

# The robust superreplication problem: a dynamic approach

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## Abstract

In the frictionless discrete time financial market of Bouchard et al.(2015) we consider a trader who, due to regulatory requirements or internal risk management reasons, is required to hedge a claim  $\xi$  in a risk-conservative way relative to a family of probability measures  $\mathcal{P}$ . We first describe the evolution of  $\pi_t(\xi)$  - the superhedging price at time  $t$  of the liability  $\xi$  at maturity  $T$  - via a dynamic programming principle and show that  $\pi_t(\xi)$  can be seen as a concave envelope of  $\pi_{t+1}(\xi)$  evaluated at today's prices. Then we consider an optimal investment problem for the trader who is rolling over her robust superhedge and phrase this as a robust maximisation problem, where the expected utility of inter-temporal consumption is optimised subject to a robust superhedging constraint. This utility maximisation is carried out under a new family of measures  $\mathcal{P}^u$ , which no longer have to capture regulatory or institutional risk views but rather represent trader's subjective views on market dynamics. Under suitable assumptions on the trader's utility functions, we show that optimal investment and consumption strategies exist and further specify when, and in what sense, these may be unique.

## 1 Introduction

We consider a discrete time financial market and an agent who needs to hedge a liability  $\xi$  maturing at a future date  $T$  in a robust and risk-conservative way. Our focus is on the interplay between the beliefs used for assessing the risks, the beliefs used for agent's investment decisions and the dynamics of agent's actions. For simplicity we assume away other factors and consider an agent who can trade in a dynamic way with no constraints or frictions in  $d$  assets available in the market at prices which are exogenous. More precisely, following the approach of Samuelson [1969] and Black and Scholes [1973], risky assets are modelled as stochastic processes and their behaviour specified by a probability measure. However, unlike the classical uni-prior approach which fixes one such measure  $P$ , we consider a multi-prior framework

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and work simultaneously under a whole family of measures  $P \in \mathcal{P}$ . This offers a robust approach which accounts for model ambiguity, also referred to as *Knightian uncertainty* after Knight [1921].

The price to pay for a robust modelling view comes through specificity of outputs: while the uni-prior setting might generate a unique fair price for a derivate contract a multi-prior setting will typically generate a relatively wide interval of no-arbitrage prices, a tradeoff first identified in the seminal paper of Merton [1973]. We consider a trader who, due to regulatory requirements or internal risk management reasons, is required to hedge  $\xi$  in a risk-conservative way relative to  $\mathcal{P}$ . This means that initially she has to allocate capital equal to  $\pi(\xi)$ , the superhedging price of  $\xi$ , i.e., the price of cheapest trading strategies which are guaranteed to cover the liability  $\xi$  under all  $P \in \mathcal{P}$ . There might be many such cheapest superhedging strategies and the trader can pick any one of them to follow until time  $T$ . This is a conservative and non-linear risk assessment: the capital the trader would be allowed to borrow against a long position in  $\xi$  is  $-\pi(-\xi)$  and is typically significantly lower than  $\pi(\xi)$ .

The superhedging price  $\pi(\xi)$  can be characterised theoretically and has been considered in a number of papers, see Bouchard and Nutz [2015] and the discussion below. To the best of our knowledge, the focus of most of these works has been on the static problem: the problem today for the horizon  $T$ . In contrast, in this paper we want to focus on the dynamics of the robust pricing and hedging problem *through time*. We ask how  $\pi(\xi)$  changes *over time* and how the trader should act optimally through time. Clearly, tomorrow she will see new prices in the market and will be able to recompute the superhedging price. If the new price is lower, she will be able to unwind her old position, buy a new position and be left with a surplus. She could then consume this (e.g., pay into her credit line if the initial capital was borrowed) or invest further if she believes the market offers suitable opportunities.

Our first main contribution is to describe the evolution of  $\pi_t(\xi)$  - the superhedging price at time  $t$  of the liability  $\xi$  at maturity  $T$ . We work in the setting of Bouchard and Nutz [2015] and consider an abstract set of priors  $\mathcal{P}$ , possibly large and in particular not dominated by a single probability measure. The measures  $P \in \mathcal{P}$  are represented as compositions of one-step kernels and to establish the dual characterisation of  $\pi_0(\xi)$  Bouchard and Nutz [2015] have essentially proven a dynamic programming principle for the dual objects. We prove that  $(\pi_t(\xi))_{0 \leq t \leq T}$  satisfy a dynamic programming principle, and that  $\pi_t(\xi)$  can be seen as a concave envelope of  $\pi_{t+1}(\xi)$  evaluated at today's prices. To the best of our knowledge, this was first suggested in the robust setting by Dupire [2010]. We also characterise  $\pi_t(\xi)$  as the wealth of a minimal superhedging strategy in the sense of Föllmer and Kramkov [1997]. These results provide natural robust extensions of classical uni-prior results, see Föllmer and Schied [2002], including a robust version of the algorithm in Carassus et al. [2006]. Further, considering  $\mathcal{P}$  which corresponds to the pointwise robust setting of Burzoni et al., we show that  $\pi_t(\xi)$  corresponds to the uniprior superhedging price for an extreme  $P \in \mathcal{P}$ . Proving our results in the robust setting requires rather lengthy and technical arguments. This is mainly due to delicate measurability questions.

Our second main contribution is to consider an optimal investment problem for a trader who is rolling over her robust superhedge. This is phrased as a problem of robust maximisation of expected utility of inter-temporal consumption subject to a robust superhedging constraint. Here the robust constraint means the superhedging has to be satisfied  $P$ -a.s. for all  $P \in \mathcal{P}$ . The robust utility maximisation means that we consider a max-min problem, where minimisation is over  $P \in \mathcal{P}^u$ . We argue that the

latter problem should be considered with respect to a different set of priors  $\mathcal{P}^u \subseteq \mathcal{P}$  than the former problem. Measures  $P \in \mathcal{P}^u$  no longer have to capture regulatory or institutional risk views but rather represent trader's subjective views on market dynamics. Under suitable assumptions on the trader's utility functions, we show that optimal investment and consumption strategies exist and further specify when, and in what sense, these may be unique. We provide examples to illustrate various pitfalls occurring when our assumptions are not satisfied.

Throughout, we work in the setup of Bouchard and Nutz [2015] who extended the classical uni-prior theory of pricing and hedging in discrete time to the robust multi-prior case, introducing a suitable notion of no-arbitrage, proving a robust version of the fundamental theorem of pricing and hedging and establishing a robust pricing-hedging duality. Numerous authors have since adopted their setup and worked on robust extensions of the classical problems in quantitative finance such as pricing and hedging of American options, utility maximisation or transaction cost theory to name just a few examples, see Nutz [2016], Blanchard and Carassus [2017], Aksamit et al. [2018], Bayraktar and Zhou [2017], Bouchard et al. and the references therein. We note that alternative ways to address model uncertainty are possible, including the pathwise, or pointwise, approach developed in Davis and Hobson [2007], Acciaio et al. [2013], Burzoni et al. [2016b,a], Burzoni et al. among others. Whilst the resulting robust framework for pricing and hedging is equipped with different notions of arbitrage and different fundamental theorems, it was recently shown by Oblój and Wiesel [2018] to be equivalent to the multi-prior approach. Thus, on an abstract level, there is no loss of generality in our choice to adopt the multi-prior approach of Bouchard and Nutz [2015]. It is important however that we work in discrete time. While in the classical setup no-arbitrage theory, including dynamic understanding of the superhedging price, is well developed in continuous time, see Föllmer and Kramkov [1997], Delbaen and Schachermayer [2006], in the robust setting an extension of abstract no-arbitrage theory, as developed in Bouchard and Nutz [2015] or Burzoni et al., to the continuous time is still open. This is despite a body of works which have achieved either particular or generic steps towards such a goal, large enough so that we can not do it justice in this introduction but refer to Avellaneda et al. [1996], Lyons [1995], Denis and Martini [2006], Cox and Oblój [2011], Denis and Kervarec [2013], Epstein and Ji [2014], Biagini et al. [2017], Hou and Oblój [2018], Beiglböck et al. [2017], Bartl et al. [2017] and the references therein.

We note that  $d$  may be large and our assets may include both primary and derivate assets. Indeed, one way of making robust outputs more specific is by including more traded assets in the analysis. This was the original motivation behind the works on the robust pricing and hedging in continuous time, going back to Hobson [1998.], where one typically assumes that the market prices of European options on the underlying assets co-maturing with our liability  $\xi$  are known. Here, we consider an abstract general setup and allow any  $d$ -tuple of traded assets, for a finite  $d$ . We may expect that the level of uncertainty regarding different assets may differ and this would be reflected in  $\mathcal{P}$ . However it is crucial that all the assets are traded dynamically. From a theoretical standpoint, this is both necessary to obtain a dynamic programming principle for the superhedging prices and without loss of generality in the sense that any Bouchard and Nutz [2015] setup where some assets are only available for trading at time 0 can be lifted to a setup with dynamic trading in all assets in a way which does not introduce arbitrage and does not affect time-0 superhedging prices, see Aksamit et al. [2018]. From a practical standpoint, this is not a significant

assumption as we may only consider liquidly traded assets.

The remainder of the paper is organised as follows. The next section introduces and discusses our modelling framework. Section 3 presents the results characterising the dynamics of the superhedging price. We then specialise, in section 3.2, to the pathwise setting when  $\mathcal{P}$  contains all measures with specified supports. This allows for a more intuitive interpretation of the results, easier proofs and explicit examples. Section 4 then considers the secondary utility maximisation problem for a trader who dynamically re-balances her superhedging strategy and states the existence and uniqueness results for the optimal investment and consumption strategies. Finally, proofs are presented in three appendices.

## 2 Models of Financial markets

In this section we set up the multi-prior modelling framework and give introductory definitions. Future dynamics of financial assets are modelled using probability measures but, unlike the classical case where one such measure is fixed, we typically work simultaneously under all  $P$  from a large family of measures  $\mathcal{P}$ . Our market has  $d$  traded assets, these could be stocks or options, but importantly all are traded dynamically. We do not consider statically traded assets, i.e., only available for buy-and-hold trading, as then the superhedging prices typically can not admit a dynamic programming principle across all times, see Aksamit et al. [2018].

### 2.1 Uncertainty modelling

We work in the setting of Bouchard and Nutz [2015] to which we refer for details and motivation. We only recall the main objects of interest here and refer to Bertsekas and Shreve [2004][Chapter 7] for technical details. Let  $\Omega$  be a Polish space and denote by  $\Omega^t$  its  $t$ -fold Cartesian product. We define the price process  $S$  of discounted prices of  $d$  traded stocks as a Borel measurable map  $S_t(\omega) = (S_t^1(\omega), \dots, S_t^d(\omega)) : \Omega^T \rightarrow \mathbb{R}_+^d$  for every  $\omega = (\omega_0, \dots, \omega_T)$  with the convention  $S_0(\omega) = s_0 \in \mathbb{R}_+^d$  and  $T \in \mathbb{N}$  is the time horizon. Prices are specified in discounted units and we have a riskless asset with price equal to 1 for all  $0 \leq t \leq T$ . Furthermore let  $\mathfrak{P}(\Omega^t)$  be the set of all probability measures on  $\mathcal{B}(\Omega^t)$ , the Borel- $\sigma$ -algebra on  $\Omega^t$ . We denote by  $\mathcal{F}_t^{\mathcal{U}}$  the universal completion of  $\mathcal{B}(\Omega^t)$ . We often consider  $(\Omega^t, \mathcal{F}_t^{\mathcal{U}})$  as a subspace of  $(\Omega^T, \mathcal{F}_T^{\mathcal{U}})$  and write  $\mathbb{F}^{\mathcal{U}} = (\mathcal{F}_t^{\mathcal{U}})_{t=0, \dots, T}$ . In the rest of the paper, we will use the same notation for  $P \in \mathfrak{P}(\Omega^T)$  and for its (unique) extension to  $\mathcal{F}_T^{\mathcal{U}}$ . For a given  $\mathcal{P} \subseteq \mathfrak{P}(\Omega^T)$ , a set  $N \subset \Omega^T$  is called a  $\mathcal{P}$ -polar if for all  $P \in \mathcal{P}$ , there exists some  $A_P \in \mathcal{B}(\Omega^T)$  such that  $P(A_P) = 0$  and  $N \subset A_P$ . We say that a property holds  $\mathcal{P}$ -quasi-surely (q.s.), if it holds outside a  $\mathcal{P}$ -polar set. Finally we say that a set is of  $\mathcal{P}$ -full measure if its complement is a  $\mathcal{P}$ -polar set.

To give a probabilistic description of the market we consider a family of random sets  $\mathcal{P}_t : \Omega^t \rightarrow \mathfrak{P}(\Omega)$ , for all  $0 \leq t \leq T - 1$ . The set  $\mathcal{P}_t(\omega)$  can be seen as the set of all possible models for the  $t + 1$ -th period given the path  $\omega \in \Omega^t$  at time  $t$ . In order to aggregate trading strategies on different paths in a measurable way, we assume here that the sets  $\mathcal{P}_t$  have the following property:

**Assumption 2.1** The set  $\mathcal{P}$  has Analytic Product Structure (APS), which means that

$$\mathcal{P} = \{P_0 \otimes \dots \otimes P_{T-1} \mid P_t \text{ is an } \mathcal{F}_t^{\mathcal{U}}\text{-measurable selector of } \mathcal{P}_t\},$$

where the sets  $\mathcal{P}_t(\omega) \subseteq \mathcal{P}(\Omega)$  are nonempty, convex and

$$\text{graph}(\mathcal{P}_t) = \{(\omega, P) \mid \omega \in \Omega^t, P \in \mathcal{P}_t(\omega)\}$$

is analytic.

The fact that  $\text{graph}(\mathcal{P}_t)$  is analytic allows for an application of the Jankov-von-Neumann theorem ([Bertsekas and Shreve, 2004, Prop. 7.49, p.182]), which guarantees the existence of universally measurable selectors  $P_t : \Omega^t \rightarrow \mathfrak{P}(\Omega)$ . Here  $P_0 \otimes \cdots \otimes P_{T-1}$  denotes the  $T$ -fold application of Fubini's theorem, which defines a measure on  $\mathfrak{P}(\Omega^T)$ . Indeed, analyticity of the graph of  $\mathcal{P}_t$  is of paramount importance for the preservation of measurability properties. For example the proof of a quasisure superreplication theorem (see [Bouchard and Nutz, 2015, Lemma 4.10]) uses the fact that if  $X_{t+1} : \Omega^{t+1} \rightarrow \mathbb{R}$  is upper semianalytic, then  $\sup_{P \in \mathcal{P}_t(\omega)} \mathbb{E}_P[X_{t+1}(\omega, \cdot)]$  remains upper semianalytic. Apart from assumption 2.1, we make no specific assumptions on the set of priors  $\mathcal{P}$ . It is neither assumed to be dominated by a given reference probability measure nor to be weakly compact. Some concrete examples, including when  $\mathcal{P}_t(\omega)$  are non-compact random sets, are discussed in section 3.2.

## 2.2 Trading

Trading strategies are represented by  $\mathbb{F}^{\mathcal{U}}$ -predictable  $d$ -dimensional processes  $H := \{H_t\}_{1 \leq t \leq T}$  where for all  $1 \leq t \leq T$ ,  $H_t$  represents the investor's holdings in each of the  $d$  assets at time  $t$ . The set of trading strategies is denoted by  $\mathcal{H}(\mathbb{F}^{\mathcal{U}})$ . Investors are allowed to consume and their cumulative consumption is represented by an  $\mathbb{R}$ -valued  $\mathbb{F}^{\mathcal{U}}$ -adapted process  $C = \{C_t\}_{1 \leq t \leq T}$ ,  $C_0 = 0$  and which is assumed to be non-decreasing:  $C_t \leq C_{t+1}$   $\mathcal{P}$ -q.s. The set of cumulative consumption processes is denoted by  $\mathcal{C}$ . We will use the notation  $\Delta S_t = S_t - S_{t-1}$  and  $\Delta C_t = C_t - C_{t-1}$  for  $1 \leq t \leq T$ . Given an initial wealth  $x \in \mathbb{R}$ , a trading portfolio  $H$  and a cumulative consumption process  $C$ , the wealth process  $V^{x,H,C}$  is governed by

$$\begin{aligned} V_0^{x,H,C} &= x \\ V_t^{x,H,C} &= V_{t-1}^{x,H,C} + H_t \Delta S_t - \Delta C_t \quad \text{for } 1 \leq t \leq T. \end{aligned} \quad (1)$$

The condition  $C = 0$  means that the portfolio  $H$  is self-financing and in this case we write  $V^{x,H}$  instead of  $V^{x,H,0}$ .

We are interested in superhedging of a (European) contingent claim and therefore adapt the presentation of Föllmer and Kramkov [1997] to the robust framework. A (European) contingent claim is represented by an  $\mathcal{F}_T^{\mathcal{U}}$ -measurable random variable  $\xi$  and the set of superhedging strategies for  $\xi$  is denoted by

$$\mathcal{A}(\xi) := \left\{ (x, H, C) \in \mathbb{R} \times \mathcal{H}(\mathbb{F}^{\mathcal{U}}) \times \mathcal{C} \mid V_T^{x,H,C} \geq \xi \text{ } \mathcal{P}\text{-q.s.} \right\}. \quad (2)$$

**Definition 2.2** The superreplication price  $\pi(\xi)$  of an  $\mathcal{F}_T^{\mathcal{U}}$ -measurable random variable  $\xi$  is the minimal initial capital needed for superhedging  $\xi$ , i.e.,

$$\pi(\xi) := \inf \{x \in \mathbb{R} \mid \exists (H, C) \in \mathcal{H}(\mathbb{F}^{\mathcal{U}}) \times \mathcal{C} \text{ such that } (x, H, C) \in \mathcal{A}(\xi)\}, \quad (3)$$

with  $\pi(h) = +\infty$  if  $\mathcal{A}(\xi) = \emptyset$ . A superhedging strategy  $(\hat{x}, \hat{H}, \hat{C}) \in \mathcal{A}(\xi)$  is called *minimal* if for all  $(x, H, C) \in \mathcal{A}(\xi)$   $V_t^{x,H,C} \geq V_t^{\hat{x}, \hat{H}, \hat{C}}$   $\mathcal{P}$ -q.s. for all  $0 \leq t \leq T$ .

It is easy to see that  $\hat{x} = \pi(\xi)$  for any minimal superhedging strategy  $(\hat{x}, \hat{H}, \hat{C}) \in \mathcal{A}(\xi)$ .

### 2.3 No-arbitrage condition and Pricing measures

We recall the no-arbitrage condition introduced in Bouchard and Nutz [2015].

**Assumption 2.3** There is no  $\mathcal{P}$ -quasi-sure arbitrage ( $\text{NA}(\mathcal{P})$ ) in the market if for all  $H \in \mathcal{H}(\mathbb{R}^d)$  with  $V_T^{0,H} \geq 0$   $\mathcal{P}$ -q.s. we have  $V_T^{0,H} = 0$   $\mathcal{P}$ -q.s.

