R}$ such that

$$\widehat{u}(kT, a_k) = \alpha, \tag{3.1}$$

where $\alpha \in (0, p(0))$ is a given constant. The following lemma gives the local convergence of $\widehat{u}(t+kT,x+a_k)$ around the level α as $k \to \infty$.

Lemma 3.2. For any $\alpha \in (0, p(0))$, let $(a_k)_{k \in \mathbb{N}}$ be the sequence provided by (3.1). Then the following limit exists for the topology of $C^1(\mathbb{R}^2)$:

$$\lim_{k \to \infty} \widehat{u}(t + kT, x + a_k) := w_{\infty}(t, x; \alpha). \tag{3.2}$$

The function $w_{\infty}(t, x; \alpha)$ is a positive entire solution of (1.1a) which is steeper than any other entire solution between 0 and p. Furthermore, it is either spatially homogeneous or spatially decreasing.

Proof. By parabolic estimates, the sequence $\{\widehat{u}(t+kT,x+a_k)\}_{k\in\mathbb{N}}$ is uniformly bounded along with their derivatives. Thus, it is relatively compact for the topology of $C^1(\mathbb{R}^2)$. Then there exists a subsequence $(k_j)_{j\in\mathbb{N}}$ of integers such that $k_j \to \infty$ as $j \to \infty$ and that

$$\widehat{u}(t+k_jT,x+a_{k_j})\to w_\infty(t,x;\alpha)$$
 as $j\to\infty$ in $C^1(\mathbb{R}^2)$,

where $w_{\infty}(t, x; \alpha)$ is an entire solution of (1.1a). Clearly, $w_{\infty}(t, x; \alpha)$ is an Ω -limit solution of $\widehat{u}(t, x)$ and $w_{\infty}(0, 0; \alpha) = \alpha$. It is also easily seen from the strong maximum principle that $0 < w_{\infty}(t, x; \alpha) < p(t)$ for $t \in \mathbb{R}$, $x \in \mathbb{R}$. Furthermore, by Lemma 3.1, $w_{\infty}(t, x; \alpha)$ is steeper than any other entire solution of (1.1a) lying between 0 and p.

It is easily checked from Definition 1.2 that, $w_{\infty}(t, x; \alpha)$ is the unique solution of (1.1a) that is steeper than any other entire solution between 0 and p and satisfies $w_{\infty}(0, 0; \alpha) = \alpha$. This implies that w_{∞} does not depend on the choice of (k_j) , and hence, the whole sequence $\widehat{u}(t + kT, x + a_k)$ converges to $w_{\infty}(t, x; \alpha)$ as $k \to \infty$ in $C^1(\mathbb{R}^2)$.

It remains to show that w_{∞} is either spatially homogeneous or decreasing in $x \in \mathbb{R}$. Since for each $k \in \mathbb{N}$, the function $x \mapsto \widehat{u}(t+kT,x+a_k)$ is decreasing, sending to the limit as $k \to \infty$, we have

$$\partial_x w_{\infty}(t, x; \alpha) < 0 \text{ for } t \in \mathbb{R}, x \in \mathbb{R}.$$

Then applying the strong maximum principle to the equation satisfied by $\partial_x w_{\infty}(t, x; \alpha)$, we immediately obtain that either $\partial_x w_{\infty} \equiv 0$ or $\partial_x w_{\infty}(t, x; \alpha) < 0$ for $t \in \mathbb{R}$, $x \in \mathbb{R}$. This completes the proof.

In Lemma 3.4 below, we will further prove that the limit function $w_{\infty}(t, x; \alpha)$ is either a solution of (1.5) or a periodic traveling wave. The proof will need the following spreading properties of $\widehat{u}(t, x)$:

Lemma 3.3. There exist constants $-\infty < c_* \le c^* < \infty$ such that

- (i) for each $c > c^*$, $\lim_{t\to\infty} \sup_{x>ct} \widehat{u}(t,x) = 0$;
- (ii) for each $c < c_*$, $\lim_{t \to \infty} \sup_{x < ct} |\widehat{u}(t, x) p(t)| = 0$.

It should be noted that, the above two constants c_* and c^* are independent of $a \in \mathbb{R}$ (the jumping position of the initial function p(0)H(a-x)).

Proof. The proof follows from the arguments used in the first part of the proof of [9, Lemma 2.9] with some obvious modifications, therefore we omit the details. \Box

Let us now define a sequence of real number $(l_k)_{k\in\mathbb{N}}$ as follows:

$$l_k := \begin{cases} a_k - a_{k-1}, & \text{if } k > 1, \\ a_0, & \text{if } k = 0, \end{cases}$$

where $(a_k)_{k\in\mathbb{N}}$ is the sequence given in (3.1). Clearly, for all $k\in\mathbb{N}$, $a_k=\sum_{j=0}^k l_j$. Since $\widehat{u}(kT,a_k)=\alpha\in(0,p(0))$ for each $k\in\mathbb{N}$, it follows from Lemma 3.3 that

$$c_*kT \le a_k \le c^*kT$$
 for all large $k \in \mathbb{N}$.

Thus, we have

$$c_* \le \liminf_{k \to \infty} \frac{\sum_{j=0}^k l_j}{kT} \le \limsup_{k \to \infty} \frac{\sum_{j=0}^k l_j}{kT} \le c^*. \tag{3.3}$$

Lemma 3.4. For any given $\alpha \in (0, p(0))$, let $w_{\infty}(t, x; \alpha)$ be the entire solution provided by Lemma 3.2. Then either of the following alternatives holds:

- (a) $w_{\infty}(t,x;\alpha)$ is spatially homogeneous, and it is a positive solution of (1.5);
- (b) $w_{\infty}(t, x; \alpha)$ is decreasing in $x \in \mathbb{R}$, and it is a periodic traveling wave of (1.1a). Furthermore, l_k converges to some $l_{\infty} \in [c_*T, c^*T]$ as $k \to \infty$, and l_{∞}/T is the wave speed of w_{∞} .

Proof. We split the proof into two parts, according to whether there exists some subsequence of $(l_k)_{k\in\mathbb{N}}$ converging to a finite number.

Case (1): There is a subsequence $(k_j)_{j\in\mathbb{N}}$ such that $k_j \to \infty$ as $j \to \infty$ and that $l_{k_j} \to l_{\infty}$ as $j \to \infty$ for some $l_{\infty} \in \mathbb{R}$.

In this case, by using (3.2), we have

$$w_{\infty}(t, x - l_{\infty}; \alpha) = \lim_{j \to \infty} \widehat{u}(t + k_j T, x + a_{k_j} - l_{k_j})$$

$$= \lim_{j \to \infty} \widehat{u}(t + k_j T, x + a_{k_j - 1}) = w_{\infty}(t + T, x; \alpha).$$
(3.4)

Notice from Lemma 3.2 that w_{∞} is either spatially homogeneous or spatially decreasing. Therefore, w_{∞} is a positive solution of (1.5) if $\partial_x w_{\infty}(t,x) \equiv 0$; it is a periodic traveling wave connecting two distinct solutions of (1.5) if $\partial_x w_{\infty}(t,x) < 0$.

Clearly, if the latter case occurs, then l_{∞}/T is the wave speed of the wave w_{∞} . We now show that, in such a situation, the whole sequence l_k converges to l_{∞} as $k \to \infty$. Let $\varepsilon_0 > 0$ be a sufficiently small constant. Thanks to the C^1 convergence of $\widehat{u}(kT, x + a_{k-1})$ to $w_{\infty}(T, x; \alpha)$ as $k \to \infty$ and the fact that $w_{\infty}(T, x; \alpha) < 0$ for $x \in \mathbb{R}$, there exists some large $K \in \mathbb{N}$ such that

$$\partial_x \widehat{u}(kT, l_\infty + \varepsilon + a_{k-1}) \le \frac{1}{2} \partial_x w_\infty(T, l_\infty; \alpha) \text{ for all } \varepsilon \in [0, \varepsilon_0), \ k \ge K.$$

This implies in particular that

$$\widehat{u}(kT, l_{\infty} + \varepsilon_0 + a_{k-1}) \le \widehat{u}(kT, l_{\infty} + a_{k-1}) + \frac{\varepsilon_0}{2} \partial_x w_{\infty}(T, l_{\infty}; \alpha) \text{ for all } k \ge K.$$
 (3.5)

Let δ be a positive constant given by

$$\delta := -\frac{\varepsilon_0}{4} \partial_x w_{\infty}(T, l_{\infty}; \alpha).$$

Then, replacing K by some larger integer if necessary, it follows from (3.2) that

$$\widehat{u}(kT, l_{\infty} + a_{k-1}) \le w_{\infty}(T, l_{\infty}; \alpha) + \delta \text{ for all } k \ge K.$$

Combining this with (3.5), we obtain

$$\widehat{u}(kT, l_{\infty} + \varepsilon_0 + a_{k-1}) \le w_{\infty}(T, l_{\infty}; \alpha) \text{ for all } k \ge K.$$
 (3.6)

Due to (3.4), we have $w_{\infty}(T, l_{\infty}; \alpha) = w_{\infty}(0, 0; \alpha) = \alpha$. It follows that

$$\widehat{u}(kT, l_{\infty} + \varepsilon_0 + a_{k-1}) \le \alpha \text{ for all } k \ge K.$$

Note that for each $k \in \mathbb{N}$, $\widehat{u}(kT, a_k) = \alpha$ and $\widehat{u}(kT, x)$ is nonincreasing in $x \in \mathbb{R}$. Then we have

$$a_{k-1} + l_{\infty} + \varepsilon_0 \ge a_k$$
 for all $k \ge K$.

By similar arguments to those used in showing (3.6), we can prove that

$$\widehat{u}((k-1)T, -l_{\infty} + \varepsilon_0 + a_k) \leq w_{\infty}(-T, -l_{\infty}; \alpha) = \alpha$$
 for all large k ,

and then conclude that

$$a_k - l_{\infty} + \varepsilon_0 \ge a_{k-1}$$
 for all large k .

Combining the above, we have $|a_k - a_{k-1} - l_{\infty}| \leq \varepsilon_0$ for all large $k \in \mathbb{N}$. Since $\varepsilon_0 > 0$ can be chosen arbitrarily small, this immediately gives that $l_k \to l_{\infty}$ as $k \to \infty$. Furthermore, thanks to (3.3), the limit l_{∞} belongs to the interval $[c_*T, c^*T]$.

Therefore, we can conclude that if case (1) occurs, then either case (a) or case (b) of the present lemma holds.

Case (2): there is no subsequence of $(l_k)_{k\in\mathbb{N}}$ converging to a finite number.

In this situation, we want to show that only case (a) occurs. As no subsequence of $(l_k)_{k\in\mathbb{N}}$ converges to a finite number, we observe from (3.3) that, there must exist a subsequence l_{k_j} converging to ∞ and another one $l_{k'_j}$ converging to $-\infty$ as $j\to\infty$. Let M>0 be arbitrary. Then we have $l_{k_j}>M$ and $l_{k'_j}<-M$ for all large $j\in\mathbb{N}$. Since $\widehat{u}(t,x)$ is nonincreasing in x, it follows that

$$\widehat{u}(t+k_jT, x+a_{k_j-1}) \ge \widehat{u}(t+k_jT, x+a_{k_j}-M)$$

for all $t \in \mathbb{R}$, $x \in \mathbb{R}$ and large $j \in \mathbb{N}$. Taking the limit along the sequence $j \to \infty$, we obtain

$$w_{\infty}(t+T,x;\alpha) \ge w_{\infty}(t,x-M;\alpha)$$
 for $t \in \mathbb{R}, x \in \mathbb{R}$.

Then, passing to the limit as $M \to \infty$ followed by letting $x \to \infty$ yields

$$\lim_{t \to \infty} w_{\infty}(t+T, x; \alpha) \ge \lim_{t \to -\infty} w_{\infty}(t, x; \alpha) \text{ for } t \in \mathbb{R}.$$
 (3.7)

Similarly, since

$$\widehat{u}(t + k_j'T, x + a_{k_i'-1}) \le \widehat{u}(t + k_j'T, x + M + a_{k_i'})$$

for all $t \in \mathbb{R}$, $x \in \mathbb{R}$ and large $j \in \mathbb{N}$, passing to the limits as $j \to \infty$, $M \to \infty$ and $x \to -\infty$ in order, we have

$$\lim_{x \to -\infty} w_{\infty}(t+T, x; \alpha) \le \lim_{x \to +\infty} w_{\infty}(t, x; \alpha) \text{ for } t \in \mathbb{R}.$$
 (3.8)

Combining (3.7), (3.8) and the fact that $w_{\infty}(t, x; \alpha)$ is nonincreasing in x, we see that w_{∞} is spatially homogeneous, and hence it is a solution of (1.5). The proof of Lemma 3.4 is thus complete.

3.2. Existence of a minimal terrace. Based on the preparation in the above subsection, we are now ready to construct a minimal propagating terrace under Assumption 1.6.

Proof of Theorem 1.8. Denote by $(q_m)_{0 \le m \le M}$ the decomposition between 0 and p, and for each $1 \le m \le M$, let $V_m(t,x)$ be a periodic traveling wave of (1.1a) connecting q_m to q_{m-1} . Choose $\alpha_1 > 0$ such that $q_1(0) < \alpha_1 < p(0)$ and let $w_{\infty}(t,x;\alpha_1)$ be the entire solution provided by Lemma 3.2.

Claim 3.5. $U_1(t,x) := w_{\infty}(t,x;\alpha_1)$ is a periodic traveling wave of (1.1a) connecting q_{m_1} to p for some $m_1 \in \{1, 2, \dots, M\}$.

Proof of Claim 3.5. We first prove that $U_1(t,x)$ is a periodic traveling wave. Suppose the contrary that it is not true. Then by Lemma 3.4, $w_{\infty}(t,x;\alpha_1)$ is spatially homogeneous, and $w_{\infty}(t) := w_{\infty}(t,x;\alpha_1)$ is a solution of (1.5) with $w_{\infty}(0) = \alpha_1$. Notice that $q_1 < w_{\infty} < p$, and $V_1(t,x)$ is a periodic traveling wave connecting q_1 to p. Clearly, $V_1(t,x)$ is steeper than $w_{\infty}(t)$ in the sense of Definition 1.2. This is a contradiction with the assertion stated in Lemma 3.2 that w_{∞} is steeper than any other entire solution between 0 and p. Therefore, $U_1(t,x)$ is a periodic traveling wave.

Next, we prove that

$$\lim_{x \to -\infty} U_1(t,x) = p(t) \text{ locally uniformly in } t \in \mathbb{R}.$$

Since $w_{\infty}(t, x; \alpha_1)$ is a periodic traveling wave, the limit $\lim_{x \to -\infty} w_{\infty}(t, x; \alpha_1) := w_{\infty}(t, -\infty)$ holds locally uniformly in $t \in \mathbb{R}$, and $w_{\infty}(t, -\infty)$ is a solution of (1.5). It is clear that

$$0 \le w_{\infty}(t, -\infty) \le p(t) \text{ for } t \in \mathbb{R}. \tag{3.9}$$

As $w_{\infty}(t, x; \alpha_1)$ is a steepest solution of (1.1a) between 0 and p, in view of Definition 1.2 and Lemma 2.2, we see that $w_{\infty}(t, -\infty)$ is steeper than any other entire solution between 0 and p. This implies in particular that for any $t \in \mathbb{R}$, $w_{\infty}(t, -\infty)$ and $V_1(t, x)$ cannot intersect. Since $w_{\infty}(0, -\infty) > \alpha_1$ and $V_1(0, x_0) = \alpha_1$ for some $x_0 \in \mathbb{R}$, it then follows that $w_{\infty}(t, -\infty) > V_1(0, x)$ for all $x \in \mathbb{R}$. By the comparison principle, we have

$$w_{\infty}(t, -\infty) \ge V_1(t, x)$$
 for $t \ge 0, x \in \mathbb{R}$.

Due to the *T*-periodicity of $w_{\infty}(t, -\infty)$ and the fact the $V_1(t, x)$ converges to p(t) as $x \to -\infty$, we have $w_{\infty}(t, -\infty) \ge p(t)$ for $t \in \mathbb{R}$. Combining this with (3.9), we immediately obtain $w_{\infty}(t, -\infty) \equiv p(t)$.

