WEAK* SOLUTIONS II: THE VACUUM IN LAGRANGIAN GAS DYNAMICS

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Abstract. We develop a framework in which to make sense of solutions containing the vacuum in Lagrangian gas dynamics. At and near vacuum, the specific volume becomes infinite and enclosed vacuums are represented by Dirac masses, so they cannot be treated in the usual weak sense. However, the weak* solutions recently introduced by the authors can be extended to include solutions containing vacuums. We present a definition of these *natural vacuum solutions* and provide explicit examples which demonstrate some of their features. Our examples are isentropic for clarity, and we briefly discuss the extension to the full 3×3 system of gas dynamics. We also extend our methods to one-dimensional dynamic elasticity to show that fractures cannot form in an entropy solution.

Key words. gas dynamics, vacuum, conservation laws, elasticity, fracture

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1. Introduction.

The oldest and most fundamental system of hyperbolic conservation laws is that of isentropic gas dynamics, which are the simplest analog of Newton's Law for a continuous medium. The equations can be expressed either in an Eulerian spatial frame, or in a Lagrangian or co-moving material frame. In the Lagrangian frame, the equations are

$$\partial_t v - \partial_x u = 0, \qquad \partial_t u + \partial_x p = 0, \tag{1}$$

where x is the material variable, v is the specific volume, and u and p are the fluid velocity and pressure, respectively. The system is closed by specifying a constitutive relation $p = P(v)$, a monotone decreasing function which is integrable as $v \to \infty$. Alternatively, in an Eulerian frame, the equations are

$$\partial_t \rho + \partial_y(\rho\, u) = 0, \qquad \partial_t(\rho\, u) + \partial_y(\rho\, u^2 + p) = 0,$$

representing conservation of mass and momentum, respectively, where y is the spatial variable, and $\rho = 1/v$ is the density.

The main effect of nonlinearity in a hyperbolic system is the presence of shock waves, across which the pressure and velocity are discontinuous, and the equations cannot be satisfied in the classical sense. This problem is usually solved by the use of weak solutions, which are defined by multiplying by test functions and integrating by parts. There is now a mature and largely complete theory of BV weak solutions of systems of conservation laws, provided the data is appropriately small [2].

Another effect of nonlinearity is the presence of a vacuum, which corresponds to $v = \infty$ or $\rho = 0$. The vacuum presents different difficulties depending on the frame: in an Eulerian frame, the equations degenerate and the velocity u is underdetermined, while in a Lagrangian frame the vacuum is formally described using a Dirac mass, so the class of weak solutions is not large enough. The goal of this paper is to rigorously justify the use of Dirac masses and thus present a satisfactory notion of solution which includes vacuums in a Lagrangian frame.

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In the recent paper [12], the authors introduced the notion of weak* solution, which we believe holds several advantages over weak solutions. Our approach is natural and general, and allows us to view the system as an evolutionary ODE in Banach space, which in turn confers some regularity. In addition, the "multiplication by test function and integration by parts" step is treated abstractly rather than explicitly, leading to cleaner calculations. Our approach is also general enough to handle certain extensions, including the treatment of vacuums as Dirac masses. In [12], we also proved that BV weak solutions are weak* solutions and *vice versa*, which implies that the well-known uniqueness and regularity results for BV solutions apply unchanged to weak* solutions.

To define a weak* solution of an abstract system of conservation laws,

$$\partial_t U + \partial_x F(U) = 0,$$

we begin with a normed vector space X of *spatial* test functions, and regard the solution $U(t)$ as a function taking values in the dual space X^* of X. For BV solutions, we take $X = C_0(\Omega)^n$ with $\Omega \subset \mathbb{R}$, so that $X^* = M(\Omega)^n$, the space of Radon measures. Then if $U \in BV^n_{loc}$, so is $F(U)$, and so the distributional derivative $\mathbb{D}_x F(U) \in X^*$. We then declare U to be a weak* solution if is satisfies the Banach space ODE

$$U' + \mathbb{D}_x F(U) = 0 \quad \text{in} \quad X^*_{loc} = M_{loc}(\mathbb{R})^n, \tag{2}$$

where U' is the appropriate time derivative of $U(t)$. This is the Gelfand weak* derivative, or G-derivative, defined by using the Gelfand weak* integral of functions $\phi : [0,T] \to X^*$. The associated spaces are labelled $W^{1,q}_{w*}(0,T;X^*)$.

In this paper, we extend the ideas of [12] to include the use of Dirac masses in weak* solutions. The key observation is that in (1), although v is no longer bounded, or even a function, the flux vector (u,p) remains BV, so that its spatial derivative is a measure, so lives in X^*, and the ODE (2) makes sense. Instead of treating the constitutive relation $p = P(v)$ as a pointwise function, we regard it as a map of fields,

$$P : \mathcal{A} \subset L^1(\Omega) \to L^1(\Omega) \quad \text{via} \quad p = P \circ v : \Omega \to \mathbb{R},$$

and in order to extend weak* solutions, we need only extend this to a map $\widehat{P}$ defined on positive Radon measures. Since pressure vanishes at vacuum, this extension is easily accomplished using the Lebesgue decomposition theorem. To avoid unphysical solutions, we impose a condition which we call *consistency of the medium*, and which states that the density and pressure must vanish whenever a vacuum is present; although this can be regarded as an entropy-type condition, it is distinct from the usual entropy condition which degenerates to an equality at vacuum. We refer to a vacuum weak* solution which satisfies consistency of the medium as a *natural vacuum solution*. In our framework the entropy and entropy flux are also regarded as maps on L^1 which are similarly extended to positive Radon measures. The entropy production is calculated to be a measure which is supported on shocks, and which is required to be negative. This again agrees with the entropy condition for BV weak solutions.

Once we have defined natural vacuum solutions to (1) that include Dirac masses which account for vacuums of finite extent, we present a few detailed examples. These are natural vacuum solutions but not weak solutions, and our explicit description of the solutions and calculations of norms clearly demonstrates the advantages gained by treating the test functions and integration by parts abstractly and implicitly in the spaces $W^{1,q}_{w*}(0,T;X^*)$.

We next describe the straight-forward extension of our results to the full 3×3 equations of gas dynamics in a Lagrangian frame. We again define an extension of the pressure and specific internal energy to the positive Radon measures, by declaring that the pressure and internal energy vanish at vacuum. We then define a weak* solution and the corresponding entropy condition as would be expected.

As a final application, we extend our results to the equations of one-dimensional elasticity,

$$\partial_t u - \partial_x v = 0, \qquad \partial_t v - \partial_x \tau(u) = 0,$$

where u, v and $\tau(u)$ are the strain, velocity and stress, respectively; we assume that $\tau'(u) > 0$, with a softening response, $\tau''(u) < 0$. Here we reproduce results of Giesselmann and Tzavaras [9], in which they introduce so-called *slic*-solutions to study crack formation and resolve an apparent paradox of nonuniqueness of solutions found in [15]. Following [9], we study the onset of fracture, which we represent as a Dirac mass in the strain. To do so, we obtain the natural extension of the stress to Dirac masses, namely

$$\widehat{\tau}(w_0\, \delta_{x_0}) = L_\tau\, w_0\, \delta_{x_0}, \quad \text{where} \quad L_\tau := \lim_{u\to\infty} \frac{\tau(u)}{u}.$$

Extending the stress and the energy allows us to define weak* solutions, and a brief analysis reveals that weak* solutions admitting a crack are defined if and only if $L_\tau = 0$; however, none of these solutions are entropic. These are the same conclusions as those of [9], but our results significantly extend the one-dimensional results of [9], because their analysis applies to the single example of a solution provided in [15], while ours hold for any crack in a weak* solution. In [9], slic solutions are obtained as limits of mollified approximations, and their calculation of a single example requires several integrations and error estimates. In contrast, with our approach the mollification and integration by parts is abstract, and we are able to work directly with measures, leading to a direct and exact development without the need for error estimates.

The paper is arranged as follows: in section 2, we set notation and recall the definition and properties of weak* integrable functions and the Gelfand integral, developed in our earlier paper [12]. Next we recall the definition of weak* solutions to conservation laws, and specifically to gas dynamics (1), and extend this definition to include vacuums. We derive generalized Rankine-Hugoniot jump conditions and discuss the entropy condition, while showing that it remains an identity at the vacuum. In section 4 we present some detailed examples of natural vacuum solutions which are not weak solutions. Section 5 briefly describes the extension of our methods to the full system of gas dynamics, and in section 6 we consider the onset and propagation of fractures in one-dimensional elasticity.

2. Preliminaries.

We begin by setting notation and recalling the Gelfand integral and related notions which are necessary to define weak* solutions of systems of conservation laws. For simplicity we work in a single space dimension. We refer the reader to [12] for a more detailed discussion and proofs of quoted results.

2.1. Banach spaces.

Given a vector space X with norm $\|\cdot\|_X$, we denote its dual by X^*, and recall

$$\|\phi\|_{X^*} := \sup_{x\in X, x\neq 0} \frac{\langle \phi, x\rangle}{\|x\|_X}.$$

We denote the n-fold product by $X^n := \big\{(x_1, x_2, \dots, x_n) : x_i \in X\big\}$, and equip it with the "Euclidean" norm

$$\big\|x\big\|_{X^n} := \Big(\sum_{i=1}^{n} \big\|x_i\big\|_X^2\Big)^{1/2}.$$

It follows that if we define the action of $\phi \in (X^*)^n$ on $x \in X^n$ by

$$\big\langle \phi, x\big\rangle := \sum_{i=1}^{n} \big\langle \phi_i, x_i\big\rangle,$$

then we can write $(X^*)^n = (X^n)^*$. In particular, any statements on scalar valued function spaces $X = X(\Omega) = \{f : \Omega \to \mathbb{R}\}$ extend naturally to vector-valued functions $X^n = \{F : \Omega \to \mathbb{R}^n\}$.

We recall the hierarchy of spaces that are most useful for us: first, fixing an open bounded $\Omega \subset\subset \mathbb{R}$, we have the inclusions

$$BV(\Omega) \subset L^\infty(\Omega) \subset L^p(\Omega) \subset L^1(\Omega).$$

Next, any $f \in L^1(\Omega)$ generates a measure $\mu_f = \iota(f)$, given by

$$\iota(f)(E) = \mu_f(E) := \int_E f\, dx, \quad \text{for each} \quad E \in \mathcal{B}(\Omega),$$

so we regard $\iota(L^1(\Omega)) \subset M(\Omega)$, the set of Radon measures on Ω; moreover, we have

$$\big\|\iota(f)\big\|_{M(\Omega)} = \big|\mu_f\big|(\Omega) = \int_\Omega |f|\, dx = \big\|f\big\|_{L^1(\Omega)}.$$

Note that for any $f \in L^1(\Omega)$, $\iota(f) \ll \lambda$, that is $\iota(f)$ is absolutely continuous with respect to Lebesgue measure, and indeed, $f = \frac{d\mu_f}{d\lambda}$ is the Radon-Nikodym derivative of $\iota(f)$. On the other hand, by the Lebesgue decomposition theorem, any Radon measure $\mu \in M(\Omega)$ can be uniquely decomposed into absolutely continuous and singular parts,

$$\mu = \mu_c + \mu_s \quad \text{with} \quad \mu_c \ll \lambda \quad \text{and} \quad \mu_s \perp \lambda,$$

and moreover $\frac{d\mu_c}{d\lambda} \in L^1(\Omega)$. We thus define the map

$$\Pi : M(\Omega) \to L^1(\Omega) \quad \text{by} \quad \Pi(\mu) := \frac{d\mu_c}{d\lambda} \in L^1(\Omega), \tag{3}$$

the Radon-Nikodym derivative of the absolutely continuous part of μ. It then follows that

$$\Pi \circ \iota(f) = f \quad \text{for} \quad f \in L^1(\Omega),$$

while also

$$\iota \circ \Pi(\mu) = \mu_c \quad \text{for} \quad \mu_c + \mu_s =: \mu \in M(\Omega), \tag{4}$$

so that $\iota \circ \Pi : M(\Omega) \to M(\Omega)$ is projection onto the absolutely continuous part of the measure.

Recall that the Radon measures form the dual of C_0: that is, regarding $C_0(\Omega)$ as the closure of $C_c^\infty(\Omega)$ under the sup-norm, we can regard $M(\Omega) = C_0(\Omega)^*$ under the action

$$\big\langle \mu, \varphi\big\rangle = \int_\Omega \varphi(x)\, \mu(dx), \quad \varphi \in C_0(\Omega),$$

and it is not difficult to verify that $\big\|\mu\big\|_{C_0(\Omega)^*} = \big\|\mu\big\|_{M(\Omega)}$.

Definition 1. *We say that $f \in X^*$ has an X^*-valued distributional derivative, written $\mathbb{D}_x f \in X^*$, if, for all $\phi \in C_c^\infty(\Omega) \subset X$, we have*

$$\big|\big\langle f, \phi' \big\rangle\big| \leq C\, \|\phi\|_X,$$

where we recall $C_c^\infty(\Omega)$ is dense in X, and in this case we define $\mathbb{D}_x f$ by

$$\big\langle \mathbb{D}_x f, \phi \big\rangle := -\big\langle f, \phi' \big\rangle.$$

Finally, recall that $BV(\Omega)$ is the set of functions whose distributional derivative $\mathbb{D}_x f$ is in L^1:

$$\|f\|_{BV} = \sup \sum \big|f(x_{i+1}) - f(x_i)\big| = \int_\Omega |\mathbb{D}_x f|\, dx = \big\|\mathbb{D}_x f\big\|_{L^1},$$

the supremum and sum being taken over finite ordered partitions.

We can combine the above together with product spaces, using inclusions as necessary, to get the following hierarchy of spaces:

$$BV(\Omega)^n \subset L^\infty(\Omega)^n \subset L^1(\Omega)^n \subset M(\Omega)^n = C_0(\Omega)^{*n}, \tag{5}$$

where these functions take values in $\mathbb{R}^n$. Moreover, since $\Omega \subset\subset \mathbb{R}$ is arbitrary, these inclusions extend to locally bounded functions on all of $\mathbb{R}$,

$$BV_{loc}^n \subset (L_{loc}^\infty)^n \subset (L_{loc}^1)^n \subset M_{loc}^n. \tag{6}$$

2.2. The Gelfand integral. We next recall the definition and calculus of the Gelfand integral, which we need to define weak* solutions. Again we refer the reader to [12] for more details and proofs of statements. We briefly discuss different ways to integrate functions mapping to an abstract Banach space, namely the Bochner integral and Gelfand integral.

The Bochner integral of $f : [0,T] \to X$ is obtained by approximating functions by simple functions. The function f is strongly measurable, or Bochner measurable, if $f^{-1}(E)$ is measurable for each measurable $E \subset X$. The integral of a simple measurable function f is defined in the usual way,

$$\int \sum u_i\, \mathcal{X}_{E_i}(t)\, dt = \sum u_i\, \lambda(E_i) \in X,$$

and f is Bochner integrable if there is a sequence $\{h_n\}$ of simple functions such that the Lebesgue integral $\int_0^T \|h_n - f\|\, dt \to 0$ as $n \to \infty$, and in this case we have $\int_E f\, dt = \lim \int_E h_n\, dt$.

