Spectral Filtering of Interpolant Observables for a Discrete-in-time Downscaling Data Assimilation Algorithm

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Abstract

We describe a spectrally-filtered discrete-in-time downscaling data assimilation algorithm and prove, in the context of the two-dimensional Navier–Stokes equations, that this algorithm works for a general class of interpolants, such as those based on local spatial averages as well as point measurements of the velocity. Our algorithm is based on the classical technique of inserting new observational data directly into the dynamical model as it is being evolved over time, rather than nudging, and extends previous results in which the observations were defined directly in terms of an orthogonal projection onto the large-scale (lower) Fourier modes. In particular, our analysis does not require the interpolant to be represented by an orthogonal projection, but requires only the interpolant to satisfy a natural approximation of the identity.

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1 Introduction

The goal of data assimilation is to optimally combine known information about the dynamics of a solution with low-resolution observational measurements of that solution over time to create better and better approximations of the current state. While model error in the dynamics and measurement error in the observations are significant issues with practical data assimilation, we consider here the error-free case in order to study the role played by spatial filtering. In particular, even if the observations are error free, in certain geophysical models they can contain high-frequency spillover and gravity waves which need to be controlled in order for the data assimilation to perform well. Additional issues arise because commonly used filtering techniques can lead to non-orthogonal interpolants. These issues are the focus of the current paper. Our results extend the work of Hayden, Olson and Titi [17] on discrete-in-time data assimilation from the case where the low-resolution observations are given by projection onto the low Fourier modes to both the first and second type of general interpolant observables that appear in Azouani, Olson and Titi [3], see also Bessaih, Olson and Titi [4]. To make this extension, we apply a spectral filter based on the Stokes operator to the interpolant observables and call the new method spectrally-filtered discrete-in-time downscaling data assimilation. It is worth noting that much of advances in the accuracy of present day weather forecasting have come from better filtering techniques, see for example Budd, Freitag and Nichols [5]. From this point of view, the analytic results presented here for spectral filtering may be seen as a first step towards a rigorous analysis of more complicated methods.

An alternative algorithm for discrete-in-time data assimilation based on nudging was recently studied by Foias, Mondaini and Titi in [15]. In that work it was shown that nudging works for interpolants of what is known by now as type-I, such as those which are based on local coarse spatial scale volume elements measurements without any additional filtering—the dissipation provided by the Navier–Stokes equations themselves is sufficient; however, a similar treatment for type-II interpolant observables is missing. The algorithm studied here is based on the classical technique of inserting the observational data directly into the model as it is evolved forward in time, see for example Daley [10] and references therein. When inserting the data directly into the model, the need for filtering becomes more evident. Moreover, by developing a spectrally-filtered algorithm we are also able to handle type-II interpolant observable. Although it is likely a similar technique could be applied to a nudging algorithm to handle type-II interpolant observables, we do not pursue that line of analysis here, but will be reported in future work.

The two-dimensional incompressible Navier–Stokes equations are given by

$$\frac{\partial U}{\partial t} - \nu\Delta U + \nabla P + (U\cdot\nabla)U = f, \qquad \nabla\cdot U = 0. \tag{1.1}$$

Following Constantin and Foias [9], Foias, Manley, Rosa and Temam [13], Robinson [19] and Temam [20], and in order to simplifying our presentation and fix ideas, we consider flows on the domain $\Omega = [0, L]^2$ equipped with periodic boundary conditions. Let $\mathcal{V}$ be the set of all divergence-free L-periodic trigonometric polynomials with zero spatial averages, V be the closure of $\mathcal{V}$ in $H^1(\Omega, \mathbf{R}^2)$, V^* be the dual of V, and P_σ be the orthogonal projection of $L^2(\Omega; \mathbf{R}^2)$ onto H, where H is the closure of $\mathcal{V}$ in $L^2(\Omega, \mathbf{R}^2)$. Define $A\colon V \to V^*$ and $B\colon V\times V \to V^*$ to be the unique continuous extensions for $u, v \in \mathcal{V}$ of the operators given by

$$Au = -P_\sigma\Delta u \qquad \text{and} \qquad B(u, v) = P_\sigma(u\cdot\nabla v).$$

Remark that in periodic case $A = -\Delta$, thus, the two-dimensional incompressible Navier–Stokes equations may be written as

$$\frac{dU}{dt} + \nu AU + B(U,U) = f \tag{1.2}$$

with initial condition $U_0 \in V$, at time $t = t_0$. Here $\nu > 0$ is the kinematic viscosity, and the body force $f \in L^\infty([t_0,\infty); H)$ is taken to be divergence free, but possibly time dependent.

When the force is time independent, as shown in any of the aforementioned references, equations (1.2) are well posed with unique regular solutions depending continuously on the initial conditions and which exist for all time, $t \geq t_0$. The case when the force depends on time is somewhat more delicate and we shall place further assumptions on f in Section 2, see also Appendix A, to ensure the resulting solutions have enough regularity for the subsequent analysis. In either case, we define the semi-process S as the solution operator that maps initial conditions into their subsequent time evolution by $S(t, t_0; U_0) = U(t)$ for all $t \geq t_0$.

We now describe the general interpolant observables to which our results will apply. These interpolants are inspired by the modes, nodes and volume elements of Jones and Titi [18], see also Foias and Titi [12], and are equivalent to the first and second types of general interpolant observables that appear in [3], see also [4] and the general framework presented in Cockburn, Jones and Titi [7]. In particular, we state

Definition 1.1. A linear operator $I_h\colon V \to L^2$ is said to be a *type-I interpolant observable* if there exists $c_1 > 0$ such that

$$\|U - I_hU\|_{L^2}^2 \leq c_1 h^2 \|U\|^2 \qquad \text{for all} \qquad U \in V. \tag{1.3}$$

A linear operator $I_h\colon \mathcal{D}(A) \to L^2$ is said to be a *type-II interpolant observable* if

$$\|U - I_hU\|_{L^2}^2 \leq c_1 h^2 \big(\|U\|^2 + h^2|AU|^2\big) \qquad \text{for all} \qquad U \in \mathcal{D}(A). \tag{1.4}$$

Here $\mathcal{D}(A) = H^2(\Omega) \cap V$ is the domain of A viewed as an operator into L^2. Specifically, in terms of Fourier modes, let

$$H = \left\{ \sum_{k\in\mathcal{J}} \widehat{U}_k e^{ik\cdot x} \quad : \quad \widehat{U}_k \in \mathbf{C}^2, \quad \widehat{U}_k^* = \widehat{U}_{-k}, \quad k\cdot\widehat{U}_k = 0 \quad \text{and} \quad \sum_{k\in\mathcal{J}} |\widehat{U}_k|^2 < \infty \right\},$$

where

$$\mathcal{J} = \left\{ \frac{2\pi}{L}(n_1,n_2) : n = (n_1,n_2) \in \mathbb{Z}^2 \backslash \{(0,0)\} \right\}.$$

For notational convenience assume $\widehat{U}_0 = 0$ even though this coefficient doesn't enter into the above characterization of H. We employ the notations

$$|U| = \|U\|_0, \qquad \|U\| = \|U\|_1 \qquad \text{and} \qquad |AU| = \|U\|_2$$

where

$$\|U\|_\alpha^2 = L^2 \sum_{k\in\mathcal{J}} |k|^{2\alpha} |\widehat{U}_k|^2 \qquad \text{when} \qquad U = \sum_{k\in\mathcal{J}} \widehat{U}_k e^{ik\cdot x}. \tag{1.5}$$

Further define $V_\alpha = \{\, U \in H : \|U\|_\alpha < \infty \,\}$. Consequently $\mathcal{D}(A) = V_2$ and $V = V_1$.

In Definition 1.1 we note that h is a length scale corresponding to the observation resolution and c_1 is a dimensionless constant. For example, suppose nodal measurements of the velocity are given by

$$\big(U(x_1), U(x_2), \ldots, U(x_d)\big),$$

where $x_i \in \Omega$ have been chosen in such a way that

$$\sup_{x\in\Omega} \inf \big\{\, \|x - x_j\| : j = 1, 2, \ldots, d \,\big\} \le h.$$

Then

$$I_h(U)(x) = \sum_{j=1}^{d} U(x_i)\widetilde{\chi}_j(x) \qquad \text{where} \qquad \widetilde{\chi}_j(x) = \chi_j(x) - \frac{1}{|\Omega|}\int_\Omega \chi_j$$

with

$$\chi_j(x) = \begin{cases} 1 & \text{if } \|x - x_j\| < \|x - x_i\| \text{ for all } i \neq j \\ 0 & \text{otherwise} \end{cases}$$

is a type-II interpolant observable.

It is worth reflecting that the type-II interpolant observable described above naturally results in a piecewise constant vector field which is discontinuous. Although $I_h(U) \in L^2$ as required, the Fourier transform of the resulting vector field possesses a significant high-frequency component due to the discontinuities. A similar interpolant was considered in [16] for numerical simulations of a data-assimilation method based on nudging. Those computations show that the adverse effects of the high-frequency spill over which result from the spatial discontinuities can be mitigated by appropriate convolution with a smoothing kernel. The spectral filtering considered in this work also removes the high-frequency component in spatial Fourier representation while enjoying additional approximation properties useful for the analysis of the resulting data assimilation algorithm.