The above definition gives an intuitive extension of the classical no-arbitrage condition, specified under a fixed probability measure  $P$ , to the multi-prior case of family of probability measures  $\mathcal{P}$ . The intuition is justified by the FTAP generalisation proved by [Bouchard and Nutz, 2015, Theorem 4.5]: under assumption 2.1 (recall that  $S$  is Borel-adapted)  $\text{NA}(\mathcal{P})$  is equivalent to the fact that for all  $P \in \mathcal{P}$ , there exists some  $Q \in \mathcal{Q}$  such that  $P \ll Q$  where

$$\mathcal{Q} := \{Q \in \mathfrak{P}(\Omega^T) \mid \exists P \in \mathcal{P}, Q \ll P \text{ and } S \text{ is a martingale under } Q\}. \quad (4)$$

*Remark 2.4* By the same token, further results, e.g., on the Superhedging Theorem or the worst-case expected utility maximisation (see Nutz [2016], Blanchard and Carassus [2017], Bartl [2019] and Neufeld and Sikic [2018]) provide more evidence supporting the view that  $\text{NA}(\mathcal{P})$  is a well-chosen extension of the classical no-arbitrage assumption. However, the price to pay when using  $\text{NA}(\mathcal{P})$  is related to technical measurability issues arising when one considers a one step version of the  $\text{NA}(\mathcal{P})$  (see (5) below). In Bartl [2019] a stronger version of assumption 2.3 is introduced which states that (5) below is satisfied for all  $\omega \in \Omega^t$ . In Blanchard and Carassus [2017], a stronger version of no-arbitrage is proposed ( $\text{sNA}(\mathcal{P})$ ) which states that there is no-arbitrage in the classical sense for all measures  $P \in \mathcal{P}$ . In both cases some of the measurability issues are simplified. Finally, different approaches to model uncertainty may lead to fundamentally different notions of arbitrage. In the pathwise approach, one typically asks that some subset of paths supports a feasible model – this is in contrast to the multi-prior setup in this paper where essentially *all*  $P \in \mathcal{P}$  are assumed to be feasible models. In consequence, the no-arbitrage conditions in the pathwise approach, e.g., model independent arbitrage as in Davis and Hobson [2007], Cox and Obłój [2011], Acciaio et al. [2013] or Arbitrage de la classe  $\mathcal{S}$  (see Burzoni et al. [2016b]), are much weaker than  $\text{NA}(\mathcal{P})$ , i.e., their notions of arbitrage are much stronger than the  $\mathcal{P}$ -q.s. arbitrage. To wit, negation of  $\text{sNA}(\mathcal{P})$  above gives that there is a classical arbitrage for at least one  $P \in \mathcal{P}$  while Davis and Hobson [2007] say that there is a *weak arbitrage opportunity* if there is a classical arbitrage under *all*  $P \in \mathcal{P}$ .

The one step version of the  $\text{NA}(\mathcal{P})$  is the following: for  $\omega \in \Omega^t$  fixed we say that  $\text{NA}(\mathcal{P}_t(\omega))$  condition holds if for all  $H \in \mathbb{R}^d$

$$H \Delta S_{t+1}(\omega, \cdot) \geq 0 \text{ } \mathcal{P}_t(\omega)\text{-q.s.} \quad \Rightarrow \quad H \Delta S_{t+1}(\omega, \cdot) = 0 \text{ } \mathcal{P}_t(\omega)\text{-q.s.} \quad (5)$$

It is proved in [Bouchard and Nutz, 2015, Theorem 4.5] that under the assumption that  $S$  is Borel measurable and (APS) of  $\mathcal{P}$ , the condition  $\text{NA}(\mathcal{P})$  is equivalent to the fact that for all  $0 \leq t \leq T - 1$ , there exists some  $\mathcal{P}$ -full measure set  $\Omega_{NA}^t \in \mathcal{F}_t^{\mathcal{U}}$ , such that for all  $\omega \in \Omega_{NA}^t$ ,  $\text{NA}(\mathcal{P}_t(\omega))$  holds. We also introduce the one-step versions of the set  $\mathcal{Q}$ :

$$\mathcal{Q}_t(\omega) = \{Q \in \mathfrak{P}(\Omega) \mid \exists P \in \mathcal{P}_t(\omega) \text{ such that } Q \ll P \text{ and } \mathbb{E}_Q[\Delta S_{t+1}(\omega, \cdot)] = 0\}.$$

As is shown in [Bouchard and Nutz, 2015, Lemma 4.8],  $\mathcal{Q}_t$  has an analytic graph. An application of the Jankov-von Neumann Theorem and Fubini's Theorem shows that we have

$$\mathcal{Q} = \{Q_0 \otimes \cdots \otimes Q_{T-1} \mid Q_t \text{ is } \mathcal{F}_t^{\mathcal{U}}\text{-measurable selector of } \mathcal{Q}_t \text{ for all } 0 \leq t \leq T-1\}. \quad (6)$$

### 3 Existence and characterisation of minimal superhedging strategies

The Superhedging theorem, also known as the pricing-hedging duality, is one of the fundamental results in the classical setting of  $\mathcal{P} = \{P\}$ , see Föllmer and Schied [2002], Föllmer and Kramkov [1997] and the references therein. One of the main results in Bouchard and Nutz [2015] was its extension to the multi-prior case:

$$\pi(\xi) = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\xi]. \quad (7)$$

While this duality is important and theoretically pleasing, its use for computations may be hampered by lack of a tractable characterisation of the set  $\mathcal{Q}$ . One of our aims is to give a more algorithmic approach to the above duality. To this end, we establish a suitable dynamic programming principle (DPP) for the superhedging price and also show existence of minimal superhedging strategies in the spirit of Föllmer and Kramkov [1997]. This leads to a robust generalisation of the algorithm in Carassus et al. [2006] and gives a way to handle computation of superhedging prices and, importantly, strategies.

#### 3.1 Main Result

To state our main result we need to introduce some further notation. For an upper semianalytic function  $\xi : \Omega^T \rightarrow \mathbb{R}$  let  $\{\pi_t(\xi)\}_{0 \leq t \leq T}$  denote the one step superhedging prices  $\pi_t(\xi) : \Omega^t \rightarrow \overline{\mathbb{R}}$  given by

$$\begin{aligned} \pi_T(\xi)(\omega) &= \xi(\omega), \quad \text{and for } 0 \leq t \leq T-1 \\ \pi_t(\xi)(\omega) &= \inf\{x \mid \exists H \in \mathbb{R}^d \text{ such that } x + H\Delta S_{t+1}(\omega, \cdot) \geq \pi_{t+1}(\xi)(\omega, \cdot) \mathcal{P}_t(\omega)\text{-q.s.}\}. \end{aligned} \quad (8)$$

Note that the above superhedging prices can be construed as concave envelopes. Indeed, with a slight abuse of notation we denote the one-step quasisure concave envelope  $\widehat{f} : \Omega^t \times \mathbb{R}_+^d \rightarrow \mathbb{R}$  by

$$\widehat{f}(\omega, s) = \inf\{u(s) \mid u : \mathbb{R}_+^d \rightarrow \mathbb{R} \text{ closed concave, } u(S_{t+1}(\omega, \cdot)) \geq f(\omega, \cdot) \mathcal{P}_t(\omega)\text{-q.s.}\}$$

for  $t \in \{1, \dots, T\}$  and an upper semianalytic function  $f : \Omega^t \times \Omega \rightarrow \mathbb{R}$ , where we recall that a concave function is closed, if its superlevel set is closed. As every concave function can be written as the pointwise infimum of linear functions the equality

$$\pi_t(\xi)(\omega) = \widehat{\pi_{t+1}(\xi)}(\omega, S_t(\omega)), \quad \omega \in \Omega^t, \quad 0 \leq t \leq T-1 \quad (9)$$

holds and the one-step superhedging prices can be obtained by iteratively taking concave envelopes in the coordinates of  $\Omega$ .

Let us now define the corresponding dual expressions for the one step case. For  $\omega \in \Omega^t$  and  $f : \Omega^t \times \Omega \rightarrow \overline{\mathbb{R}}$ , we define  $\mathcal{E}_t(f) : \Omega^t \rightarrow \overline{\mathbb{R}}$  by

$$\mathcal{E}_t(f)(\omega) = \sup_{Q \in \mathcal{Q}_t(\omega)} \mathbb{E}_Q[f(\omega, \cdot)].$$

Furthermore, for measurable  $\xi : \Omega^T \rightarrow \mathbb{R}$ , we define the sequences of operators

$$\mathcal{E}^T(\xi) = \xi \quad \text{and} \quad \mathcal{E}^t(\xi) = \mathcal{E}_t \circ \mathcal{E}^{t+1}(\xi), \quad 0 \leq t \leq T-1. \quad (10)$$

With notation at hand, we can state our first main result which gives existence of minimal superhedging strategies and establishes a Dynamic Programming Principle for  $\pi_t(\xi)$  and  $\mathcal{E}^t(\xi)$ .

**Theorem 3.1** *Let assumption 2.1 and NA( $\mathcal{P}$ ) hold. Let  $\xi : \Omega^T \rightarrow \mathbb{R}$  be an upper semianalytic function such that  $\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\xi^-] < \infty$ . Then:*

- (i) *there exists a minimal superhedging strategy in  $\mathcal{A}(\xi)$ ;*
- (ii) *for any minimal superhedging strategy  $(\hat{x}, \hat{H}, \hat{C}) \in \mathcal{A}(\xi)$ , its value satisfies*

$$V_t^{\hat{x}, \hat{H}, \hat{C}} = \pi_t(\xi) = \mathcal{E}^t(\xi) \quad \mathcal{P}\text{-q.s.}, \quad 0 \leq t \leq T. \quad (11)$$

*In particular,*

$$\hat{x} = \pi(\xi) = \pi_0(\xi) = \mathcal{E}^0(\xi).$$

Perhaps suprisingly the proof of the above result is technically involved and is thus relegated to Appendix B. However in the special case of the canonical setting  $\Omega = \mathbb{R}_+^d$ ,  $S_t(\omega) = \omega_t$  and  $\mathcal{P} = \{P \in \mathfrak{P}(X) \mid \text{supp}(P) \text{ is finite}\}$  for an analytic set  $X \subseteq \Omega^T$  the underlying arguments are quite intuitive and simple. We outline them in the next section.

### 3.2 Canonical space: Concave envelopes and computation of the superhedging price

In this subsection we work on the canonical space, i.e. we set  $\Omega = \mathbb{R}_+^d$  and  $S_t(\omega) = (\omega_t^1, \dots, \omega_t^d)$ . In particular  $\xi(S_1(\omega), \dots, S_T(\omega)) = \xi(\omega)$  holds.

We start by developing in more detail the special case when  $\mathcal{P}$  is obtained by specifying the support for feasible moves of the stock prices. This captures the pathwise approach but is also natural in the quasisure framework as NA( $\mathcal{P}$ ) and  $\pi(\xi)$  only depend on the polar sets of  $\mathcal{P}$ . More precisely we give the following definition:

**Definition 3.2** Assume that for  $0 \leq t \leq T-1$  we are given correspondences  $f_t : \Omega^t \rightarrow \mathbb{R}^d$ . We say that a sequence of sets  $(\mathcal{P}_t)_{0 \leq t \leq T-1}$  such that  $\mathcal{P}_t \subseteq \mathfrak{P}(\Omega)$  for all  $0 \leq t \leq T-1$  is generated by  $\{f_t\}_{0 \leq t \leq T-1}$  if

$$\mathcal{P}_t(\omega) = \{P \in \mathfrak{P}(\Omega) \mid \text{supp}(P) \subseteq f_t(\omega)\}$$

for  $0 \leq t \leq T-1$ , where  $\text{supp}(P)$  denotes the support of a measure  $P$ .

Recall that a correspondence  $f : \Omega^t \rightarrow \mathbb{R}^d$  is called measurable if  $\{\omega \in \Omega^t \mid f(\omega) \cap O \neq \emptyset\} \in \mathcal{B}(\Omega^t)$  for all open sets  $O \subseteq \mathbb{R}^d$ . We refer to [Rockafellar and Wets, 1998, 14.A, p.643ff.] for the theory of measurable correspondences.



**Lemma 3.3** *Let  $(\mathcal{P}_t)_{0 \leq t \leq T-1}$  be generated by measurable, closed valued correspondences  $\{f_t\}_{0 \leq t \leq T-1}$ . Then  $\mathcal{P}_t$  has Borel measurable graph for all  $0 \leq t \leq T-1$ .*

Under the assumptions of lemma 3.3 we can then define  $\mathcal{P} \subseteq \mathfrak{P}(\Omega^T)$  satisfying (APS) as in assumption 2.1.

*Proof.* By assumption the graph of  $f_t$  is  $\mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d) = \mathcal{B}((\mathbb{R}^d)^{t+1})$ -measurable for all  $t \in \{0, \dots, T-1\}$  (see [Rockafellar and Wets, 1998, Theorem 14.8, p.648]). Thus by [Bertsekas and Shreve, 2004, Cor. 7.25.1, p.134]  $\mathfrak{P}(\text{graph}(f_t))$  is Borel as well. Define the map

$$D : \Omega^t \times \mathfrak{P}(\mathbb{R}_+^d) \rightarrow \mathfrak{P}(\Omega^{t+1}), (\omega, P) \mapsto \delta_\omega \otimes P$$

and note that  $D$  is a homeomorphism from  $\Omega^t \times \mathfrak{P}(\mathbb{R}_+^d)$  to  $\{\delta_\omega \otimes P \mid \omega \in \Omega^t, P \in \mathfrak{P}(\mathbb{R}_+^d)\}$ . Indeed, take a sequence  $(\omega_n, P_n) \in \Omega^t \times \mathfrak{P}(\mathbb{R}_+^d)$  such that  $(\omega_n, P_n)$  converges to  $(\omega, P)$  in the product topology. Denote by  $\mathcal{L}_b^1(\Omega^{t+1})$  the bounded 1-Lipschitz functions on  $\Omega^{t+1}$ . Then

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sup_{f \in \mathcal{L}_b^1(\Omega^{t+1})} \left| \int_{\Omega^{t+1}} f d(\delta_{\omega_n} \otimes P_n) - \int_{\Omega^{t+1}} f d(\delta_\omega \otimes P) \right| \\ & \leq \lim_{n \rightarrow \infty} \left( |\omega_n - \omega| + \sup_{f \in \mathcal{L}_b^1(\Omega^{t+1})} \left| \int_{\Omega^{t+1}} f(\omega, \cdot) dP_n - \int_{\Omega^{t+1}} f(\omega, \cdot) dP \right| \right) = 0, \end{aligned}$$

so  $\delta_{\omega_n} \otimes P_n$  converges weakly to  $\delta_\omega \otimes P$ . Continuity of the inverse map follows directly from the definition of weak convergence of measures. Note also that a homeomorphism map Borel sets to Borel sets. As

$$\mathfrak{P}(\text{graph}(f_t)) \cap \{\delta_\omega \otimes P \mid \omega \in \Omega^t, P \in \mathfrak{P}(\mathbb{R}^d)\}$$

is Borel-measurable, applying the inverse map  $D^{-1}$  we conclude that

$$\text{graph}(\mathcal{P}_t) = D^{-1}(\mathfrak{P}(\text{graph}(f_t)) \cap \{\delta_\omega \otimes P \mid \omega \in \Omega^t, P \in \mathfrak{P}(\mathbb{R}^d)\})$$

is Borel. □

In fact, for such a set  $\mathcal{P}$  the condition  $\text{NA}(\mathcal{P}_t(\omega))$  is equivalent to  $0 \in \text{ri}(f_t(\omega) - S_t(\omega))$ , where  $\text{ri}(A)$  denotes the relative interior of the convex hull of  $A$ . For a proof of this result in a more general setup, see [Oblój and Wiesel, 2018, Thm. 3.3, p. 6]. This deterministic condition is called No Pointwise Arbitrage in Burzoni et al. and can be checked without resorting to the use of probability measures.

As an intuitive outline of the proof of theorem 3.1, let us now assume that  $\mathcal{P} = \{P \in \mathfrak{P}(X) \mid \text{supp}(P) \text{ is finite}\}$  and  $\text{NA}(\mathcal{P})$  holds, where  $X \subseteq \Omega^T$  is some analytic set. We can now prove the crucial equality  $\pi_t(\xi) = \mathcal{E}_t(\pi_{t+1}(\xi))$  directly using the concave envelope characterisation (9), see also Beiglböck and Nutz [2014] and the references therein. Indeed, it follows from [Oblój and Wiesel, 2018, Prop 6.1, p. 14] that  $\mathcal{P}$  satisfies assumption 2.1 in this case and

$$Q = \{Q \in \mathfrak{P}(X) \mid \text{supp}(Q) \text{ is finite and } S \text{ is a martingale under } Q\},$$

see also [Bouchard and Nutz, 2015, Example 1.2, p.827] for  $X = (\mathbb{R}^d)^T$  and [Lange, 1973, Cor. 4.6, p.151] for locally compact  $X$ . Let  $\omega = (\omega_1, \dots, \omega_t) \in \Omega^t$ . Using Jensen's inequality

$$\begin{aligned} \mathcal{E}_t(f)(\omega) &= \sup_{Q \in \mathcal{Q}_t(\omega)} \mathbb{E}_Q[f(\omega, \cdot)] \leq \sup_{Q \in \mathcal{Q}_t(\omega)} \mathbb{E}_Q[\hat{f}(\omega, \cdot)] \\ &\leq \sup_{Q \in \mathcal{Q}_t(\omega)} \hat{f}(\omega, \mathbb{E}_Q[\cdot]) = \hat{f}(\omega, \omega_t), \end{aligned} \tag{12}$$

where  $\mathbb{E}_Q[\cdot] = \int_{\mathbb{R}^d} yQ(dy)$ . To establish the “ $\geq$ ”-inequality, it suffices to observe that

$$s \mapsto \sup_{Q \ll P \text{ for some } P \in \mathcal{P}_t(\omega), \mathbb{E}_Q[\cdot]=s} \mathbb{E}_Q[f(\omega, \cdot)]$$

is concave and dominates  $f(\omega, \cdot)$  on  $S_{t+1}(\Sigma_t^\omega)$ , where  $\Sigma_t^\omega := \{\tilde{\omega} \in X \mid (\tilde{\omega}_1, \dots, \tilde{\omega}_t) = \omega\}$ . While concavity is clear in general (see [Beiglböck and Nutz, 2014, Lemma 2.2]), the domination property crucially relies on the fact that the set  $\{Q \ll P \text{ for some } P \in \mathcal{P}_t(\omega), \mathbb{E}_Q[\cdot] = s\}$  contains the Dirac measures at points  $s \in S_{t+1}(\Sigma_t^\omega)$ . For a general set  $\mathcal{P}$  this is not true: For example in the case  $\mathcal{P} = \{P\}$  for some  $P \in \mathfrak{P}(\Omega)$  in general only the set  $\{Q \ll P, \mathbb{E}_Q[\cdot] = s\}$  is non-empty for  $s$  in the relative interior of the convex hull of the support of  $P$  (see [Föllmer and Schied, 2002, Theorem 1.48, p.29]). The following definition further characterises closed-valued correspondences  $\{f_t\}_{0 \leq t \leq T-1}$  and is needed to identify an important subclass of sets  $\{\mathcal{P}_t\}_{0 \leq t \leq T-1}$  generated by  $\{f_t\}_{0 \leq t \leq T-1}$ :

**Definition 3.4** A closed-valued correspondence  $f_t : \Omega^t \rightarrow \mathbb{R}^d$  is called uniformly continuous if for all  $\epsilon > 0$  there exists  $\delta > 0$  such that for all  $\omega, \omega' \in \Omega^T$  such that  $|\omega' - \omega| \leq \delta$  we have  $d_H(f_t(\omega), f_t(\omega')) \leq \epsilon$ , where

$$d_H(A, B) := \max \left( \sup_{v \in A} \inf_{\tilde{v} \in B} |v - \tilde{v}|, \sup_{\tilde{v} \in B} \inf_{v \in A} |v - \tilde{v}| \right)$$

denotes the Hausdorff metric on closed subsets  $A, B$  of  $\Omega$ .