It remains to show

$$\lim_{x\to\infty} U_1(t,x) = q_{m_1}(t) \text{ locally uniformly in } t\in\mathbb{R}$$

for some $m_1 \in \{1, 2, \dots, M\}$. In fact, by similar arguments used as above, one can conclude that $w_{\infty}(t, \infty) := \lim_{x \to \infty} w_{\infty}(t, x; \alpha_1)$ is steeper than any other entire solution of (1.1a) between 0 and p. This already implies the desired result. Otherwise, there would exist some $\widetilde{m} \in \{2, \dots, M\}$ such that the periodic traveling wave $V_{\widetilde{m}}(t, x)$ crosses through $w_{\infty}(t, \infty)$, which is impossible. The proof of Claim 3.5 is thus complete.

For convenience, let us set $m_0 = 0$. Then $U_1(t, x)$ is a traveling wave connecting some q_{m_1} to q_{m_0} . If $m_1 = M$, i.e., $q_{m_1} \equiv 0$, then we have already obtained a propagating terrace, which consists of a single wave U_1 . Suppose, on the other hand, that q_{m_1} is positive. We can continue our iteration as follows:

Claim 3.6. Suppose that for some $m_i \in \{1, \dots, M-1\}$ and some $\alpha_i \in (q_{m_i}(0), p(0))$, the function

$$U_i(t,x) :\equiv w_{\infty}(t,x;\alpha_i)$$

is a periodic traveling wave connecting q_{m_i} to some $q_{m_{i-1}} > q_{m_i}$. Then there exists $\alpha_{i+1} \in (0, q_{m_i}(0))$ such that

$$U_{i+1}(t,x) := w_{\infty}(t,x;\alpha_{i+1})$$

is a periodic traveling wave connecting some $q_{m_{i+1}} < q_{m_i}$ to q_{m_i} .

Proof of Claim 3.6. The proof is almost the same as that of Claim 3.5, therefore we omit the details. \Box

Note that the above iteration process ends when $q_{m_i} \equiv 0$. Since there is a finite number of q_m , this clearly happens in a finite number of steps i = N for some $1 \leq N \leq M$. Therefore, we obtain a sequence of decreasing numbers $(\alpha_i)_{1 \leq i \leq N}$, a sequence of decreasing solutions $(q_{m_i})_{0 \leq i \leq N}$ of (1.5), and a sequence $(U_i)_{1 \leq i \leq N}$ such that for each $1 \leq i \leq N$, $U_i(t,x)$ a periodic traveling wave connecting q_{m_i} to $q_{m_{i-1}}$ satisfying $U_i(0,0) = \alpha_i$.

For each $1 \leq i \leq N$, let c_i be the wave speed of $U_i(t,x)$. Now, we want to prove that

$$\mathcal{T} := ((q_{m_i})_{0 \le i \le N}, (U_i, c_i)_{1 \le i \le N})$$

is a propagating terrace. According to Definition 1.1, it suffices to show that

$$c_1 \leq c_2 \leq \cdots \leq c_N$$
.

For this purpose, for each $1 \le i \le N$, let us denote by $(a_{i,k})_{k \in \mathbb{N}}$ the sequence obtained in (3.1) with α replaced by α_i . Then Lemma 3.4 provides that $\lim_{k\to\infty}(a_{i,k}-a_{i,k-1})=c_iT$. This clearly implies

$$c_i = \lim_{k \to \infty} \frac{a_{i,k}}{kT}.$$
 (3.10)

Notice that $\alpha_{i+1} < \alpha_i$ for each $1 \le i \le N-1$ and that the solution $\widehat{u}(t,x)$ is decreasing in x. We have $a_{i+1,k} > a_{i,k}$ for all $k \in \mathbb{N}$. It then follows immediately from (3.10) that $c_{i+1} \ge c_i$. The existence of propagating terrace is thus proved.

From the above construction of U_i , one sees that q_{m_i} and U_i are steeper than any other entire solution between 0 and p, which immediately gives statement (ii) of the present theorem and the minimality of the propagating terrace \mathcal{T} . By Proposition 1.5, \mathcal{T} is unique up to spatial shifts. Lastly, statement (i) follows directly from Lemma 2.6. The proof is thus complete. \square

3.3. Convergence with Heaviside type initial data. This subsection is devoted to the proof of Theorem 1.10. Namely, provided that there exists a minimal propagating terrace, we show that it attracts all solutions of (1.1) with Heaviside type initial data.

Proof of Theorem 1.10. Let $((p_i)_{0 \le i \le N}, (U_i, c_i)_{1 \le i \le N})$ be a minimal propagating terrace connecting 0 to p. By Proposition 1.5, up to spatial shifts, it is the unique minimal terrace. Moreover, thanks to Theorem 1.8, we know that each of the p_i and U_i is steeper than any other entire solution of (1.1a) between 0 and p.

Let $\widehat{u}(t,x)$ be the solution of (1.1) with a given Heaviside type initial function. For each $i \in \{1, \dots, N\}$, let $(a_{i,k})_{k \in \mathbb{N}}$ be the sequence of real numbers such that

$$\widehat{u}(kT, a_{i,k}) = U_i(0,0)$$
 for all $k \in \mathbb{N}$.

Since $U_i(t, x)$ is steeper than any other entire solution between 0 and p, it follows from Lemmas 3.2 and 3.4 that for any $t \ge 0$,

$$\widehat{u}(t+kT,x+a_{i,k}) \to U_i(t,x) \text{ as } k \to \infty \text{ in } L^{\infty}_{loc}(\mathbb{R}).$$
 (3.11)

Since

$$U_i(\cdot, \cdot) = U_i(\cdot + mT, \cdot + c_i mT) \text{ for all } m \in \mathbb{Z},$$
 (3.12)

(3.11) implies that

$$\widehat{u}(t, x + a_{i, \lfloor t/T \rfloor}) - U_i(t, x + c_i \lfloor t/T \rfloor T) \to 0 \text{ as } t \to \infty \text{ in } L^{\infty}_{loc}(\mathbb{R}),$$
 (3.13)

where $\lfloor t/T \rfloor$ is the floor function of t/T, that is, the greatest integer not larger than t/T.

For each $i \in \{1, \dots, N\}$, let $\eta_i : [0, \infty) \to \mathbb{R}$, $t \mapsto \eta_i(t)$ be a $C^1([0, \infty))$ function satisfying

$$\eta_i(t) + c_i \lfloor t/T \rfloor T - a_{i,|t/T|} \to 0 \text{ as } t \to \infty.$$
 (3.14)

Now we want to show that $(\eta_i)_{1 \leq i \leq N}$ are the desired functions satisfying statements (i)-(iii). Notice from (3.11) and (3.12) that $\lim_{k \to \infty} (a_{i,k+1} - a_{i,k}) = c_i T$. This implies that

$$\frac{c_i \lfloor t/T \rfloor T - a_{i, \lfloor t/T \rfloor}}{t} = \frac{\lfloor t/T \rfloor T \left(c_i - \frac{a_{i, \lfloor t/T \rfloor}}{T \lfloor t/T \rfloor} \right)}{t} \to 0 \text{ as } t \to \infty.$$

It then follows directly from (3.14) that $\eta_i(t)/t \to 0$ as $t \to \infty$. Thus, statement (i) is proved. Next, from (3.11) again, we see that

$$a_{i+1,k} - a_{i,k} \to \infty$$
 as $k \to \infty$ for $i \in \{1, \dots, N-1\}$.

This together with (3.14) immediately yields that

$$\eta_{i+1}(t) - \eta_i(t) \to \infty$$
 as $t \to \infty$ whenever $c_i = c_{i+1}$.

Thus, statement (ii) is obtained.

It remains to show the uniform convergence in statement (iii). Note that each U_i satisfies

$$\lim_{x \to -\infty} U_i(t, x + c_i t) = p_{i-1}(t) \quad \text{and} \quad \lim_{x \to \infty} U_i(t, x + c_i t) = p_i(t) \quad \text{uniformly in } t \in \mathbb{R}.$$

Then given any small $\varepsilon > 0$, there exists some M > 0 such that

$$U_i(t, c_i t + M) \le p_i(t) + \frac{\varepsilon}{2}, \quad U_i(t, c_i t - M) \ge p_{i-1}(t) - \frac{\varepsilon}{2} \text{ for } t \in \mathbb{R}.$$
 (3.15)

By (3.13) and (3.14), we can find $T_0 > 0$ large enough such that for $t \geq T_0$,

$$|\widehat{u}(t,x) - U_i(t,x - \eta_i(t))| \le \frac{\varepsilon}{2} \quad \text{for } c_i t + \eta_i(t) - M \le x \le c_i t + \eta_i(t) + M.$$
 (3.16)

This together with (3.15) implies that

$$\widehat{u}(t, c_i t + \eta_i(t) + M) \leq p_i(t) + \varepsilon$$
, $\widehat{u}(t, c_i t + \eta_i(t) - M) \geq p_{i-1}(t) - \varepsilon$ for $t \geq T_0$.

Since $\widehat{u}(t,x)$ is decreasing in $x \in \mathbb{R}$, it then follows that for each $i \in \{2, \dots, N\}$,

$$-\varepsilon \le \widehat{u}(t,x) - p_{i-1}(t) \le \varepsilon \quad \text{for} \quad c_{i-1}t + \eta_{i-1}(t) + M \le x \le c_i t + \eta_i(t) - M, \ t \ge T_0, \quad (3.17)$$

that

$$0 < \widehat{u}(t, x) \le \varepsilon \quad \text{for} \quad x \ge c_N t + \eta_N(t) + M, \ t \ge T_0, \tag{3.18}$$

and that

$$p(t) - \varepsilon \le \widehat{u}(t, x) < p(t) \quad \text{for} \quad x \le c_1 t + \eta_1(t) - M, \ t \ge T_0. \tag{3.19}$$

Combining the inequalities (3.16)-(3.19), we obtain that for all $t \geq T_0$ and $x \in \mathbb{R}$,

$$\left|\widehat{u}(t,x) - \left(\sum_{i=1}^{N} U_i(t,x - \eta_i(t)) - \sum_{i=1}^{N} p_i(t)\right)\right| \le N\varepsilon.$$

This immediately gives statement (iii). The proof is thus complete.

4. Asymptotic behavior of solutions with initial data satisfying (H2)

In this section, we study the asymptotic behavior of solutions of (1.1) with initial data satisfying (H2), and prove Theorems 1.11 and 1.16. Throughout this section, we always put

$$u_0(x) = p(0) \text{ for } x \in (-\infty, a_-], \quad u_0(x) = 0 \text{ for } x \in [a_+, \infty)$$
 (4.1)

for some $a_{+} > a_{-}$.

Let $((p_i)_{0 \le i \le N}, (U_i, c_i)_{1 \le i \le N})$ be the minimal propagating terrace connecting 0 to p. Since it is only unique up to spatial shifts, for definiteness, we normalize it as follows:

$$U_i(0,0) = \frac{p_{i-1}(0) + p_i(0)}{2}$$
 for each $i = 1, \dots, N$. (4.2)

With this normalization, each U_i is uniquely determined, and we will assume this in our discussion below.

The following lemma is fundamental in this section.

Lemma 4.1. Let u(t,x) be the solution of (1.1) with u_0 satisfying (H2). Then for every $w \in \Omega(u)$, either of the following alternatives holds:

- (a) $w \equiv p_i$ for some integer $0 \le i \le N$;
- (b) w(t,x) satisfies

$$U_i(t, x + \xi_0 - a_-) \le w(t, x) \le U_i(t, x + \xi_0 - a_+) \text{ for } t \in \mathbb{R}, x \in \mathbb{R},$$
 (4.3)

for some integer $1 \le i \le N$ and some $\xi_0 \in \mathbb{R}$, where $a_{\pm} \in \mathbb{R}$ are given in (4.1). Moreover, we have $\{p_i\}_{0 \le i \le N} \subset \Omega(u)$.

Proof. Let $(k_j)_{j\in\mathbb{N}}\subset\mathbb{N}$ and $(x_j)_{j\in\mathbb{N}}\subset\mathbb{R}$ be the sequences such that

$$u(t+k_jT,x+x_j) \to w(t,x) \text{ as } j \to \infty \text{ in } C^1(\mathbb{R}^2).$$
 (4.4)

Since u_0 satisfies (H2), it follows easily from the comparison principle that, for each $j \in \mathbb{N}$,

$$\widehat{u}(t+k_{j}T,x+x_{j};a_{-}) \le u(t+k_{j}T,x+x_{j}) \le \widehat{u}(t+k_{j}T,x+x_{j};a_{+})$$
 (4.5)

for $t \geq 0$ and $x \in \mathbb{R}$. Here $\widehat{u}(t, x; a_{\pm})$ denote the solutions of (1.1) with initial functions $p(0)H(a_{+}-x)$.

By standard parabolic estimates and possibly up to a subsequence, we may assume that

$$\widehat{u}(t+k_jT,x+x_j;0) \to \widehat{w}(t,x) \text{ as } j \to \infty \text{ in } C^1(\mathbb{R}^2),$$

where $\widehat{w}(t,x)$ is an entire solution of (1.1a). Clearly, $\widehat{w}(t,x)$ is an Ω -limit solution of $\widehat{u}(t,x;0)$, and

$$\widehat{u}(t+k_jT,x+x_{k_j};a_{\pm})\to\widehat{w}(t,x-a_{\pm})$$
 as $j\to\infty$ in $C^1(\mathbb{R}^2)$.

Passing to the limit as $j \to \infty$ in (4.5), we obtain

$$\widehat{w}(t, x - a_{-}) \leq w(t, x) \leq \widehat{w}(t, x - a_{+})$$
 for $t \in \mathbb{R}, x \in \mathbb{R}$.

Furthermore, it follows directly from Theorem 1.10 that

$$\widehat{w} \in \{U_i(\cdot, \cdot + \xi) : \xi \in \mathbb{R}, 1 \le i \le N\} \cup \{p_i : 0 \le i \le N\}.$$

This clearly gives the alternatives of the present lemma.

Similarly as above, one can prove that for any Ω -limit solution \widehat{w} of the solution $\widehat{u}(t, x; 0)$, there exists $\widetilde{w} \in \Omega(u)$ such that

$$\widehat{w}(t, x - a_{-}) \leq \widetilde{w}(t, x) \leq \widehat{w}(t, x - a_{+}) \text{ for } t \in \mathbb{R}, x \in \mathbb{R}.$$

In particular, if $\widehat{w} \equiv p_i$ for some $0 \le i \le N$, then $\widetilde{w} \equiv p_i$. This immediately implies the relation $\{p_i\}_{0 \le i \le N} \subset \Omega(u)$. The proof of Lemma 4.1 is thus complete.

The reaming part of this section is organized as follows. In Subsection 4.1, we will show that if w(t,x) is an Ω -limit solution satisfying case (b) of Lemma 4.1, then it is spatially decreasing. This immediately gives the first part of the conclusions of Theorem 1.11. The remaining conclusions will be proved in Subsection 4.2. Subsection 4.3 is concerned with the proof of Proposition 1.15 and Theorem 1.16.

4.1. Monotonicity of Ω -limit solutions. This subsection is devoted to the proof of the following proposition:

Proposition 4.2. Let u(t,x) be the solution of (1.1) with u_0 satisfying (H2) and let $w \in \Omega(u)$ satisfy case (b) of Lemma 4.1. Then for each $t \in \mathbb{R}$, w(t,x) is decreasing in $x \in \mathbb{R}$.

Before giving the proof, let us first show some properties of the solution u(t, x) at any finite time, which will be needed later.

Lemma 4.3. Let u(t,x) be the solution of (1.1) with u_0 satisfying (H2). Then for each t>0,

$$\partial_x u(t,x) < 0 \text{ for } x \in (-\infty, a_-) \cup (a_+, \infty),$$

where a_{\pm} are the constants given in (4.1).

Proof. This lemma can be proved by a simple reflection argument. Fix any $x_0 \in (-\infty, a_-)$ and define

$$v(t,x) := u(t,x) - u(t,2x_0 - x)$$
 for $-\infty < x < x_0, t > 0$.

Since u(t,x) is bounded, f(t,u) is C^1 -smooth in u and T-periodic in t, v(t,x) satisfies

$$v_t = v_{xx} + c(t, x)v$$
 for $-\infty < x < x_0, t > 0$,

with some bounded function c(t,x). Moreover, it is easily checked that

$$v(t, x_0) = 0$$
 for $t > 0$, $v(0, x) \ge 0$ for $x < x_0$, and $v(0, x) \ne 0$.