We now introduce the spectrally-filtered discrete-in-time data assimilation algorithm which forms the focus of our study. Let $P_\lambda \colon H \to H$ be the orthogonal projection onto the Fourier modes with wave numbers k such that $|k|^2 \le \lambda$ given by

$$P_\lambda U = \sum_{|k|^2 \le \lambda} \widehat{U}_k e^{ik\cdot x}.$$

and let $Q_\lambda = I - P_\lambda$ be the orthogonal complement of P_λ. Now, given $\lambda > 0$ and I_h define

$$J = P_\lambda P_\sigma I_h \qquad \text{and} \qquad E = I - J. \tag{1.6}$$

Note, although no additional orthogonality or regularity properties other than those appearing in Definition 1.1 have been assumed on I_h, the above spectral filtering yields an operator J which is nearly orthogonal and has a range contained in $\mathcal{D}(A)$. The downscaling data assimilation algorithm studied in this paper may now be stated as

Definition 1.2. Let U be an exact solution of (1.2) which evolves according to dynamics given by the semi-process S. Let $t_n = t_0 + n\delta$ be a sequence of times for which partial observations of U are interpolated by I_h. Then the approximating solution u given by

$$\begin{cases} u_0 = JU(t_0) \\ u_{n+1} = ES(t_{n+1}, t_n; u_n) + JU(t_{n+1}) \\ u(t) = S(t, t_n; u_n) \qquad \text{for} \qquad t \in [t_n, t_{n+1}) \end{cases}$$

is what we shall call *spectrally-filtered discrete-in-time downscaling data assimilation.*

We stress that only the spectrally filtered low-resolution observations of the exact solution represented by $JU(t_n)$ for $t_n \leq t$ are used to construct the approximating solution u at time t. Since we assume the dynamics governing the evolution of U to be known, then exact knowledge of the initial condition $U(t_0) = U_0$ would, in theory, obviate the need for data assimilation at subsequent times. Of course, knowing the exact dynamics and being able to practically compute with them are two different things. Although not the focus of the present research, the algorithm stated above may also be used to stabilize the growth of numerical error. Putting such numerical considerations aside, we view the data assimilation algorithm given in Definition 1.2 as a way of improving estimates of the unknown state of U at time t by means of known dynamics and a time-series of low-resolution observations.

Intuitively, at each time t_{n+1} a new measurement is used to kick the approximating solution towards the exact solution by constructing of an improved approximation of the current state u_{n+1} which may be seen as a combination of a prediction based on the previous approximation and a correction based on the observation. This improved approximation then serves as an initial condition from which to further evolve the approximating solution. Since $JU(t_{n+1})$ is supported on a finite number of Fourier modes, the regularity of u_{n+1} is determined by $ES(t_{n+1}, t_n; u_n)$. For type-I interpolant observables our working assumptions described after Proposition 2.2 in Section 1 shall imply that $u_{n+1} \in V$ and for type-II that $u_{n+1} \in \mathcal{D}(A)$.

Although we have taken the sequence of observation times t_n to be equally spaced, intuitively one might imagine for a suitably small value of δ that it would be sufficient for

$$0 < t_{n+1} - t_n \leq \delta, \qquad \text{with} \qquad t_n \to \infty, \quad \text{as} \quad n \to \infty. \tag{1.7}$$

Our analysis, however, makes use of a minimum distance between t_{n+1} and t_n as well as the maximum. Measurements need to be inserted frequently enough to overcome the tendency for two nearby solutions to drift apart, while at the same time the possible lack of orthogonality in our general interpolant observables means measurements should not be inserted too frequently. Specifically, we need to have enough time to elapse between each insertion to allow enough time for the use of the dynamics of the equation, i.e. integrating the Navier-Stokes equations for long enough time to correct the high modes. Our algorithm consists of two steps: *Step 1.* Inserting the coarse spatial scale measurements. *Step 2.* Integrating the Navier-Stokes equations for short time, but not too short, to recover and correct the missing high modes, i.e. the fine spatial scales of the solutions. Preliminary numerical simulations further indicate this requirement is likely physical and not merely a technical condition used by our analysis. Given times t_n that satisfy (1.7) it would be straightforward to construct a subsequence of observations t'_n such that $\delta/2 < t'_{n+1} - t'_n \leq 2\delta$ and obtain results similar to the ones presented here. We leave such a refinement to the reader.

Note that the algorithm described above reduces to the discrete data assimilation method studied in [17] by taking $I_h = P_\lambda$. In particular, when the interpolant observable itself is given by an orthogonal projection onto the large-scale Fourier modes. In this work I_h can by any interpolant operator satisfying Definition 1.1. Carefully adjusting the relationship between h and λ then allows us to prove our main result, stated as

Theorem 1.3. *Let U be a solution to the two-dimensional incompressible Navier–Stokes equations (1.2) and $u(t)$, for $t \geq t_0$ be the process given by Definition 1.2. Then, for every $\delta > 0$ there exists $h > 0$ and $\lambda > 0$ depending only on c_1, f, ν such that*

$$|u(t) - U(t)| \to 0, \quad \textit{exponentially in time, as} \quad t \to \infty.$$

Here c_1 is the constant in Definition 1.1 given by the general interpolant observables.

Since we have assumed the observational measurements to be noise-free and that the exact solution evolves according to known dynamics, it is natural to obtain a result in which the difference between the exact solution U and the approximation u converges to zero over time. We further remark that if by chance $u(t_n) = U(t_n)$ at any of the data assimilation steps, then $u(t) = U(t)$ for all $t \geq t_n$. In particular, if somehow U_0 is known exactly and we take $u_0 = U_0$ as the first step of Definition 1.2, then $u(t) = U(t)$ for all $t \geq t_0$.

This paper is organized as follows. In section 2 we set our notation, recall some facts about the Navier–Stokes equations and prove some preliminary results regarding the spectrally-filtered interpolant observables that will be used in our subsequent analysis. Section 3 proves our main result for type-I interpolant observables while section 4 treats the case of type-II interpolant observables. We finish with some concluding remarks concerning the dependency of h and λ on δ and the other physical parameters in the system.

2 Preliminaries

We begin by recalling some inequalities. Writing the smallest eigenvalue of the Stokes operator A as $\lambda_1 = (2\pi/L)^2$ we have the Poincaré inequalities

$$\lambda_1 |U|^2 \leq \|U\|^2 \qquad \text{for} \qquad U \in V \tag{2.1}$$

and

$$\lambda_1^2 |U|^2 \leq \lambda_1 \|U\|^2 \leq |AU|^2 \qquad \text{for} \qquad U \in \mathcal{D}(A). \tag{2.2}$$

An advantage of using the projection P_λ in our data assimilation algorithm, rather than a different type of spatial filtering, is that this directly leads to improved Poincaré inequalities and reverse inequalities which are, respectively, given by

$$\lambda |Q_\lambda U|^2 \leq \|Q_\lambda U\|^2 \qquad \text{and} \qquad \lambda^2 |Q_\lambda U|^2 \leq \lambda \|Q_\lambda U\|^2 \leq |AQ_\lambda U|^2 \tag{2.3}$$

and

$$\|P_\lambda U\|^2 \leq \lambda |P_\lambda U|^2 \qquad \text{and} \qquad |AP_\lambda U|^2 \leq \lambda^2 |P_\lambda U|^2. \tag{2.4}$$

All of the inequalities given in (2.1), (2.2), (2.3) and (2.4) may easily be verified via Fourier series. We also recall Agmon's inequality [1] (see also [9]) as

$$\|U\|_{L^\infty} \le C|U|^{1/2}|AU|^{1/2}. \tag{2.5}$$

Here C is a dimensionless constant depending only on the domain Ω.

As mentioned in the introduction, the spectrally filtered interpolation operator J given by (1.6) possesses approximate orthogonality and regularity properties that the original interpolant observable I_h may fail to have. We summarize these properties in

Proposition 2.1. *Let c_1 be the dimensionless constant appearing in Definition 1.1. For type-I interpolant observables setting $\varepsilon = c_1\lambda h^2$ yields*

$$|EU|^2 \le \lambda^{-1}(1+\varepsilon)\|U\|^2 \quad \textit{and} \quad \|EU\|^2 \le (1+\varepsilon)\|U\|^2, \quad \textit{for every} \quad U \in V.$$

For type-II interpolant observables setting $\varepsilon = c_1\lambda_1^{-1}\lambda^2h^2(1+\lambda_1h^2)$ yields

$$|EU|^2 \le (\lambda\lambda_1)^{-1}(1+\varepsilon)|AU|^2, \qquad \|EU\|^2 \le \lambda_1^{-1}(1+\varepsilon)|AU|^2$$

and

$$|AEU|^2 \le (1+\varepsilon)|AU|^2 \qquad \textit{for every} \qquad U \in \mathcal{D}(A).$$

Proof of Proposition 2.1. Estimate $|EU|$ for type-I interpolant observables as

$$\begin{aligned} |EU|^2 &= |U - JU|^2 = |U - P_\lambda U + P_\lambda U - P_\lambda P_\sigma I_h U|^2 \\ &\le |Q_\lambda U|^2 + |P_\lambda(U - P_\sigma I_h U)|^2 \le \lambda^{-1}\|U\|^2 + |U - P_\sigma I_h U|^2 \\ &= \lambda^{-1}\|U\|^2 + |P_\sigma(U - I_h U)|^2 \le \lambda^{-1}\|U\|^2 + \|U - I_h U\|_{L^2}^2 \\ &\le \lambda^{-1}\|U\|^2 + c_1h^2\|U\|^2 = \left(\lambda^{-1} + c_1h^2\right)\|U\|^2. \end{aligned}$$

From the definition of ε it follows that

$$|EU|^2 \le \lambda^{-1}\big(1+\varepsilon\big)\|U\|^2. \tag{2.6}$$

Similarly bound $\|EU\|$ as

$$\begin{aligned} \|EU\|^2 &= \|U - JU\|^2 = \|Q_\lambda U\|^2 + \|P_\lambda(U - P_\sigma I_h U)\|^2 \\ &\le \|U\|^2 + \lambda|U - P_\sigma I_h U|^2 \le \|U\|^2 + \lambda c_1 h^2\|U\|^2 \le (1+\varepsilon)\|U\|^2. \end{aligned}$$