The Bochner integral requires strong measurability, which is not always obvious in an abstract Banach space. The Dunford integral is a weak integral, defined using the functionals on X. For our purposes it is more convenient to use the Gelfand integral, which is defined for functions ϕ which take values in the dual space X^* of a Banach space X. The map $\phi : [0,T] \to X^*$ is weak* measurable if $\big\langle \phi(\cdot), \alpha \big\rangle : [0,T] \to \mathbb{R}$ is Lebesgue measurable for all $\alpha \in X$. Two functions ϕ and ψ are weak* equivalent if $\big\langle \phi(\cdot), \alpha \big\rangle = \big\langle \phi(\cdot), \alpha \big\rangle$ for λ-almost all t. It can be shown that any weak* measurable function ϕ is weak* equivalent to a function $\widehat{\phi}$ which is norm-measurable, by which we mean the scalar function $\|\widehat{\phi}(\cdot)\|_{X^*}$ is Lebesgue measurable. We will denote the weak* equivalence class of a weak* measurable ϕ by $[\phi]$, and a norm-measurable representative by $\widehat{\phi} \in [\phi]$, although we will often abuse notation by simply writing ϕ when there is no ambiguity.

The Gelfand integral is defined as follows. Suppose that we are given a weak*-measurable function $\phi : [0,T] \to X^*$, and suppose also that

$$\big\langle \phi(\cdot), \alpha \big\rangle \in L^1(0,T) \quad \text{for all} \quad \alpha \in X\,.$$

For a given Borel set E, we define the map $T_E : X \to L^1(0,T)$ by

$$T_E(\alpha) = \big\langle \phi(\cdot), \alpha \big\rangle\, \chi_E(\cdot) \in L^1(0,T)\,.$$

It is clear that T_E is linear, and if $\alpha_n \to \alpha$ and $T_E(\alpha_n) \to y$ in L^1, then by the Riesz-Fischer theorem, a subsequence $T_E(\alpha_{n_k})(s) \to y(s)$ a.e., while also $T_E(\alpha_n)(s) \to \big\langle \phi(s), \alpha \big\rangle\, \chi_E(s)$ for all $s \in [0,T]$. It follows that $y \in L^1(0,T)$, so T_E is closed, and further, by the closed graph theorem, it is bounded, so we can write $\|T_E(\alpha)\|_{L^1} \le \|T_E\|\,\|\alpha\|$ for all $\alpha \in X$. Since integration is a bounded linear operator of L^1 into $\mathbb{R}$, it follows that the map

$$\alpha \mapsto \int_0^T T_E(\alpha)(s)\; ds = \int_E \big\langle \phi(s), \alpha \big\rangle\; ds$$

is a bounded linear functional on X, so defines an element of the dual X^*. This functional is the Gelfand integral of ϕ over E, and we denote it by $\star\!\int_E \phi(s)\; ds \in X^*$. Thus the Gelfand integral over a measurable set E is that element of X^* defined by the condition

$$\Big\langle \star\!\int_E \phi(s)\; ds, \alpha \Big\rangle = \int_E \big\langle \phi(s), \alpha \big\rangle\; ds \quad \text{for all} \quad \alpha \in X\,. \tag{7}$$

Again it follows easily that if ϕ is Bochner integrable with values in X^*, then it is Gelfand integrable and the integrals coincide.

2.3. Gelfand-Sobolev Spaces. We now describe the X^* valued Gelfand L^q spaces, for $1 \le q \le \infty$. Given a weak* equivalence class $[\phi]$ of Gelfand integrable functions, set

$$\big|\!\big|\!\big|\, [\phi] \,\big|\!\big|\!\big|_q := \inf \big\{ \|g\|_{L^q(0,T)} \;:\; \|\widehat{\phi}(t)\| \le g(t) \;\lambda\text{-a.e.} \big\}\,,$$

where $\widehat{\phi} \in [\phi]$ is a norm-measurable element of the equivalence class. It follows that $|\!|\!|\, \cdot \,|\!|\!|_q$ is a norm, and we let $L^q_{w*}(0,T;X^*)$ be the space of equivalence classes $[\phi]$ of finite norm,

$$L^q_{w*}(0,T;X^*) := \big\{ [\phi] : \big|\!\big|\!\big|\, [\phi] \,\big|\!\big|\!\big|_q < \infty \big\}\,.$$

It is not difficult to show that $L^q_{w*}(0,T;X^*)$ is a Banach space and that the trivial inclusion of the Bochner L^q space in the Gelfand L^q space

$$L^q(0,T;X^*) \subset L^q_{w*}(0,T;X^*) \quad \text{via} \quad f \mapsto [f],$$

is a norm-preserving isomorphism. Moreover, if $\widehat{\phi} \in [\phi] \in L^q_{w*}(0,T;X^*)$ is norm-measurable, then $\|\widehat{\phi}\| \in L^q(0,T)$ and

$$\big|\!\big|\!\big|\, [\phi] \,\big|\!\big|\!\big|_q = \big\|\, \|\widehat{\phi}(\cdot)\| \,\big\|_{L^q(0,T)}\,.$$

It follows that if ϕ is Bochner integrable, then we can calculate the Gelfand integral as a Bochner integral.

Now suppose that $\phi,\ \psi : [0,T] \to X^*$ are weak* integrable, so that $[\phi],\ [\psi] \in L^1_{w*}(0,T;X^*)$. We say that ψ is the *Gelfand weak derivative* or G-weak derivative of ϕ, written $\phi'(t) = \psi(t)$ or $[\phi'] = [\psi]$, if

$$\begin{gathered} \star\!\int_0^T \phi(t)\,\eta'(t)\,dt = -\star\!\int_0^T \psi(t)\,\eta(t)\,dt, \quad \text{that is} \\ \int_0^T \big\langle \phi(t),\alpha\big\rangle\,\eta'(t)\,dt = -\int_0^T \big\langle \psi(t),\alpha\big\rangle\,\eta(t)\,dt, \end{gathered} \tag{8}$$

for all $\alpha \in X$ and scalar functions $\eta \in C_c^\infty(0,T)$.

We now define the space $W^{1;q}_{w*}\big(0,T;X^*\big)$, for $1 \le q \le \infty$, to be the set of weak* equivalence classes $[\phi] \in L^q_{w^*}\big(0,T;X^*\big)$ with G-weak derivative $[\phi'] \in L^q_{w*}\big(0,T;X^*\big)$, with norm

$$\big\|\!\big|\, [\phi] \,\big\|\!\big|_{W^{1;q}_{w*}(0,T;X^*)} := \begin{cases} \Big(\int_0^T (\|\widehat{\phi}(t)\|^q + \|\widehat{\phi}'(t)\|^q)\,dt\Big)^{1/q}, & 1 \le q < \infty \\ \operatorname{ess\,sup}_{t\in[0,T]} \big(\|\widehat{\phi}(t)\| + \|\widehat{\phi}'(t)\|\big), & q = \infty\,, \end{cases}$$

for norm-measurable representatives $\widehat{\phi}$ and $\widehat{\phi}'$.

If in addition, ϕ has values in some $Y \subset X^*$, then we write $\phi \in W^{1;q}_{w*}\big(0,T;Y,X^*\big)$, that is we set

$$W^{1;q}_{w*}\big(0,T;Y,X^*\big) = \Big\{ \phi \in W^{1;q}_{w*}(0,T;X^*) \ :\ y(t) \in Y,\ t \in [0,T] \Big\}.$$

Note that we do not assume that Y is a subspace of X^*, because we use the topology of X^* throughout.

In [12] we state and prove some basic calculus theorems for the Gelfand integral, and the interested reader is referred there for details. We summarize the main points in the following theorem, which collects parts of Theorems 3.5 and 3.7 of [12].

THEOREM 2. *If $f \in W^{1;q}_{w*}(0,T,X^*)$, then it has an absolutely continuous representative $\overline{f} : [0,T] \to X^*$, which satisfies*

$$\overline{f}(t_2) - \overline{f}(t_1) = \star\!\int_{t_1}^{t_2} f'(s)\,ds, \tag{9}$$

for all $t_1,\ t_2 \in [0,T]$. Moreover, for all $\alpha \in W^{1,p}(0,T;X)$ strongly integrable, we have the integration by parts formula

$$\int_{t_1}^{t_2} \big\langle f'(t),\alpha(t)\big\rangle\,dt = \big\langle \overline{f}(s),\overline{\alpha}(s)\big\rangle\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \big\langle f(t),\alpha'(t)\big\rangle\,dt\,. \tag{10}$$

3. Weak* solutions. In [12], the authors introduced the notion of weak* solutions to a general system of hyperbolic conservation laws in one space dimension. Given such a system,

$$\partial_t U + \partial_x F(U) = 0, \quad U(0,\cdot) = U_0, \tag{11}$$

with $U,\ F(U) : [0,T]\times\mathbb{R} \to \mathbb{R}^n$, recall that a distributional solution is a locally integrable function $U(t,x)$ satisfying

$$\int_0^\infty \int_{\mathbb{R}} \big(U(t,x)\,\partial_t\varphi(t,x) + F(U(t,x))\,\partial_x\varphi(t,x)\big)\,dx\,dt + \int_{\mathbb{R}} U_0(x)\,\varphi(0,x)\,dx = 0,$$

for all compactly supported test functions φ, and if in addition U is locally bounded, it is a weak solution. We note that the necessity of explicitly multiplying by test function and integrating by parts means that calculations are unwieldy and often error estimates must be employed when analyzing weak solutions.

On the other hand, when considering weak* solutions, we will treat the conservation law (11) as an ODE in an appropriate Banach space. Indeed, we look at (11) directly and allow this to act linearly on the Banach space X which contains C_c^∞ as a dense subspace. That is, for each t, we treat $U(t) = U(t,\cdot)$ and $\mathbb{D}_x F(U(t,\cdot))$ as living in X^*, and we regard (11) as an ODE in X^*, so that

$$U' + \mathbb{D}_x F(U(t,\cdot)) = 0, \quad U(0) = U_0, \tag{12}$$

for appropriately defined time derivative U'. The critical issue for us is to make sense of the nonlinear flux $F(U)$ and its derivative in the space X^*.

We then say that

$$U \in W^{1,q}_{w*}\big(0, T^-; Y_{loc}, X^*_{loc}\big)$$

is a *weak* solution* of the system (11) if

$$U' + \mathbb{D}_x F(U(t)) = 0 \quad \text{in} \quad L^q_{w*}(0, T; X^*_{loc}),$$

and if $\overline{U}(0) = U_0$ in X^*_{loc}, where $\overline{U}(t)$ is the continuous representative of the weak* equivalence class, and where U' is the G-weak derivative of U. Here X^*_{loc} is understood in the usual sense and we allow any $1 \le q \le \infty$.

In our previous paper [12], we used $X = C_0(\Omega)^n$, so that $X^* = M(\Omega)^n$, and we took $Y = BV(\Omega)^n$. In that paper we studied the connections between weak* solutions and weak solutions, and proved the following theorem.

THEOREM 3. *Suppose $U \in W^{1,q}_{w*}\big(0, T^-; BV^n_{loc}, M^n_{loc}\big)$ is a weak* solution to the Cauchy problem (11), with continuous representative $\overline{U}$. Then $\overline{U}$ is Hölder continuous as a function into $L^1_{loc}(\mathbb{R};\mathbb{R}^n)$, that is, $\overline{U} \in C^{0,\,1-1/q}(0, T^-; L^1_{loc})$ for $1 \le q \le \infty$. The function $\overline{U}(t,x)$ is a distributional solution of the Cauchy problem (11). In particular, if U is locally bounded, that is $U \in L^\infty_{w*}\big(0, T^-; L^\infty_{loc}(\mathbb{R};\mathbb{R}^n)\big)$, then $\overline{U}(t,x)$ is also a weak solution to the Cauchy problem (11).*

In the same paper, we showed that a distributional solution with appropriate bounds is also a weak* solution, and in particular BV weak solutions are weak* solutions. As an immediate consequence, it follows that the global weak solutions generated by Glimm's method, front tracking, and vanishing viscosity, all of which have uniformly bounded total variation, are all weak* solutions, and the uniqueness and stability results of Bressan et.al. hold unchanged in the framework of weak* solutions.

3.1. Application to Isentropic Gas Dynamics. Because of the flexibilty provided by the choices of growth rate q and spaces Y and X^*, we regard weak* solutions as more general than weak solutions. Indeed, we will generalize weak* solutions to include the vacuum in a Lagrangian frame, in which local boundedness is lost and the specific volume is allowed to be a measure.

We work with the system of gas dynamics in a Lagrangian frame, namely

$$\begin{aligned} \partial_t v - \partial_x u &= 0, \\ \partial_t u + \partial_x p &= 0, \end{aligned} \tag{13}$$

in which the pressure p is specified as a function of specific volume v by a constitutive relation of the form

$$p = P(v), \quad \text{that is} \quad p(t,x) = P(v(t,x)), \tag{14}$$

satisfying the appropriate properties: the most common such constitutive law is that of an ideal gas, for which $P(v) = A\,v^{-\gamma}$, $\gamma > 1$.

It follows immediately that as long as v remains BV, then a BV weak* solution can be defined as above. However, we want to allow solutions which include vacuums, which are represented by Dirac masses in a Lagrangian frame. To do so, we simply allow the specific volume $v(t)$ to be a Radon measure, which includes all Dirac masses. We note that the velocity u remains BV, even when v is unbounded and includes Dirac masses. We thus extend the target set Y to include Dirac masses in the first component, while still requiring that the vector of conserved quantities remain in the set $W^{1,q}(0,T;M^2_{loc})$. In order for this extension to make sense, we must extend the constitutive relation so that the pressure is defined for any specific volume, which can now be a positive Radon measure.

The constitutive relation expresses the thermodynamic pressure in terms of the specific volume, as $p = P(v)$. This extends naturally to a map of functions,

$$P : \mathcal{A} \subset L^1(\Omega) \to L^1(\Omega) \quad \text{via} \quad p = P \circ v : \Omega \to \mathbb{R}, \tag{15}$$

where $\mathcal{A}$ is the domain of P, and allows us to close (13). We now wish to extend this constitutive map to be defined on Radon measures, and use this to define vacuum solutions of (13), which will include Dirac masses which represent vacuums.

