NOWHERE-ZERO FLOWS ON
SIGNED EULERIAN GRAPHS

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Abstract

This paper is devoted to a detailed study of nowhere-zero flows on signed eulerian graphs. We generalise the well-known fact about the existence of nowhere-zero 2-flows in eulerian graphs by proving that every signed eulerian graph that admits an integer nowhere-zero flow has a nowhere-zero 4-flow. We also characterise signed eulerian graphs with flow number 2, 3, and 4, as well as those that do not have an integer nowhere-zero flow. Finally, we discuss the existence of nowhere-zero A-flows on signed eulerian graphs for an arbitrary abelian group A.

Keywords: Eulerian graph; signed graph; bidirected graph; nowhere-zero flow

1 Introduction

A nowhere-zero flow is an assignment of an orientation and a nonzero value from an abelian group A to each edge of a graph in such a way that the Kirchhoff current law is fulfilled at each vertex: the sum of in-flowing values equals the sum of out-flowing values. This concept was introduced by Tutte [10] in 1949 and has been extensively studied by many authors. There is an analogous concept of a nowhere-zero flow that uses bidirected edges instead of directed ones, first systematically developed by Bouchet [2] in 1983. A bidirected edge is an edge consisting of two half-edges which receive separate orientations; a bidirected graph is one where each edge has been bidirected. A nowhere-zero flow on a bidirected graph is formed by valuating each edge with a nonzero element of A in such a way that, for every vertex v, the sum of values on the half-edges directed to v equals the sum of values on the half-edges directed out of v. If the half-edges of each edge of G are aligned, G essentially becomes a directed graph, which implies that the bidirected variant of a flow is more general than the simply directed one.

It is obvious that the choice of an orientation in the simply directed case is immaterial: it is always possible to reverse an edge and change its flow value to the opposite value. The same is true in the bidirected case as long as both half-edges of an edge are reversed at once, leaving the partition into aligned and non-aligned edges unaltered. It is somewhat less obvious that the flow on a bidirected graph can be preserved even if this edge-partition is disturbed by reversing all the half-edges around a vertex. If we keep the flow values

intact, the Kirchhoff law will not be violated. This indicates that the invariance of flows on bidirected graphs is best understood in terms of signed graphs where the aligned edges are positive, non-aligned edges are negative, and the just described operation is the familiar vertex-switching from signed graph theory. To summarise, the existence of a flow is a property of a signed graph invariant under switching equivalence and independent of its particular bidirection. An easy switching argument further shows that the simply directed case corresponds to the case of balanced signed graphs, those where every circuit has an even number of negative edges.

The study of flows on signed (or, equivalently, bidirected) graphs seems to have its roots in the study of embeddings of graphs in nonorientable surfaces. Signed graphs provide a convenient language for description of such embeddings and bidirected graphs arise naturally as duals of directed graphs. In 1968, Youngs [13, 14] employed nowhere-zero flows on signed cubic graphs with values in cyclic groups, combined with surface duality, to construct nonorientable triangular embeddings of certain complete graphs. A duality relationship between local tensions and flows on graphs embedded in nonorientable surfaces was the main motivation for Bouchet's work [2] on integer flows in signed graphs. As in the unsigned case, the fundamental problem here is to find, for a given signed graph, a nowhere-zero flow with the smallest maximum absolute edge-value. If this value is $k-1$, we speak of a nowhere-zero k-flow. In the same paper Bouchet showed that every graph that admits a nowhere-zero integer flow has a nowhere-zero 216-flow. He also proposed the following conjecture that has become an incentive for much of the current research in the area.

Conjecture. (Bouchet's 6-Flow Conjecture) *Every signed graph with a nowhere-zero integer flow has a nowhere-zero 6-flow.*

The present status of this conjecture can be summarised as follows. Bouchet's 216-flow theorem was improved in 1987 by Zýka [18] to a nowhere-zero 30-flow, and very recently by DeVos [3] to a nowhere-zero 12-flow, which is currently the best general approximation of Bouchet's conjecture. In 2005, the existence of a nowhere-zero 6-flow was established by Xu and Zhang [12] for every 6-edge-connected signed graph with a nowhere-zero integer flow. In 2011, Raspaud and Zhu [8] proved that every 4-edge-connected signed graph admitting an integer nowhere-zero flow has a nowhere-zero 4-flow, which is best possible. Recently, Wu et al. [11] proved that every 8-edge-connected signed graph admitting an integer nowhere-zero flow has a nowhere-zero 3-flow.

In spite of these results, very little is known about exact flow numbers for various classes of signed graphs. Surprisingly, the situation is open even for signed eulerian graphs, although Xu and Zhang [12, Proposition 1.4] proved that a signed eulerian graph with an even number of negative edges has a nowhere-zero 2-flow. The purpose of the present paper is to strengthen this result by proving the following theorem which provides a classification of signed eulerian graphs according to their flow numbers. The smallest example in each of the four resulting classes is displayed in Figure 1.

Main Theorem. *Let G be a signed eulerian graph and let $\Phi(G)$ denote the flow number of G. Then*

(a) *G has no nowhere-zero flow if and only if G is unbalanced and $G-e$ is balanced for some edge e;*

(b) *$\Phi(G) = 2$ if and only if G has an even number of negative edges;*

(c) *$\Phi(G) = 3$ if and only if G can be decomposed into three eulerian subgraphs, with an odd number of negative edges each, that share a common vertex;*
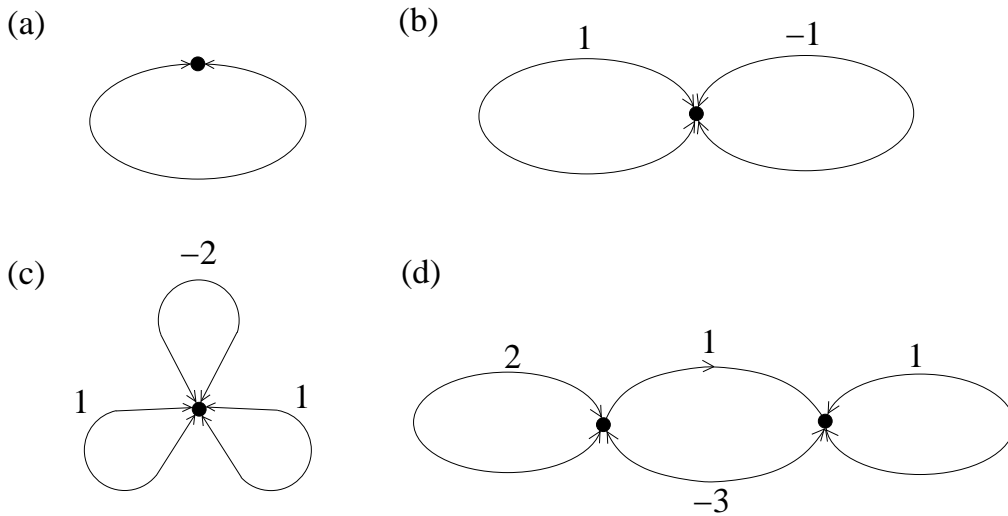
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(d) $\Phi(G) = 4$ *otherwise.*