Then the strong maximum principle implies that v(t,x) > 0 for t > 0, $x < x_0$. It further follows from the Hopf boundary lemma that $\partial_x v(t,x_0) < 0$, and hence, $\partial_x u(t,x_0) < 0$ for t > 0. Since x_0 can be chosen arbitrarily in $(-\infty,a_-)$, one obtains $\partial_x u(t,x) < 0$ for $x < a_-$, t > 0. The case $x > a_+$ can be proved in a similar way. The proof of Lemma 4.3 is complete.

Recall that $\mathcal{Z}(w)$ denotes the number of sign changes of a real-valued function w(x) defined on \mathbb{R} . The following lemma is an application of the zero-number theory introduced in Subsection 2.1.

Lemma 4.4. Let u(t,x) be the solution of (1.1) with u_0 satisfying (H2). Then for any $z \in \mathbb{R}$, we have

$$\mathcal{Z}[u(t,\cdot) - u(t,\cdot + z)] < \infty \text{ for } t > 0, \tag{4.6}$$

and

$$\mathcal{Z}[u(t+T,\cdot)-u(t,\cdot+z)]<\infty \ for \ t>0. \tag{4.7}$$

Furthermore, the above two quantities are nonincreasing in t > 0.

Proof. Let us first prove (4.6). Notice that if z = 0, then the result is trivial. Without loss of generality, we may assume that z > 0, as the case z < 0 can be argued analogously. Since both u(t,x) and u(t,x+z) are bounded solutions of (1.1a), and since f(t,u) is C^1 -smooth in u and T-periodic in t, u(t,x) - u(t,x+z) satisfies a linear equation of the form (2.1) with c(t,x) := (f(t,u(t,x)) - f(t,u(t,x+z)))/(u(t,x) - u(t,x+z)) being bounded.

Denote by $\bar{u}(t,x)$ the solution of the Cauchy problem

$$\bar{u}_t = \bar{u}_{xx} \text{ for } t > 0, x \in \mathbb{R}; \quad \bar{u}(0, x) = u_0(x) \text{ for } x \in \mathbb{R}.$$

Due to the boundedness of the solution u(t, x), we find some K > 0 such that $-Ku \le f(t, u) \le Ku$ for $t \ge 0$, $x \in \mathbb{R}$. Then, a simple comparison argument implies that

$$e^{-Kt}\bar{u}(t,x) \le u(t,x) \le e^{Kt}\bar{u}(t,x)$$
 for all $t \ge 0, x \in \mathbb{R}$.

By the assumption that $u_0(x) = 0$ on $[a_+, \infty)$, we have

$$\frac{u(x,t)}{u(x+z,t)} \ge \exp(-2Kt) \frac{\bar{u}(x,t)}{\bar{u}(x+z,t)}
= \exp(-2Kt) \frac{\int_{-\infty}^{a_+} \exp\left(-\frac{(x-y)^2}{4t}\right) u_0(y) dy}{\int_{-\infty}^{a_+} \exp\left(-\frac{(x-y+z)^2}{4t}\right) u_0(y) dy}
\ge \exp(-2Kt) \exp\left(\frac{2z(x-a_+) + z^2}{4t}\right).$$

for all $x > a_+$, t > 0. Since z > 0, passing to the limit as $x \to \infty$, we obtain that for each t > 0,

$$\frac{u(x,t)}{u(x+z,t)} \to \infty$$
 as $x \to \infty$.

Similarly, by using the assumption that $u_0(x) = p(0)$ on $(-\infty, a_-]$, we can conclude that for each t > 0,

$$\frac{p(t)-u(x,t)}{p(t)-u(x+z,t)}\to 0 \quad \text{ as } x\to -\infty.$$

Then, for any given $t_0 > 0$, it is easily checked from the above that there exists L > 0 such that

$$u(t_0, x) - u(t_0, x + z) > 0$$
 for $x \in (-\infty, -L] \cup [L, \infty)$.

Thus, by Lemma 2.1,

$$\mathcal{Z}[u(t,\cdot) - u(t,\cdot+z)] < \infty \text{ for all } t \ge t_0,$$

and this quantity is nonincreasing in $t \ge t_0$. Since $t_0 > 0$ is arbitrary, (4.6) is proved.

Let us now turn to the proof of (4.7). Similarly as above, we may assume without loss of generality that $z \ge 0$. Since f(t,u) is T-periodic in t, u(t+T,x) is also a solution of (1.1a). Then u(t+T,x)-u(t,x+z) satisfies a linear solution of the form (2.1) with bounded coefficient c(t,x).

Since $0 \le u_0 \le p(0)$, it is clear that 0 < u(t,x) < p(t) for t > 0, $x \in \mathbb{R}$. Notice that u_0 satisfies (4.1). Then we can choose $\delta > 0$ sufficiently small such that

$$u(T+t,a_{+}) > u(t,a_{+}+z) > 0$$
 and $u(T+t,a_{-}-z) < u(t,a_{-}) < p(t)$

for $0 < t \le \delta$. Since u(T, x) > u(0, x + z) for $x \ge a_+$, and u(T, x) < u(0, x + z) for $x \le a_- - z$, the comparison principle implies that

$$u(T+t,x) > u(t,x+z)$$
 for $0 < t \le \delta, x \ge a_+$

and

$$u(T+t,x) < u(t,x+z)$$
 for $0 < t \le \delta, x \le a_{-} - z$.

It then follows from Lemma 2.1 that

$$\mathcal{Z}[u(T+t,\cdot)-u(t,\cdot+z)] < \infty$$
 for all $0 < t \le \delta$,

and this quantity is nonincreasing in t > 0. This immediately implies (4.7). The proof of Lemma 4.4 is thus complete.

We are now prepared to prove Proposition 4.2.

Proof of Proposition 4.2. Let us denote $\varphi(t,x) := \partial_x w(t,x)$ for $t \in \mathbb{R}$, $x \in \mathbb{R}$. Clearly, $\varphi(t,x)$ is a solution of the following linear equation

$$\varphi_t = \varphi_{xx} + \partial_u f(t, w) \varphi$$
 for $t \in \mathbb{R}, x \in \mathbb{R}$,

where $\partial_u f(t, w) := \partial f(t, u) / \partial u|_{u=w}$ is bounded for $t \in \mathbb{R}, x \in \mathbb{R}$.

For clarity, we divide the proof into 3 steps.

Step 1: we show that for each $t \in \mathbb{R}$, $\mathcal{Z}[\varphi(t,\cdot)] < \infty$, and all zeros of $x \mapsto \varphi(t,\cdot)$ are simple. Let us first note that the function $v(t,x) := \partial_x u(t,x)$, defined on $(t,x) \in [1,\infty) \times \mathbb{R}$, solves a linear equation of the form (2.1) with bounded coefficient. Moreover, by Lemma 4.3, for each $t \geq 1$, the function $x \mapsto v(t,x)$ does not change sign on the set $(-\infty,a_-) \cup (a_+,\infty)$. It then follows from Lemma 2.1 (i) and (ii) that $\mathcal{Z}[v(t,\cdot)] < \infty$, and it is nonincreasing in $t \geq 1$. Therefore, it is a constant for all large t, and by Lemma 2.1 (iii), we have

the function
$$x \mapsto v(t,x)$$
 has only simple zeros on \mathbb{R} for all large t . (4.8)

Let $(k_j)_{j\in\mathbb{N}}\subset\mathbb{N}$ and $(x_j)_{j\in\mathbb{N}}\subset\mathbb{R}$ be the sequences such that (4.4) holds for w(t,x). Then we have

$$v(t+k_iT, x+x_i) \to \varphi(t, x) \text{ as } j \to \infty,$$
 (4.9)

where the convergence holds in $L^{\infty}_{loc}(\mathbb{R}^2)$. For any $t \in \mathbb{R}$, since $\mathcal{Z}[v(t+k_jT,\cdot)] < \infty$ for all sufficiently large k_j , it follows from the semi-continuity of \mathcal{Z} (see (2.2)) that $\mathcal{Z}[\varphi(t,\cdot)] < \infty$.

Furthermore, by standard parabolic estimates, possibly after extracting a subsequence, we know that the convergence (4.9) also takes place in $C^1(\mathbb{R}^2)$. In view of this and (4.8), and applying Lemma 2.5, we see that for each $t \in \mathbb{R}$, either $\varphi(t,x) \equiv 0$ on \mathbb{R} or $x \mapsto \varphi(t,x)$ has only simple zeros on \mathbb{R} . The former is impossible, as we have assumed that w(t,x) satisfies (4.3), thus it cannot be spatially homogeneous. Consequently, $x \mapsto \varphi(t,x)$ has only simple zeros on \mathbb{R} . This ends the proof of Step 1.

Step 2: We show that for any $z \neq 0$ and $t \in \mathbb{R}$, all zeros of $x \mapsto w(t, x) - w(t, x + z)$ are simple.

The proof follows from similar arguments to those used in Step 1, therefore we only give its outline. Let $z \neq 0$ be fixed. Thanks to (4.6), it follows from Lemma 2.1 that, for all large t > 0, the function $x \mapsto u(t,x) - u(t,x+z)$ has only finitely many zeros on \mathbb{R} and all of them are simple. Note that

$$u(t+k_iT,x+x_i)-u(t+k_iT,x+z+x_i)\to w(t,x)-w(t,x+z)$$
 as $j\to\infty$

in $C^1(\mathbb{R}^2)$, and that w(t,x) - w(t,x+z) solves a linear equation of the form (2.1) with bounded coefficient. Then by using Lemma 2.5 and the assumption that w satisfies (4.3), we obtain the desired result of Step 2.

Step 3: We show that for each $t \in \mathbb{R}$, w(t,x) is decreasing in $x \in \mathbb{R}$.

Assume by contradiction that there exists some $t_0 \in \mathbb{R}$ such that $w(t_0, x)$ is not decreasing in $x \in \mathbb{R}$. Notice that

$$\lim_{x \to \infty} w(t_0, x) = p_i(t_0) < p_{i-1}(t_0) = \lim_{x \to -\infty} w(t_0, x).$$

It then follows from Step 1 that

$$1 < \mathcal{Z}[\varphi(t_0, \cdot)] < \infty,$$

and all zeros of $x \mapsto \varphi(t_0,\cdot)$ are simple. Denote by ξ_1 the minimum of these zeros, and define

$$x_1 := \min\{x > \xi_1 : w(t_0, x) = w(t_0, \xi_1), \varphi(t_0, x) \neq 0\}.$$

Clearly, $x_1 \in (\xi_1, \infty)$ is well defined and $\varphi(t_0, x_1) < 0$. Furthermore, let ξ_2 be the maximum of the zeros of $\varphi(t_0, \cdot)$ to the left of x_1 , and let x_2 be the point such that $x_2 < \xi_1$ and $w(t_0, x_2) = w(t_0, \xi_2)$. It is then easily checked that $x_2 < \xi_1 < \xi_2 < x_1$,

$$\varphi(t_0, x) < 0 \text{ for } x \in [x_2, \xi_1) \cup (\xi_2, x_1],$$

and

$$w(t_0, x_2) = w(t_0, \xi_2), \quad w(t_0, \xi_1) = w(t_0, x_1).$$

This implies in particular that for each $x \in [x_2, \xi_1]$, there exists a unique $y \in [\xi_2, x_1]$ such that $w(t_0, x) = w(t_0, y)$. Thus, we can find a continuous function $\gamma : [x_2, \xi_1] \to [\xi_2, x_1]$ satisfying $w(t_0, x) = w(t_0, \gamma(x))$. It is easily seen that $\gamma(x_2) = \xi_2, \gamma(\xi_1) = x_1$, and that

$$\varphi(t_0, x_2) - \varphi(t_0, \gamma(x_2)) < 0$$
 and $\varphi(t_0, \xi_1) - \varphi(t_0, \gamma(\xi_1)) > 0$.

From this, it immediately follows that there exists some $x_0 \in (x_2, \xi_1)$ such that

$$\varphi(t_0, x_0) = \varphi(t_0, \gamma(x_0)).$$

Therefore, x_0 is a degenerate zero of the function $x \mapsto w(t_0, x) - w(t_0, \gamma(x_0) - x_0 + x)$, which is a contradiction with the conclusion of Step 2.

Now we can conclude that for each $t \in \mathbb{R}$, w(t,x) is decreasing in $x \in \mathbb{R}$. The proof of Proposition 4.2 is thus complete.

4.2. Completion of the proof of Theorem 1.11. In this subsection, we prove that elements of $\omega(u)$ can be classified as stated in Theorem 1.11. The key ingredient of our proof is to compare the steepness between w(t,x) and w(t+T,x), where w is any Ω -limit solution satisfying case (b) of Lemma 4.1. We will show that either w(t+T,x) is equal to a spatial shift of w(t,x), or w(t+T,x) is strictly steeper or strictly less steep than w(t,x). Once we know this, we will be able to prove that w(t,x) is either a periodic traveling wave or it is a heteroclinic solution connecting two periodic traveling waves.

Lemma 4.5. Suppose that $w \in \Omega(u)$ satisfies case (b) of Lemma 4.1. Then one of the following holds:

- (a) w(t,x) is a periodic traveling wave connecting p_i to p_{i-1} with wave speed c_i for some $1 \le i \le N$;
- (b) w(t+T,x) is strictly steeper than w(t,x);
- (c) w(t+T,x) is strictly less steep than w(t,x).

Furthermore, if case (b) (resp. case (c)) occurs, then w(t + kT, x) is strictly steeper (resp. strictly less steep) than w(t, x) for all positive integer k.

Proof. Let z be an arbitrary real number. We already know from Lemma 4.4 that $\mathcal{Z}[u(T+t,\cdot)-u(t,\cdot+z)]$ is finite for all t>0, and it is nonincreasing in t>0. By using Lemma 2.1, we see that the function $x\mapsto u(T+t,x)-u(t,x+z)$ has only simple zeros for all large t. Then, similarly as in the proof of Proposition 4.2, one can apply Lemma 2.5 to conclude that either of the following alternatives holds:

$$w(t+T,x) \equiv w(t,x+z) \tag{4.10}$$

or for each $t \in \mathbb{R}$,

the function
$$x \mapsto w(t+T,x) - w(t,x+z)$$
 has only simple zeros. (4.11)

As we have assumed that w(t,x) satisfies case (b) of Lemma 4.1, there exists some $1 \le i \le N$ such that

$$\lim_{x \to -\infty} w(t, x) = p_{i-1}(t), \quad \lim_{x \to \infty} w(t, x) = p_i(t) \text{ locally uniformly in } t \in \mathbb{R}.$$
 (4.12)

For clarity, we divide the rest of the proof into three steps.

Step 1: We show that if (4.10) holds for some $z \in \mathbb{R}$, then $z = -c_i T$ and case (a) of the present lemma occurs.

It easily seen from (4.10) and (4.12) that w(t,x) is a periodic traveling wave connecting p_i to p_{i-1} , and -z/T is the wave peed. Thus, we only need to show that $z = -c_i T$. Suppose the contrary that this is not true. We may assume without loss of generality that $z > -c_i T$. Then, since U_i is a periodic traveling wave with speed c_i , it follows that

$$U_i(mT, x - mz) \to p_{i-1}(0)$$
 as $m \to \infty$ locally uniformly in $x \in \mathbb{R}$.

Notice from (4.3) that, w satisfies

$$w(mT, -mz) \ge U_i(mT, \xi_0 - a_- - mz)$$
 for all $m \in \mathbb{Z}$.

Passing to the limit as $m \to \infty$, we obtain $\liminf_{m \to \infty} w(mT, -mz) \ge p_{i-1}(0)$. This is impossible, since $w(mT, -mz) = w(0, 0) < p_{i-1}(0)$ for each $m \in \mathbb{Z}$. Therefore, $z = -c_i T$ holds true, and hence, w(t, x) is a periodic traveling wave with speed c_i .

Step 2: We assume that (4.11) holds for all $t \in \mathbb{R}$ and $z \in \mathbb{R}$, and prove either case (b) or case (c) of the present lemma occurs.

Since w(t,x) is spatially decreasing by Proposition 4.2, for any $t \in \mathbb{R}$, we can find a C^1 function $\alpha \mapsto \zeta(\alpha;t)$ defined on $(p_i(t), p_{i-1}(t))$ such that

$$w(t,\zeta(\alpha;t)) = \alpha \text{ for } \alpha \in (p_i(t), p_{i-1}(t)). \tag{4.13}$$

Let $t_0 \in \mathbb{R}$ be an arbitrary time and let $\zeta(\alpha; t_0)$ and $\zeta(\alpha; t_0 + T)$ be the functions given as in (4.13). Since p_i and p_{i-1} are T-periodic, it is clear that $\zeta(\alpha; t_0 + T)$ is well defined on $(p_i(t_0), p_{i-1}(t_0))$ and that

$$w(t_0, \zeta(\alpha; t_0)) = w(t_0 + T, \zeta(\alpha; t_0 + T)) = \alpha \text{ for } \alpha \in (p_i(t_0), p_{i-1}(t_0)).$$
(4.14)

Next, we claim that

$$\partial_x w(t_0 + T, \zeta(\alpha; t_0 + T)) \neq \partial_x w(t_0, \zeta(\alpha; t_0))$$
 for all $\alpha \in (p_i(t_0), p_{i-1}(t_0))$.