Recalling the Lebesgue decomposition, in the notation of (3), (4), we write the measure $\mu \in M_{loc}(\mathbb{R})$ as

$$\mu = \mu_c + \mu_s, \quad \text{with} \quad \mu_c \ll \lambda, \quad \mu_s \perp \lambda,$$

so that for any Borel set A,

$$\mu(A) = \int_A \Pi(\mu)\, dx + \mu_s(A),$$

where $\Pi(\mu) = \frac{d\mu_c}{d\lambda} \in L^1_{loc}(\mathbb{R})$ is the Radon-Nikodym derivative of the absolutely continuous part of μ. Since the Lebesgue decomposition is unique, we extend the constitutive function to be defined on positive measures $M_{loc}(\mathbb{R})_+$ by

$$\widehat{P} : M_{loc}(\mathbb{R})_+ \to L^1_{loc}(\mathbb{R}), \quad \text{by} \quad \widehat{P}(\mu) = P(\Pi(\mu)),$$

since pressure vanishes at vacuum. That is, we declare that the singular part of the specific volume makes no contribution to the pressure.

When generalizing the specific volume to a positive measure, we use the following notation: given $V \in M_{loc}$ and referring to (3), (4), we write

$$v := \Pi(V) \quad \text{and} \quad \nu := V - \iota(v),$$

so that $V = \iota(v) + \nu$, with $v \in L^1_{loc}$ and $\nu \perp \lambda$. It then follows that the (generalized) pressure is

$$\widehat{P}(V) = \widehat{P}(\iota(v) + \nu) = P(v),$$

so that, as expected, the generalized pressure is the composition of the pressure function with the Radon-Nikodym derivative v of the absolutely continuous part of the measure V.

As a first attempt at defining a solution with vacuum, we again take X to be the set of continuous test functions, $X = C_0(\Omega)^2$, and we set

$$\widetilde{Y}_{loc} = M_{loc}(\mathbb{R})_+ \times BV_{loc}(\mathbb{R}) \subset M_{loc}(\mathbb{R})^2 =: X^*_{loc},$$

where $M_{loc}(\mathbb{R})_+$ denotes Radon measures that are (strictly) positive on all open sets, so that

$$\mu \in M_{loc}(\mathbb{R})_+ \quad \text{iff} \quad \mu\big((a,b)\big) > 0 \quad \forall\, a < b.$$

Definition 4. *A* vacuum weak* solution *of the p-system* (13) *is a pair*

$$(V,u) \in W^{1,q}_{w*}(0,T;\widetilde{Y}_{loc}, X^*_{loc}),$$

satisfying

$$\begin{aligned} V' - \mathbb{D}_x u &= 0 \\ u' + \mathbb{D}_x \widehat{P}(V) &= 0 \end{aligned} \qquad \text{in } L^q_{w*}(0,T;X^*_{loc}), \tag{16}$$

where $'$ denotes the G-weak derivative. When solving a Cauchy problem, the Cauchy data (V_0,u_0) must be taken on in the space X^* by the time-continuous representative $(\overline{V},\overline{u})$, that is

$$\big(\overline{V}(0),\overline{u}(0)\big) = \big(V_0,u_0\big) \quad \text{in} \quad X^*_{loc}.$$

3.2. Properties of Solutions with Vacuum. As in the general case of BV weak* solutions, we immediately observe that vacuum weak* solutions have some implicit regularity: first, the solutions have an absolutely continuous representative $\big(\overline{V}(t),\overline{u}(t)\big) \in X^*$. Also, since the flux $(-u,p)$ has a distributional derivative in $X^* = M^2_{loc}$, both $\overline{u}(t)$ and $p(t) = \widehat{P}(\overline{V}(t))$ are BV functions (of material variable x) for all t.

3.2.1. Evolution of Atomic Measures. Next, recalling that x is a material rather than spatial variable, we show that vacuums are stationary in a Lagrangian frame.

Lemma 5. *A nontrivial continuous Dirac measure is stationary: that is, a measure*

$$\mu : (a,b) \to M(\Omega) \quad \text{of the form} \quad \mu = w(t)\,\delta_{X(t)} \in M(\Omega),$$

with $w \neq 0$ and $X : (a,b) \to \Omega$, is continuous on the interval (a,b) if and only if $w(t)$ is continuous and $X(t)$ is constant on (a,b).

Proof. Recalling that $\|\delta_x\|_M = 1$, it follows easily that for x, $y \in \Omega$, and α, $\beta \in \mathbb{R}$,

$$\|\alpha\,\delta_x - \beta\,\delta_y\|_{M(\Omega)} = (|\alpha| + |\beta|)\,\mathbb{1}_{\{x\neq y\}} + |\alpha - \beta|\,\mathbb{1}_{\{x=y\}},$$

where $\mathbb{1}_E$ is the indicator function on E.

It follows immediately that if $w(t)$ is continuous on (a,b), then so is the stationary measure $\mu(t) = w(t)\,\delta_{X_0}$, for any $X_0 \in \Omega$.

Similarly, for t, $t_0 \in (a,b)$, we have

$$\big\|\mu(t) - \mu(t_0)\big\|_{M(\Omega)} = \big(|w(t)| + |w(t_0)|\big)\;\mathbb{1}_{\{X(t)\neq X(t_0)\}} + |w(t) - w(t_0)|\;\mathbb{1}_{\{X(t)=X(t_0)\}}, \tag{17}$$

both terms being non-negative. Now if $\mu \in M(\Omega)$ is continuous at t_0, then

$$\|\mu(t) - \mu(t_0)\|_{M(\Omega)} \to 0, \quad \text{as} \quad t \to t_0,$$

so, since $|w(t_0)| \neq 0$, (17) implies both

$$\lim_{t\to t_0} \mathbb{1}_{\{X(t)\neq X(t_0)\}} = 0 \quad \text{and} \quad \lim_{t\to t_0} |w(t) - w(t_0)|\, \mathbb{1}_{\{X(t)=X(t_0)\}} = 0.$$

It follows that, given any $\epsilon > 0$, there exists $\eta > 0$ such that $X(t) = X(t_0)$ for all $t \in (t_0 - \eta, t_0 + \eta)$, and moreover

$$|w(t) - w(t_0)| = \big\|\mu(t) - \mu(t_0)\big\|_{M(\Omega)} < \epsilon \quad \text{for} \quad |t - t_0| < \eta.$$

Since t_0 is an arbitrary point in (a, b), $w(t)$ is continuous on (a, b). Finally, let $(c, d) \subset (a, b)$ be the maximal interval for which $X(t) = X(t_0)$ for all $t \in (c, d)$. If $c > a$, find another ϵ_1 so that $X(t) = X(c)$ for $t \in (c - \epsilon_1, c + \epsilon_1)$ to obtain a contradiction; this implies $c = a$. Similarly, $d = b$ and the result follows. ∎

Note that in other topologies such as the Wasserstein distance used in mass transfer problems, continuity need not imply that singular measures are stationary.

3.2.2. Evolution of Unbounded Maps. We next show that integrable functions $f \in W^{1,q}_{w*}$ which are unbounded blow up on stationary sets, consistent with vacuums being stationary in a material coordinate.

To this end, let $\Omega = (c, d) \subset \mathbb{R}$ and $X \in C^1\big([a, b], \Omega\big)$, so that the curve

$$\mathcal{C} = \Big\{(x, t) \ : \ t \in [a, b],\ x = X(t)\Big\} \subset [a, b] \times \Omega, \tag{18}$$

and let $\gamma > 0$ be such that

$$\gamma < \sup_{t\in[a,b]} \Big(\min\big\{|X(t) - c|,\ |X(t) - d|\big\}\Big).$$

Also suppose that the function $f(t, x) : (a, b) \times \Omega \to (0, \infty)$ is continuous at each point of the set $\big((a, b) \times \Omega\big)\backslash\mathcal{C}$, that the possibly infinite one-sided limits $\lim_{x\to X(t)^\pm} f(t, x)$ exist for each $t \in (a, b)$, and that for some $1 < q \leq \infty$, the map

$$t \to f(t, \cdot) \in W^{1,q}_{w*}\big(a, b; L^1(\Omega), M(\Omega)\big). \tag{19}$$

Denote the sets on which f is unbounded by

$$S^\pm_\infty = \Big\{t \in [a, b] \ : \ \lim_{x\to X(t)^\pm} f(t, x) = \infty\Big\} \quad \text{and} \quad S_\infty = S^-_\infty \bigcup S^+_\infty\,.$$

We first show that f is almost uniformly unbounded on the set S_∞, in the sense of [21].

Lemma 6. *The sets S^+_∞, S^-_∞, and S_∞ are measurable, and for any $\eta > 0$, there are measurable sets $A^\pm \subset S^\pm_\infty$, with*

$$\lambda(S^-_\infty\backslash A^-) < \eta \quad \textit{and} \quad \lambda(S^+_\infty\backslash A^+) < \eta,$$

such that for every $m \in \mathbb{N}$, there exists $\delta_m > 0$ such that

$$\operatorname*{ess\,inf}_{t\in A^\pm} f\big(t, X(t) \pm \epsilon\big) > m \quad \textit{for all} \quad 0 < \epsilon < \delta_m\,.$$

Proof. For $\epsilon \in (0,\gamma)$, the functions

$$g_\epsilon^-(t) = f(t, X(t) - \epsilon) \quad \text{and} \quad g_\epsilon^+(t) = f(t, X(t) + \epsilon)$$

are defined and continuous on all of $[a,b]$.

We have

$$S_\infty^\pm = \Big\{ t \in (a,b) : \lim_{\epsilon \to 0^+} g_\epsilon^\pm(t) = \infty \Big\},$$

so we can write this as

$$S_\infty^\pm = \bigcap_{k=1}^\infty \left\{ \bigcup_{N=\lceil 1/\gamma \rceil}^\infty \bigcap_{n=N}^\infty \Big\{ t \in [a,b] \ : \ g_{\frac{1}{n}}^\pm(t) > k \Big\} \right\},$$

and continuity of $g_{\frac{1}{n}}^\pm$ yields measurability of $S_\infty^\pm$.

Now take any $\eta > 0$. By assumption $g_\epsilon^- > 0$ on (a,b) for every $\epsilon \in (0,\gamma)$, so we can write

$$S_\infty^\pm = \Big\{ t \in (a,b) \ : \ \lim_{\epsilon \to 0^+} \frac{1}{g_\epsilon^\pm(t)} = 0 \Big\}.$$

By Zakon [21], there exist measurable sets $A^\pm \subset S_\infty^\pm$, with $\lambda(S_\infty^\pm \backslash A^\pm) < \eta$, such that for every $m \in \mathbb{N}$ there exists $\delta_m > 0$ such that

$$\operatorname*{ess\,sup}_{t \in A^\pm} \frac{1}{g_\epsilon^\pm(t)} < \frac{1}{m} \quad \text{for all} \quad 0 < \epsilon < \delta_m,$$

and the proof follows. ∎

We next show that if the discontinuity $X(t)$ is non-stationary, then f is bounded almost everywhere along $\mathcal{C}$.

Lemma 7. *Let $\sigma > 0$ and suppose that the curve $\mathcal{C}$ given in (18) satisfies $X' > \sigma$ on (a,b). If f satisfies the conditions (19) given above, then*

$$\lambda(S_\infty^+) = \lambda(S_\infty^-) = \lambda(S_\infty) = 0\,.$$

The same conclusion holds if $X' < -\sigma$ on (a,b).

Proof. We shall obtain a contradiction by constructing a sequence of test functions $\psi_m \in W^{1,\infty}\big(a,b;C_0(\Omega)\big)$ for which one side of the integration by parts formula (10) is unbounded, while the other remains bounded.

Without loss of generality, we assume that that $\lambda(S_\infty^-) > 0$ and $X' > \sigma > 0$ on (a,b). According to Lemma 6, there exists a set A^- with $\lambda(A^-) > \frac{1}{2}\lambda(S_\infty^-)$, such that for every $m \in \mathbb{N}$ there exists $0 < \delta_m < \gamma$ such that

$$\operatorname*{ess\,inf}_{t \in A^-} f(t, X(t) - \epsilon) > m \quad \text{for all} \quad 0 < \epsilon < \delta_m\,. \tag{20}$$

Let $\varphi : [0,\infty) \to \mathbb{R}$ be a C^1 monotone function such that $\varphi(x) = 1$ for $x \le \frac{1}{8}$, $\varphi(x) = 0$ for $x \ge \frac{7}{8}$, $\varphi'(x) = -\frac{3}{2}$ for $\frac{1}{4} \le x \le \frac{3}{4}$, and $-\frac{3}{2} \le \varphi'(x) \le 0$ elsewhere. For each $m \in \mathbb{N}$, define

$$\phi_m(x) = \begin{cases} \varphi(-x/\delta_m), & x \le 0, \\ \varphi(x/\gamma), & x \ge 0, \end{cases} \quad \text{and} \quad \psi_m(t,x) = \phi_m(x - X(t)),$$

so that $\psi_m \in C^1(\mathcal{U})$. Moreover, $\psi_m(t,\cdot)$ and $\partial_t\psi_m(t,\cdot)$ are in $C_0(\Omega)$ for each $t \in (a,b)$, so the map $t \to \psi_m(t,\cdot)$ belongs to $W^{1,\infty}\big(a,b;C_0(\Omega)\big)$.

We now use f and ψ_m in the integration by parts formula (10): first,

$$\begin{aligned}
-\int_a^b \big\langle f(t),\psi_m'(t)\big\rangle\,dt &= \int_a^b X'(t)\int_c^{X(t)} f(x,t)\,\phi_m'(x-X(t))\,dx\,dt\\
&\quad+\int_a^b X'(t)\int_{X(t)}^d f(x,t)\,\phi_m'(x-X(t))\,dx\,dt\\
&=: I_1+I_2\,.
\end{aligned}$$

By changing variables and using Fubini's Theorem and (20), we get

$$\begin{aligned}
I_1 &\ge \int_a^b \sigma\int_{X(t)-3\,\delta_m/4}^{X(t)-\delta_m/4} f(t,x)\,\frac{3}{2\,\delta_m}\,dx\,dt\\
&= \frac{3\,\sigma}{2\,\delta_m}\int_{-3\,\delta_m/4}^{-\delta_m/4}\int_a^b f(t,X(t)+\epsilon)\,dt\,d\epsilon\\
&\ge \frac{3\,\sigma}{4}\,m\,\lambda(A^-) \ge \frac{3\,\sigma}{8}\,m\,\lambda(S_\infty^-).
\end{aligned}$$

Next, setting $\overline{\sigma} = \sup_{t\in(a,b)} X'(t)$ and using $-\frac{3}{2\,\gamma} \le \phi_m'(x) \le 0$ for $x \ge 0$, we have

$$\begin{aligned}
0 \le -I_2 &\le \frac{3\,\overline{\sigma}}{2\,\gamma}\int_a^b\int_c^d f(x,t)\,dx\,dt\\
&\le \frac{3\,\overline{\sigma}}{2\,\gamma}\int_a^b \left(1+\Big(\int_c^d f(x,t)\,dx\Big)^q\right)dt\\
&= \frac{3\,\overline{\sigma}}{2\,\gamma}\Big((b-a)+\|f\|_{L^q(a,b;L^1(\Omega))}\Big)\,,
\end{aligned}$$

which is bounded. Using $\|\phi_m\|_\infty = 1$, we estimate the other terms in (10) by

$$\begin{gathered}
\big\langle \overline{f},\overline{\psi}_m\big\rangle\Big|_a^b \le \big\|\overline{f}(b)\big\|_{L^1(\Omega)} + \big\|\overline{f}(a)\big\|_{L^1(\Omega)}, \quad \text{and}\\
\Big|\int \big\langle f'(t),\psi_m(t)\big\rangle\,dt\Big| \le \big\|f'\big\|_{L^q_{w*}(a,b;L^1(\Omega))}\,(b-a)^{\frac{1}{p}},
\end{gathered}$$

which are also bounded. Since m is arbitrary, we have a contradiction and the lemma is proved. ∎

Corollary 8. *For $\mathcal{C}$ as in (18) and f as in (19), f blows up on an essentially stationary set, that is*

$$\lambda\big(S_\infty \backslash Z_X\big) = 0, \quad \textit{where} \quad Z_X := \Big\{t\in[a,b] : X'(t) = 0\Big\}\,. \tag{21}$$

Proof. Without loss of generality, suppose that

$$\lambda\Big(S_\infty \bigcap \Big\{t\in[a,b]\ :\ X'(t)>0\Big\}\Big) > 0.$$

By continuity, we can find times $(t_1,t_2) \subset (a,b)$ and $\sigma > 0$ such that

$$\lambda\big(S_\infty \textstyle\bigcap (t_1,t_2)\big) > 0 \quad \text{and} \quad X'(t) > \sigma,$$

for all $t \in (t_1,t_2)$, contradicting Lemma 7. ∎

3.2.3. Nonphysical Solutions. Despite the regularity shown above, our definition is not yet restrictive enough due to a large number of extraneous solutions which satisfy our definition but are clearly inadmissible for physical reasons: here we present an explicit example.