Figure 1: Examples of signed eulerian graphs from Main Theorem

Our paper is organised as follows. The next section gives a brief introduction to signed graphs collecting the concepts and results to be used later in this paper; for more information on signed graphs we refer the reader to [15, 16, 17]. The third section concentrates on the problem of the existence of nowhere-zero integer flows in signed graphs and provides a characterisation of graphs that admit a nowhere-zero integer flow. A corollary of this characterisation states that a signed eulerian graph has no nowhere-zero integer flow if and only if its signature is switching-equivalent to one with a single negative edge. In Section 4 we prove that all other eulerian graphs admit a nowhere-zero 4-flow and in Section 5 we characterise those with flow number 3. Finally, the last section deals with the existence of nowhere-zero A-flows on signed eulerian graphs for an arbitrary abelian group A.

We conclude this section with a few terminological and notational remarks. All our graphs are finite and may have multiple edges and loops. A graph is called *eulerian* if it is connected, with vertices of even valency. A *circuit* is a connected 2-regular graph, and a *cycle* is a graph that has a decomposition (possibly empty) into edge-disjoint circuits. A set of edges is often identified with the subgraph it induces; this should not cause any confusion. Each walk is understood to be directed from its initial to its terminal vertex. The walk obtained from W by reversing the direction will be denoted by W^{-1}. If x and y are two vertices of W listed in the order of their appearance on W, we let $W[x, y]$ denote the portion of W initiating at x and terminating at y. Finally, if W_1 is a u-v-walk and W_2 is a v-w-walk, then $W_1 W_2$ denotes the u-w-walk where the traversal of W_1 is followed by the traversal of W_2.

2 Fundamentals of signed graphs

A *signed graph* is a graph G endowed with a *signature*, a mapping that assigns $+1$ or -1 to each edge. In our notation, the signature is usually implicit in the notation of the graph itself; only when needed, it will be denoted by σ_G, or simply by σ, if G is clear from the context. As most graphs considered here will be signed, the term *graph* will usually be used to mean a signed graph, and the adjective *signed* will be added only for emphasis.

An unsigned graph will be regarded as a signed graph having the *all-positive signature* $\sigma_G \equiv +1$.

The actual distribution of edge signs in a signed graph is not very important. What is fundamental is the product of signs on each of its circuits. This constitutes the concept of a *balance* in a signed graph: Let F be a subgraph or a set of edges of a signed graph G. We define the *sign* of F, denoted by $\sigma_G(F)$, as the product of the signs of all edges in F. Thus, every subgraph or set of edges can either be *positive* or *negative*, depending on whether its sign is $+1$ or -1. A closed walk, in particular a circuit, is said to be *balanced* if its sign is $+1$; otherwise it is *unbalanced*. A graph is *balanced* if it contains no unbalanced circuit. The collection of all balanced circuits of a signed graph is its most fundamental characteristic: signed graphs that have the same underlying graphs and the same sets of balanced circuits are considered to be *identical*, irrespectively of their actual signatures. Signatures of identical signed graphs are called *equivalent*.

Let G be a signed graph and let v be a vertex of G. It is clear that if we interchange the signs of all non-loop edges incident with v, the set of balanced circuits will not change. This operation, called *switching* at v, thus produces an equivalent signature. More generally, for a set U of vertices of G we define *switching* at U as the interchange of signs on all edges with exactly one end-vertex in U. It is easy to see that switching the signature at U has the same effect as switching at all the vertices of U in a succession. Furthermore, it is not difficult to prove that two signatures are equivalent if and only if they are *switching-equivalent*, that is, if they can be turned into each other by a sequence of vertex switchings [16, Proposition 3.2]. In particular, a signed graph is balanced if and only if its signature is switching-equivalent to the all-positive signature.

The following characterisation of balanced graphs due to Harary [5] has the same spirit as the well-known characterisation of bipartite graphs.

Theorem 2.1. (Harary's Balance Theorem [5]) *A signed graph is balanced if and only if its vertex set can be partitioned into two sets (either of which may be empty) in such a way that each edge between the sets is negative and each edge within a set is positive.*

The partition of the vertex set of a signed into two sets mentioned in the previous theorem will be called a *balanced bipartition*. It is useful to realise that the balanced bipartition of a signed graph depends on the chosen signature. However, once a signature is fixed and the graph is connected, the balanced bipartition is uniquely determined.

A signed graph G is said to be *antibalanced* if replacing its signature σ_G with $-\sigma_G$ makes it balanced. This definition immediately implies that a signed graph is antibalanced if and only if its signature is equivalent to the all-negative signature. Hence, a circuit of an antibalanced graph is balanced precisely when it has an even length. Consequently, an antibalanced graph is balanced if and only if its underlying unsigned graph is bipartite.

The following is a direct consequence of Harary's Balance Theorem.

Corollary 2.2. *A signed graph is antibalanced if and only if its vertex set can be partitioned into two sets (either of which may be empty) in such a way that each edge between the sets is positive and each edge within a set is negative.*

The partition of the vertex set of a signed into two sets mentioned in the previous corollary will be called an *antibalanced bipartition*.

3   Flows and flow-admissible signed graphs

The aim of this section is to introduce the concepts related to flows in signed graphs and to characterise signed graphs that admit a nowhere-zero integer flow.

As in the case of unsigned graphs, the definition of a flow calls for an orientation of the underlying graph. The signed version, however, requires a bidirection of edges rather than just a usual orientation. For this purpose we regard each edge e, including loops, as consisting of two half-edges, each half-edge being incident with only one end-vertex of e. An edge is *bidirected* if each of its two half-edges is individually directed from or to its associated end-vertex. Thus every edge has four possible orientations which fall into two types:

- Two of the four orientations have one half-edge directed from and the other half-edge towards its end-vertex. Such bidirected edges will be identified with the usual directed edges and called *ordinary* edges.

- The other type of bidirection has either both half-edges directed from or both half-edges oriented towards their end-vertices. In the former case, the edge is said to be *extroverted*, and in the latter case it is said to be *introverted*. Extroverted and introverted edges are collectively called *broken* edges.

A bidirected edge e incident with a vertex v is said to be directed *out* of v if its half-edge incident with v is directed out of v. Similarly, e is said to be directed *to* v if its half-edge incident with v is directed to v. For an arbitrary vertex v, the set of all edges directed out of v is denoted by $E^{\mathrm{out}}(v)$, and the set of all edges directed to v is denoted by $E^{\mathrm{in}}(v)$; the set of all edges incident with v is denoted simply by $E(v)$.