Uniformly continuous correspondences are in particular continuous (see [Rockafellar and Wets, 1998, Def. 5.4, p.152]) and thus measurable ([Rockafellar and Wets, 1998, Theorem 5.7, p.154]). It turns out, that when the correspondences fulfil this continuity condition and are compact-valued, the  $\mathcal{P}$ -q.s. superhedging price of a continuous payoff  $\xi$  coincides with the  $P$ -a.s. superhedging price of  $\xi$  for every  $P$  with support equal to the paths generated by the correspondences  $\{f_t\}_{0 \leq t \leq T-1}$ :

**Proposition 3.5** Suppose  $(\mathcal{P}_t)_{0 \leq t \leq T-1}$  is generated by closed-valued, uniformly continuous correspondences  $\{f_t\}_{0 \leq t \leq T-1}$  and that  $NA(\mathcal{P})$  holds. Furthermore assume that the function  $\xi : \Omega^T \rightarrow \mathbb{R}$  is continuous and  $\{f_t\}_{0 \leq t \leq T-1}$  are compact-valued. Take any measure  $P = P_0 \otimes \dots \otimes P_{T-1}$  such that

$$\text{supp}(P_t(\omega)) = f_t(\omega), \quad 0 \leq t \leq T-1, \quad \omega \in \Omega^t.$$

Then, for all  $0 \leq t \leq T-1$  and  $\omega \in \Omega^t$ ,

$$\pi_t(\xi)(\omega) = \inf\{x \in \mathbb{R} \mid \exists H \in \mathbb{R}^d \text{ such that } x + H\Delta S_{t+1}(\omega, \cdot) \geq \pi_{t+1}(\xi)(\omega, \cdot) \text{ } P\text{-a.s.}\}. \quad (13)$$

and  $\omega \mapsto \pi_t(\xi)(\omega)$  is continuous.

The proof of the above result is relegated to Appendix A.

We now apply this result to a one-dimensional case of particular interest, as in Carassus and Vargiolu [2018], where it is easy to explicitly compute the minimal superhedging prices:

**Proposition 3.6** Assume that for all  $0 \leq t \leq T-1$ ,  $d_{t+1} < 1 < u_{t+1}$  and that the (random) sets  $\mathcal{P}_t$  are given by

$$\mathcal{P}_t(\omega) = \{P \in \mathfrak{P}(\mathbb{R}) \mid \text{supp}(P) \subset [\omega_t d_{t+1}, \omega_t u_{t+1}]\},$$

where  $\omega = (\omega_1, \dots, \omega_t) \in \Omega^t$ . Then  $\text{NA}(\mathcal{P})$  holds. Let  $\xi : \mathbb{R}^T \rightarrow \mathbb{R}$  be convex. Then

$$\begin{aligned} \pi_T(\xi) &= \xi \\ \pi_t(\xi)(\omega) &= \alpha_{t+1}\pi_{t+1}(\xi)(\omega, \omega_t u_{t+1}) + (1 - \alpha_{t+1})\pi_{t+1}(\xi)(\omega, \omega_t d_{t+1}), \end{aligned} \quad (14)$$

where  $\alpha_t := \frac{1-d_t}{u_t-d_t}$ ,  $1 \leq t \leq T$ .

*Proof.* Noting that  $f_t(\omega) = [\omega_t d_{t+1}, \omega_t u_{t+1}]$  is a uniformly continuous compact-valued correspondence, the graph of  $\mathcal{P}_t$  is clearly non-empty, convex and Borel measurable for  $0 \leq t \leq T-1$  by lemma 3.3. As  $0 \in \text{ri}(f_t(\omega) - S_t(\omega)) = \text{ri}([- \omega_t(1-d_{t+1}), \omega_t(u_{t+1}-1)])$ ,  $\text{NA}(\mathcal{P})$  holds. We prove by induction that  $\pi_t(\xi)$  satisfies (14) and is convex: This is clear for  $t = T$ . Now we assume that for some  $0 \leq t \leq T-1$ ,  $\pi_{t+1}(\xi)$  is convex. As  $\mathcal{P}_t(\omega)$  contains the Dirac measures on  $[\omega_t d_{t+1}, \omega_t u_{t+1}]$  we conclude that

$$\pi_t(\xi)(\omega) = \inf\{x \in \mathbb{R} \mid \exists H \text{ s. t. } x + H\Delta S_{t+1}(\omega, \cdot) \geq \pi_{t+1}(\xi)(\omega, \cdot) \text{ on } [\omega_t d_{t+1}, \omega_t u_{t+1}]\}.$$

As  $\pi_t(\xi)(\omega)$  is the pointwise concave envelope of the convex function  $\pi_{t+1}(\xi)(\omega, \cdot)$ , it can be written as the unique convex combination of the extreme points of  $\pi_{t+1}(\xi)(\omega, \cdot)$  on the interval  $[\omega_t d_{t+1}, \omega_t u_{t+1}]$ , which conserves the barycentre  $\omega_t$ . Thus, we obtain (14) for  $t$ . Clearly  $\pi_t(\xi) : \mathbb{R}^t \rightarrow \mathbb{R}$  is then a linear combination of convex functions (with non-negative coefficients) and thus also a convex function.  $\square$

It is insightful to observe that the above superreplication price corresponds to the actual replication price in a Cox-Ross-Rubinstein model of Cox et al. [1979] where the stock price evolves on a binomial tree with  $S_{t+1} \in \{d_{t+1}S_t, u_{t+1}S_t\}$ .

## 4 Maximising expected utility of consumption in $\mathcal{A}(\xi)$

### 4.1 Main results

In theorem 3.1 above, we characterised the superhedging prices  $\pi_t(\xi)$  and introduced ways for computing minimal superhedging strategies. However, these are typically non-unique. Indeed, as we see from (9), if the concave envelope  $\widehat{f(\omega, \cdot)}$  of a function  $f : \Omega^{t+1} \rightarrow \mathbb{R}$  is not differentiable at  $\omega_t$ , every point  $H \in \mathbb{R}^d$  in its superdifferential constitutes a minimal superhedging strategy, see also example 4.6 below. To select the “best” among minimal superhedging strategies we propose a secondary optimisation problem of robust maximisation of expected utility with intermediate consumption, given by

$$\sup_{(H,C) \in \mathcal{A}_x} \inf_{P \in \mathcal{P}^u} \mathbb{E}_P \left[ \sum_{s=1}^T U(s, \Delta C_s) \right], \quad (15)$$

where  $\mathcal{A}_x$  is the set of investment-consumption strategies which superhedge  $\xi : \Omega^T \rightarrow \mathbb{R}$ , i.e.

$$\mathcal{A}_x := \{(H, C) \in \mathcal{H}(\mathbb{R}^d) \times \mathcal{C} \mid V_T^{x,H,C} \geq \xi \text{ } \mathcal{P}\text{-q.s.}\}$$

and the set  $\mathcal{P}^u \subseteq \mathfrak{P}(\Omega^T)$  fulfils the following condition:

**Assumption 4.1**  $\mathcal{P}^u$  satisfies (APS) and  $\mathcal{P}^u \subseteq \mathcal{P}$ .

The set  $\mathcal{P}^u$  represents the subjective views of an investor. While superhedging with respect to  $\mathcal{P}$  reflects the necessity to satisfy certain regulatory and risk requirements,  $\mathcal{P}^u$  is used to express individual preferences for the optimisation problem (15) and does not need to satisfy any further requirements than those of assumption 4.1, e.g.  $\text{NA}(\mathcal{P}^u)$  can fail. In theorem 4.3 and theorem 4.5 below, we show that (15) is well posed and admits an optimiser which, under suitable assumptions, is unique. The assumptions imposed on the utility functions  $U(t, \cdot, \cdot)$  are in line with those in Nutz [2016]:

**Assumption 4.2** For  $t = 1, \dots, T$  the utility function  $U(t, \cdot, \cdot) : \Omega^t \times [0, \infty) \rightarrow \mathbb{R}$  is lower semianalytic and bounded from above. Furthermore

1.  $\omega \mapsto U(t, \omega, x)$  is bounded from below for each  $x > 0$ .
2.  $x \mapsto U(t, \omega, x)$  is non-decreasing, concave and continuous for each  $\omega \in \Omega^t$ .

We believe that boundedness assumptions on utility functions which we make here could be weakened, similarly to Blanchard and Carassus [2017]. However, due to the overall length and already technical character of proofs, we decided to leave this extension for further research.

We remark that by 2. in assumption 4.2 it is sufficient to consider investment-consumption strategies which hedge  $\xi$ , i.e. for which  $V_T^{x,H,C} = \xi$ , since the superhedging surplus can be consumed at terminal time.

Note that by assumption 4.2 and standard results on Carathéodory functions (see [Aliprantis and Border, 2006, Lemma 4.51, p. 153]) we conclude that  $U(t, \cdot, \cdot)$  is  $\mathcal{F}_t^U \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable. We set  $U(t, x, \omega) = -\infty$  for  $x < 0$  and often write  $U(t, x)$  instead of  $U(t, x, \omega)$ .

**Theorem 4.3** Let  $U(t, \cdot, \cdot)$  be given for  $1 \leq t \leq T$  and let  $\text{NA}(\mathcal{P})$ , assumption 2.1, assumption 4.1 and assumption 4.2 hold. Then for any Borel  $\xi : \Omega^T \rightarrow \mathbb{R}$  such that  $\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\xi^-] < \infty$  there exists  $(\hat{H}, \hat{C}) \in \mathcal{A}_\pi$  such that

$$\inf_{P \in \mathcal{P}^u} \mathbb{E}_P \left[ \sum_{s=1}^T U(s, \Delta \hat{C}_s) \right] = \sup_{(H,C) \in \mathcal{A}_\pi} \inf_{P \in \mathcal{P}^u} \mathbb{E}_P \left[ \sum_{s=1}^T U(s, \Delta C_s) \right],$$

where  $\pi = \pi(\xi)$  is the  $\mathcal{P}$ -q.s. superhedging price of  $\xi$ .

In order to obtain uniqueness of the above maximiser  $(\hat{H}, \hat{C})$ , we again switch to the canonical setup  $\Omega^T = (\mathbb{R}_+^d)^T$ ,  $S_t(\omega) = \omega_t$ . In line with Denis and Kervarec [2013] we strengthen assumptions on the utility functions  $U(t, \cdot, \cdot)$  and also assume weak compactness of the set  $\mathcal{P}^u$ . This enables us to show existence of a “worst-case” measure  $\tilde{P} \in \mathcal{P}^u$ , in analogy to the argumentation in Schied and Wu [2005]. In fact, example 4.7 below shows, that one cannot expect uniqueness of maximizers in general, if  $\mathcal{P}^u$  is not weakly closed.

**Assumption 4.4** For  $t = 1, \dots, T$  the non-random utility functions  $U(t, \cdot)$  satisfy assumption 4.2 and are bounded. The mapping  $x \mapsto U(t, x)$  is strictly concave, non-decreasing and continuous. Furthermore, for  $t = 0, \dots, T-1$  and  $\mathcal{P}^u$ -q.e  $\omega \in \Omega^t$  the set  $\mathcal{P}_t^u(\omega)$  is weakly compact and the sets  $\mathcal{P}$  and  $\mathcal{P}^u$  fulfil the following continuity criteria:

1. If  $\omega, \tilde{\omega} \in \Omega^t$  and  $\epsilon > 0$ , then there exists  $\delta > 0$  such that for  $|\omega - \tilde{\omega}| \leq \delta$  and for every  $P \in \mathcal{P}_t^u(\omega)$  there exists  $\tilde{P} \in \mathcal{P}_t^u(\tilde{\omega})$  such that  $d_L(P, \tilde{P}) \leq \epsilon$ , where

$$d_L(P, \tilde{P}) = \inf \{ \epsilon \geq 0 \mid P(A) \leq \tilde{P}(A^\epsilon) + \epsilon \text{ for all } A \in \mathcal{B}(\Omega) \}$$

denotes the Levy metric on  $\mathfrak{P}(\Omega)$  and  $A^\epsilon = \{ \omega \in \Omega \mid \exists \tilde{\omega} \in A \text{ such that } |\omega - \tilde{\omega}| < \epsilon \}$ .

2. The map  $f_t(\omega) := \text{supp}(\mathcal{P}_t(\omega))$  is uniformly continuous in the sense of definition 3.4, where

$$\text{supp}(\mathcal{P}_t(\omega)) = \bigcap \{A \subseteq \Omega \text{ closed} \mid P(A) = 1 \text{ for all } P \in \mathcal{P}_t(\omega)\}$$

is the quasisure support of  $\mathcal{P}_t(\omega)$  for  $\omega \in \Omega^t$ .

**Theorem 4.5** *In the setup of theorem 4.3 assume further that assumption 4.4 holds and that the functions  $\pi_t(\xi) : \Omega^t \rightarrow \mathbb{R}$  are continuous on  $\{(\omega, v) \in \Omega^t \mid v \in f_{t-1}(\omega)\}$  for all  $1 \leq t \leq T$ . Then there exists a probability measure  $\hat{P} \in \mathcal{P}^u$  such that*

$$\sup_{(H,C) \in \mathcal{A}_\pi} \inf_{P \in \mathcal{P}^u} \mathbb{E}_P \left[ \sum_{s=1}^T U(s, \Delta C_s) \right] = \sup_{(H,C) \in \mathcal{A}_\pi} \mathbb{E}_{\hat{P}} \left[ \sum_{s=1}^T U(s, \Delta C_s) \right].$$

Furthermore, the maximising strategy  $(\hat{H}, \hat{C}) \in \mathcal{A}_\pi$  is unique in the following sense: for any two maximising strategies  $(H^1, C^1), (H^2, C^2) \in \mathcal{A}_\pi$  and for  $1 \leq t \leq T$  we have  $C_t^1 = C_t^2$  and  $H_t^1 \Delta S_t = H_t^2 \Delta S_t$   $\hat{P}$ -a.s.

The proofs of theorem 4.3 and theorem 4.5 are given in section C. We first establish theorem 4.3 in the one-period case ( $T = 1$ ) and then extend it to the general multi-step setting and consider the uniqueness.

## 4.2 Examples and comments

To illustrate the above results, we discuss several examples. We start with a simple example for non-uniqueness of minimal superhedging strategies.

### Example 4.6 (Non-Uniqueness of minimal superhedging strategies and maximizers)

We take  $\Omega = \mathbb{R}_+$ , where  $d = 1$  and  $T = 2$  as well as  $s_0 = 2$ . Furthermore  $S_t(\omega) = \omega_t$  for  $t = 1, 2$  and

$$\mathcal{P}_t(\omega) = \{P \in \mathfrak{P}(\mathbb{R}_+)\}, \quad t = 0, 1.$$

We want to superhedge the running minimum at time 2, i.e.  $\xi(\omega) = \underline{S}_2(\omega)$ . Clearly  $\mathcal{Q}_t(\omega) = \{Q \in \mathfrak{P}(\mathbb{R}_+) \mid \mathbb{E}_Q[\Delta S_{t+1}(\omega, \cdot)] = 0\}$  for all  $\omega \in \Omega^t$  and  $t = 0, 1$ . Besides it is easy to see that

$$\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\xi] = s_0 = 2,$$

so we have some degree of freedom to choose our superhedging strategy  $H \in \mathcal{H}(\mathbb{F}^u)$ . As it turns out we can choose any  $H_1 \in [0, 1]$ , which gives a wealth of  $2 + H_1(S_1 - 2)$  at time 1. For time 2 we have

$$H_2(\omega) \in \begin{cases} [0, H_1] & \text{if } S_1(\omega) \geq 2, \\ \left[0, \frac{2}{S_1(\omega)} + \frac{H_1}{S_1(\omega)}(S_1(\omega) - 2)\right] & \text{if } S_1(\omega) < 2. \end{cases}$$

Note also that the superhedging cost at time 1 is given by

$$\pi_1(\xi)(\omega) = \sup_{Q \in \mathcal{Q}_1(\omega)} \mathbb{E}_Q[\xi(\omega, \cdot)] = \begin{cases} 2 & \text{if } S_1(\omega) \geq 2, \\ S_1(\omega) & \text{if } S_1(\omega) < 2. \end{cases}$$

So according to (1) and (8) we can consume

$$C_1(\omega) \in \begin{cases} [0, H_1(S_1(\omega) - 2)] & \text{if } S_1(\omega) \geq 2, \\ [0, (H_1(\omega) - 1)(S_1(\omega) - 2)] & \text{if } S_1(\omega) < 2 \end{cases}$$

at time 1.

We now show that if assumption 4.4 is not satisfied (namely  $\mathcal{P}^u$  does not fulfil assumption 4.4.1.), then theorem 4.5 is not true in general. For this we specify the set  $\mathcal{P}^u$  and iteratively solve the optimization problem (15): We set  $U(2, \omega, x) = U(1, \omega, x) = U(x)$  for some bounded concave, non-decreasing and continuous function  $U : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  as well as  $\mathcal{P}_1^u(S_1) = \{\delta_{S_1}\}$  for  $S_1 > 2$  and  $\mathcal{P}_1^u(S_1) = \{\delta_{S_1+1}\}$  for  $S_1 \leq 2$ . Note that  $\mathcal{P}_1^u$  obviously violates assumption 4.4.1. We obtain the following optimal one-step prices, where we use notation from section C.2: For  $S_1 > 2$  and  $x \geq 2$  we find

$$\begin{aligned} U_1(S_1, x) &= \sup_{(H,c) \in \mathcal{A}_{1,x}(S_1)} \left( \mathbb{E}_{\delta_{S_1}} [U(x + H(S_2 - S_1) - \underline{S}_2 - c)] + U(c) \right) \\ &= \sup_{(H,c) \in \mathcal{A}_{1,x}(S_1)} (U(x - 2 - c) + U(c)) = 2U\left(\frac{x-2}{2}\right) \end{aligned}$$

with  $c = (x - 2)/2$  and some  $0 \leq H \leq \min(\frac{x/2+1}{S_1}, \frac{x/2-1}{S_1-2})$ . For  $S_1 \leq 2$  and  $x \geq S_1$  we have

$$\begin{aligned} U_1(S_1, x) &= \sup_{(H,c) \in \mathcal{A}_{1,x}(S_1)} \left( \mathbb{E}_{\delta_{S_1+1}} [U(x + H(S_2 - S_1) - \underline{S}_2 - c)] + U(c) \right) \\ &= \sup_{(H,c) \in \mathcal{A}_{1,x}(S_1)} (U(x + H - S_1 - c) + U(c)) \geq U(0) + U(1) \end{aligned}$$

with  $H = x/S_1$  and  $c = 0$ . Setting  $\mathcal{P}_0^u = \{\delta_x \mid x \in \mathbb{R}_+\}$  we obtain

$$\begin{aligned} U_0(2) &= \sup_{H \in \mathcal{A}_{0,2}} \inf_{P \in \mathcal{P}_0^u} \mathbb{E}_P [U_1(S_1, 2 + H(S_1 - 2))] \\ &= \sup_{H \in \mathcal{A}_{0,2}} \inf_{P \in \mathcal{P}_0^u} \mathbb{E}_P \left[ \mathbf{1}_{\{S_1 > 2\}} 2U\left(\frac{2 + H(S_1 - 2) - 2}{2}\right) \right. \\ &\quad \left. + \mathbf{1}_{\{S_1 \leq 2\}} U_1(2 + H(S_1 - 2)) \right] \\ &= 2U(0). \end{aligned}$$

Note that by the proof of theorem 4.5 under assumption 4.4 there would exist  $\hat{P} \in \mathcal{P}_0^u$  such that

$$U_0(2) = \sup_{H \in \mathcal{A}_{0,2}} \mathbb{E}_{\hat{P}} [U_1(S_1, x + H\Delta S_1)].$$

On the contrary, in our case there exists no  $\hat{P} \in \mathcal{P}_0^u$  such that

$$U_0(2) = 2U(0) = \mathbb{E}_{\hat{P}} \left[ \mathbf{1}_{\{S_1 > 2\}} 2U\left(\frac{S_1 - 2}{2}\right) + \mathbf{1}_{\{S_1 < 2\}} U_1(S_1, 2) \right]$$

as the RHS is strictly greater than  $2U(0)$  for all  $\hat{P} \in \mathcal{P}_0^u$ : Thus theorem 4.5 does not hold.

The next example shows that we cannot expect to have uniqueness of maximizers without assuming some closedness property of  $\mathcal{P}^u$ .

