THE COMPLEX AIRY OPERATOR ON THE LINE WITH A SEMI-PERMEABLE BARRIER

Denis S Grebenkov, Bernard Helffer, Raphael Henry

THE COMPLEX AIRY OPERATOR ON THE LINE WITH A SEMI-PERMEABLE BARRIER

DENIS S. GREBENKOV*, BERNARD HELFFER†, AND RAPHAEL HENRY‡

Abstract. We consider a suitable extension of the complex Airy operator, $-d^2/dx^2 + ix$, on the real line with a transmission boundary condition at the origin. We provide a rigorous definition of this operator and study its spectral properties. In particular, we show that the spectrum is discrete, the space generated by the generalized eigenfunctions is dense in L^2 (completeness), and we analyze the decay of the associated semi-group. We also present explicit formulas for the integral kernel of the resolvent in terms of Airy functions, investigate its poles, and derive the resolvent estimates.

Key words. Airy operator, transmission boundary condition, spectral theory, Bloch-Torrey equation

AMS subject classifications. 35P10, 47A10, 47A75

1. Introduction.

The transmission boundary condition which is considered in this article appears in various exchange problems such as molecular diffusion across semi-permeable membranes [37, 34, 33], heat transfer in composite materials [11, 18, 8], or transverse magnetization evolution in nuclear magnetic resonance (NMR) experiments [20]. In the simplest setting of the latter case, one considers the local transverse magnetization $G(x, y\,; t)$ produced by the nuclei that started from a fixed initial point y and diffused in a constant magnetic field gradient g up to time t. This magnetization is also called the propagator or the Green function of the Bloch-Torrey equation [39]:

$$\frac{\partial}{\partial t} G(x, y\,; t) = (D\Delta - i\gamma g x_1)\, G(x, y\,; t)\,, \tag{1.1}$$

with the initial condition

$$G(x, y\,; t = 0) = \delta(x - y), \tag{1.2}$$

where $\delta(x)$ is the Dirac distribution, D the intrinsic diffusion coefficient, $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_d^2$ the Laplace operator in $\mathbb{R}^d$, γ the gyromagnetic ratio, and x_1 the coordinate in a prescribed direction. From an experimental point of view, the local transverse magnetization is too weak to be detected but the macroscopic signal produced by all the nuclei in a sample can be measured. In other words, one can access the double integral of $G(x, y\,; t)\,\rho(y)$ over the starting and arrival points y and x, where $\rho(y)$ is the initial density of the nuclei in the sample. The microstructure of the sample, which can eventually be introduced through boundary conditions to (1.1), affects the motion of the nuclei, the magnetization $G(x, y\,; t)$, and the resulting macroscopic signal. Measuring the signal at various times t and magnetic field gradients g, one aims at inferring structural properties of the sample [19]. Although this

*Laboratoire de Physique de la Matière Condensée, CNRS–Ecole Polytechnique, University Paris-Saclay, 91128 Palaiseau, France (denis.grebenkov@polytechnique.edu, http://pmc.polytechnique.fr/pagesperso/dg/).

†Laboratoire de Mathématiques Jean Leray, Université de Nantes 2 rue de la Houssinière 44322 Nantes, France, and Laboratoire de Mathématiques, Université Paris-Sud, CNRS, Univ. Paris Saclay, France (bernard.helffer@math.u-psud.fr).

‡Laboratoire de Mathématiques, Université Paris-Sud, CNRS, Univ. Paris Saclay, France (raphael.ash@gmail.com).

non-invasive experimental technique has found numerous applications in material sciences and medicine, mathematical aspects of this formidable inverse problem remain poorly understood. Even the forward problem of relating a given microstructure to the macroscopic signal is challenging because of the non-selfadjoint character of the Bloch-Torrey operator $D\Delta - i\gamma g x_1$ in (1.1). In particular, the spectral properties of this operator were rigorously established only on the line $\mathbb{R}$ (no boundary condition) and on the half-axis $\mathbb{R}_+$ with Dirichlet or Neumann boundary conditions (see Sec. 3).

Throughout this paper, we focus on the one-dimensional situation $(d = 1)$, in which the operator

$$D_x^2 + ix = -\frac{d^2}{dx^2} + ix$$

is called the complex Airy operator and appears in many contexts: mathematical physics, fluid dynamics, time dependent Ginzburg-Landau problems and also as an interesting toy model in spectral theory (see [3]). We will consider a suitable extension $\mathcal{A}_1^+$ of this differential operator and its associated evolution operator $e^{-t\mathcal{A}_1^+}$. The Green function $G(x, y\,; t)$ is the distribution kernel of $e^{-t\mathcal{A}_1^+}$. A separate article will address this operator in higher dimensions [24].

For the problem on the line $\mathbb{R}$, an intriguing property is that this non self-adjoint operator, which has compact resolvent, has empty spectrum (see Section 3.1). However, the situation is completely different on the half-line $\mathbb{R}_+$. The eigenvalue problem

$$(D_x^2 + ix)u = \lambda u,$$

for a spectral pair (u, λ) with $u \in H^2(\mathbb{R}_+)$ and $xu \in L^2(\mathbb{R}^+)$ has been thoroughly analyzed for both Dirichlet $(u(0) = 0)$ and Neumann $(u'(0) = 0)$ boundary conditions. The spectrum consists of an infinite sequence of eigenvalues of multiplicity one explicitly related to the zeros of the Airy function (see [36, 26]). The space generated by the eigenfunctions is dense in $L^2(\mathbb{R}_+)$ (completeness property) but there is no Riesz basis of eigenfunctions.[1] Finally, the decay of the associated semi-group has been analyzed in detail. The physical consequences of these spectral properties for NMR experiments have been first revealed by Stoller, Happer and Dyson [36] and then thoroughly discussed in [15, 19, 22].

In this article, we consider another problem for the complex Airy operator on the line but with a transmission condition at 0 which reads [22]:

$$\begin{cases} u'(0_+) &= u'(0_-)\,, \\ u'(0) &= \kappa\big(u(0_+) - u(0_-)\big)\,, \end{cases} \tag{1.3}$$

where $\kappa \geq 0$ is a real parameter. In physical terms, the transmission condition accounts for the diffusive exchange between two media $\mathbb{R}_-$ and $\mathbb{R}_+$ across the barrier at 0, while κ is defined as the ratio between the barrier permeability and the bulk diffusion coefficient. This situation is particularly relevant for biological samples and applications [19, 21, 22]. The case $\kappa = 0$ corresponds to two independent Neumann problems on $\mathbb{R}_-$ and $\mathbb{R}_+$ for the complex Airy operator. When κ tends to $+\infty$, the

[1] We recall that a collection of vectors (x_k) in a Hilbert space $\mathcal{H}$ is called Riesz basis if it is an image of an orthonormal basis in $\mathcal{H}$ under some isomorphism.

second relation in (1.3) becomes the continuity condition, $u(0_+) = u(0_-)$, and the barrier disappears. As a consequence, the problem tends (at least formally) to the standard problem for the complex Airy operator on the line.
The main purpose of this paper is to define the complex Airy operator with transmission (Section 4) and then to analyze its spectral properties. Before starting the analysis of the complex Airy operator with transmission, we first recall in Section 2 the spectral properties of the one-dimensional Laplacian with the transmission condition, and summarize in Section 3 the known properties of the complex Airy operator. New properties are also established concerning the Robin boundary condition and the behavior of the resolvent for real λ going to $+\infty$. In Section 4 we will show that the complex Airy operator $\mathcal{A}_1^+ = D_x^2 + ix$ on the line $\mathbb{R}$ with a transmission property (1.3) is well defined by an appropriate sesquilinear form and an extension of the Lax-Milgram theorem. Section 5 focuses on the exponential decay of the associated semi-group. In Section 6, we present explicit formulas for the integral kernel of the resolvent and investigate its poles. In Section 7, the resolvent estimates as $|\mathrm{Im}\,\lambda| \to 0$ are discussed. Finally, the proof of completeness is reported in Section 8. In five Appendices, we recall the basic properties of Airy functions (Appendix A), determine the asymptotic behavior of the resolvent as $\lambda \to +\infty$ for extensions of the complex Airy operator on the line (Appendix B) and in the semi-axis (Appendix C), give the statement of the needed Phragmen-Lindelöf theorem (Appendix D) and finally describe the numerical method for computing the eigenvalues (Appendix E).

We summarize our main results in the following:

THEOREM 1. *The semigroup* $\exp(-t\mathcal{A}_1^+)$ *is contracting. The operator* $\mathcal{A}_1^+$ *has a discrete spectrum* $\{\lambda_n(\kappa)\}$*. The eigenvalues* $\lambda_n(\kappa)$ *are determined as (complex-valued) solutions of the equation*

$$2\pi \mathrm{Ai}'(e^{2\pi i/3}\lambda)\mathrm{Ai}'(e^{-2\pi i/3}\lambda) + \kappa = 0, \tag{1.4}$$

where $\mathrm{Ai}'(z)$ *is the derivative of the Airy function.*
For all $\kappa \geq 0$*, there exists* N *such that, for all* $n \geq N$*, there exists a unique eigenvalue of* $\mathcal{A}_1^+$ *in the ball* $B(\lambda_n^\pm, 2\kappa|\lambda_n^\pm|^{-1})$*, where* $\lambda_n^\pm = e^{\pm 2\pi i/3}a_n'$*, and* a_n' *are the zeros of* $\mathrm{Ai}'(z)$*.*
Finally, for any $\kappa \geq 0$ *the space generated by the generalized eigenfunctions of the complex Airy operator with transmission is dense in* $L^2(\mathbb{R}_-) \times L^2(\mathbb{R}_+)$*.*

REMARK 2. *Numerical computations suggest that all the spectral projections have rank one (no Jordan block) but we shall only prove in Proposition 36 that there are at most a finite number of eigenvalues with nontrivial Jordan blocks. It will be shown in [6] that the eigenvalues are actually simple. Hence one can replace "generalized eigenfunctions" by "eigenfunctions" in the Theorem 1.*

2. The free Laplacian with a semi-permeable barrier. As an enlighting exercise, let us consider in this section the case of the free one-dimensional Laplacian $-\frac{d^2}{dx^2}$ on $\mathbb{R} \setminus \{0\}$ with the transmission condition (1.3) at $x = 0$. We work in the Hilbert space

$$\mathcal{H} := L^2_- \times L^2_+ \,,$$

where $L^2_- := L^2(\mathbb{R}_-)$ and $L^2_+ := L^2(\mathbb{R}_+)$.
An element $u \in L^2_- \times L^2_+$ will be denoted by $u = (u_-, u_+)$ and we shall use the notation $H^s_- = H^s(\mathbb{R}_-)$, $H^s_+ = H^s(\mathbb{R}_+)$ for $s \geq 0$.