Otherwise, there exists some $\alpha_0 \in (p_i(t_0), p_{i-1}(t_0))$ such that the equality holds at $\alpha = \alpha_0$. This together with (4.14) implies that $x = \zeta(\alpha_0; t_0 + T)$ is a degenerate zero of the function $x \mapsto w(t_0 + T, x) - w(t_0, x + \zeta(\alpha_0; t_0) - \zeta(\alpha_0; t_0 + T))$, which is a contradiction with our assumption that (4.11) holds for all $t \in \mathbb{R}$ and $z \in \mathbb{R}$. Thus, our claim is proved.

It then follows that either

$$\partial_x w(t_0 + T, \zeta(\alpha; t_0 + T)) > \partial_x w(t_0, \zeta(\alpha; t_0))$$
 for all $\alpha \in (p_i(t_0), p_{i-1}(t_0))$,

or

$$\partial_x w(t_0 + T, \zeta(\alpha; t_0 + T)) < \partial_x w(t_0, \zeta(\alpha; t_0))$$
 for all $\alpha \in (p_i(t_0), p_{i-1}(t_0))$.

Since $\alpha \in (p_i(t_0), p_{i-1}(t_0))$ is arbitrary, we obtain that $w(t_0 + T, \cdot)$ is either strictly steeper or strictly less steep than $w(t_0, \cdot)$ in the sense of Definition 1.2. Furthermore, by Lemma 2.3 and the arbitrariness of $t_0 \in \mathbb{R}$, we can conclude that either case (b) or case (c) occurs.

Step 3: We show that if case (b) (resp. case (c)) occurs, then w(t+kT,x) is strictly steeper (resp. strictly less steep) than w(t,x) for all positive integer k.

Let us assume without loss of generality that case (b) occurs. Then, for each $k \in \mathbb{N}$, w(t+kT,x) is strictly steeper than w(t+(k-1)T,x). For any $t \in \mathbb{R}$, let $\alpha \mapsto \zeta(\alpha;t+kT)$ be the function given as in (4.13). It then follows that for each $t \in \mathbb{R}$, $\alpha \in (p_i(t), p_{i-1}(t))$ and $k \geq 1$, there holds

$$w(t+kT,\zeta(\alpha;t+kT)) = \dots = w(t+T,\zeta(\alpha;t+T)) = w(t,\zeta(\alpha;t)) = \alpha,$$

and

$$\partial_x w(t+kT,\zeta(\alpha;t+kT)) > \cdots > \partial_x w(t+T,\zeta(\alpha;t+T)) > \partial_x w(t,\zeta(\alpha;t)).$$

This implies that w(t + kT, x) is strictly steeper than w(t, x). The proof of Lemma 4.5 is thus complete.

In the following lemma, we show that if case (b) or case (c) of Lemma 4.5 holds, then w(t, x) is a heteroclinic solution connecting two periodic traveling waves.

Lemma 4.6. If $w \in \Omega(u)$ satisfies case (b) (resp. case (c)) of Lemma 4.5, then there are $V_{\pm} \in \Omega(u)$ such that

$$w(t,x) - V_{\pm}(t,x) \to 0 \text{ as } t \to \pm \infty \text{ uniformly in } x \in \mathbb{R},$$
 (4.15)

and $V_{+}(t,x)$ is strictly steeper (resp. strictly less steep) than $V_{-}(t,x)$. Furthermore, V_{\pm} are periodic traveling waves of (1.1a) connecting p_i to p_{i-1} and sharing the same speed c_i for some $i \in \{1, \dots, N\}$.

Proof. We only give the proof in the case where w(t+T,x) is strictly steeper than w(t,x), as the proof for the other case is identical.

Note that w satisfies (4.3) for some $1 \leq i \leq N$ and some $\xi_0 \in \mathbb{R}$. We can find a sequence $(z_m)_{m \in \mathbb{Z}} \subset \mathbb{R}$ such that

$$w(mT, z_m) = \frac{p_{i-1}(0) + p_i(0)}{2}$$
 for $m \in \mathbb{Z}$.

Furthermore, by the normalization of U_i in (4.2), we have

$$c_i m T - \xi_0 + a_- \le z_m \le c_i m T - \xi_0 + a_+ \text{ for } m \in \mathbb{Z}.$$
 (4.16)

For each $m \in \mathbb{Z}$, let us define

$$w_m(t,x) := w(t+mT, x+z_m)$$
 for $t \in \mathbb{R}, x \in \mathbb{R}$.

Clearly, for each $m \in \mathbb{Z}$, $w_m(0,0) = (p_{i-1}(0) + p_i(0))/2$. Moreover, by Lemma 4.5, we see that for any integer $k \geq 1$, $w_{m+k}(t,x)$ is strictly steeper than $w_m(t,x)$.

We now show the convergence of w(t,x) to a periodic traveling wave as $t \to \infty$. The convergence as $t \to -\infty$ can be proved analogously. We proceed with three steps.

Step 1: we prove the convergence of $w_m(t,x)$ as $m \to \infty$ in $L^{\infty}_{loc}(\mathbb{R}^2)$.

Let us first notice that from standard parabolic estimates, the sequence $\{w_m(t,x)\}_{m\in\mathbb{Z}}$ is uniformly bounded along with their derivatives. Therefore, it is relatively compact for the topology of $L^{\infty}_{loc}(\mathbb{R}^2)$ with respect to (t,x). Then there exist a subsequence of integers $(m_j)_{j\in\mathbb{N}}$ $(m_j \to \infty \text{ as } j \to \infty)$ and an entire solution $W_+(t,x)$ of (1.1a) such that

$$w_{m_j}(t,x) \to W_+(t,x)$$
 as $j \to \infty$ in $L^{\infty}_{loc}(\mathbb{R}^2)$.

Clearly, W_+ belongs to $\Omega(u)$, as $\Omega(u)$ is compact in $L^{\infty}_{loc}(\mathbb{R}^2)$ (see Subsection 2.2). Since for any fixed $m \in \mathbb{Z}$, $w_{m_j}(t,x)$ is strictly steeper than $w_m(t,x)$ for all large m_j , it follows from Lemma 2.2 that $W_+(t,x)$ is steeper than each $w_m(t,x)$.

Let $(\widetilde{m}_j)_{j\in\mathbb{N}}$ be another subsequence of integers such that $\widetilde{m}_j \to \infty$ as $j \to \infty$, and that

$$w_{\widetilde{m}_j}(t,x) \to \widetilde{W}_+(t,x)$$
 as $j \to \infty$ in $L^{\infty}_{loc}(\mathbb{R}^2)$

for some $\widetilde{W}_+ \in \Omega(u)$. Similarly as above, we can conclude that $\widetilde{W}_+(t,x)$ is steeper than each $w_m(t,x)$.

In particular, we have $W_+(t,x)$ is steeper than each $w_{\widetilde{m}_j}(t,x)$, and $\widetilde{W}_+(t,x)$ is steeper than each $w_{m_j}(t,x)$. Then, by using Lemma 2.2 again, we see that the two functions W_+ and \widetilde{W}_+ are steeper than each other. Furthermore, neither lies strictly above or below the other one, since

$$\widetilde{W}_{+}(0,0) = W_{+}(0,0) = \frac{p_{i-1}(0) + p_{i}(0)}{2}.$$

This implies that $\widetilde{W}_+ \equiv W_+$. Therefore, the whole sequence $w_m(t,x)$ converges to $W_+(t,x)$ as $m \to \infty$ in $L^{\infty}_{loc}(\mathbb{R}^2)$.

Step 2: we show that W_+ is a periodic traveling wave connecting p_i to p_{i-1} , and c_i is the wave speed.

From the definition of w_m and the fact that w satisfies (4.3), it is easily seen that for each $m \in \mathbb{Z}$,

$$w_m(\cdot + T, \cdot + z_{m+1} - z_m) \equiv w_{m+1}(\cdot, \cdot),$$
 (4.17)

and

$$U_i(t, x + z_m - c_i mT + \xi_0 - a_-) \le w_m(t, x) \le U_i(t, x + z_m - c_i mT + \xi_0 - a_+)$$
(4.18)

for $t \in \mathbb{R}$, $x \in \mathbb{R}$. By (4.16), the sequences $(z_{m+1} - z_m)_{m \in \mathbb{Z}}$ and $(z_m - c_i mT)_{m \in \mathbb{Z}}$ are bounded. Then there exist a subsequence $(\bar{m}_k)_{k \in \mathbb{N}}$ and $l_1 \in \mathbb{R}$, $l_2 \in \mathbb{R}$ such that $\bar{m}_k \to \infty$ as $k \to \infty$, and that

$$z_{\bar{m}_k+1} - z_{\bar{m}_k} \to l_1$$
 and $z_{\bar{m}_k} - c_i \bar{m}_k T \to l_2$ as $k \to \infty$.

Passing to the limits along the subsequence $\bar{m}_k \to \infty$ in (4.17) and (4.18), we obtain

$$W_{+}(\cdot + T, \cdot + l_1) \equiv W_{+}(\cdot, \cdot),$$

and

$$U_i(t, x + l_2 + \xi_0 - a_-) \le W_+(t, x) \le U_i(t, x + l_2 + \xi_0 - a_+) \text{ for } t \in \mathbb{R}, x \in \mathbb{R}.$$

This implies that $W_+(t,x)$ is a periodic traveling wave connecting p_i to p_{i-1} , and l_1/T is the wave speed. Furthermore, by the arguments used in Step 1 of the proof of Lemma 4.5, we have $l_1 = c_i/T$. This completes the proof of Step 2.

Step 3: we show the convergence of w(t,x) to a periodic traveling wave as $t \to \infty$ in $L^{\infty}(\mathbb{R})$. Notice that the limit $l_1 = c_i/T$ does not depend on the choice of the subsequence $(\bar{m}_k)_{k \in \mathbb{N}}$. Thus, the whole sequence $z_{m+1} - z_m$ converges to l_1 as $m \to \infty$, and hence, the whole sequence $z_m - c_i mT$ converges to l_2 as $m \to \infty$.

Let us now write

$$V_+(t,x) := W_+(t,x-l_2)$$
 for $t \in \mathbb{R}, x \in \mathbb{R}$.

Clearly, $V_+(t,x)$ is a periodic traveling wave connecting p_i to p_{i-1} with speed c_i . We want to prove that w(t,x) converges to $V_+(t,x)$ as $t\to\infty$ in $L^\infty(\mathbb{R})$.

Let $\varepsilon > 0$ be a given small number. From the asymptotics of U_i and V_+ , there exists C > 0 such that

$$p_{i-1}(t) - \frac{\varepsilon}{2} \le U_i(t,x), V_+(t,x) \le p_{i-1}(t)$$
 for all $x - c_i t \le -C, t \in \mathbb{R}$,

and

$$p_i(t) \le U_i(t,x), V_+(t,x) \le p_i(t) + \frac{\varepsilon}{2}$$
 for all $x - c_i t \ge C, t \in \mathbb{R}$.

It then follows from (4.3) that, after making C larger if necessary,

$$|w(t,x) - V_+(t,x)| \le \varepsilon \text{ for all } |x - c_i t| \ge C, t \in \mathbb{R}.$$
 (4.19)

On the other hand, we know from Step 1 that $w_m(t,x)$ converges to $V_+(t,x+l_2)$ as $m\to\infty$ in $L^{\infty}_{loc}(\mathbb{R}^2)$. Thus, we have

$$w(t, x + z_{|t/T|}) - V_{+}(t, x + c_{i}|t/T|T + l_{2}) \to 0 \text{ as } t \to \infty \text{ in } L^{\infty}_{loc}(\mathbb{R}),$$

where $\lfloor t/T \rfloor$ denotes the floor function of t/T, as introduced in the proof of Theorem 1.10. Notice that $c_i \lfloor t/T \rfloor T - z_{\lfloor t/T \rfloor} \to -l_2$ as $t \to \infty$. There exists some $T_0 > 0$ sufficiently large such that

$$|w(t,x) - V_+(t,x)| \le \varepsilon$$
 for all $t \ge T_0$, $|x - c_i t| \le C$.

Combining this with (4.19), we immediately obtain

$$|w(t,x) - V_+(t,x)| \le \varepsilon$$
 for all $t \ge T_0, x \in \mathbb{R}$.

Since $\varepsilon > 0$ is arbitrary, Step 3 is proved.

From the proof of Step 1, we see that $V_+(t,x)$ is steeper than w(t,x). Since $V_+(t,x)$ is not equal to w(t,x) up to any spatial shift, $V_+(t,x)$ is strictly steeper than w(t,x).

Similarly as above, one can prove that there exists another periodic traveling wave $V_{-}(t,x)$ connecting p_i to p_{i-1} with speed c_i such that w(t,x) converges to $V_{-}(t,x)$ as $t \to -\infty$ in $L^{\infty}(\mathbb{R})$ and that $V_{-}(t,x)$ is strictly less steep than w(t,x). Thus, (4.15) is proved. It is straightforward to check that $V_{+}(t,x)$ is strictly steeper than $V_{-}(t,x)$. This ends the proof of Lemma 4.6. \square

Clearly, Theorem 1.11 follows from Proposition 4.2 and Lemmas 4.1, 4.5, 4.6.

4.3. **Proof of Theorem 1.16.** In this subsection, we let Assumption 1.13 hold and give the proof of Proposition 1.15 and Theorem 1.16.

Proof of Proposition 1.15. The proof is similar to that of [13, Theorem 4.1] which is devoted to a spatially periodic problem. For the completeness of the present paper, we give its outline.

By Lemma 2.6, either q_2 is stable from below or q_1 is stable from above. Without loss of generality, we assume that the former case occurs. Then by Assumption 1.13, there exist some $\sigma_0 > 0$ and a T-periodic function g(t) such that

$$\int_0^T g(t)dt \le 0 \text{ and } \partial_u f(t,u) \le g(t) \text{ for all } u \in (q_2(t) - \sigma_0, q_2(t)], t \in \mathbb{R}.$$
 (4.20)

Let us first show that $c_1 \leq c_2$. Suppose the contrary that $c_1 > c_2$. By the characterization of periodic traveling waves, it is known that for each $j = 1, 2, V_j(t, x + c_j t)$ is periodic in t, decreasing in x, and it converges to $q_2(t)$ as $x \to -\infty$ uniformly in $t \in \mathbb{R}$. Then there exist some $x_0 \in \mathbb{R}$ and some $\sigma \in (0, \sigma_0)$ such that

$$V_1(t, x + c_1 t) \in (q_2(t) - \sigma_0, q_2(t)]$$
 for all $t \in \mathbb{R}, x \le x_0$,

$$V_1(t, x + c_1 t) \in [q_2(t) - \sigma_0, q_2(t) - \sigma]$$
 for all $t \in \mathbb{R}, x = x_0$,

and that for some large negative $t_0 < 0$,

$$V_2(t, x + c_1 t) \in [q_2(t) - \sigma/2, q_2(t)]$$
 for all $t \le t_0, x \le x_0$.

Let us define

$$V(t,x) := V_2(t,x+c_1t) - V_1(t,x+c_1t)$$
 for $t \le t_0, x \le x_0$.