**Example 4.7 (Non-uniqueness of maximisers for non-closed  $\mathcal{P}^u$ )** Let  $T = 1$ ,  $d = 2$ ,  $\Omega = \mathbb{R}^2$ ,  $\mathcal{P} = \mathfrak{P}(\mathbb{R}_+^2)$ ,  $S_t(\omega) = \omega_t$  and  $S_0 = (1, 1)$ . Consider  $\xi = \min(S_1^1, S_1^2)$ . Then  $\pi(\xi) = 1$  and  $H_1$  is of the form

$$H_1 = \begin{pmatrix} \lambda \\ 1 - \lambda \end{pmatrix},$$

where  $\lambda \in [0, 1]$ . Take

$$\mathcal{P}^u = \{P_n\}_{n=1}^\infty \quad \text{where } P_n = \frac{\delta_{\{S_1^1=n-\frac{1}{n}, S_1^2=n+\frac{1}{n}\}}}{2} + \frac{\delta_{\{S_1^1=0, S_1^2=0\}}}{2}.$$

Then clearly  $\mathcal{P}^u$  is not closed. We note that for  $H \in \mathcal{A}_1$

$$\begin{aligned} \mathbb{E}_{P_n}[U(1 + H\Delta S_1 - \xi)] &= \frac{1}{2}U\left(\lambda\left(n - \frac{1}{n}\right) + (1 - \lambda)\left(n + \frac{1}{n}\right) - \left(n - \frac{1}{n}\right)\right) + \frac{1}{2}U(0) \\ &= \frac{1}{2}U\left((1 - \lambda)\frac{2}{n}\right) + \frac{1}{2}U(0) \downarrow U(0), \quad n \rightarrow \infty. \end{aligned}$$

Thus we conclude

$$\sup_{H \in \mathcal{A}_1} \inf_{P \in \mathcal{P}^u} \mathbb{E}_P[U(1 + H\Delta S_1 - \xi)] = U(0),$$

in particular

$$H \mapsto \inf_{P \in \mathcal{P}^u} \mathbb{E}_P[U(1 + H\Delta S_1 - \xi)] = U(0)$$

is constant and thus the maximizer is not unique.

Finally, we illustrate that even with a compact  $\mathcal{P}^u$  we can not strengthen the sense in which the optimisers are unique in theorem 4.5.

**Example 4.8 (On uniqueness property of maximisers)** We consider a one-step version of example 4.6:  $T = 1$ ,  $d = 1$ ,  $\Omega = \mathbb{R}_+$ ,  $S_t(\omega) = \omega_t$ ,  $s_0 = 2$ ,  $\xi(S) = \underline{S}_1$ ,  $\mathcal{P} = \mathfrak{P}(\mathbb{R}_+)$ . We have  $\pi(\xi) = 2$ . We also set  $\mathcal{P}^u = \{\delta_2\}$ , where  $\delta_2$  is defined by

$$\delta_2(S_t = 2 \text{ for all } t = 0, 1) = 1.$$

Furthermore let  $U(\cdot) = U(1, \cdot, \cdot)$  such that the conditions of theorem 4.5 are satisfied. The optimisers are then non-unique in the sense that (15) is equal to  $U(0)$  and is attained for every  $H \in [0, 1]$  but are unique in the sense of theorem 4.5 since  $H\Delta S_1 = 0$   $\delta_2$ -a.s. for all  $H \in \mathbb{R}$ .

# Appendices

We now provide the proofs of proposition 3.5, theorem 3.1, theorem 4.3 and of theorem 4.5. These proofs require a number of technical lemmata which are established alongside the main proofs.

## Appendix A Proof of proposition 3.5

*Proof.* Fix  $\omega \in \Omega^{T-1}$  and  $\epsilon > 0$ . Recall that  $\xi$  is continuous and  $\{f_t\}_{0 \leq t \leq T-1}$  are compact-valued. Note that the set

$$B := \{(\tilde{\omega}, \tilde{v}) \in \Omega^{T-1} \times \mathbb{R}^d \mid \text{dist}((\omega, f_{T-1}(\omega)), (\tilde{\omega}, \tilde{v})) \leq 1\}$$

is compact, thus  $\xi$  is uniformly continuous on  $B$ , i.e. there exists  $\delta \in (0, 1)$  such that  $|\xi(\omega, v) - \xi(\tilde{\omega}, \tilde{v})| \leq \epsilon/3$  for  $|(\omega, v) - (\tilde{\omega}, \tilde{v})| \leq \delta$  for  $v \in f_{T-1}(\omega)$ ,  $(\tilde{\omega}, \tilde{v}) \in B$ . This implies  $\sup_{\{\tilde{\omega} \mid |\omega - \tilde{\omega}| \leq 1\}} \pi_{T-1}(\xi)(\tilde{\omega}) < \infty$  and that for all  $\tilde{\omega} \in \Omega^{T-1}$  with  $|\omega - \tilde{\omega}| \leq 1$  there exists  $H_T(\tilde{\omega}) \in \mathbb{R}^d$  such that

$$\epsilon/3 + \pi_{T-1}(\xi)(\tilde{\omega}) + H_T(\tilde{\omega})\Delta S_T(\tilde{\omega}, \cdot) \geq \xi(\tilde{\omega}, \cdot) \quad \text{on } f_{T-1}(\tilde{\omega}) \quad (16)$$

or equivalently the inequality (16) holds  $\mathcal{P}_{T-1}(\tilde{\omega})$ -q.s.

Note that by the uniform continuity of the correspondence  $f_{T-1}$  for any  $\tilde{\omega}$  close to  $\omega$  and for any  $v \in f_{T-1}(\omega)$  there exists  $\tilde{v} \in f_{T-1}(\tilde{\omega})$  which is close to  $v$ , thus  $|(\omega, v) - (\tilde{\omega}, \tilde{v})|$  is small. Furthermore we show below that  $H_T(\tilde{\omega})$  can be chosen bounded uniformly in  $\tilde{\omega}$  for all  $\tilde{\omega}$  close to  $\omega$ . Thus, for some  $\delta_1$  determined below,  $|\omega - \tilde{\omega}| \leq \delta_1$  implies

$$\begin{aligned} \epsilon + \pi_{T-1}(\xi)(\tilde{\omega}) + H_T(\tilde{\omega})\Delta S_T(\omega, v) &\geq \epsilon + \pi_{T-1}(\xi)(\tilde{\omega}) + H_T(\tilde{\omega})\Delta S_T(\tilde{\omega}, \tilde{v}) - \epsilon/3 \\ &\geq \epsilon/3 + \xi(\tilde{\omega}, \tilde{v}) \geq \xi(\omega, v), \end{aligned} \quad (17)$$

and thus  $\pi_{T-1}(\xi)(\omega) \leq \pi_{T-1}(\xi)(\tilde{\omega}) + \epsilon$ . Exchanging the roles of  $\omega$  and  $\tilde{\omega}$  concludes the proof of continuity of  $\omega \mapsto \pi_{T-1}(\omega)$ .

We now argue that there exists  $\delta_0 > 0$  and  $C > 0$  such that  $|H_T(\tilde{\omega})| < C$  for all  $\tilde{\omega} \in \Omega^{T-1}$  with  $|\omega - \tilde{\omega}| \leq \delta_0$  and  $H_T(\tilde{\omega}) \in \text{lin}(f_{T-1}(\tilde{\omega}) - S_{T-1}(\tilde{\omega}))$ . Assume towards a contradiction this is not the case, i.e. there exists a sequence  $(\tilde{\omega}^N)_{N \in \mathbb{N}}$  with  $|\omega - \tilde{\omega}^N| \leq 1/N$ ,  $H_T(\tilde{\omega}^N) \in \text{lin}(f_{T-1}(\tilde{\omega}^N) - S_{T-1}(\tilde{\omega}^N))$  for all  $N \in \mathbb{N}$  and  $\lim_{N \rightarrow \infty} |H_T(\tilde{\omega}^N)| = \infty$ . After passing to a subsequence (without relabelling)  $\tilde{H}^N := H_T(\tilde{\omega}^N)/|H_T(\tilde{\omega}^N)| \rightarrow \tilde{H}$  with  $|\tilde{H}| = 1$ . Note that as  $f_{T-1}(\tilde{\omega}^N)$  converges in Hausdorff distance to  $f_{T-1}(\omega)$  and as  $f_{T-1}(\omega)$  is compact, it follows by the same arguments as above that  $\sup_{f_{T-1}(\tilde{\omega}^N)} \xi(\tilde{\omega}^N, \cdot)$  and  $\pi_{T-1}(\xi)(\tilde{\omega}^N)$  are bounded uniformly in  $N \in \mathbb{N}$ . Thus dividing (16) by  $|H_T(\tilde{\omega}^N)|$  and taking limits we get

$$\tilde{H}\Delta S_T(\omega, \cdot) \geq 0 \quad \text{on } f_{T-1}(\omega).$$

By  $\text{NA}(\mathcal{P}_{T-1}(\omega))$  this yields  $\tilde{H}\Delta S_T(\omega, \cdot) = 0$  on  $f_{T-1}(\omega)$ . As  $\tilde{H} \in \text{span}(f_{T-1}(\omega) - S_{T-1}(\omega))$ ,  $\tilde{H} = 0$  follows, a contradiction.

Now we choose  $\delta_1 \leq \delta_0$  such that for  $|\omega - \tilde{\omega}| \leq \delta_1$  we have

$$d_H((\omega, f_{T-1}(\omega)), (\tilde{\omega}, f_{T-1}(\tilde{\omega}))) \leq \min(\delta, \epsilon/(3C))$$



and see that (17) holds. The proof of continuity of  $\omega \mapsto \pi_t(\xi)(\omega)$  for  $1 \leq t \leq T - 2$  follows by backward induction using dynamic programming principle and the same arguments as above. Lastly, as for any  $P \in \mathfrak{P}(\mathbb{R}^d)$  such that  $\text{supp}(P) = f_{t-1}(\omega)$

$$\pi_{t-1}(\xi)(\omega) + H_t(\omega)\Delta S_t(\omega, \cdot) \geq \pi_t(\xi)(\omega, \cdot) \quad P\text{-a.s.}$$

implies

$$\pi_{t-1}(\xi)(\omega) + H_t(\omega)\Delta S_t(\omega, \cdot) \geq \pi_t(\xi)(\omega, \cdot) \quad \text{on } f_{t-1}(\omega),$$

the claim follows.  $\square$

*Remark A.1* Note that the proof of boundedness of  $H_T(\tilde{\omega})$  above does not require that  $f_{T-1}(\tilde{\omega})$  is compact-valued.

## Appendix B Proof of theorem 3.1

**Lemma B.1** *Let  $NA(\mathcal{P})$  hold. Assume that  $\xi$  is upper semianalytic. Furthermore let  $\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\xi^-] < \infty$ . Then  $\mathcal{E}^t(\xi)$  is upper semianalytic and  $\mathcal{E}^t(\xi^-)$  is lower semianalytic for all  $0 \leq t \leq T - 1$ . Furthermore*

$$\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\mathcal{E}^t(\xi^-)] < \infty$$

and the analytic set  $\Omega_\xi^t := \{\mathcal{E}^t(\xi^-) < \infty\}$  is of full  $\mathcal{P}$ -measure. Let

$$\hat{\Omega}_\xi^t := \{\omega \in \Omega^t \mid \mathcal{E}^{t+1}(\xi)(\omega, \cdot) > -\infty, \mathcal{P}_t(\omega)\text{-q.s.}\}. \quad (18)$$

Then  $\Omega_\xi^t \subset \hat{\Omega}_\xi^t$ , in particular  $\hat{\Omega}_\xi^t$  is a  $\mathcal{P}$ -full measure set.

*Proof.* Using [Bouchard and Nutz, 2015, Lemma 4.10] recursively,  $\mathcal{E}^t(\xi)$  is upper semianalytic and  $\mathcal{E}^t(\xi^-)$  is lower semianalytic for all  $0 \leq t \leq T$ .

As  $\Omega_\xi^t = \{\mathcal{E}^t(\xi^-) < \infty\} = \bigcup_{n \geq 1} \{\mathcal{E}^t(\xi^-) \leq n\}$ ,  $\Omega_\xi^t$  is an analytic set. We now prove by induction that  $\Omega_\xi^t$  is a  $\mathcal{P}$ -full measure set and that  $\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\mathcal{E}^t(\xi^-)] < \infty$ . For  $t = T$ ,  $\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\xi^-] < \infty$  by assumption. If there exists some  $P \in \mathcal{P}$  such that  $P(\Omega_\xi^T) < 1$  then  $\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\mathcal{E}^T(\xi^-)] = \infty$ , as  $\mathcal{P}$  and  $\mathcal{Q}$  have the same polar sets (see [Bouchard and Nutz, 2015, First Fundamental Theorem, p. 828]).

Assume for some  $t \leq T - 1$  that  $\Omega_\xi^{t+1}$  is a  $\mathcal{P}$ -full measure set and that  $\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\mathcal{E}^{t+1}(\xi^-)] < \infty$ . Fix  $\epsilon > 0$ . From [Bertsekas and Shreve, 2004, Proposition 7.50 p184] (recall that  $\mathcal{Q}_t$  has an analytic graph), there exists an  $\mathcal{F}_t^{\mathcal{U}}$ -measurable function  $Q_\epsilon : \Omega^t \rightarrow \mathfrak{P}(\Omega)$ , such that  $Q_\epsilon(\omega) \in \mathcal{Q}_t(\omega)$  for all  $\omega \in \Omega^t$  and

$$\mathbb{E}_{Q_\epsilon}[\mathcal{E}^{t+1}(\xi^-(\omega, \cdot))] \geq \begin{cases} \mathcal{E}^t(\xi^-)(\omega) - \epsilon & \text{if } \omega \in \Omega_\xi^t, \\ \frac{1}{\epsilon} & \text{otherwise.} \end{cases} \quad (19)$$

Assume that  $\Omega_\xi^t$  is not a  $\mathcal{P}$ -full measure set. Then there exists some  $P \in \mathcal{P}$  such that  $P(\Omega_\xi^t) < 1$ . As  $\mathcal{P}$  and  $\mathcal{Q}$  have the same polar sets, we have that  $Q(\Omega_\xi^t) < 1$  for some  $Q \in \mathcal{Q}$ . We denote by  $Q|_{\mathcal{F}_t^{\mathcal{U}}}$  the restriction of  $Q$  to  $\mathcal{F}_t^{\mathcal{U}}$  and set  $Q^* := Q|_{\mathcal{F}_t^{\mathcal{U}}} \otimes Q_\epsilon$ . Then  $Q^* \in \mathcal{Q}|_{\mathcal{F}_{t+1}^{\mathcal{U}}}$  (see (6)) and we have that

$$\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\mathcal{E}^{t+1}(\xi^-)] \geq \mathbb{E}_{Q^*}[\mathcal{E}^{t+1}(\xi^-)] \geq \frac{1}{\epsilon}(1 - Q^*(\Omega_\xi^t)) - \epsilon Q^*(\Omega_\xi^t).$$

As the previous inequality holds for all  $\epsilon > 0$ , letting  $\epsilon$  go to 0 we obtain that

$$\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\mathcal{E}^{t+1}(\xi^-)] = \infty,$$

a contradiction. Thus  $\Omega_\xi^t$  is a  $\mathcal{P}$ -full measure set.

Now, for all  $Q \in \mathcal{Q}$ , we set  $Q^* = Q|_{\mathcal{F}_t^{\mathcal{U}}} \otimes Q_\epsilon \in \mathcal{Q}|_{\mathcal{F}_{t+1}^{\mathcal{U}}}$  (see (6)). Then, using (19) we see that

$$\mathbb{E}_Q[\mathcal{E}^t(\xi^-)] - \epsilon = \mathbb{E}_Q[\mathbb{1}_{\Omega_\xi^t} \mathcal{E}^t(\xi^-)] - \epsilon \leq \mathbb{E}_{Q^*}[\mathcal{E}^{t+1}(\xi^-)] \leq \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\mathcal{E}^{t+1}(\xi^-)].$$

Again, as this is true for all  $\epsilon > 0$  and all  $Q \in \mathcal{Q}$  we obtain that  $\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\mathcal{E}^t(\xi^-)] \leq \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\mathcal{E}^{t+1}(\xi^-)] < \infty$ .

Let  $0 \leq t \leq T - 1$  and  $\omega \in \Omega_\xi^t$ . Then for all  $Q \in \mathcal{Q}_t(\omega)$ ,  $\mathbb{E}_Q[\mathcal{E}^{t+1}(\xi^-)(\omega, \cdot)] < \infty$ , which implies that  $\mathcal{E}^{t+1}(\xi^-)(\omega, \cdot) < \infty$   $Q$ -a.s. and thus  $\mathcal{E}^{t+1}(\xi^-)(\omega, \cdot) < \infty$   $\mathcal{P}_t(\omega)$ -q.s. Assume for a moment that we have proved  $\mathcal{E}^{t+1}(\xi) \geq -\mathcal{E}^{t+1}(\xi^-)$ . Then  $-\mathcal{E}^{t+1}(\xi)(\omega, \cdot) < \infty$   $\mathcal{P}_t(\omega)$ -q.s. and  $\omega \in \hat{\Omega}_\xi^t$ . Thus  $\Omega_\xi^t \subseteq \hat{\Omega}_\xi^t$  and  $\hat{\Omega}_\xi^t$  is a  $\mathcal{P}$ -full measure set.

Let  $0 \leq t \leq T - 1$ . We now prove that  $\mathcal{E}^{t+1}(\xi) \geq -\mathcal{E}^{t+1}(\xi^-)$  by backward induction. The claim is clearly true for  $t = T - 1$ . Assume that it is true for some  $1 \leq t + 1 \leq T$ . Then for  $\omega \in \Omega^t$  we find

$$\begin{aligned} \mathcal{E}^t(\xi)(\omega) &= \sup_{Q \in \mathcal{Q}_t(\omega)} \mathbb{E}_Q[\mathcal{E}^{t+1}(\xi)(\omega, \cdot)] \geq \sup_{Q \in \mathcal{Q}_t(\omega)} \mathbb{E}_Q[-\mathcal{E}^{t+1}(\xi^-)(\omega, \cdot)] \\ &\geq \inf_{Q \in \mathcal{Q}_t(\omega)} \mathbb{E}_Q[-\mathcal{E}^{t+1}(\xi^-)(\omega, \cdot)] = -\mathcal{E}^t(\xi^-)(\omega). \end{aligned}$$

This concludes the proof.  $\square$

*Remark B.2* Recall the set  $\Omega_{\text{NA}}^t = \{\omega \in \Omega^t \mid \text{NA}(\mathcal{P}_t(\omega)) \text{ holds}\}$ , which is universally measurable and of  $\mathcal{P}$ -full measure (see [Bouchard and Nutz, 2015, Lemma 4.6, p.842]). Let  $\omega \in \Omega_{\text{NA}}^t$ . From [Bouchard and Nutz, 2015, Lemma 4.1], we know that  $\mathcal{E}^t(\xi)(\omega) = -\infty$  implies that  $\{\mathcal{E}^{t+1}(\xi)(\omega, \cdot) = -\infty\}$  is not  $\mathcal{P}_t(\omega)$ -polar i.e.  $\omega \notin \hat{\Omega}_\xi^t$ . Thus

$$\Omega_\xi^t \cap \Omega_{\text{NA}}^t \subseteq \hat{\Omega}_\xi^t \cap \Omega_{\text{NA}}^t \subseteq \{\omega \in \Omega_{\text{NA}}^t \mid \mathcal{E}^t(\xi)(\omega) > -\infty\}.$$

**Lemma B.3** *If  $\xi : \Omega^T \rightarrow \mathbb{R}$  is upper semianalytic, then  $\pi_t(\xi)$  is upper semianalytic for all  $0 \leq t \leq T - 1$ .*

*Proof.* We proceed by induction. As  $\pi_T(\xi) = \xi$  the claim is true for  $t = T$ . Assume now the  $\pi_{t+1}(\xi)$  is upper semianalytic for some  $t \in \{0, \dots, T - 1\}$ . We show that the claim is true for  $t$ . Indeed for all  $a \in \mathbb{R}$

$$\begin{aligned} &\{\omega \in \Omega^t \mid \pi_t(\xi) < a\} \\ &= \{\omega \in \Omega^t \mid \exists H \in \mathbb{R}^d, \epsilon > 0 \text{ s. t. } \forall P \in \mathcal{P}_t(\omega) P(a - \epsilon + H\Delta S_{t+1}(\omega, \cdot) \geq \pi_{t+1}(\xi)(\omega, \cdot)) = 1\} \\ &= \{\omega \in \Omega^t \mid \sup_{\epsilon \in \mathbb{Q}_+} \sup_{H \in \mathbb{Q}^d} \inf_{P \in \mathcal{P}_t(\omega)} P(a - \epsilon + H\Delta S_{t+1}(\omega, \cdot) \geq \pi_{t+1}(\xi)(\omega, \cdot)) \geq 1\} \end{aligned}$$