Given constants u_-, u_+, $v_0 > 0$ and $w_0 \geq 0$, set

$$u(t,x) = \begin{cases} u_-, & x < 0 \\ u_+, & x > 0 \end{cases}, \qquad V(t) = \iota(v_0) + (w_0 + [u]\,t)\,\delta_0, \tag{22}$$

where as usual $[u] = u_+ - u_-$; it is clear that $\widehat{P}(V) = p(v_0)$ and that the equation (16) is satisfied in $X^* = M_{loc}(\mathbb{R})^2$, for $t < -w_0/[u]$ if $[u] < 0$, or for all t otherwise. This solution represents a varying vacuum located at $x = 0$, adjacent on both sides to constant states with finite specific volume v_0. This is nonphysical because there is no rarefaction between $v_0 < \infty$, at which the pressure is positive, and the vacuum, at which $p = 0$, while there is no shock because $p = P(v_0)$ on either side of $x = 0$. The physical solution is the entropy solution of the vacuum Riemann problem, described below, or the usual Riemann problem if $w_0 = 0$.

3.3. Natural Vacuum Solution. In view of the nonphysical examples (22), it is clear that our definition is not yet restrictive enough. Moreover, entropy considerations play no part here, because $[u]$ can be arbitrary. Thus we need to rule out non-physical solutions without resorting to the entropy condition.

The key observation here is that in (22), we have allowed vacuums, corresponding to stationary singular measures in the specific volume, to occur while the projected specific volume $v = \Pi(V) \in L^1_{loc}(\Omega)$ remains bounded, so that the gas does not rarefy near the vacuum. This is clearly unphysical and should be ruled out, so we require that $v \to \infty$ as vacuum is approached. We call this property *consistency of the medium*; it can also be interpreted as a boundary condition induced by the vacuum.

We thus define the set of *positive consistent measures*,

$$M_\infty = \big\{\mu \in M_{loc+} \;:\; x \in \operatorname{supp}(\mu_s) \implies \operatorname*{ess\,lim}_{y\to x} \Pi(\mu_c(y)) = \infty\big\}, \tag{23}$$

where we have again written $\mu = \mu_c + \mu_s$ using the Lebesgue decomposition, and we set

$$Y_{loc} = M_\infty \times BV_{loc} \subset M^2_{loc} = X^*_{loc}.$$

Definition 9. *The pair $(V,u) \in W^{1,q}_{w*}(0,T; Y_{loc}, X^*_{loc})$ is a* natural vacuum solution *of the p-system (13), if it satisfies (16), namely*

$$\begin{aligned} V' - \mathbb{D}_x u = 0, \\ u' + \mathbb{D}_x \widehat{P}(V) = 0. \end{aligned} \quad \text{in} \quad L^q_{w*}(0,T; X^*_{loc}),$$

A natural vacuum solution solves the Cauchy problem with Cauchy data (V_0, u_0) if the time-continuous representative $(\overline{V}, \overline{u})$ satisfies

$$\big(\overline{V}(0), \overline{u}(0)\big) = \big(V_0, u_0\big) \quad \text{in} \quad X^*_{loc}.$$

3.4. Rankine-Hugoniot Conditions. Since $X^* = M^2_{loc}$ and (16) is satisfied in X^*, the distributional derivatives $\mathbb{D}_x u$ and $\mathbb{D}_x p$ are both measures, which in turn implies that $u(t)$ and $p(t)$ are BV functions of $x \in \Omega$ for a.e. t. Thus, for a.e. t, both $u(t)$ and $p(t)$ have well-defined left and right limits for each x with countably many jumps. In order to obtain appropriate jump conditions, we assume that the solution has a single isolated discontinuity located at $x = X(t)$.

Specifically, suppose there is an open set $\mathcal{U} = \{(t,x) : t \in (a,b),\ x \in (c,d)\}$ and that $X(t) \in (c,d)$ for all $t \in [a,b]$, so that $\mathcal{C} = \{(t,x) : x = X(t),\ t \in (a,b)\} \subset \mathcal{U}$. We assume that u and V are C^1 functions of (t,x) on the open region $\mathcal{U}\backslash\mathcal{C}$. In particular, $v = \Pi(V)$ is finite and $p > 0$ at any point of $\mathcal{U}\backslash\mathcal{C}$.

According to these assumptions, and since the discontinuity is isolated, it follows that for $(t,x) \in \mathcal{U}$, we can write

$$
\begin{aligned}
u(t,x) &= u_L(t,x)\,H\big(X(t)-x\big) + u_R(t,x)\,H\big(x-X(t)\big) \quad \text{and}\\
p(t,x) &= p_L(t,x)\,H\big(X(t)-x\big) + p_R(t,x)\,H\big(x-X(t)\big),
\end{aligned}
\tag{24}
$$

where each of u_L, u_R, p_L and p_R are in $C^1(\mathcal{U})$ and H is the Heaviside function. We denote the jump in a quantity g by

$$
[g](t) := g_R\big(t, X(t)+\big) - g_L\big(t, X(t)-\big),
$$

so that both $[u]$ and $[p]$ are differentiable functions of t. Since $V(t)$ is a Radon measure, which may contain a Dirac mass, we assume it has the form

$$
V(t) = \iota\Big(v_L(t,\cdot)\,H\big(X(t)-\cdot)\big)\Big) + \iota\Big(v_R(t,\cdot)\,H\big(\,\cdot - X(t))\big)\Big) + w(t)\,\delta_{X(t)},
$$

where $v_L(t,\cdot)$, $v_R(t,\cdot) \in L^1(\mathcal{U})$ and w are differentiable functions of t, consistent with $V \in W^{1,1}_{w*}(0,T;L^1_{loc},M_{loc})$, as in Lemma 5 and Corollary 8. Note that v_L and v_R are generally unbounded as $x \to X(t)$ so that $[v]$ is not necessarily defined. However, since $p_L,\ p_R \in C^1(\mathcal{U})$ and $p = P(v) = \widehat{P}(V)$, left and right limits of $v(t,\cdot)$ exist everywhere, although these may be infinite on the curve $\mathcal{C}$.

With these assumptions, we now calculate the appropriate derivatives and plug them in to (16). Using the distributional derivative, we have

$$
\begin{aligned}
\mathbb{D}_x p &= \iota\Big(\partial_x p_L\,H\big(X(t)-\cdot\big)\Big) + \iota\Big(\partial_x p_R\,H\big(\,\cdot - X(t)\big)\Big) + [p]\,\delta_{X(t)}, \quad \text{and}\\
\mathbb{D}_x u &= \iota\Big(\partial_x u_L\,H\big(X(t)-\cdot\big)\Big) + \iota\Big(\partial_x u_R\,H\big(\,\cdot - X(t)\big)\Big) + [u]\,\delta_{X(t)},
\end{aligned}
$$

where these are to be interpreted as measures. Next, we calculate

$$
u' = \iota\Big(\partial_t u_L\,H\big(X(t)-\cdot\big)\Big) + \iota\Big(\partial_t u_R\,H\big(\,\cdot - X(t)\big)\Big) + [u](t)\,\big(-X'(t)\big)\,\delta_{X(t)},
$$

where, since the distributional t-derivative is a measure, it coincides with the B-weak and G-weak derivatives. Equating the coefficients of the Dirac masses in $(16)_2$ then yields the first Rankine-Hugoniot condition,

$$
X'(t)\,[u] = [p], \tag{25}
$$

while away from the curve $x = X(t)$, the equation $\partial_t u + \partial_x p = 0$ holds in the classical sense.

Our assumptions combined with Corollary 8 imply that the set

$$
\begin{aligned}
S_\infty &= \big\{t \in (a,b)\ :\ p_L(t,X(t)-) = 0 \quad \text{or} \quad p_R(t,X(t)+) = 0\big\}\\
&= \big\{t \in (a,b)\ :\ v_L(t,X(t)-) = \infty \quad \text{or} \quad v_R(t,X(t)+) = \infty\big\}
\end{aligned}
$$

satisfies

$$\lambda\big(S_\infty \bigcap \{t \;:\; X'(t) \neq 0\}\big) = 0. \tag{26}$$

Thus if $X'(t) \neq 0$, then the limits $v_L\big(t, X(t)-\big)$ and $v_R\big(t, X(t)+\big)$ must be finite, and in this case the jump $[v]$ makes sense.

Differentiating the measure $V(t)$, and using (26), we get, for almost every $t \in S_\infty$,

$$V'(t) = \iota\Big(\partial_t v_L\, H\big(X(t) - \cdot\big)\Big) + \iota\Big(\partial_t v_R\, H\big(\,\cdot - X(t)\big)\Big) + w'(t)\, \delta_{X(t)}, \tag{27}$$

while for almost every $t \in (a,b)\backslash S_\infty$, we have

$$\begin{aligned} V'(t) &= \iota\big(\partial_t v_L\, H\big(X(t) - \cdot\big)\big) + \iota\big(\partial_t v_R\, H\big(\,\cdot - X(t)\big)\big) \\ &\quad + v_L\big(t, X(t)-\big)\, X'(t)\, \delta_{X(t)} - v_R\big(t, X(t)+\big)\, X'(t)\, \delta_{X(t)} \\ &\quad + w'(t)\, \delta_{X(t)} + w(t)\, X'(t)\, \mathbb{D}_x \delta_{X(t)}\,. \end{aligned} \tag{28}$$

It follows that V' in general is a distribution, defined by its action on test functions. In a weak* solution, $V'(t) \in M_{loc}$ must be a bounded measure, so that the coefficient of the last term of (28) necessarily vanishes,

$$w(t)\, X'(t) = 0, \tag{29}$$

consistent with Lemma 5 above. In addition, all coefficients of $\delta_{X(t)}$ in (28) must necessarily be bounded, so we can combine (27) and (28) into

$$V'(t) = \iota\Big(\partial_t v_L\, H\big(X(t) - \cdot\big)\Big) + \iota\Big(\partial_t v_R\, H\big(\,\cdot - X(t)\big)\Big) + \big(w'(t) - [v]\, X'(t)\big)\, \delta_{X(t)},$$

where we have used the convention that

$$\big([v]\, X'\big)(t) := 0 \quad \text{for all} \quad t \in S_\infty.$$

Equating the measures in $(16)_1$ yields the Rankine-Hugoniot condition

$$[u] = w'(t) - X'(t)\, [v], \tag{30}$$

while away from the curve $x = X(t)$, the equation $\partial_t v - \partial_x u = 0$ again holds in the classical sense.

We can regard the three conditions (25), (30) and (29) as generalized Rankine-Hugoniot conditions suitable for vacuum solutions, which naturally extend the usual conditions, as follows. First suppose that $w(t) > 0$, which corresponds to the presence of a vacuum of spatial width $w(t)$. By continuity, this condition persists in an interval $(t-\epsilon, t+\epsilon)$. Also, by (29), we have $X'(t) = 0$, so $X(t) =: X_0$ is constant in this interval, consistent with Lemma 5, and the natural vacuum condition gives $v(t, X(t)\pm) = \infty$, so that

$$w'(t) = [u], \quad \text{while also} \quad p(t, X_0\pm) = 0, \tag{31}$$

so that the pressure vanishes at vacuum as expected, while the spatial expansion rate of the vacuum is the jump in velocity. Now suppose that $X'(t) \neq 0$, so the discontinuity is not stationary. Again by continuity this holds in $(t-\epsilon, t+\epsilon)$, and we conclude from (29) that $w(t) = w'(t) = 0$. By our earlier remark, both v_L and v_R remain finite, and (25), (30) reduce to the usual Rankine-Hugoniot conditions,

$$X'(t)\, [-v] = [u], \qquad X'(t)\, [u] = [p]. \tag{32}$$

3.5. Entropy Condition. For smooth solutions, it is easy to derive an energy equation: multiplying the first equation of (13) by $-p$, the second by u, and adding gives the scalar equation

$$\partial_t\big(\tfrac{1}{2}u^2\big) - p\,\partial_t v + \partial_x(u\,p) = 0.$$

Thus, using the specific internal energy, which satisfies

$$\varepsilon = E(v) := \int_v^\infty P(\overline{v})\,d\overline{v},$$

we obtain the conservation of energy,

$$\partial_t\big(\tfrac{1}{2}u^2 + \varepsilon\big) + \partial_x(u\,p) = 0,$$

which in turn provides an entropy/flux pair for solutions with shocks.

As in (15), the internal energy E can be regarded as a map of functions,

$$E : \mathcal{A} \subset L^1(\Omega) \to L^1(\Omega) \quad \text{via} \quad \varepsilon = E \circ v : \Omega \to \mathbb{R},$$

and we again extend this to the positive measures $M_{loc}(\Omega)_+$ by

$$\widehat{E} : M_{loc}(\Omega)_+ \to L^1_{loc}(\Omega), \quad \text{by} \quad \widehat{E}(\mu) = E(\Pi(\mu)).$$

We now impose the entropy condition for a natural vacuum solution (V, u), namely, we require that the map

$$t \mapsto \big(\tfrac{1}{2}u^2 + \widehat{E}(V)\big) \in W^{1,1}_{w*}(0,T; M_{loc}, M_{loc}),$$

and that the entropy production be non-positive,

$$\big(\tfrac{1}{2}u^2 + \widehat{E}(V)\big)' + \mathbb{D}_x\big(u\,\widehat{P}(V)\big) \le 0 \quad \text{in} \quad L^1_{w*}(0,T; M_{loc}), \tag{33}$$

both terms being interpeted as a measure.