Every bidirected edge e has a well-defined *reverse* e^{-1} obtained by reversing the orientation of both constituting half-edges. In particular, if e is an ordinary edge, then e^{-1} is also ordinary but with the reversed order of its end-vertices; if e is extroverted, then e^{-1} is introverted, and vice versa.

Given a signed graph G, an *orientation* of G is an assignment of a bidirection to each edge in such a way that the following compatibility rule is fulfilled: every positive edge receives an orientation that turns it into an ordinary edge, while every negative edge receives an orientation that turns it into a broken edge. Thus, endowing G with an orientation makes G a *bidirected* graph. Conversely, every bidirected graph can be regarded as a signed graph in which ordinary edges are positive and broken edges are negative. In other words, a bidirected graph and a signed graph endowed with an orientation are equivalent concepts.

If a signed graph G is bidirected, switching its signature at an arbitrary vertex or set of vertices must cause the change of its orientation so that the compatibility rule remains fulfilled. Therefore we define *switching* at an arbitrary set U of vertices as an operation that reverses the orientation of each half-edge incident with a vertex in U and consequently changes the sign of each edge with exactly one end in U.

We now proceed to the definition of a flow on a signed graph. Let G be a signed graph which has been endowed with an arbitrary compatible orientation. A mapping $\xi \colon E(G) \to A$ with values in an abelian group A is called an *A-flow* on G provided that the following continuity condition, the *Kirchhoff law*, is satisfied at each vertex v of G:

$$\sum_{e \in E^{\mathrm{out}}(v)} \xi(e) \quad - \sum_{e \in E^{\mathrm{in}}(v)} \xi(e) = 0.$$

An A-flow ξ is said to be *nowhere-zero* if $\xi(e) \neq 0$ for each edge e of G. A *nowhere-zero k-flow* is a $\mathbb{Z}$-flow that takes its values from the set $\{\pm 1, \ldots \pm (k-1)\}$. Clearly, a signed graph that has a nowhere-zero k-flow also has a nowhere-zero $(k+1)$-flow. The smallest integer k for which a signed graph G has a nowhere-zero k-flow is called the *flow number* of G and is denoted by $\Phi(G)$.

Take an arbitrary flow ξ on a signed graph G. It is obvious that if we reverse the orientation of an arbitrary edge e of G – that is, if we replace e with e^{-1} – and set $\xi(e^{-1}) = -\xi(e)$, Kirchhoff's law remains satisfied. Similarly, if we switch the signature at some vertex v of G, we change the bidirection of each edge e incident with v to a well defined new bidirection e' whose type differs from that of e. At the same time we interchange the roles of the sets $E^{\mathrm{out}}(v)$ and $E^{\mathrm{in}}(v)$. It follows that if we set $\xi(e') = \xi(e)$, Kirchhoff's law will continue to hold at each vertex of G. The conclusion is that any flow on a signed graph G is essentially independent of the chosen orientation and the particular signature and only depends on the switching class of the signed graph itself. This makes the definition of a flow on a signed graph completely analogous to the definition of a flow on an unsigned graph.

In spite of this apparent similarity of definitions, flows on signed graphs can sometimes have a rather unexpected behaviour. For example, the following useful property has no analogue in unsigned graphs.

Lemma 3.1. *For any flow on a signed graph, the sum of values on negative edges taken in the extroverted orientation is zero.*

Proof. Take a flow on a signed graph G, redirect each negative edge of G to make it extroverted, and change the flow values accordingly. By the Kirchhoff law, the total outflow from every vertex is zero, so the sum of all outflows is again zero. Each edge contributes to this sum twice, a positive edge with opposite signs while a negative edge with the same sign. Ignoring the values on positive edges, which sum to zero, we infer that the doubled sum of values on the negative edges equals zero. The result follows. □

It is well known that an unsigned graph admits a nowhere-zero flow if and only if it is bridgeless, irrespectively of the group employed. In contrast, the existence of a nowhere-zero flow on a signed graph may depend on the chosen group, and exceptional graphs are less straightforward to describe. In the present section we focus on graphs that admit a nowhere-zero integer flow and call them *flow-admissible*. Other groups will be discussed in Section 6.

Consider a signed graph G. If G is balanced, then a simple switching argument shows that each flow on G corresponds to a flow on the corresponding unsigned graph. Therefore a balanced signed graph is flow-admissible if and only if it is bridgeless. In contrast, an unbalanced bridgeless signed graph may fail to be flow-admissible while a graph having a bridge may happen to be flow admissible. For example, Bouchet in [2, Lemma 2.4] observed that a 2-edge-connected graph with a single negative edge is never flow-admissible.

Our next aim is to characterise unbalanced flow-admissible signed graphs. Before proceeding to the result, some preparations are in order.

We describe a simple technique which is useful for construction of nowhere-zero flows on signed graphs. Consider a pair of adjacent edges e and f sharing a vertex v in a bidirected signed graph. We say that the walk ef is *consistently directed* at v if exactly one of the two half-edges incident with v is directed to v. A trail P is said to be *consistently directed* if all pairs of consecutive edges of P are consistently directed. Now let G be a signed graph carrying an A-flow ϕ, and let P be a u-v-trail in G; if ϕ has not been specified, we are assuming that $\phi = 0$. By *sending a value $b \in A$ from u to v along P* we mean the modification of ϕ into a new valuation $\phi' \colon E(G) \to A$ defined as follows. We keep the values of ϕ everywhere except on the edges of P. On P we change the orientation of edges in such a way that the initial edge is directed from u, and P is consistently directed

at each internal vertex; we change the flow values accordingly. Finally, for each edge e on P we replace the current value $\phi(e)$ with $\phi(e) + b$.

Under the new valuation ϕ', Kirchhoff's law will be satisfied at each inner vertex of P. Moreover, if P is closed – that is if $u = v$ – and the number of negative edges on P is even, Kirchhoff's law will be satisfied at u as well. Thus sending any value along a closed trail with an even number of negative edges will turn an A-flow into another A-flow.

Define a *signed circuit* as a signed graph of any of the following three types:

(1) a balanced circuit,

(2) the union of two unbalanced circuits that meet at a single vertex, or

(3) the union of two disjoint unbalanced circuits with a path that meets the circuits only at its ends,

A signed circuit falling under item (2) or (3) will be called an *unbalanced bicircuit*.

Observe that every signed circuit admits a nowhere-zero integer-flow. Indeed, a signed circuit S satisfying (1) or (2) constitutes a closed trail with an even number of negative edges, so sending value 1 along the trail produces a nowhere-zero 2-flow on S. If S is an unbalanced bicircuit satisfying (3) and consisting of unbalanced circuits C_1 and C_2 joined by a path P, we switch the signature of S to make P all-positive and construct a nowhere-zero 3-flow as follows. We send value 1 along C_1 from the end-vertex of P, value -1 along C_2 from the other end-vertex of P, and value -2 along P from the end-vertex in C_1 to the end-vertex in C_2. Since Kirchhoff's law is satisfied at every vertex of the bicircuit, the result is a nowhere-zero 3-flow.