So (1.3) reads

$$\left\{ \begin{array}{lcl} u_+'(0) & = & u_-'(0)\,, \\ u_+'(0) & = & \kappa\big(u_+(0)-u_-(0)\big)\,. \end{array} \right. \tag{2.1}$$

In order to define appropriately the corresponding operator, we start by considering a sesquilinear form defined on the domain

$$V = H^1_- \times H^1_+\,.$$

The space V is endowed with the Hilbert norm $\|\cdot\|_V$ defined for all $u=(u_-,u_+)$ in V by

$$\|u\|_V^2 = \|u_-\|^2_{H^1_-} + \|u_+\|^2_{H^1_+}\,.$$

We then define a Hermitian sesquilinear form a_ν acting on $V\times V$ by the formula

$$\begin{aligned} a_\nu(u,v) = & \int_{-\infty}^0 \Big(u_-'(x)\bar v_-'(x) + \nu\, u_-(x)\bar v_-(x)\Big)\,dx \\ & + \int_0^{+\infty} \Big(u_+'(x)\bar v_+'(x) + \nu\, u_+(x)\bar v_+(x)\Big)\,dx \\ & + \kappa\left(u_+(0)-u_-(0)\right)\left(\overline{v_+(0)-v_-(0)}\right)\,, \end{aligned} \tag{2.2}$$

for all pairs $u=(u_-,u_+)$ and $v=(v_-,v_+)$ in V. For $z\in\mathbb{C}$, $\bar z$ denotes the complex conjugate of z. The parameter $\nu\geq 0$ will be determined later to ensure the coercivity of a_ν.

Lemma 3. *The sesquilinear form a_ν is continuous on V.*

Proof

We want to show that, for any $\nu\geq 0$, there exists a positive constant c such that, for all $(u,v)\in V\times V$,

$$|a_\nu(u,v)| \leq c\,\|u\|_V\|v\|_V\,. \tag{2.3}$$

We have, for some $c_0>0$,

$$\begin{aligned} & \bigg|\int_{-\infty}^0 \Big(u_-'(x)\bar v_-'(x) + \nu\, u_-(x)\bar v_-(x)\Big)\,dx \\ & \qquad + \int_0^{+\infty} \Big(u_+'(x)\bar v_+'(x) + \nu\, u_+(x)\bar v_+(x)\Big)\,dx\bigg| \leq c_0\,\|u\|_V\|v\|_V\,. \end{aligned}$$

On the other hand,

$$|u_+(0)|^2 = -\int_0^{+\infty} (u_+\bar u_+)'(x)\,dx \leq 2\,\|u\|_{L^2}\|u'\|_{L^2}\,, \tag{2.4}$$

and similarly for $|u_-(0)|^2$, $|v_+(0)|^2$ and $|v_-(0)|^2$. Thus there exists $c_1>0$ such that, for all $(u,v)\in V\times V$,

$$\Big|\kappa\big(u_-(0)-u_+(0)\big)\big(\overline{v_-(0)-v_+(0)}\big)\Big| \leq c_1\|u\|_V\|v\|_V\,,$$

and (2.3) follows with $c=c_0+c_1$. □

The coercivity of the sesquilinear form a_ν for ν large enough is proved in the following lemma. It allows us to define a closed operator associated with a_ν by using the Lax-Milgram theorem.

LEMMA 4. *There exist $\nu_0 > 0$ and $\alpha > 0$ such that, for all $\nu \geq \nu_0$,*

$$\forall u \in V\,, \quad a_\nu(u,u) \geq \alpha \, \|u\|_V^2\,. \tag{2.5}$$

Proof
The proof is elementary for $\kappa \geq 0$. For completeness, we provide below the proof for the case $\kappa < 0$ (in which an additional difficulty occurs), but we will keep considering the physically relevant case $\kappa \geq 0$ throughout the paper.
Using the estimate (2.4) as well as the Young inequality

$$\forall e, f, \delta > 0\,, \quad ef \leq \frac{1}{2}\Big(\delta e^2 + \delta^{-1} f^2\Big)\,,$$

we get that, for all $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that, for all $u \in V$,

$$\big|u_-(0) - u_+(0)\big|^2 \leq \varepsilon \left(\int_{-\infty}^0 |u_-'(x)|^2\, dx + \int_0^{+\infty} |u_+'(x)|^2\, dx\right) + C(\varepsilon)\|u\|_{L^2}^2\,. \tag{2.6}$$

Thus for all $u \in V$ we have

$$\begin{aligned} a_\nu(u,u) \geq\ & (1 - |\kappa|\varepsilon) \left(\int_{-\infty}^0 |u_-'(x)|^2\, dx + \int_0^{+\infty} |u_+'(x)|^2\, dx\right) \\ & + \big(\nu - |\kappa| C(\varepsilon)\big)\|u\|_{L^2}^2\,. \end{aligned} \tag{2.7}$$

Choosing $\varepsilon < |\kappa|^{-1}$ and $\nu > |\kappa| C(\varepsilon)$, we get (2.5). □

The sesquilinear form a_ν being symmetric, continuous and coercive in the sense of (2.5) on $V \times V$, we can use the Lax-Milgram theorem [26] to define a closed, densely defined selfadjoint operator S_ν associated with a_ν. Then we set $\mathcal{T}_0 = S_\nu - \nu$. By construction, the domain of S_ν and $\mathcal{T}_0$ is

$$\begin{aligned} \mathcal{D}(\mathcal{T}_0) = \big\{ u \in V : v \mapsto a_\nu(u,v) \ \text{ can be extended continuously} \\ \text{on } \ L_-^2 \times L_+^2 \,\big\}\,, \end{aligned} \tag{2.8}$$

and the operator $\mathcal{T}_0$ satisfies, for all $(u,v) \in \mathcal{D}(\mathcal{T}_0) \times V$,

$$a_\nu(u,v) = \langle \mathcal{T}_0 u, v\rangle + \nu \langle u, v\rangle\,.$$

Now we look for an explicit description of the domain (2.8). The antilinear form $a(u,\cdot)$ can be extended continuously on $L_-^2 \times L_+^2$ if and only if there exists $w_u = (w_u^-, w_u^+) \in L_-^2 \times L_+^2$ such that

$$\forall v \in V\,, \quad a_\nu(u,v) = \langle w_u, v\rangle\,.$$

According to the expression (2.2), we have necessarily

$$w_u = \big(- u_-'' + \nu\, u_-, -u_+'' + \nu\, u_+\big) \in L_-^2 \times L_+^2\,,$$

where u_-'' and u_+'' are *a priori* defined in the sense of distributions respectively in $\mathcal{D}'(\mathbb{R}_-)$ and $\mathcal{D}'(\mathbb{R}_+)$. Moreover (u_-, u_+) has to satisfy conditions (1.3). Consequently we have

$$\begin{aligned} \mathcal{D}(\mathcal{T}_0) = \Big\{ u = (u_-, u_+) \in H_-^1 \times H_+^1 : (u_-'', u_+'') \in L_-^2 \times L_+^2 \\ \text{and } u \text{ satisfies conditions (1.3)} \Big\}\,. \end{aligned}$$

Finally we have introduced a closed, densely defined selfadjoint operator $\mathcal{T}_0$ acting by

$$\mathcal{T}_0\, u = -u''$$

on $(-\infty,0)\cup(0,+\infty)$, with domain

$$\mathcal{D}(\mathcal{T}_0) = \big\{u \in H^2_- \times H^2_+ : u \text{ satisfies conditions (1.3)}\big\}\,.$$

Note that at the end $\mathcal{T}_0$ is independent of the ν chosen for its construction. We observe also that because of the transmission conditions (1.3), the operator $\mathcal{T}_0$ might not be positive when $\kappa < 0$, hence there can be a negative spectrum, as can be seen in the following statement.

Proposition 5. *For all $\kappa \in \mathbb{R}$, the essential spectrum of $\mathcal{T}_0$ is*

$$\sigma_{ess}(\mathcal{T}_0) = [0,+\infty)\,. \tag{2.9}$$

Moreover, if $\kappa \geq 0$ the operator $\mathcal{T}_0$ has empty discrete spectrum and

$$\sigma(\mathcal{T}_0) = \sigma_{ess}(\mathcal{T}_0) = [0,+\infty)\,. \tag{2.10}$$

On the other hand, if $\kappa < 0$ there exists a unique negative eigenvalue $-4\kappa^2$, which is simple, and

$$\sigma(\mathcal{T}_0) = \big\{-4\kappa^2\big\} \cup [0,+\infty)\,. \tag{2.11}$$

Proof

Let us first prove that $[0,+\infty) \subset \sigma_{ess}(\mathcal{T}_0)$. This can be achieved by a standard singular sequence construction.
Let $(a_j)_{j\in\mathbb{N}}$ be a positive increasing sequence such that, for all $j \in \mathbb{N}$, $a_{j+1} - a_j > 2j+1$. Let $\chi_j \in \mathcal{C}_0^\infty(\mathbb{R})$ $(j \in \mathbb{N})$ such that

$$\text{Supp}\ \chi_j \subset (a_j - j, a_j + j)\,,\ ||\chi_j||_{L^2_+} = 1 \text{ and } \sup|\chi_j^{(p)}| \leq \frac{C}{j^p}\,,\quad p = 1,2\,,$$

for some C independent of j. Then, for all $r \geq 0$, the sequence $u_j^r(x) = \big(0, \chi_j(x)e^{irx}\big)$ is a singular sequence for $\mathcal{T}_0$ corresponding to $z = r^2$ in the sense of [17, Definition IX.1.2]. Hence according to [17, Theorem IX.1.3], we have $[0,+\infty) \subset \sigma_{ess}(\mathcal{T}_0)$.

Now let us prove that $(\mathcal{T}_0 - \mu)$ is invertible for all $\mu \in (-\infty,0)$ if $\kappa \geq 0$, and for all $\mu \in (-\infty,0)\setminus\{-4\kappa^2\}$ if $\kappa < 0$.
Let $\mu < 0$ and $f = (f_-,f_+) \in L^2_- \times L^2_+$. We are going to determine explicitly the solutions $u = (u_-,u_+)$ to the equation

$$\mathcal{T}_0\, u = \mu\, u + f\,. \tag{2.12}$$

Any solution of the equation $-u_\pm'' = \mu\, u_\pm + f_\pm$ has the form

$$u_\pm(x) = \frac{1}{2\sqrt{-\mu}}\int_0^x f_\pm(y)\big(e^{-\sqrt{-\mu}\,(x-y)} - e^{\sqrt{-\mu}\,(x-y)}\big)\,dy + A_\pm e^{\sqrt{-\mu}\,x} + B_\pm e^{-\sqrt{-\mu}\,x}\,, \tag{2.13}$$

for some $A_\pm, B_\pm \in \mathbb{R}$.
We shall now determine A_+ , A_- , B_+ and B_- such that (u_-,u_+) belongs to the domain $\mathcal{D}(\mathcal{T}_0)$. The conditions (1.3) yield

$$\left\{\begin{array}{rcl} A_+ - B_+ &=& A_- - B_-\,,\\ \sqrt{-\mu}\,\big(A_+ - B_+\big) &=& -\kappa\,\big(A_- + B_- - A_+ - B_+\big)\,. \end{array}\right.$$

Moreover, the decay conditions at $\pm\infty$ imposed by $u_\pm \in H^2_\pm$ lead to the following values for A_+ and B_- :

$$(2.14)\quad A_+ = \frac{1}{2\sqrt{-\mu}}\int_0^{+\infty} f_+(y)e^{-\sqrt{-\mu}\,y}\,dy\,, \qquad B_- = \frac{1}{2\sqrt{-\mu}}\int_{-\infty}^{0} f_-(y)e^{\sqrt{-\mu}\,y}\,dy\,.$$