In regions where the natural vacuum solution is differentiable, the entropy inequality is satisfied as an equality. On the other hand, if the solution is discontinuous on an isolated curve $X(t)$, we again describe the solution using (24). Calculating the derivatives of the measures as in the previous section, the measure in (33) becomes

$$-X'(t)\,\big([\tfrac{1}{2}u^2] + [\varepsilon]\big)\,\delta_{X(t)} + [u\,p]\,\delta_{X(t)},$$

where the absolutely continuous part cancels because the solution is classical where it is differentiable, and our entropy condition thus becomes

$$-\,X'(t)\,\big([\tfrac{1}{2}u^2] + [\varepsilon]\big) + [u\,p] \le 0. \tag{34}$$

Again there are two possibilities: first, if $X'(t) = 0$, then (31), (32) imply that $[u\,p] = 0$ and the entropy condition is satisfied as an equality, reflecting the fact that there is no shock.

On the other hand, if $X'(t) \neq 0$, we use the identity

$$[g_1\,g_2] = \overline{g}_1\,[g_2] + [g_1]\,\overline{g}_2, \quad \text{with} \quad \overline{g} := \frac{g_R + g_L}{2},$$

together with (32), to write

$$\begin{aligned} -X'(t)\,\big([\tfrac12 u^2]+[\varepsilon]\big)+[u\,p] &= -X'(t)\,\big(\overline{u}\,[u]+[\varepsilon]\big)+\overline{u}\,[p]+[u]\,\overline{p} \\ &= -X'(t)\,\big([\varepsilon]+[v]\,\overline{p}\big). \end{aligned}$$

Now note that

$$[\varepsilon]=E(v_R)-E(v_L)=\int_{v_R}^{v_L} P(\overline{v})\;d\overline{v}, \quad \text{and} \quad [v]\,\overline{p}=-\int_{v_R}^{v_L} \tfrac12\big(P(v_L)+P(v_R)\big)\;d\overline{v},$$

and so if $P(v)$ is convex, as is usually the case, then the entropy inequality holds provided $X'(t)\,(v_R-v_L)>0$. This in turn expressed the well-known fact that the pressure is greater behind the shock, and reduces to Lax's shock condition.

4. Examples of Natural Vacuum Solutions. By way of example we present some explicit examples of natural vacuum solutions, which are not weak solutions because of the presence of vacuums, but which clearly extend the class of BV weak solutions. Before writing down the examples we introduce a convenient variable and describe the elementary waves of the system.

4.1. Symmetric Variables. As in [], it is convenient to describe the solutions using a nonlinear change of thermodynamic variable, which in turn simplifies the description of waves. Recall that the usual costitutive relation is given by a pointwise function $P:(0,\infty)\to(0,\infty)$ expressing the pressure in terms of specific volume, $p=P(v)$, with the properties

$$P'(v)<0, \quad \lim_{v\to\infty} P(v)=0, \quad \text{and} \quad \int_1^\infty \sqrt{-P'(v)}\;dv<\infty. \tag{35}$$

These conditions express hyperbolicity of the system, pressureless vacuum, and possibility of vacuum formation, respectively. Hyperbolicity is the condition that allows for forward and backward nonlinear waves which propagate with (absolute) Langrangian sound speed $C(v):=\sqrt{-P'(v)}$. It is clear that each of these properties is satisfied for an ideal gas, which has constitutive function $P(v)=A\,v^{-\gamma}$, for $\gamma>1$.

We define the auxiliary function

$$H:(0,\infty)\to(0,\infty) \quad \text{by} \quad H(v):=\int_v^\infty C(\overline{v})\;d\overline{v},$$

and introduce the symmetric variable h, which defines $v=v(h)$ by

$$v=v(h):=H^{-1}(h), \quad p=p(h):=P(H^{-1}(h)), \quad \text{and} \quad c=c(h):=C(H^{-1}(h)).$$

Our assumptions (35) imply that H is monotone decreasing and thus invertible, and that H vanishes as $v\to\infty$, so the vacuum is characterized as $h=0$, a bounded state. In addition, we have $p(0)=0$ and $c(0)=0$, and we calculate

$$\begin{aligned} \frac{dv(h)}{dh} &= \Big(\frac{dH}{dv}\Big)^{-1}=\frac{-1}{C(v)}=\frac{-1}{c(h)}, \quad \text{and} \\ \frac{dp(h)}{dh} &= \frac{dP}{dv}\Big(\frac{dH}{dv}\Big)^{-1}=\frac{-C^2(v)}{-C(v)}=c(h). \end{aligned}$$

Using the symmetric variable h, we rewrite the p-system (13) as

$$\partial_t v(h)-\partial_x u=0, \qquad \partial_t u+\partial_x p(h)=0,$$

and we now regard the unknowns as (h,u). When the solution is differentiable, we can write it in the symmetric quasilinear form

$$\partial_t h + c(h)\,\partial_x u = 0, \qquad \partial_t u + c(h)\,\partial_x h = 0,$$

and it is clear that the Riemann invariants are $u \pm h$, so we can write the diagonal form

$$\partial_t(u+h) + c(h)\,\partial_x(u+h) = 0, \qquad \partial_t(u-h) - c(h)\,\partial_x(u-h) = 0. \tag{36}$$

Finally, recalling that the specific internal energy is given by

$$E(v) = \int_v^\infty P(\overline{v})\,d\overline{v},$$

we set $\varepsilon(h) = E(H^{-1}(h))$ and calculate

$$\frac{d\varepsilon(h)}{dh} = \frac{dE}{dv}\Big(\frac{dH}{dv}\Big)^{-1} = \frac{-P(v)}{-C(v)} = \frac{p(h)}{c(h)}, \quad \text{so} \quad \varepsilon(h) = \int_0^h \frac{p(\overline{h})}{c(\overline{h})}\,d\overline{h},$$

and again for differentiable solutions we get the entropy equation

$$\partial_t\big(\tfrac{1}{2}u^2 + \varepsilon(h)\big) + \partial_x\big(u\,p(h)\big) = 0,$$

which yields the usual entropy inequality for shocks.

By way of example, it is a straight-forward calculation to describe a γ-law gas, for which $P(v) = A\,v^{-\gamma}$, fully in terms of symmetric variables: up to rescaling by a constant, we have

$$c(h) = h^\beta, \quad v(h) = \frac{h^{1-\beta}}{\beta-1}, \quad p(h) = \frac{h^{1+\beta}}{\beta+1}, \quad \text{and} \quad \varepsilon(h) = \frac{h^2}{2(\beta+1)}, \tag{37}$$

where the constant $\beta := \frac{\gamma+1}{\gamma-1} > 1$.

4.2. Elementary Waves. There are two types of elementary waves, namely shocks and simple waves. Shocks satisfy the Rankine-Hugoniot conditions

$$X'(t)\,[-v] = [u], \qquad X'(t)[u] = [p],$$

which yields

$$X'(t) = \pm\sigma := \pm\sqrt{[p]/[-v]}, \qquad [u] = \pm\sigma\,[-v],$$

where $\sigma > 0$ is the absolute shock speed. For definiteness, we assume that the pressure $p = P(v)$ is convex, so the entropy condition implies that the pressure is greater behind the shock. Thus for a backward shock, $[-v] = v_L - v_R = \big|[-v]\big| > 0$, while for a forward shock $[-v] = -\big|[-v]\big| < 0$. In either case, the states are related by

$$\begin{gathered} = u_R - u_L = -\sigma\,\big|[-v]\big| = -\sqrt{[p]\,[-v]}, \\ \sigma = \sqrt{[p]/[-v]}, \quad \text{and} \quad X'(t) = \pm\sigma. \end{gathered} \tag{38}$$

A simple wave is a C^1 solution of the quasilinear system with one-dimensional image, so we can take $u = u(h)$, say. Plugging this into the Riemann invariant equations (36), we get

$$\big(u'(h)+1\big)\Big(\partial_t h + c(h)\,\partial_x h\Big) = 0 \quad \text{and} \quad \big(u'(h)-1\big)\Big(\partial_t h - c(h)\,\partial_x h\Big) = 0.$$

Thus, along the forward and backward characteristics

$$\frac{dx}{dt} = \pm c(h(t,x)), \quad \text{we have} \quad \frac{dh}{dt} = 0 \quad \text{and} \quad u'(h) = \pm 1,$$

respectively. In particular the characteristics are straight lines, $x - x_0 = \pm c(h)\,\big(t - t_0\big)$, on which we have $u(t,x) = \pm h(t,x) + K$. That is, we describe the simple wave by

$$h(t,x) = c^{-1}\Big(\pm \frac{x - x_0}{t - t_0}\Big), \quad u(t,x) \mp h(t,x) = u_* \mp h_*, \tag{39}$$

where (h_*, u_*) is a reference state, typically adjacent to the wave. Here (t_0, x_0) is a reference point for the individual characteristic, which will generally depend on the value of h; if the point (t_0, x_0) is fixed, it is the *center* of the wave. The wave is *compressive* or *rarefactive* if the absolute wavespeed c decreases or increases from behind the wave to ahead, respectively. In particular, a centered compression focusses in future time, and a centered rarefaction focusses in past time. We note that a simple wave may appear adjacent to the vacuum, if $h \to 0$ across the wave, and the corresponding characteristics approach the boundary of the vacuum, characterized by $x = X_0$ constant.

4.3. Collapse of a Vacuum. Our first example shows the collapse of a vacuum state. We consider a compressive vacuum with adjacent forward and backward compressions, all of which are centered so that they focus at the origin. There are thus no shocks for $t \le 0$, and for positive times the solution is resolved by solving a Riemann problem. The setup is graphically illustrated in Figure 1, in which two characteristic pictures are shown: on the left we show the Lagrangian material frame in which we work, and on the right the Eulerian spatial frame.

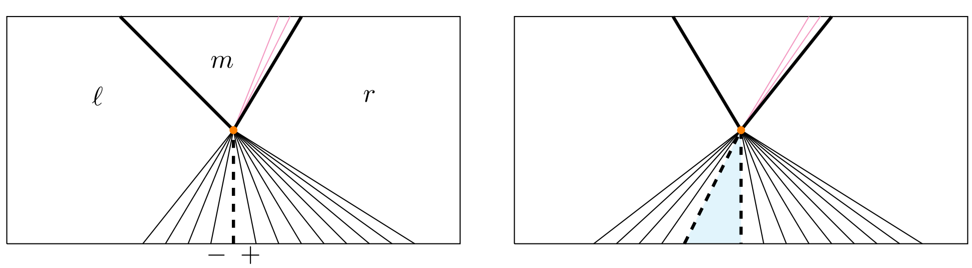

FIG. 1. *Centered collapse of a vacuum: Lagrangian and Eulerian frames*

Referring to Figure 1, we choose states subscripted by ℓ, m and r, together with velocities u_- and u_+ adjacent to the vacuum. We can specify four data, say u_-, u_+, h_ℓ and h_r, and we require that $\Delta u := u_+ - u_- < 0$, which ensures the vacuum collapses. Also, without loss of generality, we assume $h_r \ge h_\ell$. The remaining states are then deduced by (38) or (39), joining the various states by the corresponding elementary wave. Thus we have

$$u_\ell = u_- + h_\ell, \quad \text{and} \quad u_r = u_+ - h_r,$$

and the state (h_m, u_m) is found be resolving the Riemann problem, see []. Since $u_r - u_\ell < 0$, there are two cases: either $h_m > h_r$ (two shocks out) or $h_r \ge h_m > h_\ell$ (one shock out).

We use (38), (39) to write down the solution explicitly: for $t < 0$, we have

$$h(t,x) = \begin{cases} h_\ell, \\ c^{-1}(\frac{x}{t}), \\ c^{-1}(\frac{x}{-t}), \\ h_r, \end{cases} \qquad u(t,x) = \begin{cases} u_\ell, & x \le c(h_\ell)\,t \\ u_\ell - h_\ell + c^{-1}(\frac{x}{t}), & c(h_\ell)\,t \le x < 0 \\ u_r + h_r - c^{-1}(\frac{x}{-t}), & 0 < x \le -c(h_r)\,t \\ u_r, & -c(h_r)\,t \le x \end{cases},$$

and these in turn determine

$$p(t,x) = p(h(t,x)), \quad v(t,x) = v(h(t,x)), \quad \text{and} \quad V(t) = \iota(v(\cdot,t)) + \Delta u\, t\, \delta_0.$$

For the outgoing waves, there are two cases: first, if $h_m > h_r$, there are two outgoing shocks, so

$$h(t,x) = \begin{cases} h_\ell, \\ h_m, \\ h_r, \end{cases} \qquad u(t,x) = \begin{cases} u_\ell, & x < -\sigma_\ell\, t \\ u_m, & -\sigma_\ell\, t < x < \sigma_r\, t \\ u_r, & \sigma_r\, t < x \end{cases},$$

and

$$p(t,x) = p(h(t,x)), \quad v(t,x) = v(h(t,x)), \quad \text{and} \quad V(t) = \iota(v(\cdot,t)),$$

where $\sigma.$ is the (absolute) shock speed, given by $\sigma = \sqrt{[p]/[-v]}$,

On the other hand, if $h_m \le h_r$, then the right outgoing wave is a rarefaction, and

$$h(t,x) = \begin{cases} h_\ell, \\ h_m, \\ c^{-1}(\frac{x}{t}), \\ h_r, \end{cases} \qquad u(t,x) = \begin{cases} u_\ell, & x < -\sigma_\ell\, t \\ u_m, & -\sigma_\ell\, t < x \le c(h_m)\,t \\ c^{-1}(\frac{x}{t}) - h_m, & c(h_m)\,t \le x \le c(h_r)\,t \\ u_r, & c(h_r)\,t \le x \end{cases},$$

and p, v and V given as above.

Because h is monotone across each wave, it is clear that each of h, c, p and u has bounded variation as a function of x. Also, the abstract argument shows that because the characteristic and/or shock conditions hold everywhere, we have a weak* solution. It remains to check that V is a well-behaved measure, $V \in W^{1,\infty}(0,T;M_\infty)$.