It is easily checked that the function V(t,x) satisfies:

$$\begin{cases} V_t = V_{xx} + c_1 V_x + \eta(t, x) V & \text{for } t \le t_0, \ x < x_0, \\ V(t, x) \ge \sigma/2 & \text{for } t \le t_0, \ x = x_0, \\ \lim_{x \to -\infty} V(t, x) = 0 & \text{for } t \le t_0, \end{cases}$$

where

$$\eta(t,x) := \begin{cases} \frac{f(t,V_2(t,x+c_1t)) - f(t,V_1(t,x+c_1t))}{V_2(t,x+c_1t) - V_1(t,x+c_1t)} & \text{if } V(t,x) \neq 0, \\ \partial_u f(t,V_1(t,x+c_1t)) & \text{if } V(t,x) = 0. \end{cases}$$

Due to (4.20), we have $\eta(t,x) \leq g(t)$ for $t \leq t_0$, $x \leq x_0$, and $\lambda := -\frac{1}{T} \int_0^T g(t) dt \geq 0$. It is straightforward to check that for any $\kappa > 0$, the function $-\kappa \phi(t)$ is a subsolution of the equation satisfied by V(t,x), where $\phi \in C^1(\mathbb{R})$ is the solution of

$$\begin{cases}
\phi_t - g(t)\phi = \lambda \phi & \text{for } t \in \mathbb{R}, \\
\phi(t+T) = \phi(t) & \text{for } t \in \mathbb{R}, \quad \phi(0) = 1.
\end{cases}$$
(4.21)

Notice that $\liminf_{t\to-\infty}\inf_{x\leq x_0}V(t,x)\geq 0$. For any $\kappa>0$, there exists a sufficiently large negative integer k such that

$$V(t_0 + kT, x) \ge -\kappa \phi(t_0) \quad \text{for all } x \le x_0. \tag{4.22}$$

It then follows from the comparison principle that

$$V(t+kT,x) \ge -\kappa \phi(t)$$
 for all $x \le x_0, t \ge t_0$.

Since $\phi(t)$ is T-periodic, we have $V(t_0, x) \ge -\kappa \phi(t_0)$ for all $x \le x_0$. Furthermore, by the arbitrariness of $\kappa > 0$, it follows that $V(t_0, x) \ge 0$ for all $x \le x_0$, that is,

$$V_2(t_0, x + c_1t_0) \ge V_1(t_0, x + c_1t_0)$$
 for all $x \le x_0$.

Since V_1 is steeper than V_2 , $\eta(t_0,\cdot)$ must be nonpositive on the left of any zero, and hence,

$$V_2(t_0, x + c_1 t_0) \ge V_1(t_0, x + c_1 t_0)$$
 for all $x \in \mathbb{R}$.

Furthermore, by the comparison principle (applied to equation (1.1)), we obtain

$$V_2(t, x + c_1 t_0) \ge V_1(t, x + c_1 t_0)$$
 for all $t \ge t_0$ $x \in \mathbb{R}$.

This implies that V_2 has to be faster than V_1 , that is, $c_2 \ge c_1$, which is a contradiction with our assumption at the beginning of the present proof.

It remains to show that if $c_1 = c_2$, then $V_1 \equiv V_2$ up to a spatial shift. Assume by contradiction that this is not true. Then, since V_1 is steeper than V_2 , it is not difficult to find some $\xi^* \in \mathbb{R}$ and a continuous real-valued function $x^*(t)$ such that $x^*(t)$ is the only intersection of $V_1(t, \cdot - \xi^* + c_1 t) - V_2(t, \cdot + c_1 t)$, and that

$$q_2(t) - \sigma_0 < V_2(t, x + c_1 t) \le V_1(t, x - \xi^* + c_1 t) < q_2(t)$$

$$\tag{4.23}$$

for $x \leq x^*(t)$, $t \in \mathbb{R}$, where σ_0 is the constant given in (4.20). Clearly, $x^*(t)$ is T-periodic. Let us define

$$W(t,x) := V_2(t,x+c_1t) - V_1(t,x-\xi^*+c_1t) \text{ for } x \le x^*(t), t \in \mathbb{R},$$

and let $\kappa_0 > 0$ be such that $W(0, x) \ge -\kappa_0 \phi(0)$ for $x \le x^*(0)$. By similar comparison arguments to those used in showing (4.22), we can derive that

$$W(t,x) \ge -\kappa_0 \phi(t)$$
 for $t \in \mathbb{R}$, $x \le x^*(t)$,

where $\phi \in C^1(\mathbb{R})$ is the solution of (4.21). Now we can define

$$\kappa_* := \min \left\{ \kappa > 0 : W(t, x) \ge -\kappa \phi(t) \text{ for } t \in \mathbb{R}, x \le x^*(t) \right\}.$$

Since W(t,x) < 0 for $x \in (-\infty, x^*(t))$, $t \in \mathbb{R}$, due to (4.23), it is clear that $\kappa_* > 0$. Then, by using the fact that $W(t, x^*(t)) \equiv 0$ and $W(t, -\infty) \equiv 0$, we obtain $W(t, x) \geq -\kappa_* \phi(t)$ with equality at some $t_1 \in \mathbb{R}$, $x_1 \in (-\infty, x^*(t_1))$. Applying the strong maximum principle, we have $W(t,x) \equiv \kappa_* \phi(t)$, which is obviously impossible. Therefore, $V_1 \equiv V_2$ up to a spatial shift. This ends the proof of Proposition 1.15.

We are now ready to complete the proof of Theorem 1.16.

Proof of Theorem 1.16. Let u(t, x) be the solution of (1.1) with u_0 satisfying (H2). By Theorem 1.11 and Proposition 1.15, we immediately obtain that

$$\Omega(u) = \{ U_i(\cdot, \cdot + \xi) : \xi \in \mathbb{R}, 1 \le i \le N \} \cup \{ p_i : 0 \le i \le N \}.$$
(4.24)

To complete the proof, it remains to find $C^1([0,\infty))$ functions $(\eta_i)_{1\leq i\leq N}$ such that statements (i)-(iii) of Theorem 1.10 hold for the solution u(t,x) considered here. To do this, for each $i=1,\cdots,N$, let us choose a sequence $(b_{i,k})_{k\in\mathbb{N}}\subset\mathbb{R}$ such that

$$u(kT,b_{i,k}) = \frac{p_{i-1}(0) + p_i(0)}{2} \text{ for each } k \in \mathbb{N}.$$

By standard parabolic estimates, the sequence $\{u(t+kT,x+b_{i,k})\}_{k\in\mathbb{N}}$ is relatively compact for the topology of $L^{\infty}_{loc}(\mathbb{R}^2)$. Thus, it has a subsequence that converges in $L^{\infty}_{loc}(\mathbb{R}^2)$ to an element $w \in \Omega(u)$ with $w(0,0) = (p_{i-1}(0)+p_i(0))/2$. Furthermore, thanks to (4.24), $U_i(t,x)$ is the only element in $\Omega(u)$ satisfying $U_i(0,0) = (p_{i-1}(0)+p_i(0))/2$. It then follows that

$$u(t+kT, x+b_{i,k}) \to U_i(t,x)$$
 as $k \to \infty$ in $L_{loc}^{\infty}(\mathbb{R}^2)$.

This immediately implies that

$$u(t, x + b_{i, \lfloor t/T \rfloor}) - U_i(t, x + c_i \lfloor t/T \rfloor T) \to 0 \text{ as } t \to \infty \text{ in } L^{\infty}_{loc}(\mathbb{R}),$$

where |t/T| is the floor function of t/T.

For each $i=1,\cdots,N$, let $\eta_i:[0,\infty)\to\mathbb{R}$, $t\mapsto\eta_i(t)$ be a $C^1([0,\infty))$ function satisfying

$$\eta_i(t) + c_i \lfloor t/T \rfloor T - b_{i, \lfloor t/T \rfloor} \to 0 \text{ as } t \to \infty.$$

Then by modifying the proof of Theorem 1.10, one can verify that $(\eta_i)_{1 \leq i \leq N}$ are the desired functions. Indeed, in view of the above construction of $(\eta_i)_{1 \leq i \leq N}$, the same arguments as those used in the proof of Theorem 1.10 can show that $(\eta_i)_{1 \leq i \leq N}$ satisfy statements (i)-(ii), and that for any $1 \leq i \leq N$ and any large M > 0,

$$||u(t,x) - U_i(t,x - \eta_i(t))||_{L^{\infty}([c_it + \eta_i(t) - M, c_it + \eta_i(t) + M])} \to 0$$
 as $t \to \infty$.

But for the approach of u(t,x) to $p_i(t)$ on the region $[c_it + \eta_i(t) + M, c_{i+1}t + \eta_{i+1}(t) - M])$, the proof is different, since u(t,x) is not spatially decreasing any more. In such a situation, the same result can be proved by using the fact that u(t,x) can be bounded from above and below by solutions with Heaviside type initial functions and that such solutions satisfy statement (iii) of Theorem 1.10. We leave the details to interested readers. The proof of Theorem 1.16 is thus complete.

5. Convergence in a multistable case: Proof of Theorem 1.19

This section is devoted to the proof of Theorem 1.19. Throughout this section, let Assumption 1.18 hold, and let u(t, x) be the solution of (1.1) with u_0 satisfying (H3). From Assumption 1.18 and its followed discussion, it is known that there exists a minimal propagating terrace $((p_i)_{0 \le i \le N}, (U_i, c_i)_{1 \le i \le N})$ connecting 0 to p, and each p_i is linearly stable, i.e.,

$$\mu_i := -\frac{1}{T} \int_0^T \partial_u f(t, p_i(t)) dt > 0.$$
 (5.1)

5.1. Global convergence to minimal propagating terrace. In this subsection, we prove statement (i) of Theorem 1.19, that is, the solution u(t,x) converges to the minimal propagating terrace as $t \to \infty$ in $L^{\infty}(\mathbb{R})$. The key step is to show that, up to some error terms with exponential decay, u(t,x) can be bounded from above and below by solutions with Heaviside type initial data for all large times. Let us first begin with the following observation on the behavior of u(t,x) at a certain time.

Lemma 5.1. For any $\varepsilon > 0$, there exist a positive number $a_0 = a_0(\varepsilon, u_0)$ and a positive integer $k_0 = k_0(\varepsilon, u_0)$ such that

$$\widehat{u}(T, x; -a_0) - \varepsilon \le u(k_0 T, x) \le \widehat{u}(T, x; a_0) + \varepsilon \quad \text{for } x \in \mathbb{R}, \tag{5.2}$$

where $\widehat{u}(t, x; \pm a_0)$ are the solutions of (1.1) with initial functions $p(0)H(\pm a_0 - x)$.

Proof. We only show the second inequality in (5.2), as the first one can be proved analogously. Let us first set a few notations. Since $\sup_{x \in \mathbb{R}} u_0(x) \in I_+$ and $\limsup_{x \to \infty} u_0(x) \in I_-$, there exist real numbers $h_{\pm} \in I_{\pm}$ such that

$$h_{+} > \sup_{x \in \mathbb{R}} u_0(x)$$
 and $h_{-} > \limsup_{x \to \infty} u_0(x)$. (5.3)

Let $H_{\pm}(t)$ be the solutions of (1.8) with initial values h_{\pm} . It then follows that

$$\lim_{k \to \infty} H_+(t + kT) = p(t) \quad \text{and} \quad \lim_{k \to \infty} H_-(t + kT) = 0 \tag{5.4}$$

locally uniformly in $t \in \mathbb{R}$. Furthermore, since the function f(t, u) is of class $C^{1,1}$ in u uniformly for $t \in \mathbb{R}$, there exists L > 0 such that

$$|\partial_u f(t, u_1) - \partial_u f(t, u_2)| \le L|u_1 - u_2| \text{ for all } t \in \mathbb{R}, u_1, u_2 \in [0, \infty).$$
 (5.5)

We now construct a super-solution of (1.1). Set $\gamma(x) = \frac{1}{2}(1 + \tanh \frac{x}{2})$ for $x \in \mathbb{R}$. Thanks to (5.3), one finds some large number $C_1 > 0$ such that

$$u_0(x) \le h_+(1 - \gamma(x - C_1)) + h_-\gamma(x - C_1)$$
 for $x \in \mathbb{R}$.

Define

$$W(t,x) = H_{+}(t)(1 - \gamma(x - C_1 - C_2t)) + H_{-}(t)\gamma(x - C_1 - C_2t)$$

for $t \geq 0$, $x \in \mathbb{R}$, where

$$C_2 = 1 + L \sup_{t \in [0,\infty)} |H_+(t) - H_-(t)|.$$

It is clear that $u_0(x) \leq W(0,x)$ for $x \in \mathbb{R}$. Next, we check that

$$\mathcal{L}W := W_t - W_{xx} - f(t, W) \ge 0 \text{ for } t > 0, x \in \mathbb{R}.$$

Observe from (5.5) that for any t > 0, $x \in \mathbb{R}$,

$$(1 - \gamma)f(t, H_{+}(t)) + \gamma f(t, H_{-}(t)) - f(t, W)$$

$$= \gamma (1 - \gamma)(H_{+}(t) - H_{-}(t))[\partial_{u} f(t, \theta_{1} H_{+} + (1 - \theta_{1})W) - \partial_{u} f(t, \theta_{2} H_{-} + (1 - \theta_{2})W)]$$

$$> -L\gamma (1 - \gamma)(H_{+}(t) - H_{-}(t))^{2}$$

for some $\theta_1 = \theta_1(t, x), \ \theta_2 = \theta_2(t, x) \in [0, 1]$. Then, it is straightforward to compute that

$$\mathcal{L}W = (C_2 \gamma' + \gamma'')(H_+(t) - H_-(t)) - L\gamma(1 - \gamma)(H_+(t) - H_-(t))^2.$$

Notice that $\gamma' = \gamma(1 - \gamma)$ and $\gamma'' = \gamma'(1 - 2\gamma)$. Owing to the definition of C_2 , we obtain $\mathcal{L}W \geq 0$ for t > 0, $x \in \mathbb{R}$. Thus, W is a super-solution of (1.1). By the comparison principle, we have

$$u(t,x) \le W(t,x) \text{ for } t \ge 0, x \in \mathbb{R}.$$
 (5.6)

For any $\varepsilon > 0$, by using (5.4), we find some $k_0 \in \mathbb{N}$ such that

$$\sup_{x \in \mathbb{R}} W(k_0 T, x) < p(0) + \varepsilon \quad \text{and} \quad \limsup_{x \to \infty} W(k_0 T, x) < \varepsilon.$$

Since $\widehat{u}(T,x;a)$ is decreasing in $x \in \mathbb{R}$, and since

$$\lim_{a\to -\infty} \widehat{u}(T,x;a) = 0, \quad \lim_{a\to \infty} \widehat{u}(T,x;a) = p(0) \ \ \text{locally uniformly in} \ \ x\in \mathbb{R},$$

it follows that there exists $a_0 > 0$ large enough such that

$$W(k_0T, x) \leq \widehat{u}(T, x; a_0) + \varepsilon$$
 for $x \in \mathbb{R}$.

Combining this with (5.6), we immediately obtain the second inequality of (5.2). The proof of Lemma 5.1 is thus complete.

We now show the following key lemma.

Lemma 5.2. There exist positive constants ε_0 , K_0 and β_0 such that if for some $a \in \mathbb{R}$ and $\varepsilon \in (0, \varepsilon_0]$, there holds

$$u_0(\cdot) \le \widehat{u}(T, \cdot; a) + \varepsilon,$$
 (5.7)

then for all $t \geq 0$,

$$u(t,\cdot) \le \widehat{u}(t+T,\cdot -K_0\varepsilon;a) + K_0\varepsilon e^{-\beta_0 t}. \tag{5.8}$$

Analogously, if $u_0(\cdot) \geq \widehat{u}(T,\cdot;a) - \varepsilon$ for some $a \in \mathbb{R}$ and $\varepsilon \in (0,\varepsilon_0]$, then for all $t \geq 0$,

$$u(t,\cdot) \ge \widehat{u}(t+T,\cdot+K_0\varepsilon;a) - K_0\varepsilon e^{-\beta_0 t}.$$
 (5.9)

Proof. Without loss of generality, we assume that a=0, and for convenience, we write $\widehat{u}(t,x)$ instead of $\widehat{u}(t,x;a)$. Let $((p_i)_{0\leq i\leq N},(U_i,c_i)_{1\leq i\leq N})$ be the minimal propagating terrace connecting 0 to p. Then there exist $C^1([0,\infty))$ functions $(\eta_i(t))_{1\leq i\leq N}$ such that all the conclusions of Theorem 1.10 hold true. For any $\delta\in(0,1)$ and C>0, let us set

$$I_{\delta}(t) := \bigcup_{i=0}^{N} I_{\delta}^{i}(t) := \bigcup_{i=0}^{N} [p_{i}(t) - \delta, p_{i}(t) + \delta] \text{ for } t \in \mathbb{R},$$

and

$$\Pi_C(t) := \bigcup_{i=1}^N \Pi_C^i(t) := \bigcup_{i=1}^N [c_i t + \eta_i(t) - C, c_i t + \eta_i(t) + C] \text{ for } t \ge 0.$$

To prove (5.8), we will use $\hat{u}(t,x)$ to construct a suitable super-solution of the solution u(t,x). For clarity, we proceed with 3 steps.