The remaining constants A_- and B_+ have to satisfy the system

$$(2.15)\qquad \begin{cases} A_- + B_+ &= A_+ + B_-\,, \\ -\kappa A_- + \big(\sqrt{-\mu}+\kappa\big)B_+ &= \big(\sqrt{-\mu}-\kappa\big)A_+ + \kappa B_-\,. \end{cases}$$

We then notice that the equation (2.12) has a unique solution $u = (u_-, u_+)$ if and only if $\kappa \geq 0$ or $\mu \neq -4\kappa^2$.
Finally in the case $\kappa < 0$ and $\mu = -4\kappa^2$, the homogeneous equation associated with (2.12) ($i.e$ with $f \equiv 0$) has a one-dimensional space of solutions, namely

$$u(x) = \big(-Ke^{-2\kappa x}, Ke^{2\kappa x}\big)$$

with $K \in \mathbb{R}$. Consequently if $\kappa < 0$, the eigenvalue $\mu = -4\kappa^2$ is simple, and the desired statement is proved. □

The expression (2.14) along with the system (2.15) yield the values of A_- and B_+ when $\mu \notin \sigma(\mathcal{T}_0)$:

$$\begin{aligned} A_- =& \frac{2\kappa}{2\sqrt{-\mu}\big(\sqrt{-\mu}+2\kappa\big)}\int_0^{+\infty} f_+(y)e^{-\sqrt{-\mu}\,y}\,dy \\ &+ \frac{1}{2\big(\sqrt{-\mu}+2\kappa\big)}\int_{-\infty}^{0} f_-(y)e^{\sqrt{-\mu}\,y}\,dy \end{aligned}$$

and

$$\begin{aligned} B_+ =& \frac{1}{2\big(\sqrt{-\mu}+2\kappa\big)}\int_0^{+\infty} f_+(y)e^{-\sqrt{-\mu}y}\,dy \\ &+ \frac{2\kappa}{2\sqrt{-\mu}\big(\sqrt{-\mu}+2\kappa\big)}\int_{-\infty}^{0} f_-(y)e^{\sqrt{-\mu}\,y}\,dy\,. \end{aligned}$$

Using (2.13), we are then able to obtain the expression of the integral kernel of $(\mathcal{T}_0 - \mu)^{-1}$. More precisely we have, for all $f = (f_-, f_+) \in L^2_- \times L^2_+$,

$$(\mathcal{T}_0 - \mu)^{-1} = \begin{pmatrix} \mathcal{R}_\mu^{--} & \mathcal{R}_\mu^{-+} \\ \mathcal{R}_\mu^{+-} & \mathcal{R}_\mu^{++} \end{pmatrix},$$

where for $\varepsilon, \sigma \in \{-,+\}$ the operator $\mathcal{R}_\mu^{\varepsilon\sigma} : \mathbb{R}^\sigma \to \mathbb{R}^\varepsilon$ is an integral operator whose kernel (still denoted $\mathcal{R}_\mu^{\varepsilon\sigma}$) is given for all $(x,y) \in \mathbb{R}^\varepsilon \times \mathbb{R}^\sigma$ by

$$(2.16)\qquad \mathcal{R}_\mu^{\varepsilon,\sigma}(x,y) = \frac{1}{2\sqrt{-\mu}}e^{-\sqrt{-\mu}\,|x-y|} + \varepsilon\sigma\,\frac{1}{2\big(\sqrt{-\mu}+2\kappa\big)}e^{-\sqrt{-\mu}(|x|+|y|)}\,.$$

Noticing that the first term in the right-hand side of (2.16) is the integral kernel of the Laplacian on $\mathbb{R}$, and that the second term is the kernel of a rank one operator, we finally get the following expression of $(\mathcal{T}_0-\mu)^{-1}$ as a rank one perturbation of the Laplacian:

$$\begin{aligned}(\mathcal{T}_0-\mu)^{-1} &= (-\Delta-\mu)^{-1}\\ &\quad+\frac{1}{2\big(\sqrt{-\mu}+2\kappa\big)}\begin{pmatrix}\langle\,\cdot\,,\ell_\mu\rangle_-(\ell_\mu)_- & -\langle\,\cdot\,,\ell_\mu\rangle_+\,(\ell_\mu)_-\\ -\langle\,\cdot\,,\ell_\mu\rangle_-(\ell_\mu)_+ & \langle\,\cdot\,,\ell_\mu\rangle_+(\ell_\mu)_+\end{pmatrix},\end{aligned}$$

where $\ell_\mu(x)=e^{-\sqrt{-\mu}\,|x|}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_\pm$ denotes the L^2 scalar product on $\mathbb{R}^\pm$.
Here the operator $(-\Delta-\mu)^{-1}$ denotes the operator acting on $L^2_-\times L^2_+$ like the resolvent of the Laplacian on $L^2(\mathbb{R})$:

$$(-\Delta-\mu)^{-1}(u_-,u_+):=(-\Delta-\mu)^{-1}(u_-\,\mathbf{1}_{(-\infty,0)}+u_+\,\mathbf{1}_{(0,+\infty)}),$$

composed with the map $L^2(\mathbb{R})\ni v\mapsto(v_{|\mathbb{R}_-},v_{|\mathbb{R}_+})\in L^2_-\times L^2_+$.

3. Reminder on the complex Airy operator.

Here we recall relatively basic facts coming from [32, 3, 10, 26, 25, 28, 29] and discuss new questions concerning estimates on the resolvent and the Robin boundary condition. Complements will also be given in Appendices A, B and C.

3.1. The complex Airy operator on the line.

The complex Airy operator on the line can be defined as the closed extension $\mathcal{A}^+$ of the differential operator $\mathcal{A}_0^+:=D_x^2+i\,x$ on $C_0^\infty(\mathbb{R})$. We observe that $\mathcal{A}^+=(\mathcal{A}_0^-)^*$ with $\mathcal{A}_0^-:=D_x^2-i\,x$ and that its domain is

$$\mathcal{D}(\mathcal{A}^+)=\{u\in H^2(\mathbb{R})\,,\,x\,u\in L^2(\mathbb{R})\}.$$

In particular, $\mathcal{A}^+$ has a compact resolvent. It is also easy to see that $-\mathcal{A}^+$ is the generator of a semi-group S_t of contraction,

$$S_t=\exp(-t\mathcal{A}^+)\,. \tag{3.1}$$

Hence the results of the theory of semi-groups can be applied (see for example [12]).
In particular, we have, for $\mathrm{Re}\,\lambda<0$,

$$||(\mathcal{A}^+-\lambda)^{-1}||\le\frac{1}{|\mathrm{Re}\,\lambda|}\,. \tag{3.2}$$

A very special property of this operator is that, for any $a\in\mathbb{R}$,

$$T_a\,\mathcal{A}^+=(\mathcal{A}^+-ia)\,T_a\,, \tag{3.3}$$

where T_a is the translation operator: $(T_au)(x)=u(x-a)$.
As an immediate consequence, we obtain that the spectrum is empty and that the resolvent of $\mathcal{A}^+$,

$$\mathcal{G}_0^+(\lambda)=(\mathcal{A}^+-\lambda)^{-1},$$

which is defined for any $\lambda\in\mathbb{C}$, satisfies

$$||(\mathcal{A}^+-\lambda)^{-1}||=||(\mathcal{A}^+-\mathrm{Re}\,\lambda)^{-1}||\,. \tag{3.4}$$

The most interesting property is the control of the resolvent for $\mathrm{Re}\,\lambda\ge0$.

PROPOSITION 6 (W. Bordeaux-Montrieux [10]).
As $\operatorname{Re}\lambda \to +\infty$*, we have*

$$||\mathcal{G}_0^+(\lambda)|| \sim \sqrt{\frac{\pi}{2}}(\operatorname{Re}\lambda)^{-\frac{1}{4}} \exp\left(\frac{4}{3}(\operatorname{Re}\lambda)^{\frac{3}{2}}\right) , \tag{3.5}$$

where $f(\lambda) \sim g(\lambda)$ *means that the ratio* $f(\lambda)/g(\lambda)$ *tends to* 1 *in the limit* $\lambda \to +\infty$*.*

This improves a previous result (see Appendix B) by J. Martinet [32] (see also in [26, 25]) who also proved[2]

PROPOSITION 7.

$$\|\mathcal{G}_0^+(\lambda)\|_{HS} = \|\mathcal{G}_0^+(\operatorname{Re}\lambda)\|_{HS} , \tag{3.6}$$

and

$$\|\mathcal{G}_0^+(\lambda)\|_{HS} \sim \sqrt{\frac{\pi}{2}}\,(\operatorname{Re}\lambda)^{-\frac{1}{4}} \exp\left(\frac{4}{3}(\operatorname{Re}\lambda)^{\frac{3}{2}}\right) \qquad \textit{as } \operatorname{Re}\lambda \to +\infty , \tag{3.7}$$

where $\| \cdot \|_{HS}$ is the Hilbert-Schmidt norm. This is consistent with the well-known translation invariance properties of the operator $\mathcal{A}^+$, see [26]. The comparison between the HS-norm and the norm in $\mathcal{L}(L^2(\mathbb{R}))$ immediately implies that Proposition 7 gives the upper bound in Proposition 6.

3.2. The complex Airy operator on the half-line: Dirichlet case. It is not difficult to define the Dirichlet realization $\mathcal{A}^{\pm,D}$ of $D_x^2 \pm ix$ on $\mathbb{R}_+$ (the analysis on the negative semi-axis is similar). One can use for example the Lax-Milgram theorem and take as form domain

$$V^D := \{u \in H_0^1(\mathbb{R}_+) \,,\, x^{\frac{1}{2}} u \in L^2_+\} \,.$$

It can also be shown that the domain is

$$\mathcal{D}^D := \{u \in V^D \,,\, u \in H^2_+\} \,.$$

This implies

PROPOSITION 8. *The resolvent* $\mathcal{G}^{\pm,D}(\lambda) := (\mathcal{A}^{\pm,D} - \lambda)^{-1}$ *is in the Schatten class* C^p *for any* $p > \frac{3}{2}$ *(see [16] for definition), where* $\mathcal{A}^{\pm,D}$ *is the Dirichlet realization of* $D_x^2 \pm ix$*, as emphasized by the superscript* D*.*

More precisely we provide the distribution kernel $\mathcal{G}^{-,D}(x, y\,; \lambda)$ of the resolvent for the complex Airy operator $D_x^2 - ix$ on the positive semi-axis with Dirichlet boundary condition at the origin (the results for $\mathcal{G}^{+,D}(x, y\,; \lambda)$ are similar). Matching the boundary conditions, one gets

$$\mathcal{G}^{-,D}(x,y\,;\lambda) = \begin{cases} 2\pi \frac{\operatorname{Ai}(e^{-i\alpha}w_y)}{\operatorname{Ai}(e^{-i\alpha}w_0)} \big[\operatorname{Ai}(e^{i\alpha}w_x)\operatorname{Ai}(e^{-i\alpha}w_0) \\ \qquad\qquad -\operatorname{Ai}(e^{-i\alpha}w_x)\operatorname{Ai}(e^{i\alpha}w_0)\big] \quad (0 < x < y)\,, \\ 2\pi \frac{\operatorname{Ai}(e^{-i\alpha}w_x)}{\operatorname{Ai}(e^{-i\alpha}w_0)} \big[\operatorname{Ai}(e^{i\alpha}w_y)\operatorname{Ai}(e^{-i\alpha}w_0) \\ \qquad\qquad -\operatorname{Ai}(e^{-i\alpha}w_y)\operatorname{Ai}(e^{i\alpha}w_0)\big] \quad (x > y)\,, \end{cases} \tag{3.8}$$