Now we are in position to present a characterisation of unbalanced flow-admissible graphs. Much of the result has been previously known: the equivalence $(a) \Leftrightarrow (b)$ follows from the combination of Bouchet's Proposition 3.1 in [2] with Zaslavsky's characterisation of circuits in the signed graphic matroid [16, Theorem 5.1 (e)]; a direct graph-theoretical proof can be found in [6, Corollary 3.2]. The implication $(a) \Rightarrow (c)$ is a strengthening of Lemma 2.5 from [2], and a special case of the equivalence $(a) \Leftrightarrow (c)$ for antibalanced signed graphs has been proved by Akbari et al. in [1, Theorem 1] using a different terminology. Nevertheless, to our best knowledge the complete statement and the proof have never appeared in the literature.

Theorem 3.2. *The following statements are equivalent for every connected unbalanced signed graph G.*

(a) *G admits a nowhere-zero integer flow.*

(b) *The edges of G can be covered by signed circuits.*

(c) *For each edge e, the graph $G - e$ contains no balanced component.*

Proof. We only prove that $(a) \Leftrightarrow (c)$. For the proof of $(a) \Leftrightarrow (b)$ see [2] or [6].

$(a) \Rightarrow (c)$ Let ξ be a nowhere-zero integer flow on G, and suppose that, for some edge e, the graph $G - e$ has a balanced component H. Switch the signature of G to make each edge of H positive. If e was a bridge in G, then, as in the unsigned case, the sum of outflows from the vertices of H would force $\xi(e) = 0$, which is impossible. Therefore $G - e$ is connected and e is the only negative edge of G. However, by Lemma 3.1, the sum of values on the negative edges taken with the extroverted orientation is 0, so $\xi(e) = 0$, and we have a contradiction again.

$(c) \Rightarrow (a)$ Assume that for every edge e the graph $G - e$ contains only unbalanced components. To see that G is flow-admissible we first show that each edge of G belongs to either a balanced circuit or to a weak unbalanced bicircuit. We define a weak unbalanced bicircuit as a signed graph consisting of two edge-disjoint unbalanced circuits C_1 and C_2, not necessarily vertex-disjoint, joined with a path P, which may be trivial, with no edge in $C_1 \cup C_2$.

Let e be an arbitrary edge of G, and suppose that it is not contained in a balanced circuit. If e is a bridge, then both components of $G - e$ contain an unbalanced circuit. We take one in each component and connect the circuits with a path, thus creating a weak unbalanced bicircuit containing e. If e is not a bridge, then it is contained in an unbalanced circuit, say C_1. The graph $G - e$ is connected and unbalanced, so it contains an unbalanced circuit C_2. If C_1 and C_2 are edge-disjoint, then joining C_1 and C_2 with a path yields a weak unbalanced bicircuit containing e. Therefore assume that C_1 and C_2 have an edge in common, and consider the modulo 2 sum $C_1 \oplus C_2$ of C_1 and C_2. Since $C_1 \oplus C_2$ is an edge-disjoint union of circuits and e is contained in $C_1 \oplus C_2$, there is a circuit $D_1 \subseteq C_1 \oplus C_2$ containing e. As there is no balanced circuit through e, the circuit D_1 is unbalanced. However, $\sigma(C_1 \oplus C_2) = +1$, so $C_1 \oplus C_2$ has to contain an unbalanced circuit D_2 that is edge-disjoint from D_1. By connecting D_1 and D_2 with a path we obtain a weak unbalanced bicircuit containing e. Thus every edge of G belongs to either a balanced circuit of a weak unbalanced bicircuit.

Now let $\{B_1, B_2, \ldots, B_t\}$ be a covering of the edges of G such that each B_i is either a balanced circuit or a weak unbalanced bicircuit. If B_i is a balanced circuit, then it admits a nowhere-zero 2-flow. If B_i is a weak unbalanced bicircuit, then it is easy to see that it has a nowhere-zero 3-flow. In both cases, there exists a nowhere-zero 3-flow ϕ_i on each B_i. Regarding each ϕ_i as a flow on the entire G with zero values outside B_i we can form the function $\phi = \sum_{i=1}^{t} 3^{i-1} \phi_i$ which is easily seen to be a nowhere-zero 3^t-flow on G. $\qquad \square$

We say that a signed graph G is *tightly unbalanced* if it is unbalanced and there is an edge e such that $G - e$ is balanced.

Corollary 3.3. *A 2-edge-connected unbalanced signed graph is flow-admissible if and only if it is not tightly unbalanced.*

The following immediate corollary establishes part (a) of our Main Theorem.

Corollary 3.4. *A signed eulerian graph is flow-admissible if and only if it is not tightly unbalanced.*

We conclude this section with another result concerning eulerian graphs that easily follows from general results presented in this section. It is due to Xu and Zhang [12] and implies part (b) of Main Theorem. Here we provide a simple proof.

Call a signed eulerian graph *even* if it has an even number of negative edges and *odd* otherwise. Note that the parity of the number of negative edges in an eulerian graph is clearly preserved by every vertex switching. Hence, the property of being even or odd is an invariant of the switching class of an eulerian graph.

Theorem 3.5. (Xu and Zhang [12]) *A connected signed graph G admits a nowhere-zero 2-flow if and only if G is an even eulerian graph.*

Proof. The condition is clearly sufficient, because sending the value 1 along any eulerian trail produces a nowhere-zero 2-flow. For the converse, assume that G is a connected graph that admits a nowhere-zero 2-flow ξ. At every vertex, the values of all incident

edges are either $+1$ or -1 and must sum to zero, so the valency is even. By Lemma 3.1, the sum of values on negative edges with extroverted orientation is 0. Again, all the summands are $+1$ or -1, so G must have an even number of negative edges. □

4 Nowhere-zero 4-flows

In this section we show that every flow-admissible signed eulerian graph admits a nowhere-zero 4-flow. By Theorem 3.5, this is true for all even eulerian graphs, so we can restrict here to odd eulerian graphs. In fact, for odd eulerian graphs we prove a stronger result providing several equivalent statements one of which is the existence of a nowhere-zero 4-flow.

Before proceeding to the result we need a simple lemma. Let G and H be two signed graphs with intersecting vertex sets and assume that $G \cap H$ is equally signed in both G and H. Then $G \cup H$ will denote the signed graph where edges inherit their signs from G and H. If G and H are balanced graphs, we say that $G \cap H$ is a *consistent* subgraph of G and H whenever $G \cap H$ has a balanced bipartition that extends to a balanced bipartition of G as well as to a balanced bipartition of H.