Now, estimate $|EU|$ for type-II interpolant observables as

$$\begin{aligned} |EU|^2 &= |U - P_\lambda P_\sigma I_h U|^2 = |U - P_\lambda U + P_\lambda U - P_\lambda P_\sigma I_h U|^2 \\ &= |U - P_\lambda U|^2 + |P_\lambda P_\sigma(U - I_h U)|^2 \le |Q_\lambda U|^2 + \|U - I_h U\|_{L^2}^2 \\ &\le |Q_\lambda U|^2 + c_1h^2\left(\|U\|^2 + h^2|AU|^2\right) \\ &\le \left(\frac{1}{\lambda} + c_1h^2\right)\|U\|^2 + c_1h^4|AU|^2 \\ &\le \left[\frac{1}{\lambda_1}\left(\frac{1}{\lambda} + c_1h^2\right) + c_1h^4\right]|AU|^2. \end{aligned} \tag{2.7}$$

Setting $\varepsilon = c_1\lambda_1^{-1}\lambda^2 h^2(1+\lambda_1 h^2)$ yields that

$$|EU|^2 \le (\lambda\lambda_1)^{-1}(1+\varepsilon)|AU|^2.$$

Next, estimate $\|EU\|$ as

$$\begin{aligned}
\|EU\|^2 &= \|U - P_\lambda P_\sigma I_h U\|^2 = \|U - P_\lambda U + P_\lambda U - P_\lambda P_\sigma I_h U\|^2 \\
&= \|U - P_\lambda U\|^2 + \|P_\lambda P_\sigma(U - I_h U)\|^2 \\
&= \|Q_\lambda U\|^2 + \lambda |P_\sigma(U - I_h U)|^2 \\
&\le \|Q_\lambda U\|^2 + \lambda \|U - I_h U\|_{L^2}^2 \\
&\le \lambda^{-1}|AU|^2 + c_1\lambda h^2\big(\|U\|^2 + h^2|AU|^2\big) \\
&\le \lambda^{-1}|AU|^2 + c_1\lambda h^2\big(\lambda_1^{-1}|AU|^2 + h^2|AU|^2\big) \\
&\le \big(\lambda^{-1} + c_1\lambda_1^{-1}\lambda h^2(1+\lambda_1 h^2)\big)|AU|^2 \\
&\le \lambda_1^{-1}\big(1 + c_1\lambda h^2(1+\lambda_1 h^2)\big)|AU|^2 \le \lambda_1^{-1}(1+\varepsilon)|AU|^2,
\end{aligned} \tag{2.8}$$

and finally $|AEU|$ as

$$\begin{aligned}
|AEU|^2 &= |Q_\lambda AU|^2 + |AP_\lambda(U - P_\sigma I_h U)|^2 \\
&\le |AU|^2 + \lambda^2|P_\sigma(U - I_h U)|^2 \\
&\le |AU|^2 + \lambda^2\|U - I_h U\|_{L^2}^2 \\
&\le |AU|^2 + c_1\lambda^2 h^2\big(\|U\|^2 + h^2|AU|^2\big) \\
&\le \big(1 + c_1\lambda_1^{-1}\lambda^2 h^2(1+\lambda_1 h^2)\big)|AU|^2 = (1+\varepsilon)|AU|^2.
\end{aligned} \tag{2.9}$$

This completes the proof of the proposition. □

Our analysis will make use of *a priori* bounds on the solution U of (1.2). If $f \in H$ is time independent, such bounds can be inferred from bounds on the global attractor. For example, Propositions 12.2 and 12.4 in Robinson [19] may be stated as

Proposition 2.2. *If $f \in H$ is time independent, then there are absorbing sets in H, V and $\mathcal{D}(A)$ of radiuses ρ_H, ρ_V and ρ_A, respectively, depending only on $|f|$, Ω and ν such that for every $U_0 \in H$ there is a time t_A depending only on $|U_0|$ for which*

$$|U(t)| \le \rho_H, \quad \|U(t)\| \le \rho_V \quad \text{and} \quad |AU(t)| \le \rho_A \quad \text{for all} \quad t \ge t_A. \tag{2.10}$$

Moreover,

$$\int_t^{t+\delta} |AU|^2 \le \Big(\frac{1}{\nu} + \frac{\delta\lambda_1}{2}\Big)\rho_V^2 \quad \text{for all} \quad t \ge t_A. \tag{2.11}$$

Similar bounds may be found in Temam [20] and Constantin and Foias [9]. The best estimate of ρ_A to date appears in [14]. Before considering the case when f depends on time, we further note when $f \in V$ is time independent that the bounds in (2.10) are finite for $t > t_0$. Moreover, (2.11) is finite and

$$\int_t^{t+\delta} \|AU\|^2 < \infty \qquad \text{for all} \qquad t \ge t_0. \tag{2.12}$$

We remark that estimate (2.12) follows as a particular case of the proof presented in Appendix A for the time-dependent forcing term, see discussion below.

When $f \in L^\infty([t_0,\infty),H)$ depends on time, the resulting solution U does not automatically satisfy the ρ_A bound in (2.10) nor the finiteness condition (2.12). In the case of type-I interpolant observables the remaining bounds given by ρ_H and ρ_V are sufficient for our analysis. However, for type-II interpolant observables we need ρ_A as well as the finiteness condition (2.12). These bounds may be obtained in a number of different ways. For example, one could assume that $f \in L^\infty([t_0,\infty),V)$ and $df/dt \in L^\infty([t_0,\infty),V^*)$. For details see Appendix A.

Our analysis shall be made under the working assumption that ρ_H, ρ_V and ρ_A are known when needed and that the unknown initial condition U_0 in (1.2) comes from a long-time evolution prior to time t_0. Thus, we assume $t_0 \geq t_A$ and in particular that the bounds (2.10), (2.11) and (2.12) hold, in fact, for $t \geq t_0$ regardless of whether f depends on time or not. For other initial conditions we further suppose that the norms and time integrals appearing in all the above bounds are at least finite when $t > t_0$. We now state a standard result concerning the finite-time continuous dependence on initial conditions for solutions to the two-dimensional incompressible Navier-Stokes equations.

Theorem 2.3. *Under the working assumptions given above, there exists $\beta > 0$ depending only on $|f|$, L and ν such that the free-running solution satisfies*

$$|U(t) - S(t,t_*;u_*)|^2 \leq e^{\beta(t-t_*)}|U(t_*) - u_*|^2 \quad \textit{for} \quad t \geq t_* \quad \textit{and} \quad u_* \in V. \tag{2.13}$$

We remark that the above continuity result is obtained from the first Lyapunov exponent, which reflects the instability in turbulent flows. Thus, the constant β in Theorem 2.3 is very large but uniform for $u_* \in V$. The fact that β does not depend on u^* is a fact we shall make salient use of in our subsequent analysis.

We recall that the bilinear term B has the algebraic property that

$$\big\langle B(u,v),w\big\rangle = -\big\langle B(u,w),v\big\rangle \tag{2.14}$$

for $u,v,w \in V$, and consequently the orthogonality property that

$$\big\langle B(u,w),w\big\rangle = 0. \tag{2.15}$$

Here the pairing $\langle\cdot,\cdot\rangle$ denotes the dual action of V^* on V. Details may be found, e.g., in [9], [13], [19] and [20]. In the case of periodic boundary conditions the bilinear term possesses the additional orthogonality property

$$\big(B(w,w),Aw\big) = 0, \qquad \text{for every} \quad w \in \mathcal{D}(A); \tag{2.16}$$

and consequently one has

$$\big(B(u,w),Aw\big) + \big(B(w,u),Aw\big) = -\big(B(w,w),Au\big), \qquad \text{for every} \quad u,w \in \mathcal{D}(A). \tag{2.17}$$

We further recall some well-known bounds on the non-linear term which appear in [9], [20], [21] and specifically as Proposition 9.2 in [19].

Proposition 2.4. *One has*

$$|(B(u,v),w)| \le \|u\|_{L^\infty}\|v\||w|, \tag{2.18}$$

where $u \in L^\infty$, $v \in V$ *and* $w \in H$. *If* $u, v, w \in V$ *then*

$$|(B(u,v),w)| \le c|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2}, \tag{2.19}$$

and if $u \in V$ $v \in \mathcal{D}(A)$, *and* $w \in H$,

$$|(B(u,v),w)| \le c|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w|. \tag{2.20}$$

Here c *is an absolute non-dimensional constant.*

3 Type-I Interpolant Observables

In this section we treat the case when I_h is a type-I interpolant observable. While type-I interpolant observables are also of type II, the bounds we obtain in treating these two cases separately are sharper. In addition, the proof for in the type-I case is simpler and serves as a framework to help understand the more complicated type-II case teated in the subsequent section. From Definition 1.2 it follows that the approximating solution u satisfies

$$\frac{du}{dt} + \nu Au + B(u,u) = f \qquad \text{for} \qquad t \in (t_n, t_{n+1}), \tag{3.1}$$

where $u(t_n) = u_n$ is the initial condition given by

$$u_0 = JU_0 \qquad \text{and} \qquad u_{n+1} = ES(t_{n+1}, t_n; u_n) + JU(t_{n+1}).$$

Note that $u_n \in \mathcal{D}(A) \subseteq V$, for $n = 0, 1, 2, \ldots$. Consequently, the solution of (3.1) with initial data $u(t_n) = u_n$ on the interval (t_n, t_{n+1}) is a strong solution of the Navier–Stokes equations. Moreover, because of our working assumptions on f we further obtain that $u(t) \in \mathcal{D}(A)$ for $t \in [t_n, t_{n+1}]$. It follows that the estimates we make in the proof of Proposition 3.1 below, and in the results which follow, are rigorous; in particular, $v = U - u$ exists, is unique and Av makes sense at all times $t \ge t_0$.

The equations governing the evolution of v may be written as

$$\frac{dv}{dt} + \nu Av + B(v,U) + B(U,v) + B(v,v) = 0 \tag{3.2}$$

for $t \in (t_n, t_{n+1})$, with $v(t_n) = U(t_n) - u_n$, for $n = 0, 1, 2, \ldots$..