As the function  $(\omega, P, H, \epsilon) \mapsto \mathbb{E}_P[\mathbb{1}_{\{a - \epsilon + H\Delta S_{t+1}(\omega, \cdot) \geq \pi_{t+1}(\xi)(\omega, \cdot)\}}]$  is lower semianalytic, the same holds true for  $\omega \mapsto \sup_{\epsilon \in \mathbb{Q}_+} \sup_{H \in \mathbb{Q}^d} \inf_{P \in \mathcal{P}_t(\omega)} \mathbb{E}_P[\mathbb{1}_{\{a - \epsilon + H\Delta S_{t+1}(\omega, \cdot) \geq \pi_{t+1}(\xi)(\omega, \cdot)\}}]$  (see [Bertsekas and Shreve, 2004, Lemma 7.30, p.177, Prop. 7.47, p.180]), thus the set above is coanalytic. To complete the proof, we argue why

$$\begin{aligned} &\{\omega \in \Omega^t \mid \exists H \in \mathbb{R}^d, \epsilon > 0 \text{ such that } a - \epsilon + H\Delta S_{t+1}(\omega, \cdot) \geq \pi_{t+1}(\xi)(\omega, \cdot) \text{ } \mathcal{P}_t(\omega)\text{-q.s.}\} \\ &\subseteq \{\omega \in \Omega^t \mid \exists H \in \mathbb{Q}^d, \epsilon \in \mathbb{Q}_+ \text{ such that } a - \epsilon + H\Delta S_{t+1}(\omega, \cdot) \geq \pi_{t+1}(\xi)(\omega, \cdot) \text{ } \mathcal{P}_t(\omega)\text{-q.s.}\} : \end{aligned}$$

Fix  $\omega \in \Omega^t$ ,  $\tilde{H} \in \mathbb{R}^d$ ,  $\epsilon > 0$  such that  $a - \epsilon + \tilde{H}\Delta S_{t+1}(\omega, \cdot) \geq \pi_{t+1}(\omega, \cdot)$   $\mathcal{P}_t(\omega)$ -q.s. Take  $\tilde{\epsilon} \in \mathbb{Q}_+$  such that  $0 < \tilde{\epsilon} < \epsilon/2$  and  $H \in [0, \infty)^d$  such that

$$H^1 + \dots + H^d \leq \frac{\epsilon/2}{\max_{1 \leq i \leq d} S_t^i(\omega)}.$$

It follows that for  $\mathcal{P}_t(\omega)$ -q.e.  $\omega' \in \Omega$

$$\begin{aligned} a - \tilde{\epsilon} + (H + \tilde{H})\Delta S_{t+1}(\omega, \omega') &\geq a - \epsilon/2 + \tilde{H}\Delta S_{t+1}(\omega, \omega') + H\Delta S_{t+1}(\omega, \omega') \\ &\geq \pi_{t+1}(\xi)(\omega, \omega') + \epsilon/2 - HS_t(\omega) \\ &\geq \pi_{t+1}(\xi)(\omega, \omega'). \end{aligned}$$

In particular the above inequality is valid for some  $H$  such that  $\tilde{H} + H \in \mathbb{Q}^d$ .  $\square$

*Proof of theorem 3.1.*

Let

$$\Omega_{\text{NA}, \xi} := \{\omega \in \Omega^T \mid \omega \in \Omega_{\text{NA}}^t \cap \Omega_\xi^t \text{ for all } 0 \leq t \leq T-1\},$$

where the definition of  $\Omega_\xi^t$  is given in lemma B.1 and the definition of  $\Omega_{\text{NA}}^t$  in remark B.2. Then by lemma B.1 and [Bouchard and Nutz, 2015, Lemma 4.6, p. 842]  $\Omega_{\text{NA}, \xi}$  is universally measurable and of  $\mathcal{P}$ -full measure. Let  $\omega \in \Omega_{\text{NA}, \xi}$ . By [Bouchard and Nutz, 2015, Lemma 4.10], there exists a universally measurable function  $\hat{H}_{t+1}$  such that

$$\mathcal{E}^t(\xi)(\omega) + \hat{H}_{t+1}(\omega)\Delta S_{t+1}(\omega, \cdot) \geq \mathcal{E}^{t+1}(\xi)(\omega, \cdot) \quad \mathcal{P}_t(\omega)\text{-q.s.} \quad (20)$$

To see that

$$\pi_t(\xi) = \mathcal{E}^t(\xi) \quad \mathcal{P}\text{-q.s.} \quad (21)$$

for  $0 \leq t \leq T$  we argue by backwards induction. Indeed the claim is true by definition for  $t = T$ . Now we assume that the claim is true for  $t+1 \in \{1, \dots, T\}$ . By [Bouchard and Nutz, 2015, eq. (4.8) in Lemma 4.8, p.843] the correspondence

$$\mathcal{H}_t(\omega) = \{(Q, P) \in \mathfrak{P}(\Omega) \times \mathfrak{P}(\Omega) \mid \mathbb{E}_Q[\Delta S_{t+1}(\omega, \cdot)] = 0, P \in \mathcal{P}_t(\omega), Q \ll P\}$$

has analytic graph. By [Bertsekas and Shreve, 2004, Prop. 7.47, p. 179, Prop. 7.48, p. 180, Prop. 7.50, p.184]  $(\omega, Q, P) \mapsto \mathbb{E}_Q[\mathcal{E}_{t+1}(\xi)(\omega, \cdot)]$  and  $(\omega, Q, P) \mapsto \mathbb{E}_Q[\pi_{t+1}(\xi)(\omega, \cdot)]$  are upper semianalytic functions and there exists sequences  $(\hat{P}_n, \hat{Q}_n)_{n \in \mathbb{N}}$  and  $(\bar{P}_n, \bar{Q}_n)_{n \in \mathbb{N}}$  of  $\mathcal{F}_t^{\mathcal{U}}$ -measurable selectors of  $\mathcal{H}_t$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}_{\hat{Q}_n(\omega)}[\mathcal{E}^{t+1}(\xi)(\omega, \cdot)] &= \sup_{(Q, P) \in \mathcal{H}_t(\omega)} \mathbb{E}_Q[\mathcal{E}^{t+1}(\xi)(\omega, \cdot)] = \mathcal{E}^t(\xi)(\omega), \\ \lim_{n \rightarrow \infty} \mathbb{E}_{\bar{Q}_n(\omega)}[\pi_{t+1}(\xi)(\omega, \cdot)] &= \sup_{(Q, P) \in \mathcal{H}_t(\omega)} \mathbb{E}_Q[\pi_{t+1}(\xi)(\omega, \cdot)] = \mathcal{E}_t(\pi_{t+1}(\xi))(\omega). \end{aligned}$$

Define  $P_n(\omega) = (\hat{P}_n(\omega) + \bar{P}_n(\omega))/2 \in \mathcal{P}_t(\omega)$  and  $\tilde{P}_t(\omega) = \sum_{n=1}^{\infty} 2^{-n} P_n(\omega)$ . Then  $\tilde{P}_t(\omega) \in \mathfrak{P}(\Omega)$  for all  $\omega \in \Omega^t$ ,  $\omega \mapsto \tilde{P}_t(\omega)$  is  $\mathcal{F}_t^{\mathcal{U}}$ -measurable and  $\hat{P}_n(\omega), \bar{P}_n(\omega), P_n(\omega)$  are absolutely continuous with respect to  $\tilde{P}_t(\omega)$ . Furthermore for  $\omega \in \Omega_{\text{NA}}^t$

$$\begin{aligned} \mathbb{E}_{\hat{Q}_n(\omega)}[\mathcal{E}^{t+1}(\xi)(\omega, \cdot)] &\leq \sup_{Q \ll \tilde{P}_t(\omega), \mathbb{E}_Q[\Delta S_{t+1}(\omega, \cdot)] = 0} \mathbb{E}_Q[\mathcal{E}^{t+1}(\xi)(\omega, \cdot)] \\ &\leq \inf\{x \in \mathbb{R} \mid \exists H \in \mathbb{R}^d \text{ such that } x + H\Delta S_{t+1}(\omega, \cdot) \geq \mathcal{E}^{t+1}(\xi)(\omega, \cdot) \tilde{P}_t(\omega)\text{-a.s.}\} \\ &\leq \pi_t(\mathcal{E}^{t+1}(\xi))(\omega) = \mathcal{E}_t(\mathcal{E}^{t+1}(\xi))(\omega) = \mathcal{E}^t(\xi)(\omega), \end{aligned}$$

where the third inequality follows from the fact that  $P_n(\omega) \in \mathcal{P}_t(\omega)$  for  $n \in \mathbb{N}$  and the first equality follows from [Bouchard and Nutz, 2015, Theorem 3.4] as  $\omega \in \Omega_{\text{NA}}^t$ . Letting  $n \rightarrow \infty$  we conclude

$$\begin{aligned} \sup_{Q \ll \tilde{P}_t(\omega), \mathbb{E}_Q[\Delta S_{t+1}(\omega, \cdot)] = 0} \mathbb{E}_Q[\mathcal{E}^{t+1}(\xi)(\omega, \cdot)] &= \mathcal{E}^t(\xi)(\omega), \\ \sup_{Q \ll \tilde{P}_t(\omega), \mathbb{E}_Q[\Delta S_{t+1}(\omega, \cdot)] = 0} \mathbb{E}_Q[\pi_{t+1}(\xi)(\omega, \cdot)] &= \mathcal{E}_t(\pi_{t+1}(\xi)(\omega, \cdot)), \end{aligned}$$

Fix now  $P \in \mathcal{P}$  and define  $\tilde{P} = P|_{\mathcal{F}_t^{\mathcal{U}}} \otimes \tilde{P}_t$ . Then as  $P_n(\omega) \in \mathcal{P}_t(\omega)$  the induction assumption implies that  $\mathcal{E}^{t+1}(\xi) = \pi_{t+1}(\xi)$  holds  $\tilde{P}$ -a.s. and thus for  $\tilde{P}$ -a.e.  $\omega \in \Omega^t$  we have

$$\begin{aligned} \mathcal{E}^t(\xi)(\omega) &= \sup_{Q \ll \tilde{P}_t(\omega), \mathbb{E}_Q[\Delta S_{t+1}(\omega, \cdot)] = 0} \mathbb{E}_Q[\mathcal{E}^{t+1}(\xi)(\omega, \cdot)] \\ &= \sup_{Q \ll \tilde{P}_t(\omega), \mathbb{E}_Q[\Delta S_{t+1}(\omega, \cdot)] = 0} \mathbb{E}_Q[\pi_{t+1}(\xi)(\omega, \cdot)] \\ &= \mathcal{E}_t(\pi_{t+1}(\xi))(\omega) = \pi_t(\xi)(\omega), \end{aligned}$$

where the last equality again follows from [Bouchard and Nutz, 2015, Theorem 3.4] if  $\omega \in \Omega_{\text{NA}}^t$ . This concludes the proof of (21).

Let  $(x, H, C) \in \mathcal{A}(\xi)$ . Now we show that

$$V_t^{x, H, C} \geq \pi_t(\xi) \quad \mathcal{P}\text{-q.s.} \quad (22)$$

This is clearly true at  $t = T$ . Fix some  $1 \leq t \leq T$  and assume that (22) holds true for  $t$ . Then

$$V_{t-1}^{x, H, C} + H_t \Delta S_t \geq V_t^{x, H, C} \geq \pi_t(\xi) \quad \mathcal{P}\text{-q.s.}$$

Noting that  $V_{t-1}^{x, H, C}$  is  $\mathcal{F}_{t-1}^{\mathcal{U}}$ -measurable and  $\pi_t(\xi)$  is upper semianalytic and using the same reasoning as in [Bouchard and Nutz, 2015, proof of Lemma 4.10, pp.846-848] we conclude that for  $\omega \in \Omega^{t-1}$  in a  $\mathcal{P}$  full-measure set

$$V_{t-1}^{x, H, C}(\omega) + H_t(\omega) \Delta S_t(\omega, \cdot) \geq \pi_t(\xi)(\omega, \cdot) \quad \mathcal{P}_{t-1}(\omega)\text{-q.s.} \quad (23)$$

Thus  $V_{t-1}^{x, H, C}(\omega) \geq \pi_{t-1}(\xi)(\omega)$  by (8) and (22) is proved for  $t - 1$ . Next we define the consumption process  $\hat{C}$ . Let  $P = P_0 \otimes P_1 \otimes \dots \otimes P_{T-1} \in \mathcal{P}$ , where  $P_t \in \mathcal{P}_t(\omega)$  for all  $0 \leq t \leq T - 1$ . Then using eq. (20) and Fubini's Theorem (recall [Bertsekas and Shreve, 2004, Proposition 7.45 p175]), we get that

$$\mathcal{E}^{t-1}(\xi) + \hat{H}_t \Delta S_t \geq \mathcal{E}^t(\xi) \quad \mathcal{P}\text{-q.s.} \quad (24)$$

for a universally measurable function  $\hat{H}_t : \Omega^t \rightarrow \mathbb{R}^d$ . Using (24) recursively,

$$\mathcal{E}^0(\xi) + \sum_{u=1}^t \hat{H}_u \Delta S_u \geq \mathcal{E}^t(\xi) \quad \mathcal{P}\text{-q.s.} \quad (25)$$

follows. Now we set  $\hat{C}_t = \mathcal{E}^0(\xi) + \sum_{u=1}^t \hat{H}_u \Delta S_u - \mathcal{E}^t(\xi)$ . Then  $\hat{C}_t(\omega, \cdot) - \hat{C}_{t-1}(\omega) = \mathcal{E}^{t-1}(\xi)(\omega) - \mathcal{E}^t(\xi)(\omega, \cdot) + \hat{H}_t(\omega) \Delta S_t(\omega, \cdot) \geq 0$   $\mathcal{P}_{t-1}(\omega)$ -q.s. and using again Fubini's Theorem  $\hat{C}_t - \hat{C}_{t-1} \geq 0$   $\mathcal{P}$ -q.s. Thus  $\hat{C} = (\hat{C}_t)_{0 \leq t \leq T}$  is a cumulative consumption process.

Now we prove that  $\pi(\xi) = \pi_0(\xi)$ . Let  $(x, H) \in \mathcal{A}(\xi)$ . Then as  $V_{T-1}^{x,H} + H_T \Delta S_T \geq \xi$   $\mathcal{P}_{T-1}$ -q.s. it follows as in (23)

$$V_{T-1}^{x,H}(\omega) + H_T(\omega) \Delta S_T(\omega, \cdot) \geq \xi(\omega, \cdot) \quad \mathcal{P}_{T-1}(\omega)\text{-q.s.}$$

for all  $\omega \in \Omega^{T-1}$  in an  $\mathcal{F}_{T-1}^{\mathcal{U}}$ -measurable and  $\mathcal{P}$ -full measure set. From (8), we conclude that  $\pi_{T-1}(\xi)(\omega) \leq V_{T-1}^{x,H}(\omega)$ . By induction we see that  $\pi_0(\xi) \leq x$  and thus  $\pi_0(\xi) \leq \pi(\xi)$ . Conversely, using (25) and (21)

$$V_T^{\pi_0, \hat{H}} = \pi_0(\xi) + \sum_{t=1}^T \hat{H}_t \Delta S_t \geq \mathcal{E}^T(\xi) = \xi \quad \mathcal{P}\text{-q.s.}$$

and therefore  $\pi_0(\xi) \geq \pi(\xi)$ . Thus  $\mathcal{E}^0(\xi) = \pi_0(\xi) = \pi(\xi)$  by (21) and we obtain (recall (25) and the definition of  $\hat{C}$ ) that

$$V_t^{\pi(\xi), \hat{H}, \hat{C}} = \mathcal{E}^t(\xi) = \pi_t(\xi) \quad \mathcal{P}\text{-q.s.}$$

Since  $V_T^{\pi(\xi), \hat{H}, \hat{C}} = \mathcal{E}^T(\xi) = \xi$   $\mathcal{P}$ -q.s.,  $(\pi(\xi), \hat{H}, \hat{C})$  is a superhedging strategy and it is also minimal. Indeed let  $(x, H, C) \in \mathcal{A}(\xi)$  then  $V_T^{x,H,C} \geq \xi$   $\mathcal{P}$ -q.s. From (22),  $V_t^{x,H,C} \geq \pi_t(\xi) = V_t^{\pi(\xi), \hat{H}, \hat{C}}$   $\mathcal{P}$ -q.s. This concludes the proof.  $\square$

## Appendix C Proofs of theorem 4.3 and theorem 4.5

### C.1 Proof of theorem 4.3: The one-period case

We now prove theorem 4.3 in the case  $T = 1$ , where we follow arguments given in Nutz [2016]. Let  $\xi : \Omega^T \rightarrow \mathbb{R}$  be Borel. In preparation for the multi-period case we define the set

$$\mathcal{A}_{0,x} = \{(H, c) \in \mathbb{R}^d \times \mathbb{R}_+ \mid x - c + H\Delta S_1 \geq \pi_1(\xi) \text{ } \mathcal{P}\text{-q.s.}\}.$$

Recall definition  $\pi_t(\xi)$  given in (8) for  $t = 0, 1$  and note that if  $(H, c) \in \mathcal{A}_{0,x}$  then also  $(H, 0) \in \mathcal{A}_{0,x}$ . We thus often write  $H \in \mathcal{A}_{0,x}$  instead of  $(H, c) \in \mathcal{A}_{0,x}$ . Let  $U(1, \cdot, \cdot) : \Omega \times [0, \infty) \rightarrow \mathbb{R}$  be bounded from above and  $\mathcal{F}_1^{\mathcal{U}}$ -measurable. Besides let us assume that  $x \mapsto U(1, \omega, x)$  is non-decreasing, concave and continuous for each  $\omega \in \Omega$ . Furthermore let the deterministic function  $U(0, \cdot) : [0, \infty) \rightarrow \mathbb{R}$  be non-decreasing and continuous. As usual we set  $U(t, \omega, x) = -\infty$  for  $x < 0$  and  $t = 0, 1$ . Let us now state the main theorem for  $T = 1$ :

**Proposition C.1** *Let  $\text{NA}(\mathcal{P})$  hold and  $x \geq \pi_0(\xi)$ . Then*

$$u(x) := \sup_{(H,c) \in \mathcal{A}_{0,x}} \left( \inf_{P \in \mathcal{P}^u} \mathbb{E}_P[U(1, x - c + H\Delta S_1 - \pi_1(\xi))] + U(0, c) \right) < \infty$$

*and there exists  $(\hat{H}, \hat{c}) \in \mathcal{A}_{0,x}$  such that  $\inf_{P \in \mathcal{P}^u} \mathbb{E}_P[U(1, x - \hat{c} + \hat{H}\Delta S_1 - \pi_1(\xi))] + U(0, \hat{c}) = u(x)$ .*

We prove the result via a lemma. Here we denote

$$L = \text{span}(\{\text{supp}(P \circ (\Delta S_1)^{-1}) \mid P \in \mathcal{P}\}) \subseteq \mathbb{R}^d$$

and the orthogonal complement

$$L^\perp = \{H \in \mathbb{R}^d \mid HV = 0 \text{ for all } V \in L\}.$$

**Lemma C.2** *Assume  $x \geq \pi_0(\xi)$ . Under  $\text{NA}(\mathcal{P})$  the set  $K_x = \mathcal{A}_{0,x} \cap (L \times \mathbb{R}_+) \subseteq \mathbb{R}^{d+1}$  is non-empty, convex and compact.*