Lemma 4.1. *Let G and H be balanced signed graphs. If $G \cap H$ is a consistent subgraph of G and H, then $G \cup H$ is balanced.*

Proof. Let $G \cap H$ be a consistent subgraph of G and H and let $U_1 \cup U_2$ be a balanced bipartition of $G \cap H$ that can be extended to a balanced bipartition $V_1 \cup V_2$ of G and to a balanced bipartition $W_1 \cup W_2$ of H. Without loss of generality we may assume that $U_1 \subseteq V_1$ and $U_1 \subseteq W_1$. Then every negative edge in $G \cup H$ is between the sets $V_1 \cup W_1$ and $V_2 \cup W_2$ implying that $G \cup H$ is a balanced graph. □

Here is the main result of this section.

Theorem 4.2. *The following statements are equivalent for every signed eulerian graph G with an odd number of negative edges.*

(a) *G is flow-admissibble;*

(b) *G admits a nowhere-zero 4-flow;*

(c) *G contains two edge-disjoint unbalanced circuits;*

(d) *G is a union of two even eulerian subgraphs;*

(e) *G can be decomposed into three edge-disjoint odd eulerian subgraphs.*

Proof. $(a) \Rightarrow (c)$ Assume that G is a flow-admissible odd eulerian graph. We show that G contains two edge-disjoint unbalanced circuits.

Choose any unbalanced circuit N of G, and set $G_0 = G - E(N)$. If G_0 is unbalanced, then it contains an unbalanced circuit, which together with N provides two required edge-disjoint unbalanced circuits of G. Henceforth we can assume that G_0 is balanced.

Let us switch the signature of G in such a way that G_0 becomes all-positive, and consider a fixed component M of G_0. The vertices of M divide the circuit N into *segments*, pairwise edge-disjoint paths whose first and last vertex is in M and all inner vertices lie outside M. A *multisegment* is a portion of N formed by a chain of segments. Depending on the product of signs, segments can be either positive or negative. Observe that if a segment J is negative, then $M \cup J$ is unbalanced, and vice versa.

Claim 1. *Every component of G_0 determines an odd number of negative segments on N.*

Proof of Claim 1. The product of signs of all segments determined by a component of G_0 clearly equals the sign of N. Since N is unbalanced, the number of negative segments must be odd. Claim 1 is proved.

We now consider two cases.

Case 1. G_0 *is connected.* By Claim 1, G_0 determines at least one negative segment on N. Let $S_1, S_2, \ldots, S_k$ be all the segments of N, negative or not. Suppose first that only one of them is negative, say S_1. Then for each S_i with $i \geq 2$ the graph $G_0 \cup S_i$ is balanced, and hence, by Lemma 4.1, the graph $G_0 \cup S_2 \cup S_3 \ldots \cup S_k$ is also balanced. Since $G_0 \cup S_1$ is unbalanced, every unbalanced circuit in G traverses S_1. But then for every edge e of S_1 the graph $G - e$ is balanced, contradicting Corollary 3.3.

Thus G_0 determines at least three negative segments on N. We pick two of them, say S_i and S_j. Clearly, there is an unbalanced circuit C_i in $G_0 \cup S_i$ and an unbalanced circuit C_j in $G \cup S_j$. Nevertheless, C_i and C_j may have a common edge within G_0. In order to handle this problem, we extend S_i and S_j into negative multisegments S_i^+ and S_j^+, respectively, with a vertex in common and construct two unbalanced edge-disjoint circuits by utilising S_i^+ and S_j^+ rather than S_i and S_j.

Clearly, S_i and S_j have at most one vertex in common. If S_i and S_j do have a vertex in common, we can set $S_i^+ = S_i$ and $S_j^+ = S_j$. Otherwise we can express N as $S_i U_1 S_j U_2$ for suitable multisegments U_1 and U_2. If any of them, say U_1, is negative, then we set $S_i^+ = S_i$ and $S_j^+ = U_1$. So we may assume that both U_1 and U_2 are positive, and then we may set $S_i^+ = S_i$ and $S_j^+ = U_1 S_j$. In each case we have found negative multisegments S_i^+ and S_j^+ sharing precisely one common vertex.

Let a and b be the end-vertices of S_i^+ and let b and c the end-vertices of S_j^+. Since G_0 is eulerian, it can be traversed by an eulerian trail T. The trail T encounters each of vertices a, b, and c at least once. It follows that $T = T_1 T_2 T_3$ where T_1 is an a-b-subtrail and T_2 is an b-c-subtrail of T. Then $S_i^+ T_1^{-1}$ and $S_j^+ T_2^{-1}$ are two edge-disjoint unbalanced closed trails. Each of them contains an unbalanced circuit, so G contains two edge-disjoint unbalanced circuits.

Case 2. G_0 *is disconnected.* If G_0 has a component M that produces at least three negative segments, we proceed as in Case 1. We may therefore assume that each component of G_0 determines exactly one negative segment on N.

If G_0 has two components M_1 and M_2 that determine disjoint negative segments S_1 and S_2, respectively, then each of $M_1 \cup S_1$ and $M_2 \cup S_2$ contains an unbalanced circuit, and we are done. Consequently, we may assume that the negative segments coming from any two components of G_0 intersect. Clearly, their intersection will consist of either one or two paths. Since distinct components are disjoint, these paths must be nontrivial. Let $M_1, M_2, \ldots, M_n$ be the components of G_0, let S_j be the negative segment of N determined by M_j, and for $j = 1, 2, \ldots, n$ let S_j' be the complementary path on N with the same end-vertices as S_j. We call S_j' the *cosegment* of M_j.

Claim 2. *The cosegments cover all of N.*

Proof of Claim 2. Suppose, to the contrary, that there exists an edge e of N that does not belong to any cosegment. Then e belongs to every segment S_j for $i = 1, 2, \ldots, k$. Let $Q_j = (M_j \cup S_j) - e$ and let $R_j = Q_1 \cup Q_2 \cup \ldots \cup Q_j$. Clearly, each Q_j is balanced. By induction on j we next show that each R_j is balanced, and using this fact we derive a contradiction. Since $R_1 = Q_1$, the basis of induction is verified. Assume inductively that R_j is balanced for some j with $1 \leq j \leq n - 1$. To prove that R_{j+1} is balanced we apply

Lemma 4.1 to the graphs R_j and Q_{j+1}. Obviously, $Q_{j+1} \cap R_j$ is equally signed in both Q_{j+1} and R_j. Furthermore, $Q_{j+1} \cap R_j$ is contained in $N - e$ and consists of two paths, each having one end-vertex of the edge e. Since R_j and Q_{j+1} are balanced but $R_j + e$ and $Q_{j+1} + e$ are not, the ends of e in both R_j and Q_{j+1} either belong to the same partite set or to different partite sets. This immediately implies that $Q_{j+1} \cap R_j$ is a consistent subgraph of Q_{j+1} and R_j, and by Lemma 4.1, $Q_{j+1} \cup R_j = R_{j+1}$ is a balanced graph. Thus R_j is balanced for each $j = 1, 2, \ldots, n$. This means, however, that $G - e = R_n$ is balanced, contradicting Corollary 3.3. The proof of Claim 2 is complete.