Proposition 3.1. *Let* $\tilde{v}_n = U(t_n) - S(t_n, t_{n-1}; u_{n-1})$. *For every* $\delta > 0$ *there are* λ, *large enough, and* h, *small enough, for which there exists* $\gamma \in (0,1)$ *such that*

$$\|\tilde{v}_{n+1}\|^2 \le \gamma\|\tilde{v}_n\|^2, \qquad \text{for all} \qquad n = 1, 2, \ldots.$$

Proof. Multiplying (3.2) by Av and then integrating over Ω we have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \nu|Av|^2 + (B(v,U),Av) + (B(U,v),Av) + (B(v,v),Av) = 0.$$

By (2.16) and (2.17), we have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \nu|Av|^2 = (B(v,v),AU). \tag{3.3}$$

Estimate the term on the right of the previous equation. Using (2.20) and then the interpolation inequality $\|v\| \le |v|^{1/2}|Av|^{1/2}$ yields

$$\begin{aligned}|(B(v,v),AU)| &\le c|v|^{1/2}\|v\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|AU| \\ &= c|v|^{1/2}\|v\||Av|^{1/2}|AU| \\ &\le c|v|^{1/2}|v|^{1/2}|Av|^{1/2}|Av|^{1/2}|AU| \\ &= c|v||Av||AU|.\end{aligned}$$

Combining this with (3.3), we have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \nu|Av|^2 \le c|v||Av||AU|. \tag{3.4}$$

Now, apply Young's inequality to obtain

$$\frac{d}{dt}\|v\|^2 + \nu|Av|^2 \le \frac{c^2}{\nu}|v|^2|AU|^2. \tag{3.5}$$

From Poincaré's inequality (2.2) followed by (2.13), we get

$$\frac{d}{dt}\|v\|^2 + \lambda_1\nu\|v\|^2 \le \frac{c^2}{\nu}|AU|^2 e^{\beta(t-t_n)}|v_n|^2 \le \frac{c^2}{\nu}|AU|^2 e^{\beta\delta}|v_n|^2, \tag{3.6}$$

where we have assumed $t \in [t_n, t_{n+1})$. Multiply equation (3.6) by $e^{\lambda_1\nu t}$ and then integrate in time from t_n to t. Thus,

$$\|v(t)\|^2 \le e^{-\lambda_1\nu(t-t_n)}\|v_n\|^2 + \frac{c^2}{\nu}e^{-\lambda_1\nu(t-t_n)+(\beta+\lambda_1\nu)\delta}|v_n|^2\int_{t_n}^t |AU(s)|^2 ds \tag{3.7}$$

for $t \in [t_n, t_{n+1})$. Combining (3.7) with the *a priori* estimate (2.11), we have

$$\|v(t)\|^2 \le e^{-\lambda_1\nu(t-t_n)}\|v_n\|^2 + \frac{c^2\rho_V^2}{\nu}\left(\frac{1}{\nu} + \frac{\delta\lambda_1}{2}\right)e^{-\lambda_1\nu(t-t_n)+(\beta+\lambda_1\nu)\delta}|v_n|^2. \tag{3.8}$$

Since $n \ge 1$ then

$$\begin{aligned}v_n = U(t_n) - u_n &= U(t_n) - ES(t_n, t_{n-1}; u_{n-1}) - JU(t_n) \\ &= E\big(U(t_n) - S(t_n, t_{n-1}; u_{n-1})\big) = E(\tilde{v}_n),\end{aligned}$$

and by Proposition 2.1, we can estimate

$$|v_n|^2 \le \lambda^{-1}(1+\varepsilon)\|\tilde{v}_n\|^2 \qquad \text{and} \qquad \|v_n\|^2 \le (1+\varepsilon)\|\tilde{v}_n\|^2.$$

Hence (3.8) becomes

$$\|v(t)\|^2 \le (1+\varepsilon)e^{-\lambda_1\nu(t-t_n)}\left[1+\frac{c^2\rho_V^2}{\lambda\nu}\left(\frac{1}{\nu}+\frac{\delta\lambda_1}{2}\right)e^{(\beta+\lambda_1\nu)\delta}\right]\|\tilde{v}_n\|^2, \tag{3.9}$$

for $t \in [t_n, t_{n+1})$. Taking the limit as $t \nearrow t_{n+1}$ results in $\|\tilde{v}_{n+1}\|^2 \le \gamma\|\tilde{v}_n\|^2$, where

$$\gamma = (1+\varepsilon)\left[e^{-\lambda_1\nu\delta}+\frac{c^2\rho_V^2}{\lambda\nu}\left(\frac{1}{\nu}+\frac{\delta\lambda_1}{2}\right)e^{\beta\delta}\right].$$

We now show for every $\delta > 0$ that there exists λ and h such that $\gamma \in (0,1)$. First, since

$$e^{-\lambda_1\nu\delta} < 1 \qquad \text{and} \qquad \frac{c^2\rho_V^2}{\lambda\nu}\left(\frac{1}{\nu}+\frac{\delta\lambda_1}{2}\right)e^{\beta\delta} \to 0 \quad \text{as} \quad \lambda \to \infty,$$

then there is λ large enough such that

$$e^{-\lambda_1\nu\delta}+\frac{c^2\rho_V^2}{\lambda\nu}\left(\frac{1}{\nu}+\frac{\delta\lambda_1}{2}\right)e^{\beta\delta} < 1.$$

Finally, since $\varepsilon \to 0$, as $h \to 0$, while holding λ fixed, then there is h small enough such that $1+\varepsilon$ is small enough to ensure that $\gamma < 1$. □

Observe that by a more careful analysis one could find explicit choices for λ and h in terms of β, δ, λ_1, ν and ρ_V. Note also that there is a dependency between λ and h. Since h is a physical parameter related to the resolution of the observations while λ is an easily-adjusted parameter related to our spectral filter, it would be reasonable to further choose λ to minimize h. The resulting estimate on h could then be used to compare the sharpness of the above theoretical bounds to alternative approaches to the analysis, to numerical results obtained from simulation and to similar analysis for different data assimilation schemes. Such comparisons, while interesting, are outside the scope of the present work. We end this section with our main result on type-I interpolant observables.

Theorem 3.2. *If δ, h and λ are chosen appropriately as in Proposition* (3.1)*, then $\|U(t) - u(t)\| \to 0$, as $t \to \infty$. Moreover, the rate of convergence is exponential in time.*

Proof. Choose δ, h and λ as in Proposition 3.1. In reference to equation (3.9), let

$$M = (1+\varepsilon)\left[1+\frac{c^2\rho_V^2}{\lambda\nu}\left(\frac{1}{\nu}+\frac{\delta\lambda_1}{2}\right)e^{(\beta+\lambda_1\nu)\delta}\right].$$

We first bound $\tilde{v}_1$ in terms of v_0. Since

$$v_0 = U_0 - u_0 = U_0 - JU_0 = EU_0,$$

then Proposition 2.1 and the working assumptions which follow Proposition 2.2 yield that

$$|v_0|^2 = |EU_0|^2 \le \lambda^{-1}(1+\varepsilon)\|U_0\|^2 \le \lambda^{-1}(1+\varepsilon)\rho_V^2,$$

and similarly that $\|v_0\|^2 \le (1+\varepsilon)\rho_V^2$. These two bounds substituted into (3.8) for $n = 0$ imply

$$\|v(t)\|^2 \le (1+\varepsilon)e^{-\lambda_1\nu(t-t_0)}\left[1 + \frac{c^2\rho_V^2}{\lambda\nu}\left(\frac{1}{\nu} + \frac{\delta\lambda_1}{2}\right)e^{(\beta+\lambda_1\nu)\delta}\right]\rho_V^2. \tag{3.10}$$

for $t \in [t_0, t_1)$. Taking the limit as $t \nearrow t_1$ results in $\|\tilde{v}_1\|^2 \le \gamma\rho_V^2$ where $\gamma \in (0,1)$.

Now, given $t > 0$ choose n such that $t \in [t_n, t_{n+1})$. Since $n > (t-t_0)/\delta - 1$, it follows from (3.9) that

$$\|U(t) - u(t)\|^2 = \|v(t)\|^2 \le M\|\tilde{v}_n\|^2 \le M\gamma^n\rho_V^2 \le M\gamma^{-1}\rho_V^2 e^{-\alpha(t-t_0)},$$

where $\alpha = \delta^{-1}\log(\gamma^{-1})$. Note that $\gamma \in (0,1)$ implies $\alpha > 0$. It follows that $\|U(t) - u(t)\|$ converges to zero at an exponential rate. □

4 Type-II Interpolant Observables

In this section we treat the case when I_h is a type-II interpolant observable. As before let $v = U - u$ where U is the exact solution to (1.2) about which we know only limited information through the observables and u is the approximating process obtained by the spectrally-filtered discrete data assimilation algorithm given in Definition 1.2. The proof that the difference between u and U decays to zero over time is complicated by the fact that the $|Av|$ norm enters into the bounds given by Proposition 2.1 and therefore needs to be controlled. To do so, we shall employ an equation similar to (3.3) which governs the evolution of $|Av|^2$. While such an equation could be obtained by formally multiplying (3.2) by A^2u and integrating over Ω, it is easier to work with the vorticity in two-dimensions.