*Proof.* Clearly  $K_x$  is convex and closed. It remains to show that  $K_x$  is bounded: As by definition of  $\pi_0(\xi)$  clearly  $c \in [0, x - \pi_0(\xi)]$  for all  $c \in \mathcal{A}_{0,x}$  we only need to show that  $H \in K_x$  is bounded. Note that after a translation by  $(H_0, 0) \in K_x$  we have  $0 \in \tilde{K}_x := K_x - (H_0, 0)$ . Now we assume towards a contradiction that there exist  $H_n \in \tilde{K}_x$  such that  $|H_n| \rightarrow \infty$ . We define  $\delta = |H_0| + 1$ . We can extract a subsequence  $\delta H_n / |H_n|$  that converges to a limit  $H \in \mathbb{R}^d$ , so  $|H| = \delta$ . As  $\tilde{K}_x$  is convex and contains the origin we have for  $n$  large enough  $\delta H_n / |H_n| \in \tilde{K}_x$ . It follows  $H \in \tilde{K}_x$ , since  $\tilde{K}_x$  is closed. Furthermore

$$H\Delta S_1 \geq \liminf_{n \rightarrow \infty} \frac{\pi_1(\xi) - x - H_0\Delta S_1}{|H_n|/\delta} = 0 \quad \mathcal{P}\text{-q.s.}$$

By  $\text{NA}(\mathcal{P})$  this implies  $H\Delta S_1 = 0$   $\mathcal{P}$ -q.s. and thus  $H \in L^\perp$  by use of [Nutz, 2016, Lemma 2.6]. As  $H \in \tilde{K}_x$  this implies  $H_0 + H \in K_x \subseteq L$ , which means  $|H|^2 = -H_0H$ . This contradicts  $|H| = \delta$  by Cauchy-Schwarz inequality.  $\square$

*Proof of proposition C.1.* Fatou's lemma implies that for all  $P \in \mathcal{P}^u$  the function  $(H, c) \mapsto \mathbb{E}_P[U(1, x - c + H\Delta S_1 - \pi_1(\xi))] + U(0, c)$  is upper semicontinuous on  $\mathcal{A}_{0,x}$ . It follows that  $(H, c) \mapsto \inf_{P \in \mathcal{P}^u} \mathbb{E}_P[U(1, x - c + H\Delta S_1 - \pi_1(\xi))] + U(0, c)$  is upper semicontinuous and thus attains its supremum on the compact set  $K_x$ . Finally again using [Nutz, 2016, Lemma 2.6] and recalling that  $\mathcal{P}^u \subseteq \mathcal{P}$

$$\begin{aligned} & \sup_{(H,c) \in \mathcal{A}_{0,x}} \left( \inf_{P \in \mathcal{P}^u} \mathbb{E}_P[U(1, x - c + H\Delta S_1 - \pi_1(\xi))] + U(0, c) \right) \\ &= \sup_{(H,c) \in K_x} \left( \inf_{P \in \mathcal{P}^u} \mathbb{E}_P[U(1, x - c + H\Delta S_1 - \pi_1(\xi))] + U(0, c) \right). \end{aligned}$$

□

**Corollary C.3** *Under the conditions of proposition C.1 we have*

$$\begin{aligned} & \sup_{(H,c) \in \mathcal{A}_{0,x}} \left( \inf_{P \in \mathcal{P}^u} \mathbb{E}_P[U(1, x - c + H\Delta S_1 - \pi_1(\xi))] + U(0, c) \right) \\ &= \inf_{P \in \mathcal{P}^u} \left( \sup_{(H,c) \in \mathcal{A}_{0,x}} (\mathbb{E}_P[U(1, x - c + H\Delta S_1 - \pi_1(\xi))] + U(0, c)) \right). \end{aligned}$$

*Proof.* Note that  $K_x$  is compact, convex and  $\mathcal{P}^u$  is convex. Define

$$f : K_x \times \mathfrak{P}(\Omega) \rightarrow \mathbb{R} \quad (H, c, P) \mapsto \mathbb{E}_P[U(1, x - c + H\Delta S_1 - \pi_1(\xi))] + U(0, c)$$

and note that  $(H, c) \mapsto f(H, c, P)$  is upper semicontinuous and concave. Furthermore  $P \mapsto f(H, c, P)$  is convex on  $\mathcal{P}^u$ . The claim follows from Corollary 2 in Terkelsen [1973]. □

*Remark C.4* The boundedness from above of  $U(1, \cdot, \cdot)$  can be replaced by a weaker condition: Indeed it is sufficient to assume there exists a constant  $a > 0$  such that  $\omega \mapsto U(1, \omega, a/2)$  is bounded from below and

$$\mathbb{E}_P[U^+(1, x + H\Delta S_1 - \pi_1(\xi))] < \infty \quad \text{for all } H \in \mathcal{A}_{0,x} \text{ and } P \in \mathcal{P}^u$$

as well as

$$\mathbb{E}_P[U^+(1, a)] < \infty \quad \text{for all } P \in \mathcal{P}^u.$$

The proof of proposition C.1 then follows along the lines of [Rásonyi and Stettner, 2006, Lemma 1] and [Nutz, 2016, Lemma 2.8] after a translation by  $H_0 \in \text{ri}(K_x)$ .

## C.2 Proof of theorem 4.3: The multi-period case

For the rest of this section we assume  $\text{NA}(\mathcal{P})$  and that  $\xi$  is Borel measurable. Furthermore we often abbreviate  $\pi_t(\xi)$  by  $\pi_t$ . To simplify notation we assume  $U(0, \cdot, 0) = 0$ . We give the following definition:

**Definition C.5** We define  $U_T(\omega, x) = U(T, \omega, x)$  and for  $0 \leq t \leq T - 1$

$$\begin{aligned} U_t(\omega, x) := & \sup_{(H,c) \in \mathcal{A}_{t,x}(\omega)} \left( \inf_{P \in \mathcal{P}_t^u(\omega)} \mathbb{E}_P[U_{t+1}((\omega, \cdot), x + H\Delta S_{t+1}(\omega, \cdot) - c - \mathbb{1}_{\{t=T-1\}}\xi(\omega, \cdot))] \right. \\ & \left. + U(t, \omega, c) \right), \quad x \geq \pi_t(\omega) \end{aligned}$$

and  $U_t(\omega, x) = -\infty$  otherwise, where for  $x \in \mathbb{R}$  we set

$$\mathcal{A}_{0,x}(\omega) := \{(H, c) \in \mathbb{R}^d \times \{0\} \mid x + H\Delta S_1(\omega, \cdot) \geq \pi_1(\omega, \cdot) \text{ } \mathcal{P}_0(\omega)\text{-q.s.}\}$$

$$\mathcal{A}_{t,x}(\omega) := \{(H, c) \in \mathbb{R}^d \times \mathbb{R}_+ \mid x + H\Delta S_{t+1}(\omega, \cdot) - c \geq \pi_{t+1}(\omega, \cdot) \text{ } \mathcal{P}_t(\omega)\text{-q.s.}\}, \quad t \geq 1.$$

We recall from lemma B.3 that  $\pi_t(\xi)$  is upper semianalytic. This means in particular that

$$\{(\omega, x) \mid x < \pi_t(\xi)(\omega)\} = \bigcup_{q \in \mathbb{Q}} \pi_t^{-1}((q, \infty)) \times (-\infty, q)$$

is analytic. Next we show by backwards induction, that if assumption 4.2 is satisfied, then  $U_t$  has  $\mathcal{P}^u$ -q.s. the following properties:

**Condition C.6** Let  $0 \leq t \leq T - 1$ . The function  $U_t : \Omega^t \times \mathbb{R} \rightarrow [-\infty, \infty)$  is lower semianalytic and bounded from above. Furthermore the following properties hold:

1.  $\omega \mapsto U_t(\omega, x(\omega))$  is bounded from below for  $x(\omega) := \pi_t(\omega) + \epsilon$  and each  $\epsilon > 0$ .
2.  $x \mapsto U_t(\omega, x)$  is non-decreasing, concave and continuous on  $[\pi_t(\omega), \infty)$  for each  $\omega \in \Omega^t$ .

**Lemma C.7** Let  $NA(\mathcal{P})$  and assumption 2.1, assumption 4.1 and assumption 4.2 hold for  $U(t, \cdot, \cdot)$ ,  $0 \leq t \leq T$ . Then there exist functions  $\tilde{U}_t : \Omega^t \times (-\infty, \infty) \rightarrow [-\infty, \infty)$ , which satisfy condition C.6, such that  $\tilde{U}_t = U_t$   $\mathcal{P}^u$ -q.s.

*Proof.* We prove the claim by induction. Recall that  $U_T$  satisfies assumption 4.2. We now show the induction step from  $t + 1$  to  $t$  and therefore first fix  $\omega \in \Omega^t$ . For simplicity of presentation we assume  $t \leq T - 2$ .

We first state some results regarding lower semianalyticity, which lead to the definition of  $\tilde{U}_t$ : Using [Bertsekas and Shreve, 2004, Lemma 7.30, p.177, Prop. 7.47, p.179, Prop. 7.48, p.180], assumption 4.2 and the analytic graph of  $\mathcal{P}_t^u$  we see that  $\phi : \Omega^t \times (-\infty, \infty) \times \mathbb{R}^d \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$

$$\phi(\omega, x, H, c) = \inf_{P \in \mathcal{P}_t^u(\omega)} \mathbb{E}_P[U_{t+1}((\omega, \cdot), x + H\Delta S_{t+1}(\omega, \cdot) - c)] + U(t, \omega, c)$$

is lower semianalytic as  $\Delta S_{t+1}(\omega, \cdot)$  is a Borel measurable functions (and also  $\xi(\omega, \cdot)$  for  $t = T - 1$ ). Now we define the function  $\tilde{\phi} : \Omega^t \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$

$$\tilde{\phi}(\omega, x, H, c) = \begin{cases} -\infty & \text{if } (H, c) \notin \mathcal{A}_{t,x} \text{ or } x < \pi_t(\xi)(\omega) \\ \phi(\omega, x, H, c) & \text{otherwise.} \end{cases}$$

We show that  $\tilde{\phi}$  is lower semianalytic. Fix  $a \in \mathbb{R}$ . Then

$$\begin{aligned} \{\tilde{\phi} < a\} &= \{(\omega, x, H, c) \mid \phi(\omega, x, H, c) < a, (H, c) \in \mathcal{A}_{t,x}(\omega), x \geq \pi_t(\xi)(\omega)\} \\ &\quad \cup \{(\omega, x, H, c) \mid (H, c) \notin \mathcal{A}_{t,x}(\omega) \text{ or } x < \pi_t(\xi)(\omega)\} \\ &= \{\phi < a\} \cup \{(\omega, x, H, c) \mid (H, c) \notin \mathcal{A}_{t,x}(\omega)\} \\ &\quad \cup \{(\omega, x, H, c) \mid x < \pi_t(\xi)(\omega)\}. \end{aligned}$$

By the same arguments as for the lower seminanalyticity of  $\phi$  we see that

$$\begin{aligned} &\{(\omega, x, H, c) \mid (H, c) \notin \mathcal{A}_{t,x}(\omega)\} \\ &= \left\{ (\omega, x, H, c) \mid \sup_{P \in \mathcal{P}_t^u(\omega)} \mathbb{E}_P[x + H\Delta S_{t+1}(\omega, \cdot) - c - \pi_{t+1}(\omega, \cdot)]^- > 0 \right\} \end{aligned}$$



is analytic and the sets

$$\{\phi < a\} \quad \text{and} \quad \{(\omega, x, H, c) \in \Omega^t \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \mid x < \pi_t(\xi)(\omega)\}$$

are analytic, so  $\tilde{\phi}$  is lower semianalytic. Similarly to [Blanchard and Carassus, 2017, Proposition 3.27] we define

$$\tilde{U}_t(\omega, x) = \lim_{n \rightarrow \infty} \sup_{(H, c) \in \mathbb{Q}^d \times \mathbb{Q}_+} \tilde{\phi}\left(\omega, x + \frac{1}{n}, H, c\right).$$

As the limits and countable supremum of lower semianalytic functions is lower semianalytic, we conclude that  $\tilde{U}_t$  is lower semianalytic.

From the definition it is clear that  $\tilde{U}_t(\omega, \cdot)$  is non-decreasing and bounded from above. Next we argue that  $\tilde{U}_t(\omega, \cdot)$  is concave. As the infimum of concave functions is concave, it is enough to argue that  $x \mapsto \sup_{(H, c) \in \mathbb{Q}^d \times \mathbb{Q}_+} \tilde{\phi}(\omega, x, H, c)$  is concave. This follows very similarly to Rásonyi and Stettner [2006][proof of Prop. 2, p.5]: Indeed, it is enough to show midpoint-concavity of  $\sup_{(H, c) \in \mathbb{Q}^d \times \mathbb{Q}_+} \tilde{\phi}(\omega, \cdot, H, c)$ , which is immediate by use of triangle inequality. Concavity implies that  $\tilde{U}_t(\omega, \cdot)$  is continuous on  $(\pi_t(\omega), \infty)$ . By the definition of  $\tilde{U}_t$  concavity and continuity extend to  $[\pi_t(\omega), \infty)$ .

By definition we clearly have

$$\sup_{(H, c) \in \mathbb{Q}^d \times \mathbb{Q}_+} \tilde{\phi}(\omega, x, H, c) \leq \sup_{(H, c) \in \mathcal{A}_{t, x}(\omega)} \phi(\omega, x, H, c).$$

We now show equality of  $U_t(\omega, x)$  and  $\tilde{U}_t(\omega, x)$  for  $\mathcal{P}^u$ -q.e.  $\omega \in \Omega^t$ . Let us therefore fix  $x > \pi_t(\omega)$  and  $\omega \in \Omega_{\mathbb{N}\Delta}^t$ . Using [Bouchard and Nutz, 2015, Theorem 3.4] and  $\mathcal{P}_t^u(\omega) \subseteq \mathcal{P}_t(\omega)$  there exists  $\tilde{H} \in \mathbb{R}^d$  such that

$$\pi_t(\omega) + \tilde{H} \Delta S_{t+1}(\omega, \omega') \geq \pi_{t+1}(\omega, \omega') \quad \text{for } \mathcal{P}_t^u(\omega)\text{-q.e. } \omega' \in \Omega.$$

Take  $c < x - \pi_t(\omega)$  and  $H \in [0, \infty)^d$  such that

$$H^1 + \dots + H^d \leq \frac{x - \pi_t(\omega) - c}{\max_{1 \leq i \leq d} S_t^i(\omega)}.$$

It follows for  $\mathcal{P}_t^u(\omega)$ -q.e.  $\omega' \in \Omega$  that

$$\begin{aligned} x + (H + \tilde{H}) \Delta S_{t+1}(\omega, \omega') - c &= x - \pi_t(\omega) + H \Delta S_{t+1}(\omega, \omega') + \pi_t(\omega) + \tilde{H} \Delta S_{t+1}(\omega, \omega') - c \\ &\geq x - \pi_t(\omega) - H S_t(\omega) + \pi_{t+1}(\omega, \omega') - c \\ &\geq \pi_{t+1}(\omega, \omega'). \end{aligned}$$

Thus the affine hull of  $\mathcal{A}_{t, x}(\omega)$  is  $\mathbb{R}^{d+1}$  and consequently  $\text{Ri}(\mathcal{A}_{t, x}(\omega))$  is an open set in  $\mathbb{R}^{d+1}$ . This implies

$$\sup_{(H, c) \in \mathbb{Q}^d \times \mathbb{Q}_+} \tilde{\phi}(\omega, x, H, c) = \sup_{(H, c) \in \mathcal{A}_{t, x}(\omega)} \phi(\omega, x, H, c).$$

for  $x > \pi_t(\omega)$ . Equality in  $x = \pi_t(\omega)$  follows by right-continuity of  $U_t$  and  $\tilde{U}_t$ . Indeed, right-continuity of  $U_t(x, \omega)$  in  $x = \pi_t(\omega)$  follows by compactness of  $\mathcal{A}_{t, \pi_t(\omega)+1}(\omega) \cap \text{span}(\text{supp}(\{P \circ (\Delta S_{t+1}(\omega, \cdot))^{-1} \mid P \in \mathcal{P}_t(\omega)\}))$  and Fatou's Lemma.

Lastly we show boundedness of  $\tilde{U}_t$  from below: Let  $x(\omega) = \pi_t(\omega) + \epsilon$  for some  $\epsilon > 0$ .

By the above arguments there exists  $\hat{H} \in \mathbb{Q}^d$  such that  $\pi_t(\omega) + \epsilon/3 + \hat{H}\Delta S_{t+1}(\omega, \omega') \geq \pi_{t+1}(\omega, \omega')$   $\mathcal{P}_t^u(\omega)$ -a.s. Thus

$$\begin{aligned} U_t(\omega, x(\omega)) &\geq \inf_{P \in \mathcal{P}_t^u(\omega)} \mathbb{E}_P[U_{t+1}((\omega, \cdot), x(\omega) + \hat{H}\Delta S_{t+1}(\omega, \cdot) - \epsilon/3)] + U(t, \omega, \epsilon/3) \\ &\geq \inf_{P \in \mathcal{P}_t^u(\omega)} \mathbb{E}_P[U_{t+1}((\omega, \cdot), \pi_{t+1}(\omega, \cdot) + \epsilon/3)] + U(t, \omega, \epsilon/3) \end{aligned}$$

is bounded from below by the induction hypothesis and assumption 4.2. This shows the claim.

□

**Lemma C.8** *Let  $\text{NA}(\mathcal{P})$  and assumption 2.1, assumption 4.1 and assumption 4.2 hold for  $U(t, \cdot, \cdot)$ ,  $0 \leq t \leq T$ . Let  $t \in \{0, \dots, T-1\}$  and  $(H, C) \in \mathcal{A}_{\pi_0}$ . There exist universally measurable mappings  $\hat{H}_{t+1}, \hat{c}_t$  such that  $\hat{c}_t$  is non-negative,*

$$V_{t-1}^{\pi_0, H, C}(\omega) + H_t(\omega)\Delta S_t(\omega) + \hat{H}_{t+1}(\omega)\Delta S_{t+1}(\omega, \cdot) - \hat{c}_t(\omega) \geq \pi_{t+1}(\omega, \cdot) \quad \mathcal{P}_t(\omega)\text{-q.s.}$$

and

$$\begin{aligned} \inf_{P \in \mathcal{P}_t^u(\omega)} \mathbb{E}_P \left[ U_{t+1} \left( (\omega, \cdot), V_{t-1}^{\pi_0, H, C}(\omega) + H_t(\omega)\Delta S_t(\omega) + \hat{H}_{t+1}(\omega)\Delta S_{t+1}(\omega, \cdot) - \hat{c}_t(\omega) - \mathbf{1}_{\{t=T-1\}}\xi(\omega, \cdot) \right) \right] \\ + U(t, \omega, \hat{c}_t(\omega)) = U_t \left( \omega, V_{t-1}^{\pi_0, H, C}(\omega) + H_t(\omega)\Delta S_t(\omega) \right) \end{aligned}$$

for  $\mathcal{P}^u$ -a.e.  $\omega \in \Omega^t$ .