[2] The coefficient was wrong in [32] and is corrected here, see Appendix B.

where $\mathrm{Ai}(z)$ is the Airy function,

$$w_x = ix + \lambda,$$

and

$$\alpha = 2\pi/3\,.$$

The above expression can also be written as

$$\mathcal{G}^{-,D}(x,y\,;\lambda) = \mathcal{G}_0^-(x,y\,;\lambda) + \mathcal{G}_1^{-,D}(x,y\,;\lambda), \tag{3.9}$$

where $\mathcal{G}_0^-(x,y\,;\lambda)$ is the resolvent for the complex Airy operator $D_x^2 - ix$ on the whole line,

$$\mathcal{G}_0^-(x,y\,;\lambda) = \begin{cases} 2\pi \mathrm{Ai}(e^{i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_y) & (x<y), \\ 2\pi \mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{i\alpha}w_y) & (x>y), \end{cases} \tag{3.10}$$

and

$$\mathcal{G}_1^{-,D}(x,y\,;\lambda) = -2\pi \frac{\mathrm{Ai}(e^{i\alpha}\lambda)}{\mathrm{Ai}(e^{-i\alpha}\lambda)} \mathrm{Ai}\big(e^{-i\alpha}(ix+\lambda)\big)\ \mathrm{Ai}\big(e^{-i\alpha}(iy+\lambda)\big). \tag{3.11}$$

The resolvent is compact. The poles of the resolvent are determined by the zeros of $\mathrm{Ai}(e^{-i\alpha}\lambda)$, i.e., $\lambda_n = e^{i\alpha}a_n$, where the a_n are zeros of the Airy function: $\mathrm{Ai}(a_n) = 0$. The eigenvalues have multiplicity 1 (no Jordan block). See Appendix A.

As a consequence of the analysis of the numerical range of the operator, we have

PROPOSITION 9.

$$||\mathcal{G}^{\pm,D}(\lambda)|| \le \frac{1}{|\mathrm{Re}\,\lambda|}\,,\ \ \textit{if}\ \mathrm{Re}\,\lambda < 0\,; \tag{3.12}$$

and

$$||\mathcal{G}^{\pm,D}(\lambda)|| \le \frac{1}{|\mathrm{Im}\,\lambda|}\,,\ \ \textit{if}\ \mp \mathrm{Im}\,\lambda > 0\,. \tag{3.13}$$

This proposition together with the Phragmen-Lindelöf principle (Theorem 54) and Proposition 8 implies (see [2] or [16])

PROPOSITION 10. *The space generated by the eigenfunctions of the Dirichlet realization $\mathcal{A}^{\pm,D}$ of $D_x^2 \pm ix$ is dense in L_+^2.*

It is proven in [28] that there is no Riesz basis of eigenfunctions.

At the boundary of the numerical range of the operator, it is interesting to analyze the behavior of the resolvent. Numerical computations lead to the observation that

$$\lim_{\lambda\to+\infty} ||\mathcal{G}^{\pm,D}(\lambda)||_{\mathcal{L}(L_+^2)} = 0\,. \tag{3.14}$$

As a new result, we will prove

PROPOSITION 11. *When λ tends to $+\infty$, we have*

$$||\mathcal{G}^{\pm,D}(\lambda)||_{HS} \approx \lambda^{-\frac{1}{4}}(\log\lambda)^{\frac{1}{2}}\,. \tag{3.15}$$

The convention "$A(\lambda) \approx B(\lambda)$ as $\lambda \to +\infty$" means that there exist C and λ_0 such that

$$\frac{1}{C} \leq \frac{|A(\lambda)|}{|B(\lambda)|} \leq C\,, \quad \forall \lambda \geq \lambda_0\,,$$

or, in other words, $A = \mathcal{O}(|B|)$ and $B = \mathcal{O}(|A|)$.
The proof of this proposition will be given in Appendix C.
Note that, as $||\mathcal{G}^{\pm,D}(\lambda)||_{\mathcal{L}(L^2)} \leq ||\mathcal{G}^{\pm,D}(\lambda)||_{HS}$, the estimate (3.15) implies (3.14).

3.3. The complex Airy operator on the half-line: Neumann case. Similarly, we can look at the Neumann realization $\mathcal{A}^{\pm,N}$ of $D_x^2 \pm ix$ on $\mathbb{R}_+$ (the analysis on the negative semi-axis is similar).

One can use for example the Lax-Milgram theorem and take as form domain

$$V^N = \{u \in H^1_+\,,\, x^{\frac{1}{2}} u \in L^2_+\}.$$

We recall that the Neumann condition appears when writing the domain of the operator $\mathcal{A}^{\pm,N}$.
As in the Dirichlet case (Proposition 8), this implies

Proposition 12. *The resolvent $\mathcal{G}^{\pm,N}(\lambda) := (\mathcal{A}^{\pm,N} - \lambda)^{-1}$ is in the Schatten class C^p for any $p > \frac{3}{2}$.*

More explicitly, the resolvent of $\mathcal{A}^{-,N}$ is obtained as

$$\mathcal{G}^{-,N}(x,y\,;\lambda) = \mathcal{G}_0^-(x,y\,;\lambda) + \mathcal{G}_1^{-,N}(x,y\,;\lambda) \quad \text{for } (x,y) \in \mathbb{R}^2_+,$$

where $\mathcal{G}_0^-(x,y\,;\lambda)$ is given by (3.10) and $\mathcal{G}_1^{-,N}(x,y\,;\lambda)$ is

$$\mathcal{G}_1^{-,N}(x,y\,;\lambda) = -2\pi \frac{e^{i\alpha}\mathrm{Ai}'(e^{i\alpha}\lambda)}{e^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}\lambda)} \mathrm{Ai}\big(e^{-i\alpha}(ix+\lambda)\big)\ \mathrm{Ai}\big(e^{-i\alpha}(iy+\lambda)\big)\,. \tag{3.16}$$

The poles of the resolvent are determined by zeros of $\mathrm{Ai}'(e^{-i\alpha}\lambda)$, i.e., $\lambda_n = e^{i\alpha}a'_n$, where a'_n are zeros of the derivative of the Airy function: $\mathrm{Ai}'(a'_n) = 0$. The eigenvalues have multiplicity 1 (no Jordan block). See Appendix A.
As a consequence of the analysis of the numerical range of the operator, we have

Proposition 13.

$$||\mathcal{G}^{\pm,N}(\lambda)|| \leq \frac{1}{|\mathrm{Re}\,\lambda|}\,, \quad \text{if } \mathrm{Re}\,\lambda < 0\,; \tag{3.17}$$

and

$$||\mathcal{G}^{\pm,N}(\lambda)|| \leq \frac{1}{|\mathrm{Im}\,\lambda|}\,, \quad \text{if } \mp\,\mathrm{Im}\,\lambda > 0\,. \tag{3.18}$$

This proposition together with Proposition 12 and the Phragmen-Lindelöf principle implies the completeness of the eigenfunctions:

Proposition 14. *The space generated by the eigenfunctions of the Neumann realization $\mathcal{A}^{\pm,N}$ of $D_x^2 \pm ix$ is dense in L^2_+.*

At the boundary of the numerical range of the operator, we have

PROPOSITION 15. *When λ tends to $+\infty$, we have*

$$||\mathcal{G}^{\pm,N}(\lambda)||_{HS} \approx \lambda^{-\frac{1}{4}} (\log \lambda)^{\frac{1}{2}} \,. \tag{3.19}$$

Proof
Using the Wronskian (A.3) for Airy functions, we have

$$\mathcal{G}^{-,D}(x,y\,;\lambda) - \mathcal{G}^{-,N}(x,y\,;\lambda) = -ie^{i\alpha} \frac{\mathrm{Ai}(e^{-i\alpha} w_x)\mathrm{Ai}(e^{-i\alpha} w_y)}{\mathrm{Ai}(e^{-i\alpha}\lambda)\mathrm{Ai}'(e^{-i\alpha}\lambda)} \,. \tag{3.20}$$

Hence

$$||\mathcal{G}^{-,D}(x,y\,;\lambda) - \mathcal{G}^{-,N}(x,y\,;\lambda)||^2_{HS} = \frac{(\int_0^{+\infty} |\mathrm{Ai}(e^{-i\alpha} w_x)|^2\, dx)^2}{|\mathrm{Ai}(e^{-i\alpha}\lambda)|^2\, |\mathrm{Ai}'(e^{-i\alpha}\lambda)|^2} \,.$$

We will show in (8.10) that there exists $C > 0$ such that

$$\int_0^{+\infty} |\mathrm{Ai}(e^{-i\alpha} w_x)|^2\, dx \leq C\lambda^{-\frac{1}{2}} \exp\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right) \,.$$

On the other hand, using (A.5) and (A.6), we obtain, for $\lambda \geq \lambda_0$

$$|\mathrm{Ai}(e^{-i\alpha}\lambda)\mathrm{Ai}'(e^{-i\alpha}\lambda)| \geq \frac{1}{4\pi} \exp\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right)$$

(this argument will also be used in the proof of (8.7)). We have consequently obtained that there exist $C > 0$ and $\lambda_0 > 0$ such that, for $\lambda \geq \lambda_0$,

$$||\mathcal{G}^{-,D}(\lambda) - \mathcal{G}^{-,N}(\lambda)||_{HS} \leq C\, |\lambda|^{-\frac{1}{4}} \,. \tag{3.21}$$

The proof of the proposition follows from Proposition 11.