Let $W = \operatorname{curl} U$, $w = \operatorname{curl} u$, and $g = \operatorname{curl} f$ where curl has been defined such that

$$\operatorname{curl}\Phi = \frac{\partial\Phi_2(x_1,x_2)}{\partial x_1} - \frac{\partial\Phi_1(x_1,x_2)}{\partial x_2} \qquad \text{when} \qquad \Phi(x) = \big(\Phi_1(x_1,x_2), \Phi_2(x_1,x_2)\big).$$

Since u is the approximating solution described in Definition 1.2, then w is the resulting vorticity approximation of W. Written in terms of vorticity, the corresponding version of Theorem 3.2 for type-II interpolant observables is given by

Theorem 4.1. *If δ, h and λ are chosen appropriately, then $\|W - w\| \to 0$, as $t \to \infty$. Moreover, the rate of convergence is exponential in time.*

Before proving Theorem 4.1 we fix our notation by stating a few facts about the vorticity and proving a lemma containing bounds for non-linear terms that will be used later. First note, after taking the curl of (1.1), that Definition 1.2 implies W and w satisfy

$$\frac{\partial W}{\partial t} - \nu\Delta W + (U\cdot\nabla)W = g \qquad \text{and} \qquad \frac{\partial w}{\partial t} - \nu\Delta w + (u\cdot\nabla)w = g \tag{4.1}$$

on each interval (t_n, t_{n+1}). Our working assumptions in the case of type-II interpolant observables ensure that the equations (4.1) hold in the strong sense. In particular, $W = \operatorname{curl} U$ and $w = \operatorname{curl} u$ exist and $|\Delta W| = |A^{3/2}U|$ and $|\Delta w| = |A^{3/2}u|$ are finite almost everywhere. Therefore, the equations governing the evolution through the vorticity of the difference $\xi = W - w$ may be written as

$$\frac{\partial \xi}{\partial t} - \nu \Delta \xi + (v \cdot \nabla) W + (v \cdot \nabla)\xi + (U \cdot \nabla)\xi = 0, \tag{4.2}$$

where $\xi(t_n) = W(t_n) - \operatorname{curl} u_n$ and $v = \operatorname{curl}^{-1} \xi$.

Since v is divergence-free with zero average, then $\operatorname{curl}^{-1} \xi$ is well defined and may be written in terms of Fourier series as

$$\operatorname{curl}^{-1} \xi = \sum_{k \in \mathcal{J}} \frac{i(k_2, -k_1)}{|k|^2} \widehat{\xi}_k e^{ik\cdot x} \qquad \text{when} \qquad \xi = \sum_{k\in\mathcal{J}} \widehat{\xi}_k e^{ik\cdot x}.$$

Recall that the divergence-free condition $k \cdot \widehat{v}_k = 0$ implies

$$\begin{aligned} |\widehat{\xi}_k|^2 &= |ik_1 \widehat{v}_{k,2} - ik_2 \widehat{v}_{k,1}|^2 = k_1^2 |\widehat{v}_{k,2}|^2 + k_2^2 |\widehat{v}_{k,1}|^2 - k_1 k_2 \widehat{v}_{k,1} \widehat{v}^*_{k,2} - k_1 k_2 \widehat{v}^*_{k,1} \widehat{v}_{k,2} \\ &= k_1^2 |\widehat{v}_{k,2}|^2 + k_2^2 |\widehat{v}_{k,1}|^2 + k_2^2 |\widehat{v}_{k,2}|^2 + k_1^2 |\widehat{v}_{k,1}|^2 = |k|^2 |\widehat{v}_k|^2. \end{aligned}$$

Therefore

$$|\xi|^2 = L^2 \sum_{k\in\mathcal{J}} |\widehat{\xi}_k|^2 = \|v\|^2 \qquad \text{and} \qquad \|\xi\|^2 = L^2 \sum_{k\in\mathcal{J}} |k|^2 |\widehat{\xi}_k|^2 = |Av|^2.$$

To keep the notation in the present section similar to the notation appearing in the previous section, we abuse it by extending the definitions of B and A to the vorticity as

$$B(v, \xi) = (v \cdot \nabla)\xi \qquad \text{and} \qquad A\xi = -\Delta \xi.$$

Thus, equation (4.2) may be written as

$$\frac{d\xi}{dt} + \nu A\xi + B(v, W) + B(v, \xi) + B(U, \xi) = 0. \tag{4.3}$$

Equations (4.3) are similar to (3.2) in structure; however, there are no cancellations when multiplying by $A\xi$ and integrating over Ω. To bound the resulting terms we prove

Lemma 4.2. *Let U, W, v and ξ be defined as above. The following bounds hold*

$$\begin{aligned} |(B(v, W), A\xi)| &\le C \frac{4^2}{3\nu^2} |v|^2 \|W\|^3 + \frac{\nu}{6} |A\xi|^2, \\ |(B(v, \xi), A\xi)| &\le C \frac{4^2}{3\nu^2} |v|^2 \|\xi\|^3 + \frac{\nu}{6} |A\xi|^2, \end{aligned}$$

and

$$|(B(U, \xi), A\xi)| \le C \frac{5^5}{\nu^5} \|U\|^6_{L^\infty} |v|^2 + \frac{\nu}{6} |A\xi|^2$$

for almost every $t \ge t_0$.

Proof. The condition (2.12) applied to both U and u implies that $\|Av\| = |A\xi|$ is finite for almost every $t \geq t_0$. Our working assumptions further imply that the other norms appearing in the above bounds exist everywhere. For convenience denote

$$I_1 = |(B(v,W), A\xi)|, \qquad I_2 = |(B(v,\xi), A\xi)| \qquad \text{and} \qquad I_3 = |(B(U,\xi), A\xi)|.$$

We now estimate I_1, I_2 and I_3 in turn. First, estimate I_1 using (2.18) followed by Agmon's inequality to obtain

$$I_1 \leq \|v\|_{L^\infty}\|W\||A\xi| \leq C|v|^{1/2}|Av|^{1/2}\|W\||A\xi| = C|v|^{1/2}|Av|^{1/2-\theta}|Av|^{\theta}\|W\||A\xi|.$$

Since $|Av| = \|\xi\|$, we have

$$I_1 \leq C|v|^{1/2}\|\xi\|^{1/2-\theta}|Av|^{\theta}\|W\||A\xi|.$$

We now use interpolation inequality on $|Av|^\theta$ and have $|Av|^\theta \leq |v|^{\theta/3}|A\xi|^{2\theta/3}$. This yields

$$I_1 \leq C|v|^{1/2+\theta/3}\|\xi\|^{1/2-\theta}|A\xi|^{1+2\theta/3}\|W\|.$$

Using Young's inequality with powers 3 and 3/2, we have

$$I_1 \leq C\frac{16}{3\nu^2}|v|^{3/2+\theta}\|\xi\|^{3/2-3\theta}\|W\|^3 + \frac{\nu}{6}|A\xi|^{3/2+\theta}.$$

Choose $\theta = \frac{1}{2}$, then we have

$$I_1 \leq C\frac{16}{3\nu^2}|v|^2\|W\|^3 + \frac{\nu}{6}|A\xi|^2. \tag{4.4}$$

Next, estimate I_2 using (2.18) and then Agmon's inequality. We have

$$I_2 \leq C|v|^{1/2}|Av|^{1/2}\|\xi\||A\xi| = C|v|^{1/2}|Av|^{3/2-\theta}|Av|^{\theta}|A\xi|.$$

Using interpolation on $|Av|^\theta$ it follows that

$$I_2 \leq C|v|^{1/2+\theta/3}|Av|^{3/2-\theta}|A\xi|^{1+2\theta/3}.$$

Choosing $\theta = \frac{1}{2}$ and then by Young's inequality with powers 3 and 3/2, we have

$$I_2 \leq C|v|^{4/6}|Av||A\xi|^{4/3} \leq C\frac{4^2}{3\nu^2}|v|^2\|\xi\|^3 + \frac{\nu}{6}|A\xi|^2. \tag{4.5}$$

Finally, estimate I_3 using (2.18). We have

$$\begin{aligned} I_3 &\leq \|U\|_{L^\infty}\|\xi\||A\xi| = \|U\|_{L^\infty}|Av||A\xi| \\ &\leq \|U\|_{L^\infty}|v|^{1/3}|A\xi|^{5/3}. \end{aligned}$$

Using Young's inequality with powers 6 and 6/5 it follows that

$$I_3 \leq C\frac{5^5}{\nu^5}\|U\|^6_{L^\infty}|v|^2 + \frac{\nu}{6}|A\xi|^2. \tag{4.6}$$

□

Proof of Theorem 4.1. Multiplying equations (4.3) by $A\xi$ and integrating over Ω yields

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2 + \nu|A\xi|^2 + (B(v,W),A\xi) + (B(v,\xi),A\xi) + (B(U,\xi),A\xi) = 0. \tag{4.7}$$

We remark that the working assumptions for type-II interpolant observables imply both U and u and consequently their difference has the needed regularity for the above equation to make sense. These assumptions further provide *a priori* bounds on U which are uniform in time. Although the corresponding norms of u are finite, we cannot at this point assume they are uniformly bounded in time. Under the hypotheses of this theorem, however, uniform bounds on u can be inferred as a consequence of this proof

Now, plug the estimates given by Lemma 4.2 into (4.7) to obtain

$$\frac{d}{dt}\|\xi\|^2 + \nu|A\xi|^2 \le C\left(\frac{1}{\nu^2}\|W\|^3 + \frac{1}{\nu^2}\|\xi\|^3 + \frac{1}{\nu^5}\|U\|_{L^\infty}^6\right)|v|^2. \tag{4.8}$$

We again point out that C is a non-dimensional constant independent of δ, λ and h. By Poincaré's inequality (2.2) we have

$$\frac{d}{dt}\|\xi\|^2 + \lambda_1\nu\|\xi\|^2 \le \frac{C}{\nu^2}\left(\|\xi\|^3 + K\right)|v|^2, \tag{4.9}$$

where the assumption $\|W\| \le \rho_A$ combined with Agmon's inequality (2.5) allows us to take

$$K = \rho_A^3(1 + c\nu^{-3}\rho_H^3).$$

Alternatively, one could write $K' = \rho_A(1 + c\nu^{-3}\rho_H^3)$ to obtain

$$\frac{d}{dt}\|\xi\|^2 + \lambda_1\nu\|\xi\|^2 \le \frac{C}{\nu^2}\left(\|\xi\|^3 + K'|AU|^2\right)|v|^2,$$

and then estimate the integral of $|AU|^2$ using (2.11) as we did in (3.8). Unfortunately, this improvement is dominated by subsequent estimates on $\|\xi\|$ which are proportional to ρ_A. Therefore, as the differences are minimal, we continue with (4.9) for simplicity.