Step 1: we show some estimates of $\widehat{u}(t,x)$.

For each $i = 0, \dots, N$, let μ_i be the positive constant defined in (5.1). By the C^1 -regularity and the periodicity of f, there exists a small positive constant δ_0 such that

$$|\partial_u f(t, v) - \partial_u f(t, p_i(t))| \le \frac{\mu_i}{2} \text{ for all } v \in I^i_{\delta_0}(t), t \in \mathbb{R}.$$
(5.10)

We choose a large constant $C_1 > 0$ such that

$$U_i(t, c_i t \pm C_1) \subset I_{\delta_0/3}(t)$$
 for $i \in \{1, \dots, N\}, t \in \mathbb{R}$.

Since $U_i(t,x)$ is decreasing in $x \in \mathbb{R}$, we have

$$U_i(t, \mathbb{R} \setminus [c_i t - C_1, c_i t + C_1]) \subset I_{\delta_0/3}(t) \text{ for } i \in \{1, \dots, N\}, t \in \mathbb{R},$$
 (5.11)

and we can find a positive constant $\rho_1 > 0$ such that

$$\partial_x U_i(t,x) \le -2\rho_1 \text{ for } i \in \{1,\dots,N\}, x \in [c_i t - C_1 - 2, c_i t + C_1 + 2], t \in \mathbb{R}.$$
 (5.12)

Next, by using Theorem 1.10 and (5.11), we can find some $T_1 > 0$ sufficiently large such that

$$c_i t + \eta_i(t) + C_1 < c_{i+1} t + \eta_{i+1}(t) - C_1 - 2 \text{ for } t \ge T_1, i \in \{1, \dots, N-1\},$$
 (5.13)

and that

$$\widehat{u}(t, \mathbb{R} \setminus \Pi_{C_1}(t)) \subset I_{\delta_0/2}(t) \text{ for } t \ge T_1.$$
 (5.14)

Moreover, by standard parabolic estimates, we have

$$\max_{x \in \Pi^{i}_{C_{1}+2}(t)} |\partial_{x}\widehat{u}(t,x) - \partial_{x}U_{i}(t,x-\eta_{i}(t))| \to 0 \text{ as } t \to \infty \text{ for } i \in \{1,\cdots,N\}.$$

This together with (5.12) implies that there exists $T_2 > T_1$ such that

$$\partial_x \widehat{u}(t,x) \le \max_{1 \le i \le N} \{\partial_x U_i(t,x-\eta_i(t))\} + \rho_1 \le -\rho_1 \text{ for } x \in \Pi_{C_1+2}(t), t \ge T_2.$$
 (5.15)

Note that the following convergences

$$\lim_{x \to -\infty} \widehat{u}(t,x) - p(t) \to 0$$
 and $\lim_{x \to \infty} \widehat{u}(t,x) \to 0$

hold locally uniformly in $t \in [0, \infty)$. There exists some constant $C_2 > 0$ such that

$$\begin{cases}
\widehat{u}(t+T,x) \in I_{\delta_0/2}^0(t) & \text{for } x \le -C_2, \ 0 \le t \le T_2, \\
\widehat{u}(t+T,x) \in I_{\delta_0/2}^N(t) & \text{for } x \ge C_2, \ 0 \le t \le T_2.
\end{cases}$$
(5.16)

Replacing C_2 by some larger constant if necessary, we may assume that

$$C_2 \ge \max_{t \in [0,T]} \left\{ -c_1 t - \eta_1(t) + C_1, \, c_N t + \eta_N(t) + C_1 \right\}. \tag{5.17}$$

Since $\widehat{u}(t,x) \in C^1((0,\infty) \times \mathbb{R})$ and it is decreasing in x, there exists some constant $\rho_2 > 0$ such that

$$\min\left\{-\partial_x \widehat{u}(t+T,x): x \in [-C_2-2, C_2+2], t \in [0, T_2]\right\} \ge \rho_2. \tag{5.18}$$

Step 2: we introduce some notations and present our super-solution. Let $(\zeta_i(t,x))_{0 \le i \le N}$ be a sequence of $C^2([0,\infty) \times \mathbb{R})$ functions satisfying

$$\sum_{i=0}^{N} \zeta_i(0, x) \ge 1 \quad \text{for } x \in \mathbb{R}, \tag{5.19}$$

$$\zeta_0(t,x) = \begin{cases}
1, & \text{if } x \in (-\infty, c_1 t + \eta_1(t) - C_1], t \in [0,\infty), \\
0, & \text{if } x \in [c_1 t + \eta_1(t) - C_1 + 2, \infty), t \in [0,\infty),
\end{cases}$$

$$\zeta_i(t,x) = \begin{cases}
1, & \text{if } x \in [c_i t + \eta_i(t) + C_1, c_{i+1} t + \eta_{i+1}(t) - C_1], t \in [T_2,\infty), \\
0, & \text{if } x \in \mathbb{R} \setminus [c_i t + \eta_i(t) + C_1 - 2, c_{i+1} t + \eta_{i+1}(t) - C_1 + 2], t \in [T_2,\infty), \\
0, & \text{if } x \in \mathbb{R} \setminus [-C_2, C_2], t \in [0, T_2),
\end{cases}$$

for
$$i \in \{1, \dots, N-1\},\$$

$$\zeta_N(t,x) = \begin{cases}
0, & \text{if } x \in (-\infty, c_N t + \eta_N(t) + C_1 - 2], t \in [0,\infty), \\
1, & \text{if } x \in [c_N t + \eta_N(t) + C_1, \infty), t \in [0,\infty),
\end{cases}$$
(5.20)

and

$$0 \le \zeta_i \le 1, \ |\partial_t \zeta_i| \le \max_{1 \le j \le N} |c_j| + 1^1, \ |\partial_x \zeta_i| \le 1, \ |\partial_{xx} \zeta_i| \le 1$$
 (5.21)

for $(t,x) \in [0,\infty) \times \mathbb{R}$, $i \in \{0,1,\cdots,N\}$. It is easily seen from the above that

$$\zeta_i(t,x)\zeta_i(t,x)=0$$
 for $x\in\mathbb{R},\,t\geq T_2$, whenever $i\neq j$,

¹Notice that the functions $(\eta_i)_{1 \leq i \leq N}$ are not unique, since for any $C^1([0,\infty))$ functions $(\bar{\eta}_i)_{1 \leq i \leq N}$ satisfying $\bar{\eta}_i(t) \to 0$ as $t \to \infty$ for each $1 \leq i \leq N$, the sequence $(\bar{\eta}_i + \eta_i)_{1 \leq i \leq N}$ is also associated with $\widehat{u}(t,x)$ satisfying Theorem 1.10. Therefore, we may assume without loss of generality that $|\eta_i'(t)| \leq \frac{1}{2}$ for $t \geq 0$, $i = 1, \dots, N$. This allows us to choose functions $(\zeta_i)_{0 \leq i \leq N}$ satisfying (5.20) and $|\partial_t \zeta_i| \leq \max_{1 \leq j \leq N} |c_j| + 1$.

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and

$$\sum_{i=0}^{N} \zeta_i(t,x) = 1 \quad \text{for } x \in \mathbb{R} \setminus \Pi_{C_1}(t), \ t \ge T_2.$$

Define

$$A(t,x) = \sum_{i=0}^{N} \zeta_i(t,x)b_i(t) \text{ for } t \ge 0, x \in \mathbb{R},$$

and

$$B(t) = \int_0^t \max_{0 \le i \le N} \{b_i(\tau)\} d\tau \text{ for } t \ge 0,$$

where for each $i \in \{0, 1, \dots, N\}$, the function b_i is given by

$$b_i(t) = \exp\left(\frac{\mu_i t}{2} + \int_0^t \partial_u f(\tau, p_i(\tau)) d\tau\right) \quad \text{for } t \ge 0.$$
 (5.22)

Note that for each $i \in \{0, 1, \dots, N\}$,

$$0 \le b_i(t) \le M \exp\left(-\frac{\mu_i t}{2}\right) \quad \text{for} \quad t \ge 0, \tag{5.23}$$

where

$$M = \sup_{t \in [0,T], 0 \le i \le N} \exp \left(\mu_i t + \int_0^t \partial_u f(\tau, p_i(\tau)) d\tau \right).$$

This implies that $b_i(t)$ and A(t,x) converge exponentially to 0 as $t\to\infty$, and B(t) is uniformly bounded in $t \geq 0$. Set

$$K = \frac{\sum_{i=0}^{N} (\max_{1 \le j \le N} |c_j| + \mu_i/2 + 2 + 2||\partial_u f||)}{\min\{\rho_1, \rho_2\}}$$

and

$$\varepsilon_0 = \min \left\{ \frac{\delta_0}{2M}, \ \frac{1}{KB(\infty)} \right\},$$

where $\|\partial_u f\| = \max\{|\partial_u f(t,u)| : u \in [-1, p(t)+1], t \in \mathbb{R}\}$ and $B(\infty) = \lim_{t\to\infty} B(t)$. Let $\varepsilon \in (0, \varepsilon_0]$ be an arbitrary constant. We will show that

$$V(t,x) := \widehat{u}(t+T,x-\varepsilon KB(t)) + \varepsilon A(t,x) \text{ for } t \ge 0, x \in \mathbb{R}.$$

is a super-solution of (1.1).

Step 3: we check that V(t,x) is a super-solution.

When t = 0, it follows directly from (5.7) and (5.19) that

$$u_0(x) < \widehat{u}(T,x) + \varepsilon < V(0,x) \text{ for } x \in \mathbb{R}.$$

When t > 0, we calculate that

$$\mathcal{L}V := V_t - V_{xx} - f(t, V)$$

= $-\varepsilon K B'(t) \partial_x \widehat{u} + \varepsilon (A_t - A_{xx} - \partial_u f(t, \widehat{u} + \varepsilon \theta A) A)$

for some $\theta = \theta(t, x) \in [0, 1]$. Now we claim that $\mathcal{L}V \geq 0$ for all $x \in \mathbb{R}, t > 0$. We consider the following four cases.

Case 1: $x \in \mathbb{R} \setminus \Pi_{C_1+1}(t), t \geq T_2$.

For each $i \in \{0, \dots, N\}$, define

$$S_i = \{(t, x) : x \in \mathbb{R} \setminus \Pi_{C_1 + 1}(t), t \ge T_2, \zeta_i(t, x) = 1\}.$$

One easily checks from (5.13) and (5.20) that

$$S_i \neq \emptyset$$
 for each $i \in \{0, \dots, N\}$, and $S_i \cap S_j = \emptyset$ whenever $i \neq j$,

and that

$$\bigcup_{i=0}^{N} S_i = \{(t, x) : x \in \mathbb{R} \setminus \Pi_{C_1 + 1}(t), t \ge T_2\}.$$
(5.24)

Then for any fixed $i_0 \in \{0, \dots, N\}$, we compute on the set S_{i_0} and obtain that

$$\mathcal{L}V \geq \varepsilon (A_t - A_{xx} - \partial_u f(t, \widehat{u} + \varepsilon \theta A) A)$$

$$= \varepsilon (b'_{i_0}(t) - \partial_u f(t, \widehat{u} + \varepsilon \theta b_{i_0}) b_{i_0})$$

$$= \varepsilon b_{i_0} \left(\frac{\mu_{i_0}}{2} + \partial_u f(t, p_{i_0}) - \partial_u f(t, \widehat{u} + \varepsilon \theta b_{i_0}) \right),$$

where the first inequality follows from the monotonicity of \hat{u} in x. Notice from the choice of ε_0 that

$$0 \le \varepsilon KB(t) \le 1 \quad \text{for all} \quad t \ge 0,$$
 (5.25)

and $\varepsilon\theta b_{i_0}(t) \leq \varepsilon_0 M \leq \delta_0/2$ for all $t \geq 0$. This, together with (5.13) and (5.14), implies that

$$\widehat{u}(t+T,x-K\varepsilon B(t))+\varepsilon\theta b_{i_0}(t)\in I^{i_0}_{\delta_0}(t)$$
 for $(t,x)\in S_{i_0}$.

Therefore, by using (5.10), we obtain $\mathcal{L}V \geq 0$ for $(t,x) \in S_{i_0}$. Due to (5.24) and the arbitrariness of $i_0 \in \{0, \dots, N\}$, we have $\mathcal{L}V \geq 0$ for $x \in \mathbb{R} \setminus \Pi_{C_1+1}(t)$, $t \geq T_2$.

Case 2: $x \in \Pi_{C_1+1}(t), t \geq T_2$.

In this case, it follows from (5.25) that $x - \varepsilon KB(t) \in \Pi_{C_1+2}(t)$. By using (5.15), we have $\partial_x \widehat{u}(t+T, x-K\varepsilon B(t)) \leq -\rho_1$. On the other hand, direct calculation yields that

$$\begin{aligned} &|A_t - A_{xx} - \partial_u f(t, \widehat{u} + \varepsilon \theta A) A| \\ &= \left| \sum_{i=0}^N \left[(\zeta_i)_t b_i - (\zeta_i)_{xx} b_i - \partial_u f(t, \widehat{u} + \varepsilon \theta A) \zeta_i b_i + \left(\frac{\mu_i}{2} + \partial_u f(t, p_i) \right) \zeta_i b_i \right] \right| \\ &\leq \max_{0 \leq i \leq N} \{b_i\} \sum_{i=0}^N \left[|(\zeta_i)_t| + |(\zeta_i)_{xx}| + 2\|\partial_u f\|\zeta_i + \frac{\mu_i}{2} \zeta_i \right] \\ &\leq \max_{0 \leq i \leq N} \{b_i\} \sum_{i=0}^N \left[\max_{1 \leq j \leq N} |c_j| + \frac{\mu_i}{2} + 2 + 2\|\partial_u f\| \right] \end{aligned}$$

for all $x \in \mathbb{R}$, $t \ge 0$, where the last inequality follows from (5.21). Combining the above, for $x \in \Pi_{C_1+1}(t)$, $t \ge T_2$, we obtain

$$\mathcal{L}V \geq \varepsilon K \rho_1 B'(t) - \varepsilon \max_{0 \leq i \leq N} \{b_i\} \sum_{i=0}^{N} \left[\max_{1 \leq j \leq N} |c_j| + \frac{\mu_i}{2} + 2 + 2\|\partial_u f\| \right]$$
$$= \varepsilon \max_{0 \leq i \leq N} \{b_i\} \left(K \rho_1 - \sum_{i=0}^{N} \left[\max_{1 \leq j \leq N} |c_j| + \frac{\mu_i}{2} + 2 + 2\|\partial_u f\| \right] \right).$$

Hence, by the choice of K, it follows that $\mathcal{L}V \geq 0$ for $x \in \Pi_{C_1+1}(t)$, $t \geq T_2$.

Case 3: $|x| \ge C_2 + 1$, $t \in (0, T_2)$.

In this case, from (5.17) and (5.20), we observe that

$$\begin{cases} A(t,x) \equiv b_0(t) & \text{for } x \le -C_2 - 1, \ 0 < t < T_2, \\ A(t,x) \equiv b_N(t) & \text{for } x \ge C_2 + 1, \ 0 < t < T_2. \end{cases}$$

Due to (5.25), we have $|x - K\varepsilon B(t)| \ge C_2$. It then follows from (5.16) that

$$\begin{cases} \widehat{u}(t+T, x-K\varepsilon B(t)) + \varepsilon \theta b_0(t) \in \mathcal{I}^0_{\delta_0}(t) & \text{for } x \leq -C_2 - 1, \ 0 < t < T_2, \\ \widehat{u}(t+T, x-K\varepsilon B(t)) + \varepsilon \theta b_N(t) \in \mathcal{I}^N_{\delta_0}(t) & \text{for } x \geq C_2 + 1, \ 0 < t < T_2. \end{cases}$$

Thus, similar calculations to those used in the proof of Case 1 imply that $\mathcal{L}V \geq 0$ for $|x| \geq C_2 + 1$, $t \in (0, T_2)$.

Case 4: $|x| \le C_2 + 1$, $t \in (0, T_2)$.