3.4. The complex Airy operator on the half-line: Robin case. For completeness, we provide new results for the complex Airy operator on the half-line with the Robin boundary condition that naturally extends both Dirichlet and Neumann cases:

$$\left[\frac{\partial}{\partial x}\mathcal{G}^{-,R}(x,y\,;\lambda,\kappa) - \kappa\, \mathcal{G}^{-,R}(x,y\,;\lambda,\kappa)\right]_{x=0} = 0 \,, \tag{3.22}$$

with a positive parameter κ. The operator is associated with the sesquilinear form defined on $H^1_+ \times H^1_+$ by

$$a^{-,R}(u,v) = \int_0^{+\infty} u'(x)\bar{v}'(x)\, dx - i\int_0^{+\infty} x u(x)\bar{v}(x)\, dx + \kappa\, u(0)\bar{v}(0) \,. \tag{3.23}$$

The distribution kernel of the resolvent is obtained as

$$\mathcal{G}^{-,R}(x,y\,;\lambda) = \mathcal{G}^{-}_0(x,y\,;\lambda) + \mathcal{G}^{-,R}_1(x,y\,;\lambda,\kappa) \quad \text{for } (x,y) \in \mathbb{R}^2_+,$$

where

$$\begin{aligned} \mathcal{G}^{-,R}_1(x,y\,;\lambda,\kappa) = -2\pi \frac{ie^{i\alpha}\mathrm{Ai}'(e^{i\alpha}\lambda) - \kappa \mathrm{Ai}(e^{i\alpha}\lambda)}{ie^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}\lambda) - \kappa \mathrm{Ai}(e^{-i\alpha}\lambda)} \\ \times \mathrm{Ai}\big(e^{-i\alpha}(ix+\lambda)\big)\; \mathrm{Ai}\big(e^{-i\alpha}(iy+\lambda)\big) \,. \end{aligned} \tag{3.24}$$

Setting $\kappa = 0$, one retrieves (3.16) for the Neumann case, while the limit $\kappa \to +\infty$ yields (3.11) for the Dirichlet case, as expected. As previously, the resolvent is compact and actually in the Schatten class $\mathcal{C}^p$ for any $p > \frac{3}{2}$ (see Proposition 8). Its poles are determined as (complex-valued) solutions of the equation

$$f^R(\kappa, \lambda) := ie^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}\lambda) - \kappa \mathrm{Ai}(e^{-i\alpha}\lambda) = 0\,. \tag{3.25}$$

Except for the case of small κ, in which the eigenvalues might be localized close to the eigenvalues of the Neumann problem (see Section 4 for an analogous case), it does not seem easy to localize all the solutions of (3.25) in general. Nevertheless one can prove that the zeros of $f^R(\kappa, \cdot)$ are simple. If indeed λ is a common zero of f^R and $(f^R)'$, then either $\lambda + \kappa^2 = 0$, or $e^{-i\alpha}\lambda$ is a common zero of Ai and Ai$'$. The second option is excluded by the properties of the Airy function, whereas the first option is excluded for $\kappa \geq 0$ because the spectrum is contained in the positive half-plane.

As a consequence of the analysis of the numerical range of the operator, we have

Proposition 16.

$$||\mathcal{G}^{\pm,R}(\lambda, \kappa)|| \leq \frac{1}{|\mathrm{Re}\,\lambda|}\,, \quad \textit{if}\ \mathrm{Re}\,\lambda < 0\,; \tag{3.26}$$

and

$$||\mathcal{G}^{\pm,R}(\lambda, \kappa)|| \leq \frac{1}{|\mathrm{Im}\,\lambda|}\,, \quad \textit{if}\ \mp \mathrm{Im}\,\lambda > 0\,. \tag{3.27}$$

This proposition together with the Phragmen-Lindelöf principle (Theorem 54) and the fact that the resolvent is in the Schatten class $\mathcal{C}^p$, for any $p > \frac{3}{2}$, implies

Proposition 17. *For any $\kappa \geq 0$, the space generated by the eigenfunctions of the Robin realization $\mathcal{A}^{\pm,R}$ of $D_x^2 \pm ix$ is dense in L^2_+.*

At the boundary of the numerical range of the operator, it is interesting to analyze the behavior of the resolvent. Equivalently to Propositions 11 or 15, we have

Proposition 18. *When λ tends to $+\infty$, we have*

$$||\mathcal{G}^{\pm,R}(\lambda, \kappa)||_{HS} \approx \lambda^{-\frac{1}{4}} (\log \lambda)^{\frac{1}{2}}\,. \tag{3.28}$$

Proof

The proof is obtained by using Proposition 15 and computing, using (A.3),

$$\begin{aligned} ||\mathcal{G}^{-,N}(\lambda) - \mathcal{G}^{-,R}(\kappa, \lambda)||^2_{HS} = & \left(\int_0^{+\infty} |\mathrm{Ai}(e^{-i\alpha} w_x)|^2 dx\right)^2 \\ & \times \frac{\kappa}{2\pi}\, \frac{1}{|ie^{-i\alpha}\,\mathrm{Ai}'(e^{-i\alpha}\lambda) - \kappa \mathrm{Ai}(e^{-i\alpha}\lambda)|^2\, |\mathrm{Ai}'(e^{-i\alpha}\lambda)|^2}. \end{aligned}$$

As in the proof of Proposition 15, we show that for any $\kappa_0 > 0$, there exist $C > 0$ and λ_0 such that, for $\lambda \geq \lambda_0$ and $\kappa \in [0, \kappa_0]$,

$$||\mathcal{G}^{-,N}(\lambda) - \mathcal{G}^{-,R}(\lambda, \kappa)||_{HS} \leq C|\kappa|\lambda^{-\frac{3}{4}}\,.$$

4. The complex Airy operator on the line with a semi-permeable barrier: definition and properties. In comparison with Section 2, we now replace the differential operator $-\frac{d^2}{dx^2}$ by $\mathcal{A}_1^+ = -\frac{d^2}{dx^2} + ix$ but keep the same transmission condition. To give a precise mathematical definition of the associated closed operator, we consider the sesquilinear form a_ν defined for $u = (u_-, u_+)$ and $v = (v_-, v_+)$ by

$$
\begin{aligned}
a_\nu(u,v) = & \int_{-\infty}^{0} \Big(u_-'(x)\bar v_-'(x) + i\,x u_-(x)\bar v_-(x) + \nu\, u_-(x)\bar v_-(x) \Big)\, dx \\
& + \int_{0}^{+\infty} \Big(u_+'(x)\bar v_+'(x) + i\,x u_+(x)\bar v_+(x) + \nu\, u_+(x)\bar v_+(x) \Big)\, dx \\
& + \kappa\big(u_+(0) - u_-(0)\big)\big(\overline{v_+(0) - v_-(0)}\big)\,,
\end{aligned}
\tag{4.1}
$$

where the form domain $\mathcal{V}$ is

$$
\mathcal{V} := \Big\{ u = (u_-, u_+) \in H^1_- \times H^1_+ : |x|^{\frac12} u \in L^2_- \times L^2_+ \Big\}\,.
$$

The space $\mathcal{V}$ is endowed with the Hilbert norm

$$
\|u\|_{\mathcal{V}} := \Big(\|u_-\|^2_{H^1_-} + \|u_+\|^2_{H^1_+} + \||x|^{\frac12} u\|^2_{L^2} \Big)^{\frac12}\,.
$$

We first observe

LEMMA 19. *For any $\nu \geq 0$, the sesquilinear form a_ν is continuous on $\mathcal{V}$.*

Proof
The proof is similar to that of Lemma 3, the additional term

$$
i \left(\int_{-\infty}^{0} x\, u_-(x)\, \bar v_-(x)\, dx + \int_0^{+\infty} x\, u_+(x)\, \bar v_+(x)\, dx \right)
$$

being obviously bounded by $\|u\|_V \|v\|_V$. □

Let us notice that, if u and v belong to $H^2_- \times H^2_+$ and satisfy the boundary conditions (1.3), then an integration by parts yields

$$
\begin{aligned}
a_\nu(u,v) = & \int_{-\infty}^{0} \big(- u_-''(x) + ixu_-(x) + \nu\, u_-(x)\big) \bar v_-(x)\, dx \\
& + \int_{0}^{+\infty} \big(- u_+''(x) + ixu_+(x) + \nu\, u_+(x)\big) \bar v_+(x)\, dx \\
& + \big(u_+'(0) + \kappa(u_-(0) - u_+(0))\big)\big(\overline{v_-(0) - v_+(0)}\big) \\
= & \langle \left(-\frac{d^2}{dx^2} + ix + \nu \right) u, v \rangle_{L^2_- \times L^2_+}\,.
\end{aligned}
$$

Hence the operator associated with the form a_ν, once defined appropriately, will act as $-\frac{d^2}{dx^2} + ix + \nu$ on $C_0^\infty(\mathbb{R} \setminus \{0\})$.

As the imaginary part of the potential ix changes sign, it is not straightforward to determine whether the sesquilinear form a_ν is coercive, i.e., whether there exists ν_0 such that for $\nu \geq \nu_0$ the following estimate holds:

$$
\exists \alpha > 0\,,\ \forall u \in \mathcal{V}\,, \quad |a_\nu(u,u)| \geq \alpha\, \|u\|^2_{\mathcal{V}}\,.
\tag{4.2}
$$

Let us show that it is indeed not true. Consider for instance the sequence

$$
u_n(x) = (\chi(x+n), \chi(x-n))\,, \qquad n \geq 1\,,
$$

where $\chi \in \mathcal{C}_0^\infty(-1,1)$ is an even function such that $\chi(x) = 1$ for $x \in [-1/2, 1/2]$. Then $\|u_n'\|_{L^2(-\infty,0)}$ and $\|u_n'\|_{L^2(0,+\infty)}$ are bounded, and

$$\int_{\mathbb{R}} x\,|u_n(x)|^2\,dx = 0\,,$$

since $x \mapsto x|u_n(x)|^2$ is odd, whereas $\||x|^{\frac12} u_n\|_{L^2} \longrightarrow +\infty$ as $n \to +\infty$. Consequently

$$\frac{|a_\nu(u_n,u_n)|}{\|u_n\|_{\mathcal{V}}^2} \longrightarrow 0 \text{ as } n \to +\infty\,,$$

and (4.2) does not hold.

Due to the lack of coercivity, the standard version of the Lax-Milgram theorem does not apply. We shall instead use the following generalization introduced in [4].

THEOREM 20. *Let $\mathcal{V} \subset \mathcal{H}$ be two Hilbert spaces such that $\mathcal{V}$ is continuously embedded in $\mathcal{H}$ and $\mathcal{V}$ is dense in $\mathcal{H}$. Let a be a continuous sesquilinear form on $\mathcal{V}\times\mathcal{V}$, and assume that there exist $\alpha > 0$ and two bounded linear operators Φ_1 and Φ_2 on $\mathcal{V}$ such that, for all $u \in \mathcal{V}$,*

$$\left\{\begin{array}{lcl} |a(u,u)| + |a(u,\Phi_1 u)| & \geq & \alpha\,\|u\|_{\mathcal{V}}^2\,, \\ |a(u,u)| + |a(\Phi_2 u,u)| & \geq & \alpha\,\|u\|_{\mathcal{V}}^2\,. \end{array}\right. \tag{4.3}$$

Assume further that Φ_1 extends to a bounded linear operator on $\mathcal{H}$. Then there exists a closed, densely-defined operator S on $\mathcal{H}$ with domain

$$\mathcal{D}(S) = \big\{u \in \mathcal{V} : v \mapsto a(u,v) \;\; \text{can be extended continuously on} \;\; \mathcal{H}\,\big\}\,,$$

such that, for all $u \in \mathcal{D}(S)$ and $v \in \mathcal{V}$,

$$a(u,v) = \langle Su, v\rangle_{\mathcal{H}}\,.$$

Now we want to find two operators Φ_1 and Φ_2 on $\mathcal{V}$ such that the estimates (4.3) hold for the form a_ν defined by (4.1). First we have, as in (2.7),

$$\begin{aligned} \operatorname{Re} a_\nu(u,u) \geq (1 - |\kappa|\varepsilon)\left(\int_{-\infty}^0 |u_-'(x)|^2\,dx + \int_0^{+\infty} |u_+'(x)|^2\,dx\right) \\ + \big(\nu - |\kappa| C(\varepsilon)\big)\|u\|_{L^2}^2\,. \end{aligned}$$