Claim 3. *Let $\mathcal{S}'$ be a minimal covering of N by cosegments. Then*

  (i) *every component of the intersection of any two cosegments is a nontrivial path;*

  (ii) *each edge of N is covered by either one cosegment or by two cosegments.*

Proof of Claim 3. Part (i) is trivial. To prove (ii) suppose, to the contrary, that there exists an edge of N that belongs to three cosegments J', K', and L' from $\mathcal{S}'$. Then $J' \cup K' \cup L'$ is either a path or the whole of N. In the former case, the end-vertices are in at most two of J', K', and L'. The remaining cosegment is therefore contained in the union of the other two, implying the covering $\mathcal{S}'$ is not minimal. If $J' \cup K' \cup L' = N$ and, say, $J' \cup K' \neq N$, then L' must have a disconnected intersection with one of J' and K', say J'. But then $K' \subseteq J' \cup L'$, again contradicting the minimality of $\mathcal{S}'$. This proves Claim 3.

Fix a cyclic ordering of the vertices of N and let $\mathcal{S}' = \{S_1', S_2', \ldots, S_m'\}$ where $S_i' = N[u_i, v_i]$ is the portion of N with end-vertices u_i and v_i following this cyclic ordering for each $i \in \{1, 2, \ldots, m\}$. By Claim 2, we can assume that the members of $\mathcal{S}'$ are arranged in such a way that each S_i' only intersects its predecessor S_{i-1}' and its successor S_{i+1}', and the vertices u_i and v_i occur on N in the cyclic ordering $u_1, v_m, u_2, v_1, \ldots, u_m, v_{m-1}, u_1$.

We now construct two edge-disjoint unbalanced circuits in G. Consider an arbitrary component M_i with $i \in \{1, 2, \ldots, m\}$. We can arrange the edges of each M_i into two edge-disjoint u_i-v_i-trails X_i and Y_i such that $X_i Y_i^{-1}$ is an eulerian trail of M_i. Since $M_i \cup S_i'$ is balanced, we have

$$\sigma(X_i) = \sigma(Y_i) = \sigma(S_i') = \sigma(N[u_i, v_i]). \tag{1}$$

If m is even, the following two closed trails T_1 and T_2 in G are clearly edge-disjoint:

$$T_1 = X_1 N[v_1, u_3] X_3 N[v_3, u_5] \ldots X_{m-1} N[v_{m-1}, u_1],$$
$$T_2 = X_2 N[v_2, u_4] X_4 N[v_4, u_6] \ldots X_m N[v_m, u_2].$$

From (1) we get

$$\sigma(T_1) = \sigma(N[u_1, v_1]) \sigma(N[v_1, u_3]) \sigma(N[u_3, v_3]) \ldots \sigma(N[u_{m-1}, v_{m-1}]) \sigma(N[v_{m-1}, u_1])$$
$$= \sigma(N) = -1,$$

and similarly $\sigma(T_2) = -1$. Hence both T_1 and T_2 are unbalanced, and therefore each of them contains an unbalanced circuit. This provides the two required edge-disjoint unbalanced circuits in G.

If m is odd, we have the following two edge-disjoint closed trails T_1 and T_2 in G:

$$T_1 = X_1 N^{-1}[v_1, u_2] X_2 N[v_2, u_4] X_4 \ldots X_{m-1} N[v_{m-1}, u_1],$$
$$T_2 = Y_2 N^{-1}[v_2, u_3] X_3 N[v_3, u_5] X_5 \ldots X_m N[v_m, u_2].$$

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From (1) we obtain

$$\sigma(X_1) = \sigma(N[u_1, v_1]) = \sigma(N[u_1, u_2])\sigma(N[u_2, v_1]),$$
$$\sigma(X_2) = \sigma(N[u_2, v_2]) = \sigma(N[u_2, v_1])\sigma(N[v_1, v_2]).$$

Hence

$$\sigma(X_1 N_1^{-1}[v_1, u_2] X_2) = \sigma(N[u_1, u_2])\sigma(N[u_2, v_1])\sigma(N[v_1, v_2]) = \sigma(N[u_1, v_2]), \qquad (2)$$

and analogously,

$$\sigma(Y_2 N^{-1}[v_2, u_3] X_3) = \sigma(N[u_2, v_3]). \qquad (3)$$

Employing equations (1)-(3) in a similar fashion as for m even we derive that $\sigma(T_1) = \sigma(T_2) = -1$. Therefore each of T_1 and T_2 contains an unbalanced circuit, providing two edge-disjoint unbalanced circuits in G, as required. This completes the proof of $(a) \Rightarrow (c)$.

$(c) \Rightarrow (e)$ Let G be an odd eulerian graph containing two edge-disjoint unbalanced circuits. We show that it can be decomposed into three edge-disjoint odd eulerian subgraphs.

Clearly, G admits a circuit decomposition $\mathcal{K}$ that contains the two unbalanced circuits as its members. Since G is odd, the decomposition $\mathcal{K}$ will have an odd number of unbalanced circuits, and therefore at least three unbalanced circuits. Let us consider the incidence graph $J(\mathcal{K})$ of $\mathcal{K}$; its vertices are the elements of $\mathcal{K}$ and edges join pairs of elements that have a vertex of G in common. Since G is connected, so is $J(\mathcal{K})$.

It is obvious that every connected induced subgraph of $J(\mathcal{K})$, with vertex set a subset $\mathcal{L} \subseteq \mathcal{K}$, uniquely determines an eulerian subgraph of G. The latter subgraph will have an odd number of negative edges whenever $\mathcal{L}$ contains an odd number of unbalanced circuits. Thus to finish the proof it is enough to show that $\mathcal{K}$ can be partitioned into three subsets, each containing an odd number of unbalanced circuits and each inducing a connected subgraph of $J(\mathcal{K})$. In fact, we may assume that $J(\mathcal{K})$ is a tree as the general case follows immediately with the partition of $\mathcal{K}$ obtained from a spanning tree of $J(\mathcal{K})$.

Thus, let $J(\mathcal{K}) = T$ be a tree. We may think of the vertices of T as being coloured in two colours: *black*, if the corresponding circuit in $\mathcal{K}$ is unbalanced, and *white*, if the corresponding circuit is balanced. In this terminology, it remains to prove the following.