To this end, we note that, if $t < 0$ and $\Omega = [-a,b]$, say,

$$V = \iota(v) + \Delta u\, t\, \delta_0, \quad \text{and} \quad \|V\|_{M(\Omega)} = |V|\big([-a,b]\big),$$

and since $v \to \infty$ as $x \to 0$, $V \in M_\infty$ as long as it is bounded as a measure. Since $h \ge 0$ and $\Delta u < 0$, for $t < 0$ we calculate

$$\begin{aligned} \|V(t)\|_{M(\Omega)} &= \Delta u\, t + \int_{-a}^{b} v\big(h(t,x)\big)\, dx \\ &= \Delta u\, t + v(h_r)\,\big(b - (-c(h_r)\,t)\big) + v(h_\ell)\,\big(c(h_\ell)\,t - (-a)\big) \\ &\qquad + \int_{c(h_\ell)\,t}^{0} v\big(c^{-1}(\tfrac{x}{t})\big)\, dx + \int_0^{-c(h_r)\,t} v\big(c^{-1}(\tfrac{x}{-t})\big)\, dx \\ &= \Delta u\, t + b\, v(h_r) + a\, v(h_\ell) + t\,\big(v(h_\ell)\, c(h_\ell) + v(h_r)\, c(h_r)\big) \\ &\qquad + t \int_{c(h_\ell)}^{0} v\big(c^{-1}(y)\big)\, dy + t \int_{c(h_r)}^{0} v\big(c^{-1}(y)\big)\, dy \\ &= b\, v(h_r) + a\, v(h_\ell) + t\,\big(\Delta u - h_\ell - h_r\big), \end{aligned}$$

where we have used

$$
\begin{aligned}
v(h)\,c(h) + \int_{c(h)}^{0} v\big(c^{-1}(y)\big)\;dy &= v(h)\,c(h) + \int_h^\infty v(z)\,c'(z)\;dz \\
&= -\int_h^\infty c(z)\,v'(z)\;dz = -h,
\end{aligned}
\tag{40}
$$

having integrated by parts, and used $v(h)\,c(h) \to 0$ as $h \to 0$.

Similarly, for $t \geq 0$, if $h_m > h_r$, so two shocks emerge,

$$
\begin{aligned}
\|V(t)\|_{M(\Omega)} &= \int_{-a}^{b} v\big(h(t,x)\big)\;dx \\
&= v(h_\ell)\,\big(a - \sigma_\ell\,t\big) + v(h_m)\,\big(\sigma_r\,t + \sigma_\ell\,t\big) + v(h_r)\,\big(b - \sigma_r\,t\big) \\
&= b\,v(h_r) + a\,v(h_\ell) + t\,\big(\sigma_\ell\,(v(h_m) - v(h_\ell)) + \sigma_r\,(v(h_m) - v(h_r))\big),
\end{aligned}
$$

while for $h_r \geq h_m$, the right outgoing wave is a rarefaction and

$$
\begin{aligned}
\|V(t)\|_{M(\Omega)} &= v(h_\ell)\,\big(a - \sigma_\ell\,t\big) + v(h_m)\,\big(c(h_m)\,t + \sigma_\ell\,t\big) \\
&\qquad + \int_{c(h_m)\,t}^{c(h_r)\,t} v\big(c^{-1}(\tfrac{x}{t})\big)\;dx + v(h_r)\,(b - c(h_r)\,t) \\
&= b\,v(h_r) + a\,v(h_\ell) + t\,\Big(\sigma_\ell\,\big(v(h_m) - v(h_\ell)\big) + h_r - h_m\Big),
\end{aligned}
$$

again using (40). It is now clear that $\|V(t)\|_{M(\Omega)}$ is bounded, and indeed it is Lipschitz, as expected. Piecewise linear dependence on t occurs in this instance because all waves are centered, and scale invariance implies rank one homogeneity.

4.4. Centered Waves and the Vacuum. Our next example consists of a collapsing vacuum between two centered simple waves, these being centered at different points, with one being a compression and the other a rarefaction wave, as illustrated in Figure 2. One can pose this as a Cauchy problem by taking the trace of the solution at time $t = 0$. We choose the data as in the previous case, so that the initial compression and vacuum collapse at the same point.

For short times the solution contains the vacuum and five other waves: first, adjacent to the vacuum are the focussing compression, and the centered rarefaction; next, a shock and a centered rarefaction emerge from the point of collapse of the vacuum; and finally, as the shock interacts with the original rarefaction it changes strength and a backwards compression is transmitted behind the shock, as drawn in the figure. Of course, at some later time this reflected compression will collapse to form a shock, which will lead to the generation of more (ever weaker) waves, in a process which continues indefinitely.

The main issue in resolving the solution for short times is an exact description of the states and trajectory of the shock wave; once we know these, it is routine to describe the simple waves via characteristics using (39). We briefly describe the process for exactly resolving the shock wave before secondary interactions occur. For simplicity, we assume a γ-law gas, given by (37).

The shock trajectory is a curve in the plane, and it is convenient to parameterize it by the state h ahead of the shock, which is also part of the centered rarefaction in the data. Thus the shock lies on the curve $\big(x(h), t(h)\big)$, and using (38) for the trajectory and (39) for the centered rarefaction, we have

$$
\frac{dx}{dh} = \sigma(h)\,\frac{dt}{dh}, \quad \text{while also} \quad x(h) = c(h)\,t(h),
$$

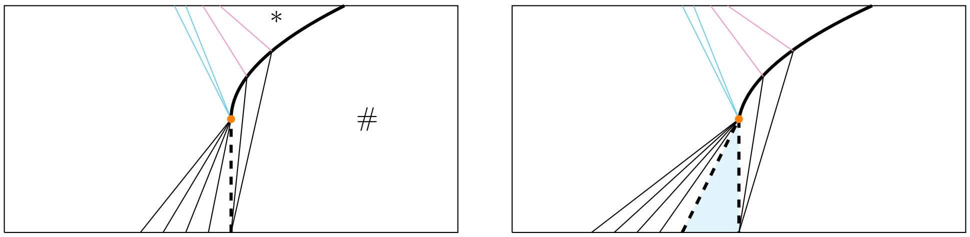

FIG. 2. *Vacuum Adjacent to Centered Waves: Lagrangian and Eulerian frames*

where $c(h) = h^\beta$ is the speed of the characteristic from the origin, and $\sigma(h)$ is the shock speed at the point $\big(x(h), t(h)\big)$. In particular, we need to show that the shock curve can be defined up to the point of vacuum collapse, that is, that the limit exists as $h \to 0+$. Combining these relations yields the linear differential equation

$$\frac{dc}{dh}\, t(h) + c(h)\, \frac{dt}{dh} = \sigma(h)\, \frac{dt}{dh},$$

which we can solve to get

$$t = t_\# \exp\Big(\int_{h_\#}^{h} \frac{c'(h)\, dh}{\sigma(h) - c(h)} \Big), \quad \text{and} \quad x(h) = c(h)\, t(h).$$

It follows that provided the integral converges as $h \to 0$, we can choose $t_\#$ so that the shock begins at the appropriate point. Using (37), we can write this integral as

$$I = \int_{h_\#}^{h} \frac{c'(h)\, dh}{\sigma(h) - c(h)} = \beta \int_{h_\#}^{h} \frac{1}{\sigma/c - 1}\, \frac{dh}{h}. \tag{41}$$

We now consider the states on either side of the shock, using the following notation: parameterizing the ahead (right) state by h, we write the behind state as $h_b = z(h)\, h$, so $z(h) > 1$ is defined to be the ratio of behind state to ahead state. Again using (37), we then write the jump across the state as

$$= p(h_b) - p(h) = \frac{(z\, h)^{\beta+1} - h^{\beta+1}}{\beta + 1} = h^{\beta+1}\, z^{\beta+1}\, q_{\beta+1}(z) \quad \text{and}$$
$$[-v] = v(h) - v(h_b) = \frac{h^{1-\beta} - (z\, h)^{1-\beta}}{\beta - 1} = h^{1-\beta}\, q_{\beta-1}(z),$$

where we have set

$$q_n(z) := \frac{1 - z^{-n}}{n}.$$

Using this notation, the shock relations (38) simplify to

$$[u] = -h\, z^{\frac{\beta+1}{2}}\, r(z) \quad \text{and} \quad [\sigma] = h^\beta\, z^{\frac{\beta+1}{2}}\, s(z) = c(h)\, z^{\frac{\beta+1}{2}}\, s(z), \tag{42}$$

where we have defined

$$r(z) := \sqrt{q_{\beta+1}(z)\, q_{\beta-1}(z)} \quad \text{and} \quad s(z) := \sqrt{q_{\beta+1}(z)/q_{\beta-1}(z)}.$$

It remains to find $z(h)$, which will in turn completely determine the shock trajectory and states. We do this by exactly resolving the interaction of the shock and centered interaction wave. Referring to Figure 2, we label states as follows: the reference ahead state is $h_\#$, with corresponding behind state $h_* = z_\# \, h_\#$, and the varying ahead state is h, with corresponding behind state $h_b = z\, h$. Across the shock, we have

$$u_\# - u_* = -h_\# \, z_\#^{\frac{\beta+1}{2}} \, r(z_\#) \quad \text{and} \quad u_h - u_b = -h\, z^{\frac{\beta+1}{2}} \, r(z),$$

while the waves joining the other states are simple, so we use (39) to write

$$u_\# - u_h = h_\# - h \quad \text{and} \quad u_* - u_b = h_b - h_* = z\, h - z_\# \, h_\#.$$

Eliminating u, we get

$$u_\# - u_b = h_\# - h - h\, z^{\frac{\beta+1}{2}} \, r(z) = -h_\# \, z_\#^{\frac{\beta+1}{2}} \, r(z_\#) + z\, h - z_\# \, h_\#,$$

which simplifies to

$$h\left(1 + z + z^{\frac{\beta+1}{2}} \, r(z)\right) = h_\# \left(1 + z_\# + z_\#^{\frac{\beta+1}{2}} \, r(z_\#)\right) =: A, \tag{43}$$

where A is a reference constant. This last relation determines $z(h)$ implicitly, but we can work explicitly by changing variables: it is clear that this equality is monotone in z, so that $z \to \infty$ as $h \to 0$, and differentiating, we get

$$dh\left(1 + z + z^{\frac{\beta+1}{2}} \, r(z)\right) + h\left(1 + z + z^{\frac{\beta+1}{2}} \, r(z)\right)' dz = 0,$$

so the integral in (41) becomes explicit,

$$I = -\beta \int_{z_\#}^{z} \frac{\left(1 + z + z^{\frac{\beta+1}{2}} \, r(z)\right)' dz}{\left(1 + z + z^{\frac{\beta+1}{2}} \, r(z)\right)\left(z^{\frac{\beta+1}{2}} \, s(z) - 1\right)}.$$

Since the integrand is of order $z^{\frac{\beta-1}{2}} / z^{\beta+1} = z^{-\frac{\beta-3}{2}}$ for z large, the integral converges as $z \to \infty$ and the shock emerges from the collapse as required. Moreover, using (43), (42) we write

$$h = \frac{A}{1 + z + z^{\frac{\beta+1}{2}} \, r(z)} \quad \text{and} \quad \sigma = \frac{A^\beta \, z^{\frac{\beta+1}{2}} \, s(z)}{\left(1 + z + z^{\frac{\beta+1}{2}} \, r(z)\right)^\beta} = O(1)\, z^{\frac{1-\beta^2}{2}},$$

so we have both $\frac{dh}{dz} < 0$ and $\frac{d\sigma}{dz} < 0$, so $\frac{d\sigma}{dh} > 0$, and the shock trajectory is initially concave, as drawn.

4.5. Vacuum Riemann Problem. For our final example, we introduce a generalization of the Riemann problem, which allows for the presence of an embedded vacuum of finite spatial width in the initial data. That is, our data consists of bounded constant left and right states (h_ℓ, u_ℓ) and (h_r, u_r), together with an initial spatial width $w_0 \geq 0$ of a vacuum located at $x = 0$; a zero width $w_0 = 0$ reduces to the usual Riemann problem. The solution of the vacuum Riemann problem then provides a building block for the construction of general solutions which contain vacuums.

If $w_0 > 0$, then there is a vacuum in the solution, which must have simple waves adjacent to it, and these must be centered at the origin, so are rarefactions. The left rarefaction connects (h_ℓ, u_ℓ) to $(0, u_-)$, and the right rarefaction connects $(0, u_+)$ to (h_r, u_r), where u_- and u_+ are the velocities at the edge of the vacuum, and are given by

$$u_- = u_\ell + h_\ell \quad \text{and} \quad u_+ = u_r - h_r.$$

The quantity

$$\Delta u := u_+ - u_- = u_r - h_r - u_\ell - h_\ell$$

determines whether the vacuum is compressive or rarefactive: if $\Delta u \geq 0$, the vacuum persists for all times $t \geq 0$, and the solution is

$$h(t,x) = \begin{cases} h_\ell, \\ c^{-1}(\frac{-x}{t}), \\ c^{-1}(\frac{x}{t}), \\ h_r, \end{cases} \qquad u(t,x) = \begin{cases} u_\ell, & x \leq -c(h_\ell)\,t \\ u_\ell + h_\ell - c^{-1}(\frac{-x}{t}), & -c(h_\ell)\,t \leq x < 0 \\ u_r - h_r + c^{-1}(\frac{x}{t}), & 0 < x \leq c(h_r)\,t \\ u_r, & c(h_r)\,t \leq x \end{cases}, \tag{44}$$

and these in turn determine

$$p(t,x) = p(h(t,x)), \quad v(t,x) = v(h(t,x)) \quad \text{and} \quad V(t) = \iota(v(\cdot,t)) + \big(w_0 + \Delta u\, t\big)\,\delta_0.$$

We note that the functions $u(t,x)$, $h(t,x)$ and $v = \Pi(V)$ are self-similar, but as long as $w_0 > 0$, neither the solution nor data are self-similar.

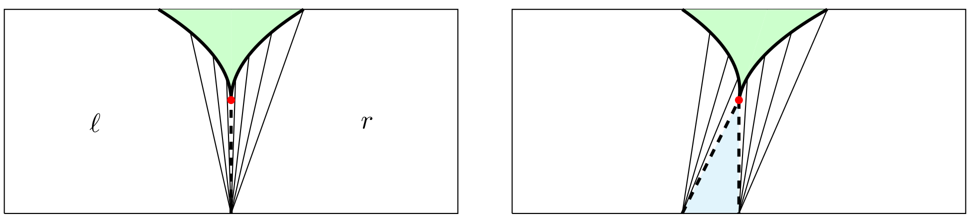

FIG. 3. *Collapsing Vacuum Riemann Problem: Lagrangian and Eulerian frames*

On the other hand, if the vacuum is compressive, so $\Delta u < 0$, (44) provides a solution for only finite times: indeed, the vacuum that was initially in the data collapses at time $T = -w_0/\Delta u$. As the vacuum collapses, two shocks emerge from the point $(T, 0)$, and the solution evolves non-trivially in the region behind these shocks, as shaded in Figure 3. For short times after collapse, this can be resolved as in the previous example with slight modifications.