In this case, we have $|x - K\varepsilon B(t)| \le C_2 + 2$, whence by (5.18), there holds $\partial \widehat{u}(t + T, x - K\varepsilon B(t)) \le -\rho_2$. Then, following the lines of the proof of Case 2, we obtain that for $|x| \le C_2 + 1$, $t \in (0, T_2)$,

$$\mathcal{L}V \ge \varepsilon \max_{0 \le i \le N} \{b_i\} \left(K\rho_2 - \sum_{i=0}^N \left[\max_{1 \le j \le N} |c_j| + \frac{\mu_i}{2} + 2 + 2\|\partial_u f\| \right] \right) \ge 0.$$

In all cases, we have $\mathcal{L}V \geq 0$, and hence, V(t,x) is a super-solution of (1.1). Then the comparison principle implies that

$$u(t,x) \le V(t,x)$$
 for $x \in \mathbb{R}, t \ge 0$.

Taking

$$K_0 = \max\{KB(\infty), (N+1)M\} \text{ and } \beta_0 = \frac{1}{2} \min_{0 \le i \le N} \{\mu_i\},$$

we immediately obtain (5.8). The proof of (5.9) is analogous and we omit the details.

Let ε_0 , K_0 and β_0 be the positive constants provided by Lemma 5.2. It follows from Lemmas 5.1 and 5.2 that, there exist a positive integer k_0 and a positive number a_0 such that

$$\widehat{u}(t+T,x+K_0\varepsilon_0;-a_0) - K_0\varepsilon_0 e^{-\beta_0 t} \le u(t+k_0T,x)$$

$$\le \widehat{u}(t+T,x+K_0\varepsilon_0;a_0) + K_0\varepsilon_0 e^{-\beta_0 t}$$
(5.26)

for all $(t,x) \in [0,\infty) \times \mathbb{R}$. In order to further show that u(t,x) approaches the minimal propagating terrace as $t \to \infty$, we need the following Liouville type result.

Lemma 5.3. Let W(t,x) be an entire solution of (1.1a) satisfying that for some $\xi_- < \xi_+$ and some $i \in \{1, \dots, N\}$,

$$U_i(t, x - \xi_-) \le W(t, x) \le U_i(t, x - \xi_+)$$
 for $t \in \mathbb{R}, x \in \mathbb{R}$.

Then $W \equiv U_i$ up to a spatial shift.

Proof. This lemma follows directly from [4, Lemma 4.3] by a sliding method. Let us mention that it can also be proved by a dynamical system approach used in an earlier work [20] (see Corollary 8.3 and Proposition B.2 in Appendix 2 of [20]).

Proof of statement (i) of Theorem 1.19. Let w be an arbitrary element of $\Omega(u)$. Because of (5.26), the same arguments as those used in showing Lemma 4.1 imply that either $w \equiv p_i$ for some $0 \le i \le N$, or there exist some integer $1 \le i \le N$ and some $\xi_0 \in \mathbb{R}$ such that

$$U_i(t, x + \xi_0 + a_0) \le w(t, x) \le U_i(t, x + \xi_0 - a_0)$$
 for $t \in \mathbb{R}, x \in \mathbb{R}$.

Moreover, if the later case occurs, then it follows directly from Lemma 5.3 that $w \equiv U_i$ up to a spatial shift. Thus, we have

$$\Omega(u) = \{U_i(\cdot, \cdot + \xi) : \xi \in \mathbb{R}, 1 \le i \le N\} \cup \{p_i : 0 \le i \le N\}.$$

The remaining proof is similar to that of Theorem 1.16, therefore we do not repeat the details here. \Box

5.2. Exponential convergence to minimal propagating terrace. The aim of this subsection is to prove statement (ii) of Theorem 1.19, that is, under the additional assumption that $c_1 < c_2 < \cdots < c_N$, the drift functions $(\eta_i(t))_{1 \le i \le N}$ are convergent, and the solution u(t,x) converges to the minimal terrace as $t \to \infty$ with an exponential rate. The strategy of the proof, which is inspired by [23, 24] for autonomous equations/systems, can be described as follows. Let $(\bar{c}_i)_{0 \le i \le N}$ be a sequence of real numbers given by

$$\bar{c}_0 := c_1 - 1, \quad \bar{c}_i := \frac{c_i + c_{i+1}}{2} \quad \text{for } i = 1, \dots, N - 1, \quad \bar{c}_N := c_N + 1.$$
 (5.27)

Since c_i , $i = 1, \dots, N$, are mutually distinct, it is clear that $\bar{c}_{i-1} < c_i < \bar{c}_i$ for each $i = 1, \dots, N$. We will show that, as $t \to \infty$, u(t,x) approaches a spatial shift of the periodic traveling wave U_i uniformly in $\bar{c}_{i-1}t \le x \le \bar{c}_it$, and the approach is exponentially fast. In the remaining regions, i.e., $x \le \bar{c}_0t$ and $x \ge \bar{c}_Nt$, we will prove that u(t,x) converges exponentially to p(t) and 0, respectively.

We will proceed by a sequence of lemmas. The first lemma is a simple extension of the well known Fife-McLeod type super/sub-solutions result for bistable equations (see [11, 1]). To state our lemma, we need a few more notations. Let $\zeta(x)$ be any $C^2(\mathbb{R})$ function satisfying

$$\zeta(x) = 0 \text{ in } [3, \infty), \quad \zeta(x) = 1 \text{ in } (-\infty, 0], \quad -1 \le \zeta'(x) \le 0 \text{ and } |\zeta''(x)| \le 1 \text{ in } \mathbb{R}.$$
 (5.28)

For each $i = 1, \dots, N$, define

$$A_i(t,x) = \zeta(x)b_{i-1}(t) + (1 - \zeta(x))b_i(t) \text{ for } t \ge 0, x \in \mathbb{R},$$
 (5.29)

where $(b_i)_{0 \le i \le N}$ are the functions defined in (5.22).

Lemma 5.4. Let $i \in \{1, \dots, N\}$ be any fixed integer. If $c > c_i$, then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ and $K \in \mathbb{R}$,

$$\bar{W}_i(t,x) := U_i(t,x + c_i t - c t + K) + \varepsilon A_i(t,x - c t)$$

satisfies

$$\partial_t \bar{W}_i \ge \partial_{xx} \bar{W}_i + f(t, \bar{W}_i) \text{ for } x \in \mathbb{R}, t > 0.$$

Similarly, if $c < c_i$, then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ and $K \in \mathbb{R}$,

$$W_i(t,x) := U_i(t,x+c_it-ct+K) - \varepsilon A_i(t,x-ct)$$

satisfies

$$\partial_t \underline{W}_i \leq \partial_{xx} \underline{W}_i + f(t, \underline{W}_i) \text{ for } x \in \mathbb{R}, t > 0.$$

Proof. This lemma can be proved by slightly modifying the arguments used in [1, Lemma 3.2]. For the sake of completeness, and also for the convenience of later applications, we include the details below. We only give the proof in the case $c > c_i$, since the proof for the other case is identical

Remember that $\partial_t U_i = \partial_{xx} U_i + f(t, U_i)$ in $(t, x) \in \mathbb{R}^2$. Direct calculation gives that for t > 0, $x \in \mathbb{R}$,

$$\mathcal{L}\bar{W}_i := \partial_t \bar{W}_i - \partial_{xx} \bar{W}_i - f(t, \bar{W}_i)$$

= $(c_i - c)\partial_x U_i + \varepsilon(\partial_t A_i - \partial_{xx} A_i - c\partial_x A_i - \partial_u f(t, U_i + \varepsilon\theta A_i) A_i)$

for some $\theta = \theta(t,x) \in [0,1]$. Let $(\mu_i)_{0 \le i \le N}$ be the positive constants given in (5.1), and let $\delta_0 > 0$, $C_1 > 0$, $\rho_1 > 0$ and M > 0 be the real numbers such that (5.10), (5.11), (5.12) and (5.23) hold. Set

$$\varepsilon_0 = \min \left\{ \frac{\delta_0}{2M}, \frac{2(c - c_i)\rho_1}{M(\mu_i/2 + \mu_{i-1}/2 + 1 + |c| + 2\|\partial_u f\|)} \right\},$$

where $\|\partial_u f\| = \max\{|\partial_u f(t, u)| : u \in [p_i(t) - 1, p_{i-1}(t) + 1], t \in \mathbb{R}\}$. We will show that, for any $0 < \varepsilon \le \varepsilon_0$, $\mathcal{L}\bar{W}_i \ge 0$ for t > 0, $x \in \mathbb{R}$.

Let us first check $\mathcal{L}\overline{W}_i \geq 0$ when $x - ct + K \geq C_1$. Replacing C_1 by some larger constant if necessary, we may assume that $C_1 \geq K + 3$. Then we have $\zeta(x - ct) \equiv 0$, whence $A_i \equiv b_i$ and $\partial_x A_i = \partial_{xx} A_i = 0$. Since $\partial_x U_i < 0$, it follows that

$$\mathcal{L}\bar{W}_i \ge \varepsilon(\partial_t A_i - \partial_u f(t, U_i + \varepsilon \theta A_i) A_i)$$

$$= \varepsilon b_i \left(\frac{\mu_i}{2} + \partial_u f(t, p_i(t)) - \partial_u f(t, U_i + \varepsilon \theta b_i) \right).$$

By (5.10), (5.11) and the fact that $0 \le \varepsilon \theta b_i \le \delta_0/2$, we obtain $\mathcal{L}\bar{W}_i \ge 0$ when $x - ct + K \ge C_1$. In a similar way, one can conclude that $\mathcal{L}\bar{W}_i \ge 0$ when $x - ct + K \le -C_1$.

For the remaining values of x and t, i.e., $-C_1 \le x - ct + K \le C_1$, we have

$$|\partial_t A_i - \partial_{xx} A_i - c \partial_x A_i - \partial_u f(t, U_i + \varepsilon \theta A_i) A_i|$$

$$\leq \max\{b_{i-1}(t), b_i(t)\} \left(\mu_i/2 + \mu_{i-1}/2 + 1 + |c| + 2\|\partial_u f\|\right).$$

It then follows from (5.12) and (5.23) that

$$\mathcal{L}\bar{W}_i \ge 2\rho_1(c - c_i) - \varepsilon M \left(\mu_i/2 + \mu_{i-1}/2 + 1 + |c| + 2\|\partial_u f\|\right) \ge 0.$$

This ends the proof of the lemma.

Next we show that, in the regions where the graph of u(t,x) is flat, u(t,x) converges to the platforms $(p_i)_{0 \le i \le N}$ with an exponential rate as $t \to \infty$.

Lemma 5.5. Let $(\bar{c}_i)_{0 \leq i \leq N}$ be the constants given in (5.27) and let ϱ be any positive constant satisfying

$$0 < \varrho \le \frac{1}{4} \min_{1 \le i \le N} \{ c_i - \bar{c}_{i-1}, \ \bar{c}_i - c_i \}. \tag{5.30}$$

Then there are positive constants $\nu > 0$, $t_0 > 0$ and C > 0 such that

$$\begin{cases}
 u(t,x) \le p(t) + Ce^{-\nu t}, & \text{for } x \in \mathbb{R}, \ t \ge t_0, \\
 u(t,x) \le p_i(t) + Ce^{-\nu t}, & \text{for } x \ge (\bar{c}_i - \varrho)t, \ t \ge t_0, \ i = 1, \dots, N,
\end{cases}$$
(5.31)

and

$$\begin{cases} u(t,x) \ge p_i(t) - Ce^{-\nu t}, & \text{for } x \le (\bar{c}_i + \varrho)t, t \ge t_0, i = 0, 1, \dots, N-1, \\ u(t,x) \ge -Ce^{-\nu t}, & \text{for } x \in \mathbb{R}, t \ge t_0. \end{cases}$$

Proof. We only prove the estimates stated in (5.31), as the proof for the others is similar.

Let $H(t; h_0)$ be the solution of (1.8) with initial value $h_0 = \sup_{x \in \mathbb{R}} u_0(x)$. Since $h_0 \in I_+$, it is clear that $H(t; h_0) - p(t) \to 0$ as $t \to \infty$. Moreover, by a simple comparison argument applied to (1.8), one finds some C > 0 and $\nu \in (0, \mu_0)$ (μ_0 is the constant provided by (5.1)) such that

$$H(t; h_0) \le p(t) + Ce^{-\nu t}$$
 for $x \in \mathbb{R}, t > 0$.

On the other hand, applying the comparison principle to the equation satisfied by $u(t,x) - H(t; h_0)$, we deduce

$$u(t,x) \le H(t;h_0)$$
 for $x \in \mathbb{R}, t > 0$.

Combining the above two inequalities, we immediately obtain that the first inequality of (5.31) holds for all t > 0, $x \in \mathbb{R}$.

Let us now turn to prove the second inequality of (5.31). Let $1 \le i \le N$ be any fixed integer and let M_i be a large positive constant such that

$$U_i(t, c_i t - M_i) \ge \frac{p_{i-1}(t) + p_i(t)}{2} \quad \text{for all } t \in \mathbb{R}.$$
 (5.32)

Remember that the solution u(t,x) satisfies statement (i) of Theorem 1.19. One finds some $k_i \in \mathbb{N}$ and a C^1 function $\xi_i(t)$ on $[k_i T, \infty)$ such that $\xi_i(t)/t \to c_i$ as $t \to \infty$ and that

$$u(t,x) \le \frac{p_{i-1}(t) + p_i(t)}{2}$$
 for all $x \ge \xi_i(t), t \ge k_i T$. (5.33)

Since $\varrho \leq \frac{1}{4}(\bar{c}_i - c_i)$, replacing k_i by some larger integer if necessary, we may assume

$$\xi_i(t) \le (\bar{c}_i - 2\varrho)t - M_i \text{ for all } t \ge k_i T.$$
 (5.34)

Let $\varepsilon \in (0, \varepsilon_0]$ be a fixed real number, where ε_0 is the positive constant determined in the first statement of Lemma 5.4 with $c = \bar{c}_i - 2\varrho$ (one easily sees from the proof of Lemma 5.4 that, after making some adjustment, ε_0 can be chosen independent of i). We claim that there exists some large constant $K_i > 0$ such that

$$u(k_i T, x) \le U_i(0, x - (\bar{c}_i - 2\varrho)k_i T - K_i) + \varepsilon \quad \text{for all } x \ge \xi_i(k_i T). \tag{5.35}$$

Indeed, in the case $1 \leq i \leq N-1$, this claim can be easily proved by using (5.33), the monotonicity of $U_i(0,x)$ in x, and the fact that $\limsup_{x\to\infty} u(k_iT,x) < p_i(0)$. In the case i=N, since $\lim_{t\to\infty} \limsup_{x\to\infty} u(t,x)=0$, replacing k_i by some larger integer if necessary, we may assume $\limsup_{x\to\infty} u(k_iT,x) \leq \varepsilon$. Then the same reasoning as above implies (5.35).

Let us define

$$\bar{W}_i(t,x) = U_i(t,x + c_i t - (\bar{c}_i - 2\varrho)(t + k_i T) - K_i) + \varepsilon A_i(t,x - (\bar{c}_i - 2\varrho)t)$$

for $x \ge \xi_i(t + k_i T)$, $t \ge 0$, where A_i is the function defined in (5.29). Clearly, (5.35) implies

$$u(k_iT, x) \leq \bar{W}_i(0, x)$$
 for all $x \geq \xi_i(k_iT)$.

It is also easily seen from Lemma 5.4 that

$$\partial_t \bar{W}_i \ge \partial_{xx} \bar{W}_i + f(t, \bar{W}_i) \text{ for } x > \xi_i(t + k_i T), t > 0.$$

Moreover, by (5.32), (5.33) and the T-periodicity of p_{i-1} , p_i , we have

$$u(t + k_i T, \xi_i(t + k_i T)) \le U_i(t, c_i t - M_i)$$
 for all $t \ge 0$.

It further follows from (5.34) and the monotonicity of $U_i(t,x)$ in x that

$$u(t + k_i T, \xi_i(t + k_i T)) \le U_i(t, \xi_i(t + k_i T) - (\bar{c}_i - 2\rho)(t + k_i T) + c_i t) \le \bar{W}_i(t, \xi_i(t + k_i T))$$

for all t > 0. Then, the comparison principle implies that

$$u(t + k_i T, x) \leq \overline{W}_i(t, x)$$
 for all $x \geq \xi_i(t + k_i T), t \geq 0$.

In particular, there exists some large time $t_0 \ge k_i T$ such that

$$u(t,x) \leq U_i(t,\rho t + c_i t - K_i) + \varepsilon b_i(t)$$
 for all $t \geq t_0, x \geq (\bar{c}_i - \rho)t$.

Notice from [1, Theorem 2.2] that $U_i(t, \varrho t + c_i t - K_i)$ approaches $p_i(t)$ as $t \to \infty$ with an exponential rate. Moreover, we know from (5.23) that $b_i(t)$ converges to 0 as $t \to \infty$ exponentially. Thus, making some adjustment to the constants C and ν if necessary, we obtain the second estimate of (5.31). This ends the proof of Lemma 5.5.