*Proof.* We show that  $\tilde{U}_t$  is  $\mathcal{F}_t^{\mathcal{U}} \otimes \mathcal{B}(\mathbb{R})$ -measurable: Indeed, we know that  $\omega \mapsto \tilde{U}_t(\omega, x)$  is lower semianalytic and in particular universally measurable. Also  $x \mapsto \tilde{U}_t(\omega, x)$  is continuous on  $[\pi_t(\omega), \infty)$ , bounded from above and  $\tilde{U}_t(\omega, x) = -\infty$  for  $x < \pi_t(\omega)$ . Thus it is concave and upper semicontinuous on  $\mathbb{R}$  and the claim follows from [Blanchard and Carassus, 2017, Lemma A.35, p. 1889]. Next we show that the function

$$\phi(\omega, x, H, c) = \inf_{P \in \mathcal{P}_t^u(\omega)} \mathbb{E}_P[\tilde{U}_{t+1}((\omega, \cdot), x + H\Delta S_{t+1}(\omega, \cdot) - c)] + U(t, \omega, c)$$

is  $\mathcal{F}_t^{\mathcal{U}} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$ -measurable: As we have argued in lemma C.7  $\omega \mapsto \phi(\omega, x, H, c)$  is lower semianalytic and in particular universally measurable. On the other hand,  $x \mapsto \tilde{U}_{t+1}(\omega, x)$  is upper semicontinuous and concave for any  $\omega \in \Omega^t$ . Since  $\tilde{U}_{t+1}$  is bounded from above, an application of Fatou's lemma yields that  $(x, H, c) \mapsto \phi(\omega, x, H, c)$  is upper semicontinuous and concave for each  $\omega \in \Omega^t$ . Again by [Blanchard and Carassus, 2017, Lemma A.35, page 1889] it follows that  $\phi$  is  $\mathcal{F}_t^{\mathcal{U}} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$ -measurable. Now we define the correspondence

$$\begin{aligned} \Phi(\omega) &:= \{(H', c') \in \mathbb{R}^d \times \mathbb{R}_+ \mid \phi(\omega, V_{t-1}^{\pi_0, H, C}(\omega) + H_t(\omega)\Delta S_t(\omega) - \mathbf{1}_{\{t=T-1\}}\xi(\omega, \cdot), H', c') \\ &= \tilde{U}_t(\omega, V_{t-1}^{\pi_0, H, C}(\omega) + H_t(\omega)\Delta S_t(\omega))\}, \quad \omega \in \Omega^t. \end{aligned}$$

Then its graph is in  $\mathcal{F}_t^{\mathcal{U}} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$ . Next we define the function

$$\Upsilon : \omega \mapsto \mathcal{A}_{t, V_{t-1}^{\pi_0, H, C}(\omega) + H_t(\omega)\Delta S_t(\omega)}(\omega).$$

By a slight variation of the arguments given in Bouchard and Nutz [2015][proof of Lemma 4.10, pp.846-848] the graph of  $\Upsilon$  is  $\mathcal{F}_t^{\mathcal{U}} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$ -measurable and thus  $\text{graph}(\Upsilon) \cap ((\Omega_{NA}^t \cap \Omega_\xi^t) \times \mathbb{R}^d \times \mathbb{R}) \in \mathcal{F}_t^{\mathcal{U}} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$ . Then also the graph of

$$\tilde{\Phi}(\omega) = \begin{cases} \mathcal{A}_{t, V_{t-1}^{\pi_0, H, C}(\omega) + H_t(\omega)\Delta S_t(\omega)}(\omega) \cap \Phi(\omega) & \omega \in \Omega_{NA}^t \cap \Omega_\xi^t \\ \emptyset & \text{otherwise} \end{cases}$$

is in  $\mathcal{F}_t^{\mathcal{U}} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$  and  $\tilde{\Phi}$  admits an  $\mathcal{F}_t^{\mathcal{U}}$ -measurable selector  $(\hat{H}_{t+1}, \hat{c}_t)$  on the universally measurable set  $\{\tilde{\Phi} \neq \emptyset\} \in \mathcal{F}_t^{\mathcal{U}}$  by the Neumann-Aumann theorem ([Sainte-Beuve, 1974, Cor.1, p.120]). We extend  $(\hat{H}_{t+1}, \hat{c}_t)$  by setting  $\hat{H}_{t+1} = \hat{c}_t = 0$  on  $\{\tilde{\Phi} = \emptyset\}$ . Moreover the one-period case given in proposition C.1 applied with  $x = V_{t-1}^{\pi_0, \hat{H}, \hat{C}}(\omega) + H_t(\omega)\Delta S_t(\omega)$ , lemma B.1, remark B.2 as well as existence of superhedging strategies as stated in [Bouchard and Nutz, 2015, Theorem 3.4] show that  $\tilde{\Phi}(\omega) \neq \emptyset$  for  $\mathcal{P}^u$ -q.e.  $\omega \in \Omega^t$ . This shows the claim as  $U_t = \tilde{U}_t$   $\mathcal{P}^u$ -q.s.  $\square$

*Proof of theorem 4.3.* Let  $(\hat{H}_1, 0)$  be an optimal strategy for

$$\inf_{P \in \mathcal{P}_0^u} \mathbb{E}_P(U_1[\pi_0 + H_1\Delta S_1])$$

as in lemma C.8. Proceeding recursively, we use lemma C.8 to define the strategy  $\omega \mapsto (\hat{H}_{t+1}, \hat{c}_t)(\omega)$  for

$$\begin{aligned} & \inf_{P \in \mathcal{P}_t^u(\omega)} \mathbb{E}_P[U_{t+1}((\omega, \cdot), V_{t-1}^{\pi_0, \hat{H}, \hat{C}}(\omega) + \hat{H}_t(\omega)\Delta S_t(\omega) + H_{t+1}(\omega)\Delta S_{t+1}(\omega, \cdot) - c_t(\omega) - \mathbb{1}_{\{t=T-1\}}\xi(\omega, \cdot))] \\ & + U(t, c_t(\omega)) \end{aligned}$$

where  $1 \leq t \leq T-1$  and define  $\hat{C}_t = \sum_{s=1}^t \hat{c}_s$  as well as  $\Delta \hat{C}_T = V_{T-1}^{\pi_0, \hat{H}, \hat{C}} + \hat{H}_T\Delta S_T - \xi$ . By construction we then have  $(\hat{H}, \hat{C}) \in \mathcal{A}_{\pi_0}$ . To establish that  $(\hat{H}, \hat{C})$  is optimal we first show that

$$\inf_{P \in \mathcal{P}^u} \mathbb{E}_P \left[ \sum_{s=1}^T U(s, \Delta \hat{C}_s) \right] \geq U_0(\pi_0). \quad (26)$$

Let  $0 \leq t \leq T-1$ . By definition of  $(\hat{H}, \hat{C})$  we have

$$\begin{aligned} & \inf_{P' \in \mathcal{P}_t^u(\omega)} \mathbb{E}_{P'}[U_{t+1}((\omega, \cdot), V_t^{\pi_0, \hat{H}, \hat{C}}(\omega) + \hat{H}_{t+1}(\omega)\Delta S_{t+1}(\omega, \cdot) - \mathbb{1}_{\{t=T-1\}}\xi(\omega, \cdot))] \\ & + U(t, \omega, \Delta \hat{C}_t(\omega)) = U_t(\omega, V_{t-1}^{\pi_0, \hat{H}, \hat{C}}(\omega) + \hat{H}_t(\omega)\Delta S_t(\omega)) \end{aligned}$$

for all  $\omega \in \Omega^t$  outside a  $\mathcal{P}^u$ -polar set. Let  $P \in \mathfrak{P}$ , then  $P = P_0 \otimes \dots \otimes P_{T-1}$  for some selectors  $P_t$  of  $\mathcal{P}_t^u$ ,  $t = 0, \dots, T-1$  and we conclude via Fubini's theorem that

$$\begin{aligned} & \mathbb{E}_P \left[ U_{t+1} \left( V_t^{\pi_0, \hat{H}, \hat{C}} + \hat{H}_{t+1}\Delta S_{t+1} - \mathbb{1}_{\{t=T-1\}}\xi \right) + \sum_{s=1}^t U(s, \Delta \hat{C}_s) \right] \\ & = \mathbb{E}_{(P_0 \otimes \dots \otimes P_{t-1})(d\omega)} \left( \mathbb{E}_{P_t(\omega)} \left[ U_{t+1} \left( (\omega, \cdot), V_t^{\pi_0, \hat{H}, \hat{C}}(\omega) + \hat{H}_{t+1}(\omega)\Delta S_{t+1}(\omega, \cdot) - \mathbb{1}_{\{t=T-1\}}\xi(\omega, \cdot) \right) \right] \right. \\ & \left. + \sum_{s=1}^t U(s, \omega, \Delta \hat{C}_s(\omega)) \right) \\ & \geq \mathbb{E}_{P_0 \otimes \dots \otimes P_{t-1}} \left[ U_t \left( V_{t-1}^{\pi_0, \hat{H}, \hat{C}} + \hat{H}_t\Delta S_t \right) + \sum_{s=1}^{t-1} U(s, \Delta \hat{C}_s) \right] \\ & = \mathbb{E}_P \left[ U_t \left( V_{t-1}^{\pi_0, \hat{H}, \hat{C}} + \hat{H}_t\Delta S_t \right) + \sum_{s=1}^{t-1} U(s, \Delta \hat{C}_s) \right]. \end{aligned}$$

A repeated application of this inequality shows (26). To conclude that  $(\hat{H}, \hat{C})$  is optimal, it remains to prove that

$$U_0(\pi_0) \geq \sup_{(H, C) \in \mathcal{A}_{\pi_0}} \inf_{P \in \mathcal{P}^u} \mathbb{E}_P \left[ \sum_{s=1}^T U(s, \Delta C_s) \right] =: v(\pi_0).$$

To this end we fix an arbitrary  $(H, C) \in \mathcal{A}_{\pi_0}$  and first show that

$$\begin{aligned} & \inf_{P \in \mathcal{P}^u} \mathbb{E}_P \left[ U_t \left( V_{t-1}^{\pi_0, H, C} + H_t \Delta S_t \right) + \sum_{s=1}^{t-1} U(s, \Delta C_s) \right] \tag{27} \\ & \geq \inf_{P \in \mathcal{P}^u} \mathbb{E}_P \left[ U_{t+1} \left( V_t^{\pi_0, H, C} + H_{t+1} \Delta S_{t+1} - \mathbb{1}_{\{t=T-1\}} \xi \right) + \sum_{s=1}^t U(s, \Delta C_s) \right], \quad t = 1, \dots, T-1. \end{aligned}$$

Let  $\epsilon > 0$ . As in the proof of lemma C.7

$$(\omega, P) \mapsto \mathbb{E}_P \left[ U_{t+1}((\omega, \cdot), V_t^{\pi_0, H, C}(\omega) + H_{t+1} \Delta S_{t+1} - \mathbb{1}_{\{t=T-1\}} \xi(\omega, \cdot)) \right] + \sum_{s=1}^t U(s, \omega, \Delta C_s(\omega)),$$

is lower semianalytic. Using [Bertsekas and Shreve, 2004, Prop. 7.50, p. 184] and [Bertsekas and Shreve, 2004, Prop. 7.44, p.172] for  $\omega \in \Omega^t$  outside a  $\mathcal{P}^u$ -polar set we have for some universally measurable  $\epsilon$ -optimal selector  $P_t^\epsilon$  that

$$\begin{aligned} & \mathbb{E}_{P_t^\epsilon(\omega)} \left[ U_{t+1} \left( (\omega, \cdot), V_t^{\pi_0, H, C}(\omega) + H_{t+1}(\omega) \Delta S_{t+1}(\omega, \cdot) - \mathbb{1}_{\{t=T-1\}} \xi(\omega, \cdot) \right) \right] + \sum_{s=1}^t U(s, \omega, \Delta C_s(\omega)) - \epsilon \\ & \leq (-\epsilon)^{-1} \vee \left( \inf_{P \in \mathcal{P}_t^u(\omega)} \mathbb{E}_P \left[ U_{t+1} \left( (\omega, \cdot), V_t^{\pi_0, H, C}(\omega) + H_{t+1}(\omega) \Delta S_{t+1}(\omega, \cdot) - \mathbb{1}_{\{t=T-1\}} \xi(\omega, \cdot) \right) \right] \right. \\ & \quad \left. + \sum_{s=1}^t U(s, \omega, \Delta C_s(\omega)) \right) \\ & \leq (-\epsilon)^{-1} \vee \left( \sup_{(H', c') \in \mathcal{A}_{t, V_{t-1}^{\pi_0, H, C}(\omega) + H_t(\omega) \Delta S_t(\omega)}} \inf_{P \in \mathcal{P}_t^u(\omega)} \mathbb{E}_P \left[ U_{t+1} \left( (\omega, \cdot), V_{t-1}^{\pi_0, H, C}(\omega) \right. \right. \right. \\ & \quad \left. \left. + H_t(\omega) \Delta S_t(\omega) - c' + H' \Delta S_{t+1}(\omega, \cdot) - \mathbb{1}_{\{t=T-1\}} \xi(\omega, \cdot) \right) \right] + \sum_{s=1}^{t-1} U(s, \omega, \Delta C_s(\omega)) + U(t, c') \right) \\ & = (-\epsilon)^{-1} \vee \left( U_t(\omega, V_{t-1}^{\pi_0, H, C}(\omega) + H_t(\omega) \Delta S_t(\omega)) + \sum_{s=1}^{t-1} U(s, \omega, \Delta C_s(\omega)) \right). \end{aligned}$$

Given  $P \in \mathcal{P}^u$  we thus have

$$\begin{aligned} & \mathbb{E}_P \left[ (-\epsilon)^{-1} \vee \left( U_t(V_{t-1}^{\pi_0, H, C} + H_t \Delta S_t) + \sum_{s=1}^{t-1} U(s, \Delta C_s) \right) \right] \\ & \geq \mathbb{E}_{P \otimes P_t^\epsilon} \left[ U_{t+1}(V_t^{\pi_0, H, C} + H_{t+1} \Delta S_{t+1} - \mathbb{1}_{\{t=T-1\}} \xi) + \sum_{s=1}^t U(s, \Delta C_s) \right] - \epsilon \\ & \geq \inf_{P' \in \mathcal{P}^u} \mathbb{E}_{P'} \left[ U_{t+1}(V_t^{\pi_0, H, C} + H_{t+1} \Delta S_{t+1} - \mathbb{1}_{\{t=T-1\}} \xi) + \sum_{s=1}^t U(s, \Delta C_s) \right] - \epsilon. \end{aligned}$$

As  $\epsilon > 0$  and  $P \in \mathcal{P}^u$  were arbitrary (27) follows. Noting that  $U_0(\pi_0) = \inf_{P \in \mathcal{P}^u} \mathbb{E}_P[U_0(V_0^{\pi_0, H, C})]$  a repeated application of (27) yields

$$\begin{aligned} U_0(\pi_0) & \geq \inf_{P \in \mathcal{P}^u} \mathbb{E}_P[U_1(\pi_0 + H_1 \Delta S_1)] \geq \dots \geq \inf_{P \in \mathcal{P}^u} \mathbb{E}_P \left[ U_T(V_{T-1}^{\pi_0, H, C} + H_T \Delta S_T - \xi) + \sum_{s=1}^{T-1} U(s, \Delta C_s) \right] \\ & = \inf_{P \in \mathcal{P}^u} \mathbb{E}_P \left[ \sum_{s=1}^T U(s, \Delta C_s) \right]. \end{aligned}$$

As  $(H, C) \in \mathcal{A}_{\pi_0}$  was arbitrary, it follows that  $U_0(\pi_0) \geq v(\pi_0)$ . This concludes the proof, since  $\pi_0 = \pi(\xi)$ .  $\square$

### C.3 Proof of theorem 4.5

*Proof.* Existence of an optimal investment consumption strategy follows from theorem 4.3. We now show uniqueness of optimisers. We fix  $0 \leq t \leq T - 1$  and recall the definition of  $\tilde{U}_t$  given in lemma C.7. Note that one can show that the function

$$(\omega, P) \mapsto \sup_{(H, c) \in \mathcal{A}_{t, x}(\omega)} \mathbb{E}_P[\tilde{U}_{t+1}((\omega, \cdot), x + H\Delta S_{t+1}(\omega, \cdot) - c - \mathbb{1}_{\{t=T-1\}}\xi(\omega, \cdot))] + U(t, c)$$

is lower semianalytic by reducing the above expression to a supremum over a countable set as in the proof of lemma C.7. Recall that again by lemma C.7 there exists a set of full  $\mathcal{P}^u$  measure on which  $\tilde{U}_t = U_t$  for all  $0 \leq t \leq T$ . For the rest of the proof we take  $\omega$  in the intersection of this set with  $\Omega_{\text{NA}}^t$ . Using the same Jankov-von-Neumann argument as in the proof of theorem 4.3 and corollary C.3 we conclude that for each  $t = 0, \dots, T - 1$  there exists a sequence  $P_t^n : \Omega^t \rightarrow \mathfrak{P}(\Omega)$  of universally measurable kernels such that  $P_t^n(\omega) \in \mathcal{P}_t^u(\omega)$  and for  $x \geq \pi_t(\xi)(\omega)$

$$\sup_{(H, c) \in \mathcal{A}_{t, x}(\omega)} \mathbb{E}_{P_t^n(\omega)}[\tilde{U}_{t+1}((\omega, \cdot), x + H\Delta S_{t+1}(\omega, \cdot) - c - \mathbb{1}_{\{t=T-1\}}\xi(\omega, \cdot))] + U(t, c) \downarrow \tilde{U}_t(\omega, x).$$

Since  $\mathcal{P}_t^u(\omega)$  is compact, there exists a probability measure  $\hat{P}_t(\omega) \in \mathcal{P}_t^u(\omega)$  and a subsequence  $\{n_k(\omega)\}_{k \in \mathbb{N}}$  such that  $\lim_{k \rightarrow \infty} P_t^{n_k(\omega)}(\omega) = \hat{P}_t(\omega)$ . We now show, that for  $\mathcal{P}^u$ -q.e.  $\omega \in \Omega^t$  and  $x \geq \pi_t(\omega)$  the functions

$$U_t(\omega, x) = \sup_{(H, c) \in \mathcal{A}_{t, x}(\omega)} \inf_{P \in \mathcal{P}_t^u(\omega)} \mathbb{E}_P[U_{t+1}((\omega, \cdot), x + H\Delta S_{t+1}(\omega, \cdot) - c - \mathbb{1}_{\{t=T-1\}}\xi(\omega, \cdot))] + U(t, \omega, c)$$

have a unique optimizer  $(H, c) \in \mathcal{A}_{t, x}(\omega)$ . For notational convenience we assume that  $0 \leq t \leq T - 2$ . We note that by concavity of  $\tilde{U}_{t+1}$  and  $U(t, \cdot)$  the function

$$(H, c) \mapsto \inf_{P \in \mathcal{P}_t^u(\omega)} E_P \left( \tilde{U}_{t+1}((\omega, \cdot), y + H\Delta S_{t+1}(\omega, \cdot) - c) \right) + U(t, c)$$

is concave. Now assume that there are  $(H^1, c^1), (H^2, c^2) \in \mathcal{A}_{t, x}(\omega)$  such that

$$\begin{aligned} & \inf_{P \in \mathcal{P}_t^u(\omega)} E_P \left( \tilde{U}_{t+1}((\omega, \cdot), x + H^1\Delta S_t(\omega, \cdot) - c^1) \right) + U(t, c^1) \\ &= \inf_{P \in \mathcal{P}_t^u(\omega)} E_P \left( \tilde{U}_{t+1}((\omega, \cdot), x + H^2\Delta S_t(\omega, \cdot) - c^2) \right) + U(t, c^2) \\ &= \tilde{U}_t(\omega, x). \end{aligned}$$

Note that for the strategy  $(H^3, c^3) := ((H^1 + H^2)/2, (c^1 + c^2)/2) \in \mathcal{A}_{t, x}(\omega)$  we have by concavity

$$\begin{aligned} & \inf_{P \in \mathcal{P}_t^u(\omega)} E_P \left( \tilde{U}_{t+1}((\omega, \cdot), x + H^3\Delta S_t(\omega, \cdot) - c^3) \right) + U(t, c^3) \\ & \geq \frac{1}{2} \left( \inf_{P \in \mathcal{P}_t^u(\omega)} E_P \left( \tilde{U}_{t+1}((\omega, \cdot), (x + H^1\Delta S_t(\omega, \cdot) - c^1)) \right) + U(t, c^1) \right. \\ & \left. + \inf_{P \in \mathcal{P}_t^u(\omega)} E_P \left( \tilde{U}_{t+1}((\omega, \cdot), y + H^2\Delta S_{t+1}(\omega, \cdot) - c^2) \right) + U(t, c^2) \right) = \tilde{U}_t(\omega, x). \end{aligned}$$

We thus conclude

$$\inf_{P \in \mathcal{P}_t^u(\omega)} E_P \left( \tilde{U}_{t+1}((\omega, \cdot), x + H^3 \Delta S_t(\omega, \cdot) - c^3) \right) + U(t, c^3) = \tilde{U}_t(\omega, x).$$

Furthermore, for any  $x \geq \pi_t(\omega)$  and any maximizer  $(\tilde{H}, \tilde{c}) \in \mathcal{A}_{t,x}(\omega)$  of  $\tilde{U}_t(\omega, x)$  we have

$$\begin{aligned} & \sup_{(H,c) \in \mathcal{A}_{t,x}(\omega)} \left( \mathbb{E}_{P_t^{n_k}(\omega)}[\tilde{U}_{t+1}((\omega, \cdot), x + H \Delta S_{t+1}(\omega, \cdot) - c)] + U(t, c) \right) \\ & \geq \mathbb{E}_{P_t^{n_k}(\omega)}[\tilde{U}_{t+1}((\omega, \cdot), x + \tilde{H} \Delta S_{t+1}(\omega, \cdot) - \tilde{c})] + U(t, \tilde{c}) \\ & \geq \inf_{P \in \mathcal{P}_t^u(\omega)} \mathbb{E}_P[\tilde{U}_{t+1}((\omega, \cdot), x + \tilde{H} \Delta S_{t+1}(\omega, \cdot) - \tilde{c})] + U(t, \tilde{c}) = \tilde{U}_t(\omega, x), \end{aligned} \quad (28)$$

so taking limits in (28) we find

$$\lim_{k \rightarrow \infty} \mathbb{E}_{P_t^{n_k}(\omega)}[\tilde{U}_{t+1}((\omega, \cdot), x + \tilde{H} \Delta S_{t+1}(\omega, \cdot) - \tilde{c})] + U(t, \tilde{c}) = \tilde{U}_t(\omega, x).$$

Furthermore we note that by assumption and lemma C.7  $\tilde{U}_t(\omega, y)$  is bounded by some  $C$  on  $\{(\omega, x) \in \Omega^t \times \mathbb{R} \mid x \geq \pi_t(\xi)(\omega)\}$ , non-decreasing as well as continuous in  $y$  and  $\xi$  is continuous. Note the superhedging prices  $\omega \mapsto \pi_t(\xi)(\omega)$  are continuous on  $\{(\omega, v) \in \Omega^t \mid v \in f_{t-1}(\omega)\}$  by assumption.