5. Compressible Euler Equations. With minor modifications, we can apply our methods to the full 3×3 system of the compressible Euler equations in one space dimension. The system is obtained by using a more general constitutive law, which satisfies the Second Law of Thermodynamics. This introduces a second thermodynamic variable, so requires another conservation law, which is conservation of energy. The thermodynamic quantities of interest are the specific volume v, pressure p, internal energy ε, specific entropy s and temperature $T > 0$, and they are related by the Second Law,

$$T\,ds = d\varepsilon + p\,dv,$$

so, when using v and s as independent variables, we write

$$\varepsilon = E(v,s), \quad p = -\frac{\partial E}{\partial v} \quad \text{and} \quad T = \frac{\partial E}{\partial s}.$$

In a Lagrangian frame, the equations are

$$v_t - u_x = 0, \qquad u_t + p_x = 0, \qquad (\tfrac{1}{2}u^2 + \varepsilon)_t + (u\,p)_x = 0, \tag{45}$$

and for smooth solutions we easily derive the entropy equation $s_t = 0$. The most familiar constitutive law is that of an ideal polytropic gas,

$$E(v,s) = \frac{A}{\gamma-1}\, v^{1-\gamma}\, e^{s/c_v}, \quad P(v,s) = A\, v^{-\gamma}\, e^{s/c_v}, \quad \text{with} \quad \gamma > 1.$$

Because both ε and p vanish at vacuum, we can define natural vacuum solutions for the Euler equations just as for the p-system. Using (23), we set

$$Y = M_\infty \times BV_{loc} \times BV_{loc} \subset M^3_{loc} = X^*,$$

and say that a triple

$$(V,u,s) \in W^{1,q}_{w*}(0,T;Y,X^*),$$

is a natural vacuum solution of the Euler equations (45), if it satisfies

$$V' - \mathbb{D}_x u = 0, \qquad u' + \mathbb{D}_x \widehat{P}(V,s) = 0, \qquad \big(\tfrac{1}{2}u^2 + \widehat{E}(V,s)\big)' + \mathbb{D}_x\big(u\,\widehat{P}(V,s)\big) = 0,$$

in X^* for almost all $t > 0$. The solution solves the Cauchy problem, with Cauchy data (V_0, u_0, s_0) if the time-continuous representative $(\overline{V}, \overline{u}, \overline{s})$ satisfies

$$\big(\overline{V}(0), \overline{u}(0), \overline{s}(0)\big) = \big(V_0, u_0, s_0\big) \quad \text{in} \quad X^*.$$

We extend the jump conditions without difficulty; in particular, where the solution is differentiable it satisfies the quasilinear form of the equation, and at jumps the generalized Rankine-Hugoniot relations hold, namely

$$X'(t)\,[u] = [p], \qquad X'(t)\,[\tfrac{1}{2}u^2 + \varepsilon] = [u\,p],$$
$$[u] = w'(t) - X'(t)\,[v], \quad \text{and} \quad w(t)\,X'(t) = 0,$$

using the same notation as before.

As is well known, the entropy field s is a linearly degenerate contact field with vanishing characteristic speed, and across which $[u] = [p] = 0$. Since the vacuum also propagates with zero speed, it follows that the entropy s can jump arbitrarily across a vacuum; however, the occurrence of the vacuum is detected by any change in the velocity u across the jump: if $[u] = 0$ and v is finite, the jump is a contact, while if $[u] \neq 0$ and $X'(t) = 0$, the jump is a vacuum with expansion rate $w'(t) = [u]$.

The solution is an entropy solution if it satisfies

$$s' \leq 0 \quad \text{in} \quad M_{loc},$$

where we recall $s = s(t)$ is regarded as a measure, and this is the G-weak derivative. As in the 2×2 case, this measure is supported only on shocks, and the entropy equality $s' = 0$ holds in the presence of vacuums, as long as no shocks are present.

One can write down explicit examples as above, with the addition of entropy jumps where necessary, while noting that in a varying solution, the interaction of a shock with any non-trivial solution changes the trailing entropy field.

6. Fracture in Elasticity. The time-dependent displacement or *motion* $y(t,x)$ of an isentropic elastic material in one space dimension satisfies the second-order nonlinear equation

$$y_{tt} - \tau(y_x)_x = 0. \tag{46}$$

By introducing the *strain* $u := y_x$ and *velocity* $v = y_t$, we write this as the 2×2 equations of elasto-dynamics,

$$u_t - v_x = 0, \qquad v_t - \tau_x = 0, \tag{47}$$

which is closed by prescribing the *stress* τ by a stress-strain relation $\tau = \tau(u)$, which serves as a constitutive function. This system closely resembles the p-system, and for smooth solutions admits an energy inequality, namely

$$\big(\tfrac{1}{2}v^2 + W(u)\big)_t - \big(v\,\tau(u)\big)_x = 0, \quad \text{where} \quad W(u) := \int^u \tau(s)\,ds$$

is the elastic energy. We do not allow interpenetration of matter, so we require $u > 0$, and we assume that $\tau'(u) > 0$, which implies hyperbolicity of the system. In terms of energy, these are implied by $W(0+) = \infty$ and convexity of $W(u)$, respectively. In particular, we assume the existence of a unique strain u_0 such that

$$u_0 > 0 \quad \text{and} \quad \tau(u_0) = W'(u_0) = 0,$$

that is at which the stress vanishes and the energy is minimized; without loss of generality we may also assume that the energy vanishes there, $W(u_0) = 0$. We generally assume also that the material is *softening*, which means that $\tau''(u) < 0$.

We are interested in extending the notion of weak* solutions to this system, in order to understand the onset of fractures or cavities in the material. This will again be represented as a Dirac mass in the strain, or a discontinuity in the motion. Although the model will break down before an actual fracture occurs, a consistent picture of the behavior indicated by the model provides insights into the process of crack initiation.

6.1. Weak* Solutions. Following our development for the p-system, we regard the stress-strain relation as providing a map

$$\tau : \mathcal{A} \subset L^1(\Omega) \to L^1(\Omega),$$

and we wish to extend this to a map

$$\widehat{\tau} : M_{loc}(\Omega)_+ \to M(\Omega).$$

Assuming for now that this extension has been defined, we again choose the set of test functions to be $X = C_0(\Omega)^2$, and we set

$$\widetilde{Y} = M_{loc}(\Omega)_+ \times BV_{loc}(\Omega) \subset M^2_{loc}(\Omega) = X^*.$$

Definition 10. *A weak* solution of (47) is a pair* $(U, v) \in W^{1,q}_{w*}(0, T; \widetilde{Y}, X^*)$ *satisfying*

$$U' - \mathbb{D}_x v = 0, \qquad v' - \mathbb{D}_x \widehat{\tau}(U) = 0, \quad \textit{in} \quad L^q_{w*}(0, T; X^*). \tag{48}$$

When solving a Cauchy problem, the Cauchy data (U^0, v^0) *must be taken on in the space* X^* *by the time-continuous representative* $(\overline{U}, \overline{v})$*, that is*

$$\big(\overline{U}(0), \overline{v}(0)\big) = \big(U^0, v^0\big) \quad \textit{in} \quad X^*.$$

It remains to extend the stress-strain relation to Radon measures. We again follow our development for the p-system. Writing $U \in M_{loc+}$, we write

$$u := \Pi(U) \quad \text{and} \quad \mu := U - \iota(u), \quad \text{so that} \quad U = \iota(u) + \mu,$$

with $u \in L^1_{loc}$ and $\mu \perp \lambda$. Moreover, the singular measure μ is uniquely decomposed into singular continuous and atomic parts, that is

$$\mu = \mu_s + \mu_a, \quad \text{with} \quad \mu_a = \sum w_i\, \delta_{x_i},$$

where $w_i > 0$ and x_i are distinct. It is natural to require the generalized stress to satisfy

$$\widehat{\tau}(U) = \iota\big(\tau(u)\big) + \sum \widehat{\tau}\big(w_i\, \delta_{x_i}\big) \in M,$$

so we need only extend τ to a single Dirac mass.

Let ϕ denote a standard mollifier,

$$\phi \in C_c^\infty(\mathbb{R}), \quad \text{with} \quad \phi(\cdot) \geq 0 \quad \text{and} \quad \int \phi(x)\, dx = 1,$$

and set $\phi_\epsilon(x) := \phi(x/\epsilon)/\epsilon$. Recalling that the action of the Dirac mass $\delta_{x_0} \in M$ on a continuous function $g \in C_0$ can be written as

$$\big\langle \delta_{x_0}, g \big\rangle = \lim_{\epsilon \to 0} \int \phi_\epsilon(x - x_0)\, g(x)\, dx = g(x_0),$$

we define the action of $\widehat{\tau}(w_0\, \delta_{x_0})$ by

$$\big\langle \widehat{\tau}(w_0\, \delta_{x_0}), g \big\rangle = \lim_{\epsilon \to 0} \int \tau\big(u_0 + w_0\, \phi_\epsilon(x - x_0)\big)\, g(x)\, dx.$$

Here we include the offset u_0 so that the integrand again has compact support. We calculate

$$\begin{aligned}
\lim_{\epsilon \to 0} &\int \tau\big(u_0 + w_0\, \phi_\epsilon(x - x_0)\big)\, g(x)\, dx \\
&= \lim_{\epsilon \to 0} \int_{\{x :\, \phi_\epsilon > 0\}} \frac{\tau(u_0 + w_0\, \phi_\epsilon)}{u_0 + w_0\, \phi_\epsilon}\, \big(u_0 + w_0\, \phi_\epsilon\big)\, g(x)\, dx \\
&= \Big(\lim_{u \to \infty} \frac{\tau(u)}{u} \Big)\, w_0\, g(x_0)\,,
\end{aligned}$$

because $\lambda\big(\{x :\, \phi_\epsilon > 0\}\big) \to 0$. Thus we have

$$\begin{gathered}
\widehat{\tau}(\mu_a) = \sum \widehat{\tau}(w_i\, \delta_{x_i}) = \sum L_\tau\, w_i\, \delta_{x_i} = L_\tau\, \mu_a, \\
\text{where} \quad L_\tau := \lim_{u \to \infty} \frac{\tau(u)}{u},
\end{gathered} \tag{49}$$

and this completes our definition of the extension. We note that the softening condition $\tau''(u) < 0$ implies that $L_\tau < \infty$ exists and is finite.

Similarly, in order to extend the notion of entropy solutions, we need to extend the potential energy so that it is also defined on measures. As above, it suffices to define the extension $\widehat{W}(w_0\, \delta_{x_0})$, for $w_0 > 0$; having done so, and writing

$$U = \iota(u) + \mu_a + \mu_s, \quad u := \Pi(U), \quad \mu_a = \sum \widehat{\tau}(w_i\, \delta_{x_i}),$$

where μ_a is singular atomic and μ_s singular continuous, we set

$$\widehat{W}(U) = \iota\big(W(u)\big) + \sum \widehat{W}(w_i\,\delta_{x_i}).$$

Exactly as for the stress above, we extend the energy via a mollifier,

$$\begin{aligned}\big\langle \widehat{W}(w_0\,\delta_{x_0}), g\big\rangle &= \lim_{\epsilon\to 0}\int W\big(u_0 + w_0\,\phi_\epsilon(x-x_0)\big)\,g(x)\,dx\\ &= \Big(\lim_{u\to\infty}\frac{W(u)}{u}\Big)\,w_0\,g(x_0),\end{aligned}$$

so we can write

$$\begin{gathered}\widehat{W}(\mu_a) = \sum \widehat{W}(w_i\,\delta_{x_i}) = \sum L_W\,w_i\,\delta_{x_i} = L_W\,\mu_a,\\ \text{provided}\quad L_W := \lim_{u\to\infty}\frac{W(u)}{u} < \infty,\end{gathered}\tag{50}$$

and $\widehat{W}(\mu_a) = \infty$ if $L_W = \infty$; note that because the energy W is convex, the (possibly infinite) limit L_W always exists.

Having extended the energy, we say that a solution satisfying (48) is an entropy weak* solution if the entropy production is non-positive,

$$\big(\tfrac12 v^2 + \widehat{W}(U)\big)' - \mathbb{D}_x\big(v\,\widehat{\tau}(U)\big) \le 0 \quad\text{in}\quad L^q_{w*}(0,T;M_{loc}),\tag{51}$$

both terms being interpeted as Radon measures.

6.2. Properties of Weak* Solutions. Having defined weak* solutions, we now examine their properties and develop conditions that allow for a consistent model of fracture initiation. To begin, we examine the jump conditions as we did for the p-system. Thus we assume that we have a solution which is differentiable off of a discontinuity curve $x = X(t)$.

Following (24), we assume that the velocity v and strain U have the form

$$\begin{aligned}v(t,x) &= v_L(t,x)\,H\big(X(t)-x\big) + v_R(t,x)\,H\big(x-X(t)\big)\quad\text{and}\\ U(t) &= \iota\Big(u_L(t,\cdot)\,H\big(X(t)-\cdot)\big)\Big) + \iota\Big(u_R(t,\cdot)\,H\big(\cdot-X(t)\big)\Big) + w(t)\,\delta_{X(t)},\end{aligned}\tag{52}$$

so that, according to (49), we also have
(53)

$$\widehat{\tau}\big(U(t)\big) = \iota\Big(\tau\big(u_L(t,\cdot)\big)\,H\big(X(t)-\cdot)\big)\Big) + \iota\Big(\tau\big(u_R(t,\cdot)\big)\,H\big(\cdot-X(t)\big)\big)\Big) + L_\tau\,w(t)\,\delta_{X(t)}.$$

Differentiating, we get

$$\begin{aligned}\mathbb{D}_x v &= \iota\Big(\partial_x v_L\,H\big(X(t)-\cdot\big)\Big) + \iota\Big(\partial_x v_R\,H\big(\cdot-X(t)\big)\Big) + [v]\,\delta_{X(t)}\quad\text{and}\\ v' &= \iota\Big(\partial_t v_L\,H\big(X(t)-\cdot\big)\Big) + \iota\Big(\partial_t v_R\,H\big(\cdot-X(t)\big)\Big) + [v](t)\,\big(-X'(t)\big)\,\delta_{X(t)},\end{aligned}$$

where we have set $[g](t) := g_R\big(t,X(t)+\big) - g_L\big(t,X(t)-\big)$, and similarly

$$\begin{aligned}U' &= \iota\Big(\partial_t u_L\,H\big(X(t)-\cdot\big)\Big) + \iota\Big(\partial_t u_R\,H\big(\cdot-X(t)\big)\Big)\\ &\qquad + [u](t)\,\big(-X'(t)\big)\,\delta_{X(t)} + w'(t)\,\delta_{X(t)} + w(t)\,X'(t)\,\mathbb{D}_x\delta_{X(t)},\\ \mathbb{D}_x\widehat{\tau}(U) &= \iota\Big(\partial_x\tau(u_L)\,H\big(X(t)-\cdot\big)\Big) + \iota\Big(\partial_x\tau(u_R)\,H\big(\cdot-X(t)\big)\Big)\\ &\qquad + [\tau(u)]\,\delta_{X(t)} + L_\tau\,w(t)\,\mathbb{D}_x\delta_{X(t)}.\end{aligned}$$

It follows immediately that in order for (48) to make sense in X^*, the coefficients of $\mathbb{D}_x \delta_{X(t)}$ must vanish, so that

$$w(t)\, X'(t) = 0 \quad \text{and} \quad L_\tau\, w(t) = 0. \tag{54}$$

Assuming these conditions, we then again obtain a generalized jump condition,

$$w'(t) - X'(t)\,[u] = [v], \qquad -X'(t)\,[v] = [\tau(u)], \tag{55}$$

while the system holds in the classical sense where the solution is differentiable. It follows that in order to admit any fracture solution in $W^{1,1}_{w*}(0, T; Y, X^*)$, we must have

$$L_\tau = \lim_{u\to\infty} \frac{\tau(u)}{u} = 0,$$

and if this holds then any crack must necessarily be stationary, $X'(t) = 0$. These conditions are consistent with gas dynamics, for which $p(\infty) = 0$.