Thus by choosing ε and ν appropriately we get, for some $\alpha_1 > 0$,

$$|a_\nu(u,u)| \geq \alpha_1 \left(\int_{-\infty}^0 |u_-'(x)|^2\,dx + \int_0^{+\infty} |u_+'(x)|^2\,dx + \|u\|_{L^2}^2\right)\,. \tag{4.4}$$

It remains to estimate the term $\||x|^{\frac12} u\|_{L^2}$ appearing in the norm $\|u\|_{\mathcal{V}}$. For this purpose, we introduce the operator

$$\rho : (u_-, u_+) \longmapsto (-u_-, u_+)\,,$$

which corresponds to the multiplication operator by the function $\operatorname{sign} x$. It is clear that ρ maps $\mathcal{H}$ onto $\mathcal{H}$ and $\mathcal{V}$ onto $\mathcal{V}$. Then we have

$$\operatorname{Im} a_\nu(u, \rho u) = \; \||x|^{\frac12} u\|_{L^2}^2\,. \tag{4.5}$$

Thus using (4.4), there exists α_0 such that, for all $u \in \mathcal{V}$,

$$|a_\nu(u,u)| + |a_\nu(u,\rho u)| \geq \alpha \|u\|^2_{\mathcal{V}}\,.$$

Similarly, for all $u \in \mathcal{V}$,

$$|a_\nu(u,u)| + |a_\nu(\rho u,u)| \geq \alpha \|u\|^2_{\mathcal{V}}\,.$$

In other words, the estimate (4.3) holds, with $\Phi_1 = \Phi_2 = \rho$. Hence the assumptions of Theorem 20 are satisfied, and we can define a closed operator $\mathcal{A}_1^+ := S - \nu$, which is given by the identity

$$\forall u \in \mathcal{D}(\mathcal{A}_1^+)\,,\ \forall v \in \mathcal{V}\,,\quad a_\nu(u,v) = \langle \mathcal{A}_1^+ u + \nu u, v\rangle_{L^2_- \times L^2_+}$$

on the domain

$$\mathcal{D}(\mathcal{A}_1^+) = \mathcal{D}(S) = \big\{u \in \mathcal{V} : v \mapsto a_\nu(u,v) \ \text{ can be extended continuously}$$
$$\text{on } \ L^2_- \times L^2_+ \big\}\,.$$

Now we shall determine explicitly the domain $\mathcal{D}(\mathcal{A}_1^+)$.

Let $u \in \mathcal{V}$. The map $v \mapsto a_\nu(u,v)$ can be extended continuously on $L^2_- \times L^2_+$ if and only if there exists some $w_u = (w_u^-, w_u^+) \in L^2_- \times L^2_+$ such that, for all $v \in \mathcal{V}$, $a_\nu(u,v) = \langle w_u, v\rangle_{L^2}$. Then due to the definition of $a_\nu(u,v)$, we have necessarily

$$w_u^- = -u_-'' + ixu_- + \nu u_- \quad \text{ and } \quad w_u^+ = -u_+'' + ixu_+ + \nu u_+$$

in the sense of distributions respectively in $\mathbb{R}_-$ and $\mathbb{R}_+$, and u satisfies the conditions (1.3). Consequently, the domain of $\mathcal{A}_1^+$ can be rewritten as

$$\mathcal{D}(\mathcal{A}_1^+) = \Big\{u \in \mathcal{V} : (-u_-'' + ixu_-, -u_+'' + ixu_+) \in L^2_- \times L^2_+$$
$$\text{and } \ u \text{ satisfies conditions (1.3)}\Big\}\,.$$

We now prove that $\mathcal{D}(\mathcal{A}_1^+) = \widehat{\mathcal{D}}$ where

$$\widehat{\mathcal{D}} = \Big\{u \in \mathcal{V} : (u_-, u_+) \in H^2_- \times H^2_+\,,\, (xu_-, xu_+) \in L^2_- \times L^2_+$$
$$\text{and } \ u \text{ satisfies conditions (1.3)}\Big\}\,.$$

It remains to check that this implies $(u_-, u_+) \in H^2_- \times H^2_+$. The only problem is at $+\infty$. Let u_+ be as above and let χ be a nonnegative function equal to 1 on $[1,+\infty)$ and with support in $(\frac{1}{2},+\infty)$. It is clear that the natural extension by 0 of χu_+ to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$ and satisfies

$$\left(-\frac{d^2}{dx^2} + ix\right)(\chi u_+) \in L^2(\mathbb{R})\,.$$

One can apply for χu_+ a standard result for the domain of the accretive maximal extension of the complex Airy operator on $\mathbb{R}$ (see for example [26]).

Finally, let us notice that the continuous embedding

$$\mathcal{V} \hookrightarrow L^2(\mathbb{R}; |x|dx) \cap \big(H^1_- \times H^1_+\big)$$

implies that $\mathcal{A}_1^+$ has a compact resolvent; hence its spectrum is discrete.

Moreover, from the characterization of the domain and its inclusion in $\widehat{D}$, we deduce the stronger

PROPOSITION 21. *There exists* λ_0 *(*$\lambda_0 = 0$ *for* $\kappa > 0$*) such that* $(\mathcal{A}_1^+ - \lambda_0)^{-1}$ *belongs to the Schatten class* $\mathcal{C}^p$ *for any* $p > \frac{3}{2}$*.*

Note that if it is true for some λ_0 it is true for any λ in the resolvent set.

REMARK 22. *The adjoint of* $\mathcal{A}_1^+$ *is the operator associated by the same construction with* $D_x^2 - ix$*.* $\mathcal{A}_1^- + \lambda$ *being injective, this implies by a general criterion [26] that* $\mathcal{A}_1^+ + \lambda$ *is maximal accretive, hence generates a contraction semigroup.*

The following statement summarizes the previous discussion.

PROPOSITION 23. *The operator* $\mathcal{A}_1^+$ *acting as*

$$u \mapsto \mathcal{A}_1^+ u = \left(-\frac{d^2}{dx^2} u_- + ixu_-, \, -\frac{d^2}{dx^2} u_+ + ixu_+ \right)$$

on the domain

$$\mathcal{D}(\mathcal{A}_1^+) = \big\{ u \in H_-^2 \times H_+^2 : xu \in L_-^2 \times L_+^2 \quad \text{and } u \text{ satisfies conditions (1.3)} \big\} \tag{4.6}$$

is a closed operator with compact resolvent.
There exists some positive λ *such that the operator* $\mathcal{A}_1^+ + \lambda$ *is maximal accretive.*

REMARK 24. *We have*

$$\Gamma \mathcal{A}_1^+ = \mathcal{A}_1^- \Gamma \,, \tag{4.7}$$

where Γ *denotes the complex conjugation:*

$$\Gamma(u_- \,, u_+) = (\bar{u}_- \,, \bar{u}_+) \,.$$

This implies that the distribution kernel of the resolvent satisfies:

$$\mathcal{G}(x, y\,; \lambda) = \mathcal{G}(y, x; \lambda) \,, \tag{4.8}$$

for any λ *in the resolvent set.*

REMARK 25 (PT-Symmetry). *If* (λ, u) *is an eigenpair, then* $(\bar{\lambda}, \bar{u}(-x))$ *is also an eigenpair. Let indeed* $v(x) = \bar{u}(-x)$*. This means* $v_-(x) = \bar{u}_+(-x)$ *and* $v_+(x) = \bar{u}_-(-x)$*. Hence we get that* v *satisfies (2.1) if* u *satisfies the same condition:*

$$v_-'(0) = -\bar{u}_+'(0) = \kappa(\bar{u}_-(0) - \bar{u}_+(0)) = +\kappa\,(v_+(0) - v_-(0)) \,.$$

Similarly one can verify that

$$\begin{aligned} \left(-\frac{d^2}{dx^2} + ix \right) v_+(x) &= \overline{\left(-\frac{d^2}{dx^2} - ix \right) u_-(-x)} \\ &= \overline{\left(\left(-\frac{d^2}{dx^2} + ix \right) u_- \right) (-x)} \\ &= \bar{\lambda}\, v_+(x) \,. \end{aligned}$$

5. Exponential decay of the associated semi-group. In order to control the decay of the associated semi-group, we follow what has been done for the Neumann or Dirichlet realization of the complex Airy operator on the half-line (see for example [26] or [28, 29]).

THEOREM 26. *Assume $\kappa > 0$, then for any $\omega < \inf\{\operatorname{Re}\sigma(\mathcal{A}_1^+)\}$, there exists M_ω such that, for all $t \geq 0$,*

$$|| \exp(-t\mathcal{A}_1^+)||_{\mathcal{L}(L^2_- \times L^2_+)} \leq M_\omega \exp(-\omega t)\,,$$

where $\sigma(\mathcal{A}_1^+)$ is the spectrum of $\mathcal{A}_1^+$.

To apply the quantitative Gearhart-Prüss theorem (see [26]) to the operator $\mathcal{A}_1^+$, we should prove that

$$\sup_{\operatorname{Re} z \leq \omega} \|(\mathcal{A}_1^+ - z)^{-1}\| \leq C_\omega\,,$$

for all $\omega < \inf \operatorname{Re}\sigma(\mathcal{A}_1^+) := \omega_1$.
First we have by accretivity (remember that $\kappa > 0$), for $\operatorname{Re}\lambda < 0$,

$$||(\mathcal{A}_1^+ - \lambda)^{-1}|| \leq \frac{1}{|\operatorname{Re}\lambda|}\,. \tag{5.1}$$

So it remains to analyze the resolvent in the set

$$0 \leq \operatorname{Re}\lambda \leq \omega_1 - \epsilon\,, \quad |\operatorname{Im}\lambda| \geq C_\epsilon > 0\,,$$

where $C_\epsilon > 0$ is sufficiently large. Let us prove the following lemma.

LEMMA 27. *For any $\alpha > 0$, there exist $C_\alpha > 0$ and $D_\alpha > 0$ such that for any $\lambda \in \{\omega \in \mathbb{C} : \operatorname{Re}\omega \in [-\alpha, +\alpha]$ and $|\operatorname{Im}\omega| > D_\alpha\}$,*

$$||(\mathcal{A}_1^\pm - \lambda)^{-1}|| \leq C_\alpha\,. \tag{5.2}$$

Proof
Without loss of generality, we treat the case when $\operatorname{Im}\lambda > 0$. As in [9], the main idea of the proof is to approximate $(\mathcal{A}_1^+ - \lambda)^{-1}$ by a sum of two operators: one of them is a good approximation when applied to functions supported near the transmission point, while the other one takes care of functions whose support lies far away from this point.