By (2.13), we have

$$\frac{d}{dt}\|\xi\|^2 + \lambda_1\nu\|\xi\|^2 \le \frac{C}{\nu^2}\left(\|\xi\|^3 + K\right)e^{\beta(t-t_n)}|v_n|^2. \tag{4.10}$$

Note that equation (4.10) is similar to (3.6) except for the additional term involving $\|\xi\|^3$ on the right. Fortunately, this term can be controlled for times of size δ by our choosing h small and λ large. This complicates the proof and is the main reason why the type-I interpolant observables were treated separately in the previous section.

Continue as in the type-I case. First, multiply (4.10) by $e^{\lambda_1\nu t}$, integrate from t_n to t and simplify as in (3.7) to obtain

$$\|\xi\|^2 \le \|\xi_n\|^2 e^{-\lambda_1\nu(t-t_n)} + \frac{C}{\nu^2\beta}\Big(\sup_{s\in[t_n,t)}\|\xi(s)\|^3 + K\Big)e^{\beta(t-t_n)}|v_n|^2.$$

When $n = 0$ it follows from Proposition 2.1 that

$$|v_0|^2 = |EU_0|^2 \le (\lambda\lambda_1)^{-1}(1+\varepsilon)|AU_0|^2 \le (\lambda\lambda_1)^{-1}(1+\varepsilon)\rho_A^2,$$

and

$$\|\xi_0\|^2 = |Av_0|^2 = |AEU_0|^2 \le (1+\varepsilon)|AU_0|^2 \le (1+\varepsilon)\rho_A^2.$$

Therefore when $t \in [t_0, t_1)$ we have

$$\|\xi\|^2 \le (1+\varepsilon)\Big\{e^{-\lambda_1\nu(t-t_0)} + \frac{C}{\lambda\lambda_1\nu^2\beta}\Big(\sup_{s\in[t_0,t)}\|\xi(s)\|^3 + K\Big)e^{\beta(t-t_0)}\Big\}\rho_A^2. \tag{4.11}$$

Let $\delta > 0$ be arbitrary and define

$$\gamma = (1+\varepsilon)\Big\{e^{-\lambda_1\nu\delta} + \frac{C}{\lambda\lambda_1\nu^2\beta}\Big(8\rho_A^3 + K\Big)e^{\beta\delta}\Big\}.$$

As in the the proof of Proposition 3.1, since

$$e^{-\lambda_1\nu\delta} < 1 \qquad \text{and} \qquad \frac{C}{\lambda\lambda_1\nu^2\beta}\Big(8\rho_A^3 + K\Big)e^{\beta\delta} \to 0 \quad \text{as} \quad \lambda \to \infty,$$

then there is λ large enough such that

$$e^{-\lambda_1\nu\delta} + \frac{C}{\lambda\lambda_1\nu^2\beta}\Big(8\rho_A^3 + K\Big)e^{\beta\delta} < 1. \tag{4.12}$$

Furthermore, since $\varepsilon \to 0$ as $h \to 0$ while holding λ fixed, then there is h small enough such that $1 + \varepsilon < 2$ and moreover small enough to ensure that $\gamma < 1$.

For the choice of δ, h and λ given above, let

$$M = \sup_{s\in[0,\delta]}(1+\varepsilon)\Big\{e^{-\lambda_1\nu s} + \frac{C}{\lambda\lambda_1\nu^2\beta}\Big(8\rho_A^3 + K\Big)e^{\beta s}\Big\},$$

and note (4.12) along with the fact that $1+\varepsilon < 2$ implies $M < 4$. We claim that $\|\xi(s)\| < 2\rho_A$ for $s \in [t_0, t_1)$. For contradiction, suppose not. Since $\|\xi\|$ is continuous on $[t_0, t_1)$ and

$$\|\xi(t_0)\| = \|\xi_0\| \le (1+\varepsilon)^{1/2}\rho_A < 2^{1/2}\rho_A < 2\rho_A,$$

then this would imply the existence of $t_* \in (t_0, t_1)$ such that

$$\|\xi(t_*)\| = 2\rho_A \qquad \text{and} \qquad \|\xi(s)\| < 2\rho_A \quad \text{for} \quad s \in [t_0, t_*).$$

However, if this were true, then inequality (4.11) would imply

$$\|\xi(t_*)\|^2 \le (1+\varepsilon)\Big\{e^{-\lambda_1\nu(t_*-t_0)} + \frac{C}{\lambda\lambda_1\nu^2\beta}\Big(8\rho_A^3 + K\Big)e^{\beta(t_*-t_0)}\Big\}\rho_A^2 \le M\rho_A^2 < 4\rho_A^2,$$

which is a contradiction. Therefore $\|\xi(s)\| < 2\rho_A$ for $s \in [t_0, t_1)$. Consequently

$$\sup_{s\in[t_0,t_1)}\|\xi(s)\|^3 \le 8\rho_A^3,$$

and taking the limit of (4.11) as $t \to t_1$ results in $\|\tilde{\xi}_1\|^2 \le \gamma \rho_A^2$.

We proceed by induction. Let $n \ge 1$ and suppose

$$\|\tilde{\xi}_n\|^2 \le \gamma^n \rho_A^2.$$

By Proposition 2.1 it follows that

$$|v_n|^2 = |E\tilde{v}_n|^2 \le (\lambda\lambda_1)^{-1}(1+\varepsilon)|A\tilde{v}_n|^2 = (\lambda\lambda_1)^{-1}(1+\varepsilon)\|\tilde{\xi}_n\|^2,$$

and

$$\|\xi_n\|^2 = |Av_n|^2 = |AE\tilde{v}_n|^2 \le (1+\varepsilon)|A\tilde{v}_n|^2 = (1+\varepsilon)\|\tilde{\xi}_n\|^2,$$

where $\tilde{\xi}_n = \operatorname{curl} \tilde{v}_n$. Since $1+\varepsilon < 2$ we obtain

$$\|\xi(t_n)\| = \|\xi_n\| \le (1+\varepsilon)^{1/2}\|\tilde{\xi}_n\| \le 2^{1/2}\gamma^{n/2}\rho_A < 2\rho_A.$$

Following the same arguments as before, we obtain that

$$\sup_{s\in[t_n,t_{n+1})} \|\xi(s)\|^3 \le 8\rho_A^3,$$

and taking limits as $t \nearrow t_{n+1}$ conclude that

$$\|\tilde{\xi}_{n+1}\|^2 \le \gamma\|\tilde{\xi}_n\|^2 \le \gamma^{n+1}\rho_A^2,$$

which completes the induction.

Given $t > 0$ choose n such that $t \in [t_n, t_{n+1})$. It follows that

$$\|W - w\|^2 = \|\xi\|^2 \le M\|\tilde{\xi}_n\|^2 \le M\gamma^n\rho_A^2 \to 0 \qquad \text{as} \qquad t \to \infty.$$

Therefore, the same argument used in the proof of Theorem 3.2 now implies

$$\|W - w\| \to 0 \qquad \text{exponentially as} \qquad t \to \infty,$$

and finishes the proof of Theorem 4.1. □

5 Conclusions

In this paper we have shown that spectrally-filtered discrete data assimilation as described in Definition 1.2 results in an approximating solution u that converges to the reference solution U over time for any general interpolant observable of type-I or type-II when δ, λ and h are chosen appropriately. In particular, when observations of U are made using nodal points of the velocity field, we obtain a type-II interpolant observable which our analysis is able to handle. We note that this analysis relies crucially on properties of the spectral filter and would not have been possible if the unfiltered interpolants were used instead. Specifically, our analysis makes use of the fact that the filtered interpolant E can be made to have norm near unity when viewed as linear operator on the functional space implied by the bounds on the original interpolant. This fact is characterized by the respective inequalities

$$\|EU\|^2 \le (1+\varepsilon)\|U\|^2 \qquad \text{and} \qquad |AEU|^2 \le (1+\varepsilon)|AU|^2,$$

for the type-I and type-II interpolant observables given in Proposition 2.1. Different filtering methods which satisfy similar inequalities should also be effective. As a number of advances in practical data assimilation have resulted from improved filtering, we find these analytic results to be interesting and relevant.

While it may seem anticlimactic that the technique crucial for our analysis relies on spectrally projecting the interpolant observable in Fourier space, since the linear term is responsible for the dissipation, it is natural that a spectral basis with respect to that linearity provides a convenient framework in which to analyze the synchronization properties of our data assimilation algorithm. Furthermore, using this basis as a means of spatial filtering not only has the advantage of being simple, but is intrinsically compatible with the reliance of our analysis on the dissipation.

Note that the functional dependency of h and λ on δ and the other physical parameters in the system appearing in Theorem 1.3 depend on knowing an *a priori* bound ρ_A on the norm $|AU|$ in terms of those other parameters. While suitable theoretical bounds appear in the literature, these bounds are, in general, not sharp compared to *a posteriori* bounds obtained through numerical simulation. Moreover, the algorithm may continue to work with values of h much larger and values of λ much smaller than required by our analysis. For example, computational experiments performed by [16] for a different spatially filtered continuous data assimilation method based on nudging show that the method performs far better than the analytical estimates suggest. We conjecture similar numerical effectiveness for the discrete data assimilation method described in the present paper. Therefore, we refrain from determining an explicit theoretical relation between h and the Grashof number in this work, though such could be obtained from our analysis, and save such comparisons for the context of a future numerical study.

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A Estimates for Time Dependent Forcing

In this appendix we present *a priori* estimates on the solution U to the two-dimensional incompressible Navier–Stokes equations (1.2) in the case where the body force f depends on time. While these results are straight forward, we could not find suitable references in the literature and have therefore included them here for completeness of our presentation. Note that the first bound, stated as Theorem A.2 below, will be sufficient for our analysis in the case of type-I interpolant operators. The second bound, Theorem A.3, will be used for type-II interpolant operators.