For  $n \in \mathbb{N}_+$  we define the shifted utility function

$$U^{1/n}(T, x) := U(T, x + 1/n).$$

Furthermore we inductively define the corresponding one-step versions for the multiperiod case  $U_T^{1/n}(\omega, x) := U^{1/n}(T, x)$  and

$$U_t^{1/n}(\omega, x) := \sup_{(H,c) \in \mathcal{A}_{t,x}(\omega)} \inf_{P \in \mathcal{P}_t^u(\omega)} \mathbb{E}_P[U_{t+1}^{1/n}((\omega, \cdot), x + 1/n + H \Delta S_{t+1}(\omega, \cdot) - c)] + U(t, c)$$

for  $1 \leq t \leq T - 1$ . Note that in particular  $U^{1/n}(t, x)$  fulfils assumption 4.4 for all  $n \in \mathbb{N}$  and  $1 \leq t \leq T$ . Denote their lower semianalytic versions  $\tilde{U}_t^{1/n}(\omega, x)$ . Again by lemma C.7 there exists a set of full  $\mathcal{P}^u$ -measure, such that  $\tilde{U}_t^{1/n}(\omega, x) = U_t^{1/n}(\omega, x)$  for all  $n \in \mathbb{N}$  and  $1 \leq t \leq T$  and we fix  $\omega$  in this set from now on. We now show by backwards induction that for all  $n \in \mathbb{N}$  the function  $(\omega, x) \mapsto \tilde{U}_t^{1/n}(\omega, x + 1/n)$  is continuous in every point of the set  $\{(\omega, x) \in \Omega^t \times \mathbb{R} \mid x \geq \pi_t(\xi)(\omega)\}$ : Let us assume the hypothesis is true for  $t + 1$  and fix  $n \in \mathbb{N}$ ,  $x \geq \pi_t(\xi)(\omega)$ . For any  $(\tilde{\omega}, \tilde{x}) \in \Omega^t \times \mathbb{R}$  we have

$$\begin{aligned} \left| \tilde{U}_t^{1/n}(\omega, x + 1/n) - \tilde{U}_t^{1/n}(\tilde{\omega}, \tilde{x} + 1/n) \right| & \leq \left| \tilde{U}_t^{1/n}(\omega, x + 1/n) - \tilde{U}_t^{1/n}(\omega, \tilde{x} + 1/n) \right| \\ & \quad + \left| \tilde{U}_t^{1/n}(\omega, \tilde{x} + 1/n) - \tilde{U}_t^{1/n}(\tilde{\omega}, \tilde{x} + 1/n) \right|. \end{aligned}$$

As  $x \mapsto \tilde{U}_t^{1/n}(\omega, x + 1/n)$  is continuous on  $[\pi_t(\xi)(\omega) - 1/n, \infty)$ , there exists  $\delta > 0$  such that the first summand can be bounded by  $\epsilon/2$  if  $|x - \tilde{x}| \leq \delta$ . Thus it is sufficient to show that there exists  $\tilde{\delta} > 0$  such that for all  $|\tilde{\omega} - \omega| \leq \tilde{\delta}$  we have

$$\left| \tilde{U}_t^{1/n}(\omega, \tilde{x} + 1/n) - \tilde{U}_t^{1/n}(\tilde{\omega}, \tilde{x} + 1/n) \right| \leq \epsilon/2.$$

Indeed, note first that by remark A.1 and the same contradiction argument as in the proof of proposition 3.5 choosing  $\tilde{\delta} > 0$  small enough we can assume that for any

superhedging strategy  $(H, c) \in \mathcal{A}_{t, \pi_t(\xi)(\tilde{\omega})}(\tilde{\omega})$  we have  $|(H, c)| \leq \tilde{C}$  for some  $\tilde{C} > 0$  independent of  $\tilde{\omega}$ . Furthermore we can choose  $\tilde{\delta} > 0$ , such that  $|\pi_t(\xi)(\omega) - \pi_t(\xi)(\tilde{\omega})| \leq 1/n$ .

Next we make the following observation: As  $\mathcal{P}_t^u(\omega)$  is weakly compact by assumption, there exists a compact set  $[0, K]^d \subseteq \Omega$ , such that  $P([0, K]^d)^c \leq \epsilon/(48C)$  for all  $P \in \mathcal{P}_t^u(\omega)$ . By the induction hypothesis  $(v, y) \mapsto \tilde{U}_{t+1}^{1/n}(v, y + 1/n)$  is continuous in every point of the set  $\{(v, y) \in \Omega^{t+1} \times \mathbb{R} \mid y \geq \pi_{t+1}(\xi)(v)\}$  and thus uniformly continuous on a compact subset. There exists  $1/n > \delta_0 > 0$  such that for  $v, \tilde{v} \in B_1(\omega) \times \{u \in \Omega \mid \inf_{\tilde{u} \in [0, K]^d} |u - \tilde{u}| \leq \delta_0\}$ ,  $y \in [\pi_{t+1}(\xi)(v), 2CK]$  and  $|(v, y) - (\tilde{v}, \tilde{y})| \leq \delta_0$  we have

$$\left| \tilde{U}_{t+1}^{1/n}(v, y + 1/n) - \tilde{U}_{t+1}^{1/n}(\tilde{v}, \tilde{y} + 1/n) \right| \leq \epsilon/24. \quad (29)$$

By assumption 4.4.(1) and by adapting  $\tilde{\delta}$  accordingly, for all  $\tilde{\omega} \in \Omega^t$  such that  $|\omega - \tilde{\omega}| < \tilde{\delta}$  and for all  $P \in \mathcal{P}_t^u(\omega)$ , there exists  $\tilde{P} \in \mathcal{P}_t^u(\tilde{\omega})$  such that  $d_L(P, \tilde{P}) \leq \tilde{\epsilon} := \delta_0/(2\tilde{C}) \wedge \epsilon/(48C)$ . It follows by Strassen's theorem that there exists a measure  $\pi \in \mathfrak{P}(\mathbb{R}^d \times \mathbb{R}^d)$  and two random variables  $X \sim P \circ (S_{t+1})^{-1}(\tilde{\omega}, \cdot)$  and  $\tilde{X} \sim \tilde{P} \circ (S_{t+1})^{-1}(\tilde{\omega}, \cdot)$  such that  $\pi(|X - \tilde{X}| \geq \tilde{\epsilon}) \leq \tilde{\epsilon}$ . Thus we conclude that for  $y, \tilde{y} : \Omega \rightarrow \mathbb{R}$  with  $|y(x) - \tilde{y}(\tilde{x})| \leq \delta_0$  whenever  $\pi_{t+1}(\tilde{\omega}) \leq \tilde{y}(\tilde{x}) \leq 2CK$  and  $|x - \tilde{x}| \leq \tilde{\epsilon}$

$$\begin{aligned} & \left| \mathbb{E}_P \left[ \tilde{U}_{t+1}^{1/n}((\tilde{\omega}, \cdot), 1/n + y(\cdot)) \right] - \mathbb{E}_{\tilde{P}} \left[ \tilde{U}_{t+1}^{1/n}((\tilde{\omega}, \cdot), 1/n + \tilde{y}(\cdot)) \right] \right| \\ &= \left| \mathbb{E}_\pi \left[ \tilde{U}_{t+1}^{1/n}((\tilde{\omega}, X), 1/n + y(X)) - \tilde{U}_{t+1}^{1/n}((\tilde{\omega}, \tilde{X}), 1/n + \tilde{y}(\tilde{X})) \right] \right| \\ &\leq \mathbb{E}_\pi \left[ \left| \tilde{U}_{t+1}^{1/n}((\tilde{\omega}, X), 1/n + y(X)) - \tilde{U}_{t+1}^{1/n}((\tilde{\omega}, \tilde{X}), 1/n + \tilde{y}(\tilde{X})) \right| \mathbb{1}_{\{|X - \tilde{X}| \leq \tilde{\epsilon}\}} \right] + \frac{C\epsilon}{12C} \\ &\leq \epsilon/12 + \epsilon/12 = \epsilon/6. \end{aligned} \quad (30)$$

Now we modify  $\tilde{\delta} > 0$  such that  $|\pi_t(\xi)(\omega) - \pi_t(\xi)(\tilde{\omega})| \leq \delta_0$  if  $|\omega - \tilde{\omega}| \leq \tilde{\delta}$ . Furthermore applying proposition C.1 for the function  $(\omega, x + 1/n) \mapsto \tilde{U}_t^{1/n}(\omega, x + 1/n)$  there exists a maximiser  $(H', c') \in \mathcal{A}_{t, \tilde{x}+1/n}(\tilde{\omega})$  of

$$\sup_{(H, c) \in \mathcal{A}_{t, \tilde{x}+1/n}(\tilde{\omega})} \inf_{P \in \mathcal{P}_t^u(\tilde{\omega})} \mathbb{E}_P[\tilde{U}_{t+1}^{1/n}((\tilde{\omega}, \cdot), \tilde{x} + 1/n + H\Delta S_{t+1}(\tilde{\omega}, \cdot) - c)] + U(t, c)$$

and a strategy  $(H, c' - \beta) \in \mathcal{A}_{t, \tilde{x}+1/n}(\omega)$ , where  $\beta := c' \wedge |\pi_t(\xi)(\omega) - \pi_t(\xi)(\tilde{\omega})| \leq \delta_0/2$ . Furthermore there exists  $P \in \mathcal{P}_t^u(\omega)$  such that

$$\tilde{U}_t^{1/n}(\omega, \tilde{x} + 1/n) \geq \mathbb{E}_P \left[ \tilde{U}_{t+1}^{1/n}((\omega, \cdot), \tilde{x} + 2/n + H\Delta S_{t+1}(\omega, \cdot) - c' + \beta) \right] + U(t, c' - \beta) - \epsilon/6.$$

Note that we can modify  $\tilde{\delta} > 0$  such that  $|(\omega, HS_t(\omega)) - (\tilde{\omega}, HS_t(\tilde{\omega}))| \leq (\tilde{C} + 2)\tilde{\delta} \leq \delta_0/2$ . Now by (29) with  $y(\cdot) = \tilde{x} + 1/n + H\Delta S_{t+1}(\omega, \cdot) - c' + \beta$  and  $\tilde{y}(\cdot) = \tilde{x} + 1/n + H\Delta S_{t+1}(\tilde{\omega}, \cdot) - c'$

$$\begin{aligned} & \mathbb{E}_P[\tilde{U}_{t+1}^{1/n}((\omega, \cdot), \tilde{x} + 2/n + H\Delta S_{t+1}(\omega, \cdot) - c' + \beta)] + U(t, c' - \beta) - \epsilon/6 \\ &\geq \mathbb{E}_P[\tilde{U}_{t+1}^{1/n}((\tilde{\omega}, \cdot), \tilde{x} + 2/n + H\Delta S_{t+1}(\tilde{\omega}, \cdot) - c')] + U(t, c') - \epsilon/3 \end{aligned}$$

follows and by (30) with  $y(\cdot) = \tilde{x} + 1/n + H\Delta S_{t+1}(\tilde{\omega}, \cdot) - c'$ ,  $\tilde{y}(\cdot) = \tilde{x} + 1/n + H'\Delta S_{t+1}(\tilde{\omega}, \cdot) - c'$  and noting that  $|H - H'| \leq 2\tilde{C}$

$$\begin{aligned} & \mathbb{E}_P \left[ \tilde{U}_{t+1}^{1/n}((\tilde{\omega}, \cdot), \tilde{x} + 2/n + H\Delta S_{t+1}(\tilde{\omega}, \cdot) - c') \right] + U(t, c') - \epsilon/3 \\ &\geq \mathbb{E}_{\tilde{P}} \left[ \tilde{U}_{t+1}^{1/n}((\tilde{\omega}, \cdot), \tilde{x} + 2/n + H'\Delta S_{t+1}(\tilde{\omega}, \cdot) - c') \right] + U(t, c') - \epsilon/2 \\ &\geq \tilde{U}_t^{1/n}(\tilde{\omega}, \tilde{x}) - \epsilon/2. \end{aligned}$$

Exchanging the roles of  $\omega$  and  $\tilde{\omega}$  concludes the proof of the induction step.

This shows in particular continuity of  $\omega' \mapsto \tilde{U}_{t+1}^{1/n}((\omega, \omega'), x + 1/n + \tilde{H}\Delta S_{t+1}(\omega, \omega') - \tilde{c})$  as  $\omega' \mapsto x + \tilde{H}\Delta S_{t+1}(\omega, \omega') - \tilde{c}$  is continuous. As this function is also  $\mathcal{P}_t^u(\omega)$ -q.s. bounded by lemma C.7 (recall that  $(\tilde{H}, \tilde{c}) \in \mathcal{A}_{t,x}(\omega)$ ), we conclude by use of the Portmanteau theorem that

$$\begin{aligned} \tilde{U}_t(\omega, x) &= \inf_{n \in \mathbb{N}} \tilde{U}_t^{1/n}(\omega, x) \\ &= \inf_{n \in \mathbb{N}} \liminf_{k \rightarrow \infty} \mathbb{E}_{P_t^{n_k}(\omega)}[\tilde{U}_{t+1}^{1/n}((\omega, \cdot), x + 1/n + \tilde{H}\Delta S_{t+1}(\omega, \cdot) - \tilde{c})] + U(t, \tilde{c}) \\ &\geq \inf_{n \in \mathbb{N}} \mathbb{E}_{\hat{P}_t(\omega)}[\tilde{U}_{t+1}^{1/n}((\omega, \cdot), x + 1/n + \tilde{H}\Delta S_{t+1}(\omega, \cdot) - \tilde{c})] + U(t, \tilde{c}) \\ &= \mathbb{E}_{\hat{P}_t(\omega)}[\tilde{U}_{t+1}((\omega, \cdot), x + \tilde{H}\Delta S_{t+1}(\omega, \cdot) - \tilde{c})] + U(t, \tilde{c}) \\ &\geq \inf_{P \in \mathcal{P}_t^u(\omega)} \mathbb{E}_P[\tilde{U}_{t+1}((\omega, \cdot), x + \tilde{H}\Delta S_{t+1}(\omega, \cdot) - \tilde{c})] + U(t, \tilde{c}), \end{aligned}$$

which yields for  $x \geq \pi_t(\omega)$

$$\tilde{U}_t(\omega, x) = \mathbb{E}_{\hat{P}_t(\omega)}[\tilde{U}_{t+1}((\omega, \cdot), x + \tilde{H}\Delta S_{t+1}(\omega, \cdot) - \tilde{c})] + U(t, \tilde{c}).$$

In particular for  $i = 1, 2$

$$\begin{aligned} &\mathbb{E}_{\hat{P}_t(\omega)}[\tilde{U}_{t+1}((\omega, \cdot), x + H^3\Delta S_{t+1}(\omega, \cdot) - c^3)] + U(t, c^3) \\ &= \mathbb{E}_{\hat{P}_t(\omega)}[\tilde{U}_{t+1}((\omega, \cdot), x + H^i\Delta S_{t+1}(\omega, \cdot) - c^i)] + U(t, c^i). \end{aligned}$$

Now since

$$(H, c) \mapsto \mathbb{E}_{\hat{P}_t(\omega)}[\tilde{U}_{t+1}((\omega, \cdot), x + H\Delta S_{t+1}(\omega, \cdot) - c)] + U(t, c)$$

is concave and strictly concave in  $c$ , we need to have  $c^1 = c^2$  and

$$H^1\Delta S_{t+1}(\omega, \cdot) = H^2\Delta S_{t+1}(\omega, \cdot) \quad \hat{P}_t(\omega) - \text{a.s.}$$

Lastly denote by  $\Xi_t$  the correspondence

$$\Xi_t(\omega) = \left\{ P \in \mathcal{P}_t^u(\omega) \mid \tilde{U}_t(x, \omega) = \sup_{(H,c) \in \mathcal{A}_{t,x}(\omega)} \mathbb{E}_P[\tilde{U}_{t+1}((\omega, \cdot), x + H\Delta S_{t+1}(\omega, \cdot) - c)] + U(t, c) \right\}$$

for  $x \geq \pi_t(\omega)$  and note that by measurable selection arguments as in Bouchard and Nutz [2015][proof of Lemma 4.10, p. 848] the set

$$\left\{ (\omega, P) \in \text{graph}(\mathcal{P}_t^u) \mid \sup_{(H,c) \in \mathcal{A}_{t,x}(\omega)} \mathbb{E}_P[\tilde{U}_{t+1}((\omega, \cdot), x + H\Delta S_{t+1}(\omega, \cdot) - c)] + U(t, c) - \tilde{U}_t(x, \omega) \leq 0 \right\}$$

is an element of  $\mathbf{A}(\mathcal{F}_t^{\mathcal{U}} \otimes \mathcal{B}(\mathfrak{P}(\Omega)))$ , where  $\mathbf{A}(\mathcal{F}_t^{\mathcal{U}} \otimes \mathcal{B}(\mathfrak{P}(\Omega)))$  is the set of all nuclei of Suslin schemes on  $\mathcal{F}_t^{\mathcal{U}} \otimes \mathcal{B}(\mathfrak{P}(\Omega))$ . In consequence there exists an  $\mathcal{F}_t^{\mathcal{U}}$ -measurable function  $\hat{P}_t : \Omega^t \rightarrow \mathfrak{P}(\Omega)$  such that  $\text{graph}(\hat{P}_t) \subseteq \text{graph}(\Xi_t)$ . This concludes the proof.  $\square$

*Remark C.9* If we assume that  $H^1 - H^2 \in \text{span}_{\hat{P}_t(\omega)}(\Delta S_{t+1}(\omega, \cdot))$ , then  $H^1 = H^2$ .



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