Claim 4. *Let T be a tree whose vertices are partitioned into two subsets, white vertices and black vertices, such that the number of black vertices is odd and at least 3. Then the vertex set of T can be partitioned into three subsets such that each contains an odd number of black vertices and induces a subtree of T.*

Proof of Claim 4. We proceed by induction on the number of vertices of T. The conclusion is obvious if T has only three vertices, each of them black. This constitutes the basis of induction. For the induction step we assume that T has four or more vertices. It follows that at least two of them, say v_1 and v_2, are leaves of T. If both v_1 and v_2 are black, then the set $\{\{v_1\}, \{v_2\}, V(T) - \{v_1, v_2\}\}$ is the required decomposition. If one of them is white, say v_1, then by the induction hypothesis $T - v_1$ has the required decomposition $\{V_1, V_2, V_3\}$. One of these sets, say V_1, contains a neighbour of v_1, and then $\{V_1 \cup \{v_1\}, V_2, V_3\}$ is the required decomposition for T. This concludes the induction step and as well as the proof the implication $(c) \Rightarrow (e)$.

$(e) \Rightarrow (d)$ Assume that G has a decomposition $\{G_1, G_2, G_3\}$ into three odd eulerian subgraphs. Without loss of generality we may assume that G_1 and G_2 share a vertex u

and that G_2 and G_3 share a vertex v; possibly $u = v$. Then $G_1 \cup G_2$ and $G_2 \cup G_3$ are even eulerian subgraphs that cover G.

$(d) \Rightarrow (b)$ Assume that G is an odd eulerian graph that has a covering $\{H_1, H_2\}$ by two even eulerian subgraphs. By Theorem 3.5, there exists a nowhere-zero 2-flow ϕ_1 on H_1 and a nowhere-zero 2-flow ϕ_2 on H_2. Regarding each ϕ_i as a flow on the entire G with zero values outside H_i, we can set $\phi = \phi_1 + 2\phi_2$. It is obvious that ϕ is a nowhere-zero 4-flow on G.

$(b) \Rightarrow (a)$ This implication is trivial. $\qquad\square$

Theorem 4.2 and Corollary 3.4 have the following interesting consequence.

Corollary 4.3. *Let G be an signed eulerian graph with an odd number of negative edges. If any two unbalanced circuits of G have an edge in common, then there exists an edge e which is contained in all unbalanced circuits. In particular, G is tightly unbalanced.*

5   Nowhere-zero 3-flows

The aim of this section is to establish part (c) of our Main Theorem. We have to show that a signed eulerian graph G has flow number three if and only if it can be decomposed into three odd eulerian subgraphs G_1, G_2, and G_3 that have a vertex in common. If G has such a decomposition, we say that it is *triply odd*. The decomposition $\{G_1, G_2, G_3\}$ will itself be called *triply odd*. Obviously, a triply odd signed eulerian graph is odd.

It is immediate that a signed eulerian graph is triply odd if and only if its edges can be arranged into three unbalanced closed trails originating at the same vertex. The following fact is a direct consequence of this observation.

Proposition 5.1. *Let G be a triply odd signed eulerian graph. Then $\Phi(G) = 3$.*

Proof. Let $\{G_1, G_2, G_3\}$ be a triply odd decomposition of G where G_1, G_2, and G_3 share a vertex v. For $i \in \{1, 2, 3\}$ let T_i be an eulerian trail in G_i starting at v. If we send from v the value 1 along T_1 and T_2, and the value -2 along T_3, the resulting valuation will clearly be a nowhere-zero 3-flow on G. Since G is odd, Theorem 3.5 implies that $\Phi(G) \geq 3$. Hence $\Phi(G) = 3$. $\qquad\square$

The rest of this section is devoted to proving the reverse implication. The proof has two main ingredients: a reduction of the general case to antibalanced 6-regular graphs and a verification that the result holds for connected antibalanced 6-regular graphs with an odd number of edges (or equivalently, with an odd number of vertices). The latter fact is nontrivial and can be derived from the following theorem which is the main result of [7].

Theorem 5.2. *Every connected 6-regular graph of odd order can be decomposed into three eulerian subgraphs sharing a vertex such that each of them has an odd number of edges.*

As regards the reduction procedure, we begin with analysing signed eulerian graphs that carry a special type of a 3-flow under which the same edge-value does not enter and simultaneously leave any vertex. The formal definition uses an orientation where each edge is assigned the direction with positive flow value. We call this orientation a *positive orientation of G with respect to* a given nowhere-zero flow. Now let ϕ be a nowhere-zero 3-flow on a signed eulerian graph G, and let G be positively oriented with respect to ϕ.

We say that ϕ is *stable at a vertex* v if for any two edges e and f incident with v such that $\phi(e) = \phi(f)$, either both e and f are directed towards v or both are directed out of v. A nowhere-zero 3-flow ϕ is said to be *stable* if it is stable at every vertex.

Lemma 5.3. *Let G be a signed eulerian graph that admits a stable nowhere-zero 3-flow. Then G is antibalanced and the valency of every vertex is a multiple of 6.*

Proof. Let ϕ be a stable nowhere-zero 3-flow on G which is positively oriented. Consider an arbitrary vertex v of G. Clearly, all the edges from $E(v)$ that carry the same value $a \in \{1, 2\}$ under ϕ have the same orientation with respect to v – either to or from v. Thus the situation at v is that either all edges from $E(v)$ with value 1 are directed to v and those with value 2 are directed out of v (*Type* 1), or vice versa (*Type* 2).

By the Kirchhoff law, there must be an integer m such that $E(v)$ has m edges with value 2 and $2m$ edges with value 1. Hence, the valency of v equals $3m$, but since $3m$ is an even integer, we infer that the valency of v is $6n$ for some n.

Finally, observe that an edge joining two vertices of the same type is negative whereas an edge joining two vertices of different types is positive. By Corollary 2.2, the partition of the vertex set of G into the vertices of Type 1 and those of Type 2 is an antibalanced bipartition. Hence, G is antibalanced, as claimed. $\qquad\square$

We are now in the position to prove the charactarisation of signed eulerian graphs with flow number three.

Theorem 5.4. *Let G be a connected signed eulerian graph. Then $\Phi(G) = 3$ if and only if G is triply odd.*

Proof. In Proposition 5.1 we have proved that the condition is sufficient. It remains to prove its necessity. Let G be a signed eulerian graph with $\Phi(G) = 3$. By Theorem 3.5, G is unbalanced and odd. To show that G is triply odd we proceed by induction on the cycle rank $\beta(G)$ of G. Recall that the cycle rank of a graph is the dimension its cycle space, and that a graph G with k components has $\beta(G) = |E(G)| - |V(G)| + k$.

If G contains a 2-valent vertex incident with edges e_1 and e_2, we may suppress the vertex and form a new edge e whose sign equals the product of signs of e_1 and e_2. The result is a signed eulerian graph G' with same cycle rank and the same flow number. This allows us to assume, whenever convenient, that the valency of each vertex of G is at least 4. With this additional assumption we obtain that $\beta(G) \geq |V(G)| + 1$, further implying that up to a homeomorphism the only signed eulerian graph G with $\Phi(G) = 3$ and $\beta(G) \leq 3$ is the bouquet of three unbalanced loops. Its cycle rank is 3, and for this graph the result clearly holds. This verifies the basis of induction.