Next, in order to avoid spurious singular measure solutions, we need to again impose a consistency of the medium condition: this is a modelling condition dependent on physical effects at the crack after the crack has been initiated. If there is a nontrivial crack, it must be stationary and by (55) we also have $[\tau] = 0$. This means that the stress, and so also the strain, has a single limit, say

$$\tau\big(u(t, x)\big) \to \tau(u_*), \quad \text{or} \quad u(t, x) \to u_*, \quad \text{as} \quad x \to X(t), \tag{56}$$

and is analogous to the natural vacuum condition. For example, the simplest condition is that once the crack has been initiated, it imposes no extra force on the interior material, so that the crack boundary is stress-free, $u_* = u_0$.

Finally, we consider the effect of entropy: again using (52), (53), and referring to (51), we write the entropy as

$$\widehat{\eta} = \iota\Big(\big(\tfrac12 v_L^2 + W(u_L)\big)\, H\big(X(t) - \cdot\big)\Big) + \iota\Big(\big(\tfrac12 v_R^2 + W(u_R)\big)\, H\big(\cdot - X(t)\big)\Big) + L_W\, w(t)\, \delta_{X(t)},$$

while, using (54), the entropy flux becomes

$$-v\, \widehat{\tau}(U) = -\iota\Big(\big(v_L\, \tau(u_L)\big)\, H\big(X(t) - \cdot\big)\Big) - \iota\Big(\big(v_R\, \tau(u_R)\big)\, H\big(\cdot - X(t)\big)\Big).$$

Again differentiating, and using (54), the entropy production (51) becomes

$$\begin{aligned}
&\big(\tfrac12 v^2 + \widehat{W}(U)\big)' - \mathbb{D}_x\big(v\, \widehat{\tau}(U)\big)\\
&\qquad = \iota\Big(\big(\partial_t\big(\tfrac12 v_L^2 + W(u_L)\big) - \partial_x\big(v_L\, \tau(u_L)\big)\big)\, H\big(X(t) - \cdot\big)\Big)\\
&\qquad\quad + \iota\Big(\big(\partial_t\big(\tfrac12 v_R^2 + W(u_R)\big) - \partial_x\big(v_R\, \tau(u_R)\big)\big)\, H\big(\cdot - X(t)\big)\Big)\\
&\qquad\quad + \Big(-X'(t)\big[\tfrac12 v^2 + W(U)\big] + L_W\, w'(t) - \big[v\, \tau(u)\big]\Big)\, \delta_{X(t)},
\end{aligned}$$

where again $[g] = g_R\big(t, X(t)+\big) - g_L\big(t, X(t)-\big)$. The first two terms cancel because the solution is differentiable, so the entropy condition becomes non-positivity of the coefficient, so

$$-X'(t)\big[\tfrac12 v^2 + W(U)\big] + L_W\, w'(t) - \big[v\, \tau(u)\big] \le 0. \tag{57}$$

In order to make sense of this inequality, we require that $L_W < \infty$, which in turn implies

$$L_W = \lim_{u\to\infty} \frac{W(u)}{u} = \lim_{u\to\infty} \tau(u) =: \tau_\infty < \infty, \quad \text{so also} \quad L_\tau = 0.$$

Note that (57) is consistent with (34) for a gas, for which $p_\infty = 0$.

If there is no crack, then $w(t) = 0$, and (57) reduces to the usual entropy condition for shocks. On the other hand, if there is a crack, so $w(t) > 0$, then $X'(t) = 0$ and, using (55), (56), (57) simplifies as

$$\begin{aligned} L_W\, w'(t) - \big[v\,\tau(u)\big] &= L_W\, w'(t) - [v]\,\tau(u_*) \\ &= \big(\tau_\infty - \tau(u_*)\big)\, w'(t) \le 0. \end{aligned}$$

It follows that $w'(t) \le 0$, so that any crack satisfying the entropy condition must be pre-existing with $w(0) > 0$. This says that the elasticity of the material prevents crack formation in any solution.

In summary, we have shown the following.

THEOREM 11. *The space $W^{1,q}_{w*}(0,T;X^*)$ supports weak* solutions which admit fractures if and only if the limit $L_\tau = 0$. However, any nontrivial fracture fails to satisfy the entropy condition unless it has finite size in the initial data.*

6.3. Comparison to Slic-solutions. In [9], Giesselmann and Tzavaras introduce a notion of *slic*-solution in order to study the formation of cavities and fractures in dynamic elasticity. In doing so they work primarily with the second-order system (46), namely

$$y_{tt} - \tau(y_x)_x = 0,$$

and they study discontinuities in the motion y. Their main goal is to settle a question of nonuniqueness in earlier examples, in which cavitating solutions apparently had a lower energy than smooth solutions [15]. To do this, they mollify the discontinuous solution, and declare y to be a slic (Singular Limiting Induced from Continuum) solution, if for any mollifier $\phi \in C^\infty(\mathbb{R})$, the mollified solution $y^\epsilon = y * \phi_\epsilon$ satisfies

$$f^\epsilon := y^\epsilon_{tt} - \tau(y^\epsilon_x)_x \to 0\,.$$

Specifically, for the particular solutions found in [15], the authors discover the limiting value of the energetic cost of opening up a cavity, and inclusion of this term implies that the cavities are non-entropic.

Here we compare our methods and results to those of [9]. We begin with their one-dimensional example, which is a slic solution, and show that it is a weak* solution. We similarly calculate the entropy production, and show that this agrees with the limit obtained in [9]. Our approach has several advantages: by extending the stress and entropy via (49) and (50), respectively, we do not need to work directly with mollifiers; our calculations are exact so there are no approximation errors; because we are working in the space of measures, we do not need to integrate explicitly; our approach is general and yields a localized description of all waves; and our calculations themselves are much shorter.

We begin with the solution studied in [9]: this is a self-similar discontinuous motion $y(t,x)$ that represents a shearing motion with the fracture that is initiated at time $t = 0$ and then propagates outwards, behind an expanding shock wave. Explicitly

y is given by

$$
\begin{aligned}
y(t,x) &= \lambda\, x\,\big(1 - H(x+\sigma\, t)\big) \\
&\quad + (\alpha\, x - Y_0\, t)\,\big(H(x+\sigma\, t) - H(x)\big) \\
&\quad + (\alpha\, x + Y_0\, t)\,\big(H(x) - H(x-\sigma\, t)\big) \\
&\quad + \lambda\, x\, H(x-\sigma t).
\end{aligned} \tag{58}
$$

Here Y_0 is the velocity of the crack, σ is the shock speed, λ is the initial stretching and $\alpha < \lambda$ is a free parameter repressnting the strain at the cavity surface, $\alpha = u_*$ in (56). The parameters are related via the Rankine-Hugoniot conditions,

$$
Y_0 = \sigma(\lambda - \alpha), \qquad \sigma^2(\lambda-\alpha) = \tau(\lambda) - \tau(\alpha)\,. \tag{59}
$$

As above, we work with the associated first order system (47). For the given motion, the components $U \in C^1\big(0,T;M(\mathbb{R})\big)$ and $v \in C^1\big(0,T;BV(\mathbb{R})\big)$ are easily computed to be

$$
\begin{aligned}
U(t) = y_x &= 2\, t\, Y_0\, \delta_0 + \lambda\,(1 - H_{-\sigma\, t}) + \alpha\,(H_{-\sigma\, t} - H_{\sigma\, t}) + \lambda\, H_{\sigma\, t}, \\
v(t) = y_t &= -Y_0\,(H_{-\sigma\, t} - H_0) + Y_0\,(H_0 - H_{\sigma\, t}),
\end{aligned} \tag{60}
$$

where H_a stands for the shifted Heaviside function $x \to H(x-a)$, and for convenience we have dropped the inclusion $\iota : L^1 \to M$. It is easy to check that the generalized Rankine-Hugoniot conditions (55) reduce to (59) for this solution.

A direct computation reveals that

$$
U_t = 2\, Y_0\, \delta_0 - \sigma\,(\lambda-\alpha)\,(\delta_{-\sigma\, t} + \delta_{\sigma\, t}) \quad \text{and} \quad v_x = 2\, Y_0\, \delta_0 - Y_0\,(\delta_{-\sigma\, t} + \delta_{\sigma\, t}),
$$

so that $(47)_1$ holds in $C^1(0,T;M(\mathbb{R}))$. While in [9] the authors use mollifiers and slic solutions to deal with the nonlinear term $\tau(y_x)$, we calculate directly using the extension (49). Using (49) in (60), we get

$$
\widehat{\tau}(U) = L_\tau\, 2\, t\, Y_0\, \delta_0 + \tau(\lambda)\,(1 - H_{-\sigma\, t}) + \tau(\alpha)\,(H_{-\sigma\, t} - H_{\sigma\, t}) + \tau(\lambda)\, H_{\sigma\, t},
$$

so that

$$
\widehat{\tau}(U)_x = L_\tau\, 2\, t\, Y_0\, \mathbb{D}_x \delta_0 + \big(\tau(\lambda) - \tau(\alpha)\big)\,(\delta_{\sigma\, t} - \delta_{-\sigma\, t}),
$$

while from (60),

$$
v_t = \sigma\, Y_0\,(\delta_{\sigma\, t} - \delta_{-\sigma\, t})\,.
$$

Using (59), $(47)_1$ holds in $C^1(0,T;M(\mathbb{R}))$, so we conclude that y is a weak* solution, if and only if $L_\tau = 0$. This is consistent with Theorem 11 and with the (necessary and sufficient) condition in [9] for slic solutions. Thus, the approximation procedure of [9] applied to their crack initiation example can be replaced by extending the stress $\widehat{\tau}$ and working directly with singular measures.

We now compute the entropy and entropy production of the solution (60). Setting $L_\tau = 0$ and recalling the entropy is $\eta = \frac{1}{2}v^2 + W(u)$ with entropy flux $q = v\,\tau(u)$, we again extend and write

$$
\begin{aligned}
\widehat{q} &= -v\,\widehat{\tau}(U) = \Big(Y_0\big(H_{-\sigma\, t} - H_0\big) - Y_0\big(H_0 - H_{\sigma\, t}\big)\Big)\,\tau(\alpha)\,, \quad \text{and} \\
\widehat{\eta} &= \tfrac{1}{2} Y_0^2\,\big(H_{-\sigma t} - H_{\sigma t}\big) + L_W\, 2\, t\, Y_0\, \delta_0 \\
&\qquad + W(\lambda)\,\big((1 - H_{-\sigma\, t}) + H_{\sigma\, t}\big) + W(\alpha)\,\big(H_{-\sigma\, t} - H_{\sigma\, t}\big)\,.
\end{aligned}
$$

Differentiating, we get

$$\partial_x \widehat{q} = Y_0\, \tau(\alpha)\, \big(\delta_{-\sigma\, t} + \delta_{\sigma\, t} - 2\, \delta_0\big), \quad \text{and}$$
$$\partial_t \widehat{\eta} = \tfrac{1}{2} Y_0^2\, \sigma\, \big(\delta_{-\sigma t} + \delta_{\sigma t}\big) + L_W\, 2\, Y_0\, \delta_0 - \sigma\, \big(W(\lambda) - W(\alpha)\big)\, \big(\delta_{-\sigma t} + \delta_{\sigma t}\big),$$

so that the entropy production is

$$\partial_t \widehat{\eta} + \partial_x \widehat{q} = \theta\, \big(\delta_{-\sigma\, t} + \delta_{\sigma\, t}\big) + 2\, Y_0\, \big(\tau_\infty - \tau(\alpha)\big)\, \delta_0, \quad \text{where}$$
$$\theta := \sigma\big(\tfrac{1}{2} Y_0^2 + W(\alpha) - W(\lambda)\big) + \tau(\alpha)\, Y_0\,,$$

where we have used $L_W = \tau_\infty$. Finally, using (59) and manipulating, we get

$$\begin{aligned} \theta &= \sigma\, \Big(\tfrac{1}{2}\big(\tau(\lambda) + \tau(\alpha)\big)\, (\lambda - \alpha) - \big(W(\lambda) - W(\alpha)\big)\Big) \\ &= \sigma \int_\alpha^\lambda \Big(\tfrac{1}{2}\big(\tau(\lambda) + \tau(\alpha)\big) - \tau(s)\Big)\, ds < 0, \end{aligned} \tag{61}$$

because $\tau'' < 0$ for a stress with softening response.

It is now clear that the shocks with speed $\pm\sigma$ have negative entropy production, as needed, but the crack at the origin does not, so is not entropic. This again mirrors the results of [9] and Theorem 11.

Finally, in [9], the authors define the total mechanical energy of the slic-solution y on the interval $I \subset \mathbb{R}$ via the limit

$$E_{slic}(y; I) := \lim_{\epsilon \to 0} \int_I \Big(\tfrac{1}{2}(y_t^\epsilon)^2 + W(y_x^\epsilon)\Big)\, dx.$$

However, in our framework the total energy on the interval is simply

$$\widehat{\eta}(I) = \int_I \Pi(\widehat{\eta})\, dx + \widehat{\eta}_s(I),$$

where as usual $\widehat{\eta}_s$ is the singular part. In particular, if

$$(-\sigma\, t, \sigma\, t) \subset I, \quad \text{we have} \quad \iota\big(H_{-\sigma\, t} - H_{\sigma\, t}\big)(I) = 2\, \sigma\, t,$$

so we immediately obtain

$$\begin{aligned} \widehat{\eta}(I) &= W(\lambda)\, (b - a) + L_W\, 2\, t\, Y_0 + \big(\tfrac{1}{2} Y_0^2 + W(\alpha) - W(\lambda)\big)\, 2\, \sigma\, t \\ &= \widehat{\eta}_{nc}(I) + 2\, t\, \big(\theta + Y_0\, (\tau_\infty - \tau(\alpha))\big), \end{aligned}$$

where $\widehat{\eta}_{nc}$ is the entropy of the crack-free solution $y = \lambda\, x$. Using (61) and (59), we calculate

$$\theta + Y_0\, (\tau_\infty - \tau(\alpha)) = \sigma \int \tfrac{1}{2}\big(\tau(\lambda) - \tau(\alpha)\big) + \big(\tau_\infty - \tau(s)\big)\, ds > 0,$$

so that the crack-free solution has lower energy, as noted in [9]. Once again we see the advantage of working directly with measures, and avoiding explicit integrations.

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