In addition to the facts and inequalities from Section 2 this appendix makes use of Ladyzhenskaya's inequality, which in two-dimensions interpolates L^4 as

$$\|U\|_{L^4} \le C_0|U|^{1/2}\|U\|^{1/2}, \tag{A.1}$$

where C_0 is a non-dimensional constant depending only on Ω. We also make use of the following L^2 and H^1 bounds on the nonlinear term.

Lemma A.1. *If $U \in V$ then*

$$|B(U,U)| \le C_0^2|U|^{1/2}\|U\||AU|^{1/2}. \tag{A.2}$$

Furthermore, if $U \in \mathcal{D}(A)$ then

$$\|B(U,U)\| \le C_1\|U\||AU| + C_2|U|^{1/2}|AU|^{3/2}. \tag{A.3}$$

Here C_0, C_1 and C_2 are non-dimensional constants depending only on Ω.

Proof. Given $U \in V$ apply (A.1) to obtain

$$|B(U,U)| \le \|U\|_{L^4}\|\nabla U\|_{L^4} \le C_0^2|U|^{1/2}\|U\||AU|^{1/2},$$

which is the first inequality.

Now suppose that $U \in \mathcal{D}(A)$. Define

$$\Psi_\alpha = \sum_{k\in\mathcal{J}} |k|^\alpha|\widehat{U}_k|e^{ik\cdot x} \qquad \text{where} \qquad U = \sum_{k\in\mathcal{J}} \widehat{U}_k e^{ik\cdot x}.$$

Further define $\mathcal{J}_0 = \mathcal{J} \cup \{(0,0)\}$ and recall the notational convention that $\widehat{U}_0 = 0$. Note that $\|\Psi_\alpha\|_{L^2} = \|U\|_\alpha$ for all $\alpha \le 2$. Moreover, (2.5) and (A.1) imply that

$$\|\Psi_0\|_{L^\infty} \le C\|\Psi_0\|_{L^2}^{1/2}\|\Psi_2\|_{L^2}^{1/2} \qquad \text{and} \qquad \|\Psi_1\|_{L^4} \le C_0\|\Psi_1\|_{L^2}^{1/2}\|\Psi_2\|_{L^2}^{1/2}.$$

Since

$$U\cdot\nabla U = i\sum_{k,\ell\in\mathcal{J}} (\widehat{U}_k\cdot\ell)\widehat{U}_\ell\, e^{i(k+\ell)\cdot x} = i\sum_{\ell,m\in\mathcal{J}_0} \big(\widehat{U}_{m-\ell}\cdot\ell\big)\widehat{U}_\ell\, e^{im\cdot x},$$

it follows that

$$\begin{aligned}\|B(U,U)\|^2 &\le L^2 \sum_{m\in\mathcal{J}_0} |m|^2 \Big| \sum_{\ell\in\mathcal{J}_0} \big(\widehat{U}_{m-\ell}\cdot \ell\big)\widehat{U}_\ell\Big|^2 \le L^2 \sum_{m\in\mathcal{J}_0} \Big| \sum_{\ell\in\mathcal{J}_0} |m||\widehat{U}_{m-\ell}||\ell||\widehat{U}_\ell|\Big|^2 \\ &\le 2L^2 \sum_{m\in\mathcal{J}_0} \Big| \sum_{\ell\in\mathcal{J}_0} |m-\ell||\widehat{U}_{m-\ell}||\ell||\widehat{U}_\ell|\Big|^2 + 2L^2 \sum_{m\in\mathcal{J}_0} \Big| \sum_{\ell\in\mathcal{J}_0} |\widehat{U}_{m-\ell}||\ell|^2|\widehat{U}_\ell|\Big|^2 \\ &= 2\|\Psi_1^2\|_{L^2}^2 + 2\|\Psi_0\Psi_2\|_{L^2}^2 \le 2\|\Psi_1\|_{L^4}^4 + 2\|\Psi_0\|_{L^\infty}^2\|\Psi_2\|_{L^2}^2 \\ &\le 2C_0^4\|\Psi_1\|_{L^2}^2\|\Psi_2\|_{L^2}^2 + 2C^2\|\Psi_0\|_{L^2}\|\Psi_2\|_{L^2}\|\Psi_2\|_{L^2}^2 \\ &= 2C_0^4\|U\|^2|AU|^2 + 2C^2|U||AU|^3.\end{aligned}$$

Taking $C_1 = \sqrt{2}C_0^2$ and $C_2 = \sqrt{2}C$ finishes the proof of the lemma. □

Theorem A.2. *Suppose $f \in L^\infty([t_0,\infty);H)$ is time-dependent and define*

$$F = \operatorname{ess\,sup}\big\{|f(t)|^2 : t\in[t_0,\infty)\big\}.$$

Then there are absorbing sets in H and V of radiuses ρ_H and ρ_V, respectively, depending only on F, Ω and ν such that for every $U_0\in H$ there is a time t_V depending further on $|U_0|$ and t_0 for which

$$|U(t)| \le \rho_H \quad \text{and} \quad \|U(t)\| \le \rho_V \qquad \text{for all} \qquad t\ge t_V.$$

Moreover,

$$\int_t^{t+\delta} |AU(s)|^2 ds \le \Big(\frac{1}{\nu} + \frac{\delta\lambda_1}{2}\Big)\rho_V^2 \qquad \text{for all} \qquad t\ge t_V.$$

Proof. The proof is essentially the same as the time-independent case appearing in [20], [9] or [19] with F is substituted for $|f|$ throughout. For sake of brevity we present formal estimates which could be rigorously justified by means of the Galerkin method if desired.

First, take inner product of (1.2) with U and apply Cauchy's inequality followed by Young's inequality to obtain

$$\frac{1}{2}\frac{d}{dt}|U|^2 + \nu\|U\|^2 \le |f||U| \le \frac{\lambda_1\nu}{2}|U|^2 + \frac{1}{2\lambda_1\nu}|f|^2.$$

Collecting terms and applying the Poincaré inequality (2.1) gives

$$\frac{d}{dt}|U|^2 + \nu\|U\|^2 \le \frac{F}{\lambda_1\nu}. \tag{A.4}$$

Again applying (2.1), multiplying by $e^{\lambda_1\nu t}$ and integrating in time from t_0 to t yields

$$|U(t)|^2 \le e^{-\lambda_1\nu(t-t_0)}|U_0|^2 + \frac{F}{\lambda_1^2\nu^2}\big(1-e^{-\lambda_1\nu(t-t_0)}\big).$$

Upon taking t_H so large that

$$e^{-\lambda_1\nu(t_H-t_0)}|U_0|^2 \le \frac{F}{\lambda_1^2\nu^2},$$

it follows that

$$|U(t)| \le \rho_H \quad \text{for} \quad t \ge t_H \qquad \text{where} \qquad \rho_H^2 = \frac{2F}{\lambda_1^2 \nu^2}.$$

Returning to (A.4) for $t \ge t_H$ and simply integrating both sides from t to $t+\delta$ gives

$$|U(t+\delta)|^2 - |U(t)|^2 + \nu \int_t^{t+\delta} \|U(s)\|^2 ds \le \frac{\delta F}{\lambda_1 \nu}.$$

Consequently,

$$\int_t^{t+\delta} \|U(s)\|^2 ds \le \frac{1}{\nu}|U(t)|^2 + \frac{\delta}{\lambda_1 \nu^2} F \le \Big(\frac{1}{\nu} + \frac{\delta \lambda_1}{2}\Big)\rho_H^2. \tag{A.5}$$

Second, take inner product of (1.2) with AU and apply Cauchy's inequality followed by Young's inequality to obtain

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \nu |AU|^2 \le |f||AU| \le \frac{\nu}{2}|AU|^2 + \frac{1}{2\nu}|f|^2.$$

Collecting terms gives

$$\frac{d}{dt}\|U\|^2 + \nu |AU|^2 \le \frac{F}{\nu}. \tag{A.6}$$

Again applying (2.2), multiplying by $e^{\lambda_1 \nu t}$ and integrating in time from s to t yields

$$\|U(t)\|^2 \le e^{-\lambda_1 \nu (t-s)} \|U(s)\|^2 + \frac{F}{\lambda_1 \nu^2}\big(1 - e^{-\lambda_1 \nu (t-s)}\big).$$

Integrate with respect to s from t_H to $t_H + \delta$ using the fact that $e^{-\lambda_1 \nu (t-s)} \le e^{-\lambda_1 \nu (t - t_H - \delta)}$ to obtain

$$\delta \|U(t)\|^2 \le e^{-\lambda_1 \nu (t - t_H - \delta)} \Big(\frac{1}{\nu} + \frac{\delta \lambda_1}{2}\Big)\rho_H^2 + \frac{F\delta}{\lambda_1 \nu^2}.$$

Setting $\delta = 1/(\lambda_1 \nu)$ yields

$$\|U(t)\|^2 \le \frac{3\lambda_1}{2} e^{-\lambda_1 \nu (t - t_H) + 1} \rho_H^2 + \frac{F}{\lambda_1 \nu^2}.$$

Upon taking $t_V \ge t_H$ so large that

$$\frac{3\lambda_1}{2} e^{-\lambda_1 \nu (t_V - t_H) + 1} \rho_H^2 \le \frac{F}{\lambda_1 \nu^2},$$

it follows that

$$\|U(t)\| \le \rho_V \quad \text{for} \quad t \ge t_V \qquad \text{where} \qquad \rho_V^2 = \frac{2F}{\lambda_1 \nu^2}.$$

Returning to (A.6) for $t \ge t_V$ and simply integrating both sides from t to $t+\delta$ gives

$$\|U(t+\delta)\|^2 - \|U(t)\|^2 + \nu \int_t^{t+\delta} |AU(s)|^2 ds \le \frac{F\delta}{\nu}.$$

Consequently,

$$\int_t^{t+\delta} |AU(s)|^2 ds \le \frac{1}{\nu}\|U(t)\|^2 + \frac{F\delta}{\nu^2} \le \Big(\frac{1}{\nu} + \frac{\delta \lambda_1}{2}\Big)\rho_V^2.$$

This completes the proof. □

Theorem A.3. *Suppose $f \in L^\infty([t_0,\infty),V)$ and $df/dt \in L^\infty([t_0,\infty),V^*)$ and define*

$$G = \operatorname{ess\,sup}\big\{\|f(t)\|^2 : t \in [t_0,\infty)\big\} \quad \text{and} \quad F_* = \operatorname{ess\,sup}\big\{\|df/dt\|_{-1}^2 : t \in [t_0,\infty)\big\}.$$

Then there is an absorbing set in $\mathcal{D}(A)$ of radius ρ_A depending only on G, F^*, Ω and ν such that for every $U_0 \in H$ there is a time t_A depending further on $|U_0|$ and t_0 for which

$$|AU(t)| \le \rho_A \qquad \text{for all} \qquad t \ge t_A.$$

Moreover

$$\int_t^{t+1/(\lambda_1\nu)} \|AU(s)\|^2 ds < \infty \qquad \text{for all} \qquad t \ge t_A.$$

Proof. For convenience write $U_t = dU/dt$ and $f' = df/dt$. We again present our estimates in a formal manner with the remark that they could be rigorously justified if desired.