For the induction step let G be a signed eulerian graph with $\Phi(G) = 3$ and cycle rank $\beta(G) > 3$, and assume that the assertion holds for all signed eulerian graphs with cycle rank smaller than the cycle rank of G. Suppose to the contrary that for G the assertion fails. Then G is a minimum counterexample, and our aim is to derive a contradiction for G.

Claim 1. *A nowhere-zero 3-flow on a minimum counterexample is stable.*

Proof of Claim 1. Let ϕ be a nowhere-zero 3-flow on G. Suppose that there exists a vertex v at which ϕ is not stable, and let G be positively directed with respect to ϕ. Then $E(v)$ contains a pair of edges e and f with $\phi(e) = \phi(f)$ such that e is directed to v and f is directed out of v.

If e and f coincide, then e must be a positive loop. It follows that $G - e$ is an odd eulerian graph carrying a nowhere-zero 3-flow, so $\Phi(G - e) = 3$, by Theorem 3.5. Since

$\beta(G - e) < \beta(G)$, the graph $G - e$ has a triply odd decomposition $\{G_1, G_2, G_3\}$. By adding e to any G_i we obtain a triply odd decomposition for G, a contradiction.

Now suppose that e and f are distinct edges. In this case we remove e and f from G and replace the path ef with a single edge g whose sign equals the product of signs of e and f. In the resulting graph G', all vertices have an even valency and the number of negative edges has the same parity as before. Due to the consistent orientation of the inner half-edges of the path ef, the new edge g has a natural bidirection determined by the outer half-edges of ef and this bidirection is consistent with the sign of g. Furthermore, setting $\phi(g) = \phi(e) = \phi(f)$ turns ϕ into a nowhere zero 3-flow on G'.

If G' is connected, then it is an odd eulerian graph, implying that $\Phi(G') = 3$. As $\beta(G') < \beta(G)$, we can find a triply odd decomposition $\{G'_1, G'_2, G'_3\}$ of G' with common vertex v. One of the subgraphs contains the edge g; we may assume that this graph is G_1. Then $G_1 = (G'_1 - g) \cup \{e, f\}$ is an odd eulerian subgraph of G, and consequently $\{G_1, G'_2, G'_3\}$ is a triply odd decomposition of G with common vertex v, a contradiction.

If G' is disconnected, then it has exactly two components H and K, both eulerian, one of which, say H, is odd. Since H carries a nowhere-zero 3-flow, we have $\Phi(H) = 3$. However, $\beta(H) < \beta(G)$, so H has a triply odd decomposition $\{H_1, H_2, H_3\}$. Reinserting the vertex v into the edge g restores the edges e and f without changing the parity of the number negative edges in the component of G' containing g. It is therefore possible convert the decomposition $\{H_1, H_2, H_3\}$ of H into a triply odd decomposition of G, a contradiction again.

In all possible cases the assumption that the 3-flow ϕ is unstable produces a contradiction. It follows that ϕ is stable, as claimed.

Claim 2. *A minimum counterexample is an antibalanced 6-regular graph.*

Proof of Claim 2. Again, let G be a minimum counterexample. As we have just shown, G carries a stable nowhere-zero 3-flow. By Lemma 5.3, G is antibalanced and the valency of every vertex is a positive multiple of 6. It remains to prove G is 6-regular.

Suppose that G contains a vertex v of valency $6n$ for some $n > 1$. Replace v with two new vertices v' and v'' and join the edges originally incident with v to the two new vertices in such a way that v' becomes 6-valent, v'' becomes $6(n-1)$-valent, and the resulting graph G' continues to carry a stable nowhere-zero 3-flow. This is clearly possible. Since G' is odd, we have $\Phi(G') = 3$.

If G' is connected, then $\beta(G') < \beta(G)$, and by the induction hypothesis G' has a triply odd decomposition. This decomposition readily induces one for G, a contradiction. If G' is disconnected, then it has two components G_1 and G_2, only one of which, say G_1, is odd. Since $\Phi(G_1) = 3$ and $\beta(G_1) < \beta(G)$, there exists a triply odd decomposition in G_1. By extending one of the subgraphs of the decomposition with G_2 we obtain a similar decomposition for G, a contradiction again.

All this shows that G cannot have a vertex of valency $6n$ for $n > 1$. Therefore G is 6-regular, and the proof of Claim 2 is complete.

Now we can finish the induction step. As above, let G be a minimum counterexample with $\Phi(G) = 3$. From Claim 2 we know that G is an antibalanced 6-regular graph. Since G is odd, it has an odd number of edges, and hence an odd order. In this situation we can apply Theorem 5.2 to infer that G is triply odd contradicting the assumption that G is a counterexample. This concludes the induction step as well as the proof of the theorem. $\qquad\square$

6 Group-valued flows

We conclude this paper with a brief discussion of nowhere-zero flows with values in abelian groups other than the group of integers. Our final result provides a complete characterisation of signed eulerian graphs that admit a nowhere-zero A-flow for a given abelian group $A \neq 0$.

Theorem 6.1. *Let G be a signed eulerian graph and let A be a nontrivial abelian group. The following statements hold true.*

(a) *If A contains an involution, then G admits a nowhere-zero A-flow.*

(b) *If $A \cong \mathbb{Z}_3$, then G admits a nowhere-zero A-flow if and only if G is triply odd.*

(c) *Otherwise, G has a nowhere-zero A-flow if and only if G is not tightly unbalanced.*

Proof. (a) If A contains an involution, then a nowhere-zero A-flow can be produced by simply valuating each edge of G by the same involution.

(b) Assume that $A \cong \mathbb{Z}_3$. By [12, Theorem 1.5], a 2-edge-connected signed graph admits a nowhere-zero $\mathbb{Z}_3$-flow if and only if it admits a nowhere-zero 3-flow. Our Theorem 5.4 now implies that G has a nowhere-zero A-flow if and only if it is triply odd.

(c) Assume that A contains no involution and is not isomorphic to $\mathbb{Z}_3$. Then either A contains a subgroup B isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$ or has an element of order at least 4. If G is an even eulerian graph, then sending any nontrivial element of A along an eulerian trail of G will produce a nowhere-zero A-flow on G. Let G be odd but not tightly unbalanced. By Theorem 4.2 (d), we can cover G with two even eulerian subgraphs, say T_1 and T_2. If A contains $B \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, we take any generating set $\{b_1, b_2\}$ for B and send the value b_1 along an eulerian trail in T_1 and the value b_2 along an eulerian trail in T_2. If A has an element of order at least 4, say a, we proceed similarly with a in T_1 and $2a$ in T_2. In both cases we get a nowhere-zero A-flow on G. Finally, let G be tightly unbalanced. It is easy to see that the argument from the proof of Lemma 3.1 extends to any group with no involution, and hence applies to A. It follows that G has no nowhere-zero A-flow, and the proof is complete. $\square$

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