The first operator $\check{\mathcal{A}}$ is associated with the sesquilinear form $\check{a}$ defined for $u = (u_-, u_+)$ and $v = (v_-, v_+)$ by

$$\begin{aligned}\check{a}(u,v) = &\int_{-\operatorname{Im}\lambda/2}^{0} \Big(u_-'(x)\bar{v}_-'(x) + i\,x u_-(x)\bar{v}_-(x) + \lambda\, u_-(x)\bar{v}_-(x)\Big)\,dx \\ &+ \int_0^{\operatorname{Im}\lambda/2} \Big(u_+'(x)\bar{v}_+'(x) + i\,x u_+(x)\bar{v}_+(x) + \lambda\, u_+(x)\bar{v}_+(x)\Big)\,dx \\ &+ \kappa\,\big(u_+(0) - u_-(0)\big)\big(\overline{v_+(0) - v_-(0)}\big)\,,\end{aligned} \tag{5.3}$$

where u and v belong to the following space:

$$\mathbb{H}_0^1(\mathcal{S}_\lambda, \mathbb{C}) := \big(H^1(S_\lambda^-) \times H^1(S_\lambda^+)\big) \cap \{u_-(-\operatorname{Im}\lambda/2) = 0\,,\, u_+(\operatorname{Im}\lambda/2) = 0\}\,,$$

with $\mathcal{S}_\lambda^- := (-\operatorname{Im}\lambda/2, 0)$ and $\mathcal{S}_\lambda^+ := (0, +\operatorname{Im}\lambda/2)$.
The domain $\mathcal{D}(\check{\mathcal{A}})$ of $\check{\mathcal{A}}$ is the set of $u \in H^2(\mathcal{S}_\lambda^-) \times H^2(\mathcal{S}_\lambda^+)$ such that $u_-(-\operatorname{Im}\lambda/2) = 0\,, u_+(\operatorname{Im}\lambda/2) = 0$ and u satisfies conditions (1.3). Denote the resolvent of $\check{\mathcal{A}}$ by $R_1(\lambda)$ in $\mathcal{L}(L^2(\mathcal{S}_\lambda^-, \mathbb{C}) \times L^2(\mathcal{S}_\lambda^+, \mathbb{C}))$ and observe also that $R_1(\lambda) \in \mathcal{L}(L^2(\mathcal{S}_\lambda^-, \mathbb{C}) \times L^2(\mathcal{S}_\lambda^+, \mathbb{C}), \mathbb{H}_0^1(\mathcal{S}_\lambda, \mathbb{C}))$.

We easily obtain (looking at the imaginary part of the sesquilinear form) that

$$\|R_1(\lambda)\| \leq \frac{2}{\operatorname{Im}\lambda}\,. \tag{5.4}$$

Furthermore, we have, for $u = R_1(\lambda)\, f$ (with $u = (u_-, u_+)$, $f = (f_-, f_+)$)

$$\begin{aligned}\|D_x R_1(\lambda) f\|^2 &= \|D_x u\|^2\\ &\leq \|(\mathcal{A}_1^+ - \lambda)u\|\|u\| + \operatorname{Re}\lambda\|u\|^2\\ &\leq \|f\|\|R_1(\lambda)f\| + |\alpha|\|\mathcal{R}_1(\lambda)f\|^2\\ &\leq \left(\frac{2}{|\operatorname{Im}\lambda|} + \frac{4|\alpha|}{|\operatorname{Im}\lambda|^2}\right)\|f\|^2.\end{aligned}$$

Hence there exists $C_0(\alpha)$ such that, for $\operatorname{Im}\lambda \geq 1$ and $\operatorname{Re}\lambda \in [-\alpha, +\alpha]$,

$$\|D_x R_1(\lambda)\| \leq C_0(\alpha)\, |\operatorname{Im}\lambda|^{-\frac{1}{2}}\,. \tag{5.5}$$

Far from the transmission point 0, we approximate by the resolvent $\mathcal{G}_0^+$ of the complex Airy operator $\mathcal{A}^+$ on the line. Denote this resolvent by $R_2(\lambda)$ when considered as an operator in $\mathcal{L}(L^2_- \times L^2_+)$. We recall from Section 3 that the norm $\|R_2(\lambda)\|$ is independent of $\operatorname{Im}\lambda$. Since $R_2(\lambda)$ is an entire function of λ, we easily obtain a uniform bound on $\|R_2(\lambda)\|$ for $\operatorname{Re}\lambda \in [-\alpha, +\alpha]$. Hence,

$$\|R_2(\lambda)\| \leq C_1(\alpha)\,. \tag{5.6}$$

As for the proof of (5.5), we then show

$$\|D_x R_2(\lambda)\| \leq C(\alpha)\,. \tag{5.7}$$

We now use a partition of unity in the x variable in order to construct an approximate inverse $R^{\mathrm{app}}(\lambda)$ for $\mathcal{A}_1^+ - \lambda$. We shall then prove that the difference between the approximation and the exact resolvent is well controlled as $\operatorname{Im}\lambda \to +\infty$. For this purpose, we define the following triple (ϕ_-, ψ, ϕ_+) of cutoff functions in $C^\infty(\mathbb{R}, [0,1])$ satisfying

$$\begin{aligned}&\phi_-(t) = 1 \ \text{ on } (-\infty, -1/2]\,, \quad \phi_-(t) = 0 \ \text{ on } [-1/4, +\infty)\\ &\psi(t) = 1 \ \text{ on } [-1/4, 1/4]\,, \quad \psi(t) = 0 \text{ on } (-\infty, -1/2] \cup [1/2, +\infty)\,,\\ &\phi_+(t) = 1 \ \text{ on } [1/2, +\infty)\,, \quad \phi_+(t) = 0 \ \text{ on } (-\infty, 1/4],\\ &\phi_-(t)^2 + \psi(t)^2 + \phi_+(t)^2 = 1 \text{ on } \mathbb{R}\,,\end{aligned}$$

and then set

$$\phi_{\pm,\lambda}(x) = \phi_\pm\Big(\frac{x}{\operatorname{Im}\lambda}\Big)\,, \quad \psi_\lambda(x) = \psi\Big(\frac{x}{\operatorname{Im}\lambda}\Big)\,.$$

The approximate inverse $R^{\mathrm{app}}(\lambda)$ is then constructed as

$$R^{\mathrm{app}}(\lambda) = \phi_{-,\lambda} R_2(\lambda)\phi_{-,\lambda} + \psi_\lambda R_1(\lambda)\psi_\lambda + \phi_{+,\lambda} R_2(\lambda)\phi_{+,\lambda}\,, \tag{5.8}$$

where $\phi_{\pm,\lambda}$ and ψ_λ denote the operators of multiplication by the functions $\phi_{\pm,\lambda}$ and ψ_λ. Note that ψ_λ maps $L^2_- \times L^2_+$ into $L^2(\mathcal{S}_\lambda^-) \times L^2(\mathcal{S}_\lambda^+)$. In addition,

$$\begin{aligned}\psi_\lambda :\ &\mathcal{D}(\check{\mathcal{A}}) \to \mathcal{D}(\mathcal{A}_1^+),\\ \phi_\lambda :\ &\mathcal{D}(\mathcal{A}^+) \to \mathcal{D}(\mathcal{A}_1^+)\,,\end{aligned}$$

where we have defined $\phi_\lambda(u_-, u_+)$ as $(\phi_{-,\lambda}\, u_-\,,\, \phi_{+,\lambda}\, u_+)$. From (5.4) and (5.6) we get, for sufficiently large $\operatorname{Im}\lambda$,

$$\|R^{\rm app}(\lambda)\| \le C_3(\alpha). \tag{5.9}$$

Note that

$$|\phi_\lambda'(x)| + |\psi_\lambda'(x)| \le \frac{C}{|\operatorname{Im}\lambda|}, \quad |\phi_\lambda''(x)| + |\psi_\lambda''(x)| \le \frac{C}{|\operatorname{Im}\lambda|^2}\,. \tag{5.10}$$

Next, we apply $\mathcal{A}_1^+ - \lambda$ to $R^{\rm app}$ to obtain that

$$(\mathcal{A}_1^+ - \lambda) R^{\rm app}(\lambda) = I + [\mathcal{A}_1^+\,,\, \psi_\lambda] R_1(\lambda)\psi_\lambda + [\mathcal{A}_1^+\,,\, \phi_\lambda] R_2(\lambda)\phi_\lambda\,, \tag{5.11}$$

where I is the identity operator on $L^2_- \times L^2_+$, and

$$\begin{aligned}
[\mathcal{A}_1^+\,,\, \phi_\lambda] &:= \mathcal{A}_1^+\phi_\lambda - \phi_\lambda\mathcal{A}_1^+ \\
&= [D_x^2\,,\, \phi_\lambda] \\
&= -\frac{2i}{\operatorname{Im}\lambda}\phi'\Big(\frac{x}{\operatorname{Im}\lambda}\Big) D_x \;-\; \frac{1}{(\operatorname{Im}\lambda)^2}\phi''\Big(\frac{x}{\operatorname{Im}\lambda}\Big)\,.
\end{aligned} \tag{5.12}$$

A similar relation holds for $[\mathcal{A}_1^+, \psi_\lambda]$. Here we have used (5.8), and the fact that

$$(\mathcal{A}_1^+ - \lambda) R_1(\lambda)\psi_\lambda u = \psi_\lambda u\,, \quad (\mathcal{A}_1^+ - \lambda) R_2(\lambda)\phi_\lambda u = \phi_\lambda u\,, \quad \forall u \in L^2_- \times L^2_+\,.$$

Using (5.4), (5.5), (5.7), and (5.12) we then easily obtain, for sufficiently large $\operatorname{Im}\lambda$,

$$\|[\mathcal{A}_1^+, \psi_\lambda]\, R_1(\lambda)\| + \|[\mathcal{A}_1^+, \phi_\lambda]\, R_2(\lambda)\| \le \frac{C_4(\alpha)}{|\operatorname{Im}\lambda|}\,. \tag{5.13}$$

Hence, if $|\operatorname{Im}\lambda|$ is large enough then $I + [\mathcal{A}_1^+, \psi_\lambda] R_1(\lambda)\psi_\lambda + [\mathcal{A}_1^+, \phi_\lambda] R_2(\lambda)\phi_\lambda$ is invertible in $\mathcal{L}(L^2_- \times L^2_+)$, and

$$\Big\| \Big(I + [\mathcal{A}_1^+, \psi_\lambda] R_1(\lambda)\psi_\lambda + [\mathcal{A}_1^+, \phi_\lambda] R_2(\lambda)\phi_\lambda \Big)^{-1} \Big\| \le C_5(\alpha)\,. \tag{5.14}$$

Finally, since

$$(\mathcal{A}_1^+ - \lambda)^{-1} = R^{\rm app}(\lambda) \circ \big(I + [\mathcal{A}_1^+, \psi_\lambda] R_1(\lambda)\psi_\lambda + [\mathcal{A}_1^+, \phi_\lambda] R_2(\lambda)\phi_\lambda \big)^{-1}\,,$$

we have

$$\|(\mathcal{A}_1^+ - \lambda)^{-1}\| \le \|R^{\rm app}(\lambda)\| \big\| \big(I + [\mathcal{A}_1^+, \psi_\lambda] R_1(\lambda)\psi_\lambda + [\mathcal{A}_1^+, \phi_\lambda] R_2(\lambda)\phi_\lambda \big)^{-1} \big\|\,.$$

Using (5.9) and (5.14) we conclude that (5.2) is true. ☐

Remark 28. *One could alternatively use more directly the expression of the kernel $\mathcal{G}^+(x, y\,; \lambda)$ of $(\mathcal{A}_1^+ - \lambda)^{-1}$ in terms of* Ai *and* Ai′*, together with the asymptotic expansions of the Airy function, see Appendix A and the discussion at the beginning of Section 7.*