First, take inner product of (1.2) with U_t and apply Agmon's inequality (2.5) followed by Young's inequality to obtain

$$\begin{aligned}|U_t|^2 + \frac{1}{2}\frac{d}{dt}\|U\|^2 &= -\big(B(U,U),U_t\big) + (f,U_t) \le \|U\|_{L^\infty}\|U\||U_t| + |f||U_t|\\ &\le C|U|^{1/2}\|U\||AU|^{1/2}|U_t| + F^{1/2}|U_t| \le \frac{1}{2}|U_t|^2 + C^2|U|\|U\|^2|AU| + F.\end{aligned}$$

Collecting terms, assuming $t \ge t_V$ and applying the results of Theorem A.2 yields

$$|U_t|^2 + \frac{d}{dt}\|U\|^2 \le 2C^2\rho_H\rho_V^2|AU| + 2F.$$

Integrate from t to $t+\delta$ and apply the Cauchy-Schwartz inequality to obtain

$$\begin{aligned}\int_t^{t+\delta} |U_t|^2 + \|U(t+\delta)\|^2 &\le \|U(t)\|^2 + 2C^2\rho_H\rho_V^2 \int_t^{t+\delta} |AU(s)|ds + 2\delta F\\ &\le \rho_V^2 + 2C^2\delta^{1/2}\rho_H\rho_V^3\Big(\frac{1}{\nu} + \frac{\delta\lambda_1}{2}\Big)^{1/2} + 2\delta F.\end{aligned}$$

Setting $\delta = 1/(\lambda_1\nu)$ yields

$$\int_t^{t+1/(\lambda_1\nu)} |U_t(s)|^2 ds \le \sigma_H^2 \quad \text{where} \quad \sigma_H^2 = \rho_V^2 + \frac{1}{\lambda_1\nu}\big(6^{1/2}C^2\lambda_1^{1/2}\rho_H\rho_V^3 + 2F\big).$$

Second, differentiate (1.2) with respect to t to get

$$U_{tt} + \nu AU_t + B(U_t,U) + B(U,U_t) = f'.$$

Take inner product with U_t, note the orthogonality $\big(B(U,U_t),U_t\big) = 0$ and apply Ladyzhenskaya's inequality (A.1) followed by Young's inequality to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|U_t|^2 + \nu\|U_t\|^2 &= -\big(B(U_t,U),U_t\big) + (f',U_t) \le \|U_t\|_{L^4}^2\|U\| + \|f'\|_{-1}\|U_t\|\\ &\le C_0^2|U_t|\|U_t\|\|U\| + F_*^{1/2}\|U_t\| \le \frac{\nu}{2}\|U_t\|^2 + \frac{C_0^4}{\nu}\|U\|^2|U_t|^2 + \frac{F_*}{\nu}.\end{aligned}$$

Collecting terms yields

$$\frac{d}{dt}|U_t|^2 + \nu\|U_t\|^2 \le \kappa\|U\|^2|U_t|^2 + \frac{2F_*}{\nu} \qquad \text{where} \qquad \kappa = \frac{2C_0^4}{\nu}. \tag{A.7}$$

Multiply by

$$\Phi(t) = \exp\Big(-\kappa\int_s^t \|U(\tau)\|^2 d\tau\Big),$$

and integrate from s to $t+\delta$ to obtain

$$\Phi(t+\delta)|U_t(t+\delta)|^2 - |U_t(s)|^2 \le \frac{2F_*}{\nu}\int_s^{t+\delta}\Phi(\sigma)d\sigma,$$

or equivalently

$$\begin{aligned}|U_t(t+\delta)|^2 \le |U_t(s)|^2 \exp\Big(\kappa\int_s^{t+\delta}\|U(\tau)\|^2 d\tau\Big)\\ + \frac{2F_*}{\nu}\int_s^{t+\delta}\exp\Big(\kappa\int_\sigma^{t+\delta}\|U(\tau)\|^2 d\tau\Big)d\sigma.\end{aligned}$$

Integrate with respect to s from t to $t+\delta$. Since $t \ge t_V \ge t_H$ inequality (A.5) implies

$$\begin{aligned}\delta|U_t(t+\delta)|^2 &\le \int_t^{t+\delta}\Big\{|U_t(s)|^2\exp\Big(\kappa\int_s^{t+\delta}\|U(\tau)\|^2 d\tau\Big)\\ &\qquad + \frac{2F_*}{\nu}\int_s^{t+\delta}\exp\Big(\kappa\int_\sigma^{t+\delta}\|U(\tau)\|^2 d\tau\Big)d\sigma\Big\}ds\\ &\le \int_t^{t+\delta}\Big\{|U_t(s)|^2\exp\Big(\kappa\int_t^{t+\delta}\|U(\tau)\|^2 d\tau\Big)\\ &\qquad + \frac{2F_*}{\nu}\int_t^{t+\delta}\exp\Big(\kappa\int_t^{t+\delta}\|U(\tau)\|^2 d\tau\Big)d\sigma\Big\}ds\\ &\le \Big(\int_t^{t+\delta}|U_t(s)|^2 ds + \frac{2F_*\delta^2}{\nu}\Big)\exp\Big\{\kappa\Big(\frac{1}{\nu}+\frac{\delta\lambda_1}{2}\Big)\rho_H^2\Big\}.\end{aligned}$$

Setting $t_A = t_V + \delta$ with $\delta = 1/(\lambda_1\nu)$ yields

$$|U_t(t)| \le R_H \quad \text{for} \quad t \ge t_A \quad \text{where} \quad R_H^2 = \Big(\lambda_1\nu\sigma_H^2 + \frac{2F_*}{\lambda_1\nu^2}\Big)\exp\Big(\frac{3C_0^4\rho_H^2}{\nu^2}\Big).$$

We are now ready to estimate $|AU|$. Upon taking L^2 norms of (1.2) and applying (A.2) from Lemma A.1 followed by Young's inequality we obtain

$$\begin{aligned}\nu|AU| &\le |U_t| + |B(U,U)| + |f| \le |U_t| + C_0^2|U|^{1/2}\|U\||AU|^{1/2} + |f|\\ &\le |U_t| + \frac{C_0^4}{2\nu}|U|\|U\|^2 + \frac{\nu}{2}|AU|^{1/2} + |f|.\end{aligned}$$

Therefore,

$$|AU| \le \rho_A \quad \text{for} \quad t \ge t_A \qquad \text{where} \qquad \rho_A = \frac{2}{\nu}R_H + \frac{C_0^4}{\nu^2}\rho_H\rho_V^2 + \frac{2F^{1/2}}{\nu}.$$

To finish the proof, return to (A.7) for $t \geq t_A$ and simply integrate both sides from t to $t+\delta$ to obtain

$$|U_t(t+\delta)|^2 + \nu \int_t^{t+\delta} \|U_t\|^2 \leq |U_t(t)|^2 + \kappa \int_t^{t+\delta} \|U\|^2 |U_t|^2 + \frac{2\delta F_*}{\nu}.$$

Setting $\delta = 1/(\lambda_1 \nu)$ and applying the previous bounds for $t \geq t_A$ yields

$$\int_t^{t+1/(\lambda_1\nu)} \|U_t\|^2 \leq \sigma_V^2 \qquad \text{where} \qquad \sigma_V^2 = \frac{R_H^2}{\nu} + \frac{3C_0^4 \rho_H^2 R_H^2}{\nu^3} + \frac{2F_*}{\lambda_1 \nu^3}.$$

Now, upon taking H^1 norms of (1.2) and applying (A.3) from Lemma A.1 followed by Young's inequality we obtain

$$\begin{aligned}
\nu\|AU\| &\leq \|U_t\| + \|B(u,u)\| + \|f\| \\
&\leq \|U_t\| + C_1\|U\||AU| + C_2|U|^{1/2}|AU|^{3/2} + G^{1/2}.
\end{aligned}$$

Consequently

$$\|AU\|^2 \leq 4\nu^{-2}\big(\|U_t\|^2 + C_1^2\|U\|^2|AU|^2 + C_2^2|U||AU|^3 + G\big)$$

implies

$$\int_t^{t+1/(\lambda_1\nu)} \|AU\|^2 \leq \frac{4\sigma_V^2}{\nu^2} + 4C_1^2\Big(\frac{3\rho_V^2}{2\nu^3}\Big)\rho_V^2 + 4C_2^2\rho_H\rho_A\Big(\frac{3\rho_V^2}{2\nu^3}\Big) + \frac{4G}{\lambda_1\nu^3}.$$

Since the above bound is finite, this finishes the proof. □