Since U_i is a periodic traveling wave connecting two linearly stable solutions of (1.5), it is known from [1, 4] that U_i is global and exponential stable with asymptotic phase. In the following lemma, we show that this stability remains valid when there is an exponentially decaying inhomogeneity in the equation. Similar results can be found in [23, Lemma 6.23] and [24, Theorem 3.1] for autonomous equations/systems.

Lemma 5.6. Assume that g(t,x) is a continuous function on $[0,\infty)\times\mathbb{R}$ such that for some positive constants K>0 and $\gamma>0$, there holds

$$|g(t,x)| \le Ke^{-\gamma t} \text{ for all } x \in \mathbb{R}, t \ge 0.$$
 (5.36)

Let w(t,x) be a solution of

$$w_t = w_{xx} + f(t, w) + g(t, x)$$
 for $x \in \mathbb{R}, t > 0$

satisfying

$$\inf_{\eta \in \mathbb{R}} \|w(t, \cdot) - U_i(t, \cdot - \eta)\|_{L^{\infty}(\mathbb{R})} \to 0 \quad as \ t \to \infty, \tag{5.37}$$

for some $1 \leq i \leq N$. Then there exist $\nu > 0$, $\bar{\eta}_i \in \mathbb{R}$ and C > 0 such that

$$||w(t,\cdot) - U_i(t,\cdot - \bar{\eta}_i)||_{L^{\infty}(\mathbb{R})} \le Ce^{-\nu t}$$
 for all $t > 0$.

To prove this lemma, we need the following local stability of U_i .

Lemma 5.7. For each $i=1,\dots,N,\ U_i$ is local stable in the following sense: there exist $\delta^* \in (0,1),\ \mu^* \in (0,1)$ and $k^* \in \mathbb{N}$ such that for any $\psi \in C(\mathbb{R})$ satisfying

$$\min_{\eta \in \mathbb{R}} \|\psi(\cdot) - U_i(0, \cdot - \eta)\|_{L^{\infty}(\mathbb{R})} \le \delta^*,$$

there holds

$$\min_{\eta \in \mathbb{R}} \|v(k^*T, \cdot; \psi) - U_i(0, \cdot - \eta)\|_{L^{\infty}(\mathbb{R})} \le \mu^* \min_{\eta \in \mathbb{R}} \|\psi(\cdot) - U_i(0, \cdot - \eta)\|_{L^{\infty}(\mathbb{R})},$$

where $v(t,\cdot;\psi)$ denotes the solution of (1.1) with u_0 replaced by ψ .

Proof. This lemma follows directly from the proof of [1, Theorem 3.6]. \Box

Proof of Lemma 5.6. Let $\delta^* \in (0,1)$, $\mu^* \in (0,1)$ and $k^* \in \mathbb{N}$ be the constants provided by Lemma 5.7. Making $\mu^* \in (0,1)$ larger if necessary, we may assume that

$$\mu^* e^{\gamma k^* T} > 1, \tag{5.38}$$

where $\gamma > 0$ is the exponential decay rate of g in (5.36). Due to the assumption (5.37), one finds some $j^* \in \mathbb{N}$ such that

$$\min_{\eta \in \mathbb{R}} \|w(t, \cdot) - U_i(t, \cdot - \eta)\|_{L^{\infty}(\mathbb{R})} \le \delta^* \text{ for all } t \ge j^*T.$$

For each $j \geq j^*$, set

$$Z_i(t,x) = w(t,x) - v(t-jT,x;w(jT,\cdot))$$
 for $x \in \mathbb{R}, t \ge jT$,

where $v(t, x; w(jT, \cdot))$ is the solution of (1.1) with $u_0(\cdot)$ replaced by $w(jT, \cdot)$. It is clear that Z_i satisfies the following inhomogeneous linear parabolic equation

$$\begin{cases} \partial_t Z_j = \partial_{xx} Z_j + c_j(t, x) Z_j + g(t, x), & x \in \mathbb{R}, \ t > jT, \\ Z_j(jT, x) = 0, & x \in \mathbb{R}, \end{cases}$$

for some bounded function $c_j(t,x)$. One easily checks that

$$|c_i(t,x)| \leq C_1$$
 for $x \in \mathbb{R}, t > jT, j \geq j^*$,

where $C_1 = \max\{|\partial_u f(t, u)| : u \in [-1, p(t) + 1], t \in \mathbb{R}\}.$

We claim that

$$||Z_j(t,\cdot)||_{L^{\infty}(\mathbb{R})} \le C_2 e^{-\gamma jT} \quad \text{for all } jT \le t \le (j+k^*)T, \tag{5.39}$$

for some positive constant C_2 independent of $j \geq j^*$. Since g(t, x) satisfies (5.36), it follows from the comparison principle that

$$H_{-}(t) \le Z_{j}(t,x) \le H_{+}(t)$$
 for $x \in \mathbb{R}$, $jT \le t \le (j+k^{*})T$,

where H_{\pm} are the solutions of the following ODEs

$$\frac{dH_{\pm}}{dt} = \pm C_1 H_{\pm} \pm K e^{-\gamma t}$$
 for $jT < t \le (j + k^*)T$; $H_{\pm}(jT) = 0$.

Making some adjustment to C_1 if necessary, we may assume that $C_1 > \gamma$. Then direct calculation yields

$$-\frac{K}{C_1 - \gamma} e^{-\gamma k^* T} e^{-\gamma j T} \le Z_j(t, x) \le \frac{K}{C_1 + \gamma} e^{C_1 k^* T} e^{-\gamma j T}$$

for all $x \in \mathbb{R}$, $jT \le t \le (j + k^*)T$. This immediately implies that (5.39) holds with

$$C_2 = \max \left\{ \frac{K}{C_1 - \gamma} e^{-\gamma k^* T}, \frac{K}{C_1 + \gamma} e^{C_1 k^* T} \right\}.$$

Next, we prove that

$$\min_{\eta \in \mathbb{R}} \|w(mk^*T, \cdot) - U_i(0, \cdot - \eta)\|_{L^{\infty}(\mathbb{R})} \le C_3(\mu^*)^m \quad \text{for all } m \in \mathbb{N}$$
 (5.40)

for some positive constant C_3 independent of m. Clearly, for each $m \geq 1$, we have

$$\begin{split} & \min_{\eta \in \mathbb{R}} \| w(mk^*T, \cdot) - U_i(0, \cdot - \eta) \|_{L^{\infty}(\mathbb{R})} \\ & \leq \| Z_{(m-1)k^*}(mk^*T, \cdot) \|_{L^{\infty}(\mathbb{R})} + \min_{\eta \in \mathbb{R}} \| v(k^*T, \cdot; w((m-1)k^*T, \cdot) - U_i(0, \cdot - \eta) \|_{L^{\infty}(\mathbb{R})}. \end{split}$$

Let m^* be the least integer such that $m^*k^* \ge j^*$. It then follows from Lemma 5.7 and (5.39) that for all $m > m^*$,

$$\min_{\eta \in \mathbb{R}} \|w(mk^*T, \cdot) - U_i(0, \cdot - \eta)\|_{L^{\infty}(\mathbb{R})}$$

$$\leq C_2 e^{-\gamma(m-1)k^*T} + \mu^* \min_{\eta \in \mathbb{R}} \|w((m-1)k^*T, \cdot) - U_i(0, \cdot - \eta)\|_{L^{\infty}(\mathbb{R})}.$$

Notice that $||w(m^*k^*T,\cdot) - U_i(0,\cdot -\eta)||_{L^{\infty}(\mathbb{R})} \leq \sigma^*$. Then by a simple induction argument, we deduce that for all $m > m^*$,

$$\min_{\eta \in \mathbb{R}} \|w(mk^*T, \cdot) - U_i(0, \cdot - \eta)\|_{L^{\infty}(\mathbb{R})} \le \sum_{l=1}^{m-m^*} C_2 e^{-\gamma(m-l)k^*T} (\mu^*)^{l-1} + \sigma^*(\mu^*)^{m-m^*}.$$

By using (5.38), we obtain that for all $m > m^*$,

$$\min_{\eta \in \mathbb{R}} \|w(mk^*T, \cdot) - U_i(0, \cdot - \eta)\|_{L^{\infty}(\mathbb{R})} \le C_2 \frac{e^{-\gamma(m^*-1)k^*T}}{\mu^* e^{\gamma k^*T} - 1} (\mu^*)^{m-m^*} + \sigma^*(\mu^*)^{m-m^*}.$$

This implies that (5.40) holds with some $C_3 > 0$ (independent of m).

Finally, choosing $\nu = -\ln \mu^*/k^*T$, we see from (5.40) that

$$\min_{\eta \in \mathbb{R}} \|w(mk^*T, \cdot) - U_i(0, \cdot - \eta)\|_{L^{\infty}(\mathbb{R})} \le C_3 e^{-\nu mk^*T} \text{ for all } m \in \mathbb{N}.$$

Then, similar comparison arguments to those used in proving (5.39) imply that, for each $m \in \mathbb{N}$, there exist positive constants C_4 and C_5 (both are independent of $m \in \mathbb{N}$) such that

$$\min_{\eta \in \mathbb{R}} \|w(t, \cdot) - U_i(t, \cdot - \eta)\|_{L^{\infty}(\mathbb{R})} \le C_4 e^{-\nu m k^* T} + C_5 e^{-\gamma m k^* T}$$

for all $mk^*T \le t \le (m+1)k^*T$. Since $\nu < \gamma$ because of (5.38), one easily derives that

$$\min_{\eta \in \mathbb{R}} \| w(t, \cdot) - U_i(t, \cdot - \eta) \|_{L^{\infty}(\mathbb{R})} \le (C_4 + C_5) e^{\nu k^* T} e^{-\nu t} \text{ for all } t > 0.$$

This ends the proof of Lemma 5.6.

Now we can complete the proof of Theorem 1.19 (ii) by showing the following lemma:

Lemma 5.8. Let $(\bar{c}_i)_{0 \le i \le N}$ be the constants given in (5.27). There exist C > 0, $\nu > 0$ and $t_0 > 0$ such that

$$|u(t,x) - U_i(t,x - \bar{\eta}_i)| \le Ce^{-\nu t} \text{ for } \bar{c}_{i-1}t \le x \le \bar{c}_it, t \ge t_0, i = 1, \dots, N,$$
 (5.41)

for some $\bar{\eta}_i \in \mathbb{R}$.

Proof. Let $i = 1, \dots, N$ be any fixed integer and let ϱ be a positive constant satisfying (5.30). We first choose some large $t_0 > 0$ such that

$$\begin{cases} (\bar{c}_{i-1}t - 3, \ \bar{c}_{i-1}t] \subset ((\bar{c}_{i-1} - \varrho)t, \ (\bar{c}_{i-1} + \varrho)t) \\ [\bar{c}_it, \ \bar{c}_it + 3) \subset ((\bar{c}_i - \varrho)t, \ (\bar{c}_i + \varrho)t) \end{cases}$$
for all $t \ge t_0$.

Since $u - p_{i-1}$ and $u - p_i$ are solutions of linear parabolic equations, by standard parabolic estimates, we obtain some $C_1 > 0$ such that

$$\begin{cases} |u_x(t,x)| \le C_1 |u(t,x) - p_{i-1}(t)| & \text{for all } \bar{c}_{i-1}t - 3 \le x \le \bar{c}_{i-1}t, \ t \ge t_0, \\ |u_x(t,x)| \le C_1 |u(t,x) - p_i(t)| & \text{for all } \bar{c}_i t \le x \le \bar{c}_i t + 3, \ t \ge t_0. \end{cases}$$

It then follows from Lemma 5.5 that there exist $\nu > 0$ and $C_2 > 0$ such that, possibly after replacing t_0 by some larger constant,

$$|u_x(t,x)| \le C_2 e^{-\nu t}$$
 for all $x \in [\bar{c}_{i-1}t - 3, \bar{c}_{i-1}t] \cup [\bar{c}_i t, \bar{c}_i t + 3], t \ge t_0.$ (5.42)

Next, we define a function w(t,x) on $[t_0,\infty)\times\mathbb{R}$ by

$$w(t,x) = \begin{cases} \zeta(x - (\bar{c}_{i-1}t - 3))p_{i-1}(t) + (1 - \zeta(x - (\bar{c}_{i-1}t - 3)))u(t,x) & \text{for } x \leq c_i t, \\ \zeta(x - \bar{c}_i t)u(t,x) + (1 - \zeta(x - \bar{c}_i t))p_i(t) & \text{for } x \geq c_i t, \end{cases}$$
 where $\zeta(x)$ is a $C^2(\mathbb{R})$ function satisfying (5.28). It is easily seen that $w \in C^{1,2}([t_0, \infty) \times \mathbb{R})$,

and that

$$w(t,x) = \begin{cases} p_{i-1}(t) & \text{if } x \leq \bar{c}_{i-1}t - 3, t \geq t_0, \\ u(t,x) & \text{if } \bar{c}_{i-1}t \leq x \leq \bar{c}_it, t \geq t_0, \\ p_i(t) & \text{if } x \geq \bar{c}_it + 3, t \geq t_0, \end{cases}$$
(5.43)

Set

$$g(t,x) = w_t - w_{xx} - f(t,w)$$
 for $x \in \mathbb{R}, t \ge t_0$.

Clearly, g(t,x) is continuous on $[t_0,\infty)\times\mathbb{R}$ and

$$g(t,x) = 0$$
 for $x \in (-\infty, \bar{c}_{i-1}t - 3] \cup [\bar{c}_{i-1}t, \bar{c}_it] \cup [\bar{c}_it + 3, \infty), t \ge t_0$.

We claim that there exists some constant $C_3 > 0$ such that

$$|g(t,x)| \le C_3 \mathrm{e}^{-\nu t} \tag{5.44}$$

for all $x \in [\bar{c}_{i-1}t - 3, \bar{c}_{i-1}t] \cup [\bar{c}_it, \bar{c}_it + 3], t \ge t_0$. Indeed, when $x \in [\bar{c}_{i-1}t - 3, \bar{c}_{i-1}t], t \ge t_0$, it is straightforward to calculate that

$$g(t,x) = (\zeta'' + \bar{c}_{i-1}\zeta')(u - p_{i-1}) + (1 - \zeta)(f(t,u) - f(t,p_{i-1})) + (f(t,p_{i-1}) - f(t,w)) + 2\zeta' u_x(t,x),$$

where ζ , ζ' and ζ'' stand for $\zeta(x-(\bar{c}_{i-1}t-3))$, $\zeta'(x-(\bar{c}_{i-1}t-3))$ and $\zeta''(x-(\bar{c}_{i-1}t-3))$, respectively. Due to the C^1 -regularity and the T-periodicity of f, it then follows from Lemma 5.5 and (5.42) that (5.44) holds for $x \in [\bar{c}_{i-1}t - 3, \bar{c}_{i-1}t], t \geq t_0$. The proof for $x \in [\bar{c}_i t, \bar{c}_i t + 3],$ $t \geq t_0$ is analogous, therefore we omit the details.

Finally, let $\eta_i(t)$ be the $C^1([0,\infty))$ function provided by Theorem 1.19 (i). By our choice of $(\bar{c}_i)_{0 \le i \le N}$, it is easily checked that

$$||u(t,\cdot) - U_i(t,\cdot - \eta_i(t))||_{L^{\infty}([\bar{c}_{i-1}t,\bar{c}_it])} \to 0 \text{ as } t \to \infty.$$

Then, by using Lemma 5.5, (5.43) and the asymptotics of $U_i(t, c_i t + x)$ as $x \to \pm \infty$, we deduce

$$||w(t,\cdot) - U_i(t,\cdot - \eta_i(t))||_{L^{\infty}(\mathbb{R})} \to 0 \text{ as } t \to \infty.$$

Therefore, all the conditions of Lemma 5.6 are fulfilled. Consequently, there exist $\nu > 0$, $\bar{\eta}_i \in \mathbb{R}$ and C > 0 such that

$$||w(t,\cdot) - U_i(t,\cdot - \bar{\eta}_i)||_{L^{\infty}(\mathbb{R})} \le Ce^{-\nu t}$$
 for all $t > 0$.

This together with (5.43) immediately gives (5.41). The proof of Lemma 5.8 is thus complete.

It is clear that Theorem 1.19 (ii) follows directly from Lemmas 5.5 and 5.8.

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