6. Integral kernel of the resolvent and its poles. Here we revisit some of the computations of [22, 23] with the aim to complete some formal proofs. We are looking for the distribution kernel $\mathcal{G}^-(x,y;\lambda)$ of the resolvent $(\mathcal{A}_1^- - \lambda)^{-1}$ which satisfies in the sense of distribution

$$\left(-\lambda - ix - \frac{\partial^2}{\partial x^2}\right)\mathcal{G}^-(x,y\,;\lambda) = \delta(x-y), \tag{6.1}$$

as well as the boundary conditions

$$\begin{aligned}\left[\frac{\partial}{\partial x}\mathcal{G}^-(x,y\,;\lambda)\right]_{x=0^+} &= \left[\frac{\partial}{\partial x}\mathcal{G}^-(x,y\,;\lambda)\right]_{x=0^-} \\ &= \kappa\left[\mathcal{G}^-(0^+,y;\lambda) - \mathcal{G}^-(0^-,y;\lambda)\right].\end{aligned} \tag{6.2}$$

Sometimes, we will write $\mathcal{G}^-(x,y\,;\lambda,\kappa)$, in order to stress the dependence on κ. Note that one can easily come back to the kernel of the resolvent of $\mathcal{A}_1^+$ by using

$$\mathcal{G}^+(x,y\,;\lambda) = \overline{\mathcal{G}^-(y,x;\bar{\lambda})}\,. \tag{6.3}$$

Using (4.8), we also get

$$\mathcal{G}^+(x,y\,;\lambda) = \overline{\mathcal{G}^-(x,y\,;\bar{\lambda})}\,. \tag{6.4}$$

We search for the solution $\mathcal{G}^-(x,y\,;\lambda)$ in three subdomains: the negative semi-axis $(-\infty,0)$, the interval $(0,y)$, and the positive semi-axis $(y,+\infty)$ (here we assumed that $y>0$; the opposite case is similar). For each subdomain, the solution is a linear combination of two Airy functions:

$$\mathcal{G}^-(x,y\,;\lambda) = \begin{cases} A^-\mathrm{Ai}(e^{-i\alpha}w_x) + B^-\mathrm{Ai}(e^{i\alpha}w_x) & (x<0)\,, \\ A^+\mathrm{Ai}(e^{-i\alpha}w_x) + B^+\mathrm{Ai}(e^{i\alpha}w_x) & (0<x<y)\,, \\ C^+\mathrm{Ai}(e^{-i\alpha}w_x) + D^+\mathrm{Ai}(e^{i\alpha}w_x) & (x>y)\,, \end{cases} \tag{6.5}$$

with six unknown coefficients (which are functions of $y>0$). We recall that

$$\alpha = \frac{2\pi}{3}$$

and

$$w_x = ix + \lambda\,.$$

The boundary conditions (6.2) read as

$$\begin{aligned}&B^- ie^{i\alpha}\mathrm{Ai}'(e^{i\alpha}w_0) \\ &= A^+ ie^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}w_0) + B^+ ie^{i\alpha}\mathrm{Ai}'(e^{i\alpha}w_0) \\ &= \kappa\left[A^+\mathrm{Ai}(e^{-i\alpha}w_0) + B^+\mathrm{Ai}(e^{i\alpha}w_0) - B^-\mathrm{Ai}(e^{i\alpha}w_0)\right],\end{aligned} \tag{6.6}$$

where $w_0=\lambda$ and we set $A^-=0$ and $D^+=0$ to ensure the decay of $\mathcal{G}^-(x,y\,;\lambda)$ as $x\to-\infty$ and as $x\to+\infty$, respectively.
We now look at the condition at $x=y$ in order to have (6.1) satisfied in the distribution sense. We write the continuity condition,

$$A^+\mathrm{Ai}(e^{-i\alpha}w_y) + B^+\mathrm{Ai}(e^{i\alpha}w_y) = C^+\mathrm{Ai}(e^{-i\alpha}w_y)\,,$$

and the discontinuity jump of the derivative,

$$A^+ ie^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}w_y) + B^+ ie^{i\alpha}\mathrm{Ai}'(e^{i\alpha}w_y) = C^+ ie^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}w_y) + 1\,.$$

This can be considered as a linear system for A^+ and B^+. Using the Wronskian (A.3), one expresses A^+ and B^+ in terms of C^+:

$$A^+ = C^+ - 2\pi\mathrm{Ai}(e^{i\alpha}w_y)\,,\quad B^+ = 2\pi\mathrm{Ai}(e^{-i\alpha}w_y)\,. \tag{6.7}$$

We can rewrite (6.6) in the form

$$B^- = e^{-2i\alpha}\frac{\mathrm{Ai}'(e^{-i\alpha}w_0)}{\mathrm{Ai}'(e^{i\alpha}w_0)}A^+ + B^+\,, \tag{6.8}$$

and

$$\begin{aligned} &A^+ ie^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}w_0) + B^+ ie^{i\alpha}\mathrm{Ai}'(e^{i\alpha}w_0) \\ &\qquad = \kappa\, A^+\left[\mathrm{Ai}(e^{-i\alpha}w_0) - e^{-2i\alpha}\mathrm{Ai}(e^{i\alpha}w_0)\frac{\mathrm{Ai}'(e^{-i\alpha}w_0)}{\mathrm{Ai}'(e^{i\alpha}w_0)}\right]. \end{aligned} \tag{6.9}$$

Using again the Wronskian (A.3), we obtain

$$A^+\mathrm{Ai}'(e^{-i\alpha}w_0) + B^+ e^{2i\alpha}\mathrm{Ai}'(e^{i\alpha}w_0) = -\kappa A^+\frac{1}{2\pi\mathrm{Ai}'(e^{i\alpha}w_0)}\,,$$

that is

$$A^+(f(\lambda)+\kappa) + B^+(2\pi)e^{2i\alpha}\left(\mathrm{Ai}'(e^{i\alpha}w_0)\right)^2 = 0\,,$$

where

$$f(\lambda) := 2\pi\mathrm{Ai}'(e^{-i\alpha}\lambda)\mathrm{Ai}'(e^{i\alpha}\lambda)\,. \tag{6.10}$$

So we now get

$$A^+ = -\frac{1}{f(\lambda)+\kappa}(2\pi)^2 e^{2i\alpha}\left(\mathrm{Ai}'(e^{i\alpha}w_0)\right)^2\mathrm{Ai}(e^{-i\alpha}w_y)\,, \tag{6.11}$$

$$B^- = 2\pi\mathrm{Ai}(e^{-i\alpha}w_y) - 2\pi\frac{f(\lambda)}{f(\lambda)+\kappa}\mathrm{Ai}(e^{-i\alpha}w_y)\,, \tag{6.12}$$

and

$$C^+ = 2\pi\mathrm{Ai}(e^{i\alpha}w_y) - 4\pi^2\frac{e^{2i\alpha}[\mathrm{Ai}'(e^{i\alpha}\lambda)]^2}{f(\lambda)+\kappa}\,\mathrm{Ai}(e^{-i\alpha}w_y)\,. \tag{6.13}$$

Combining these expressions, one finally gets

$$\mathcal{G}^-(x,y\,;\lambda,\kappa) = \mathcal{G}_0^-(x,y\,;\lambda) + \mathcal{G}_1(x,y\,;\lambda,\kappa)\,, \tag{6.14}$$

where $\mathcal{G}_0^-(x,y\,;\lambda)$ is the distribution kernel of the resolvent of the operator $\mathcal{A}_0^* := -\frac{d^2}{dx^2} - ix$ on the line (given by Eq. (3.10)), whereas $\mathcal{G}_1(x,y\,;\lambda,\kappa)$ is given by the following expressions

$$\mathcal{G}_1(x,y\,;\lambda,\kappa) = \begin{cases} -4\pi^2\dfrac{e^{2i\alpha}[\mathrm{Ai}'(e^{i\alpha}\lambda)]^2}{f(\lambda)+\kappa}\mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_y)\,, & (x>0)\,, \\ -2\pi\dfrac{f(\lambda)}{f(\lambda)+\kappa}\mathrm{Ai}(e^{i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_y)\,, & (x<0)\,, \end{cases} \tag{6.15}$$

for $y > 0$, and

$$(6.16)\quad \mathcal{G}_1(x,y\,;\lambda,\kappa) = \begin{cases} -2\pi \dfrac{f(\lambda)}{f(\lambda)+\kappa} \mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{i\alpha}w_y)\,, & (x>0)\,, \\ -4\pi^2 \dfrac{e^{-2i\alpha}[\mathrm{Ai}'(e^{-i\alpha}\lambda)]^2}{f(\lambda)+\kappa} \mathrm{Ai}(e^{i\alpha}w_x)\mathrm{Ai}(e^{i\alpha}w_y)\,, & (x<0)\,, \end{cases}$$

for $y < 0$. Hence the poles are determined by the equation

$$(6.17)\quad f(\lambda) = -\kappa\,,$$

with f defined in (6.10).

REMARK 29. *For $\kappa = 0$, one recovers the conjugated pairs associated with the zeros a_n' of Ai'. We have indeed as poles*

$$(6.18)\quad \lambda_n^+ = e^{i\alpha}a_n'\,, \quad \lambda_n^- = e^{-i\alpha}a_n'\,,$$

where a_n' is the n-th zero (starting from the right) of Ai'. Note that $a_n' < 0$ so that $\mathrm{Re}\,\lambda_n^\pm > 0$, as expected.
In this case, the restriction of $\mathcal{G}_1(x,y\,;\lambda,0)$ to $\mathbb{R}_+^2$ is the kernel of the resolvent of the Neumann problem in $\mathbb{R}_+$.

We also know that the eigenvalues for the Neumann problem are simple. Hence by the local inversion theorem we get the existence of a solution close to each $\lambda_n^\pm$ for κ small enough (possibly depending on n) if we show that $f'(\lambda_n^\pm) \neq 0$. For λ_n^+, we have, using the Wronskian relation (A.3) and $\mathrm{Ai}'(e^{-i\alpha}\lambda_n^+) = 0\,,$

$$(6.19)\quad \begin{aligned} f'(\lambda_n^+) &= 2\pi\, e^{-i\alpha}\ \mathrm{Ai}''(e^{-i\alpha}\lambda_n^+)\mathrm{Ai}'(e^{i\alpha}\lambda_n^+) \\ &= 2\pi e^{-2i\alpha}\lambda_n^+\mathrm{Ai}(e^{-i\alpha}\lambda_n^+)\mathrm{Ai}'(e^{i\alpha}\lambda_n^+) \\ &= -i\lambda_n^+\,. \end{aligned}$$

Similar computations hold for λ_n^-. We recall that

$$\lambda_n^+ = \overline{\lambda_n^-}\,.$$

The above argument shows that $f'(\lambda_n) \neq 0$, with $\lambda_n = \lambda_n^+$ or $\lambda_n = \lambda_n^-$. Hence by the holomorphic inversion theorem we get that, for any $n \in \mathbb{N}^*$ (with $\mathbb{N}^* = \mathbb{N}^* \setminus \{0\}$), and any ϵ, there exists $h_n(\epsilon)$ such that for $|\kappa| \leq h_n(\epsilon)$, we have a unique solution $\lambda_n(\kappa)$ of (6.17) such that $|\lambda_n(\kappa) - \lambda_n| \leq \epsilon$.

We would like to have a control of $h_n(\epsilon)$ with respect to n. What we should do is inspired by the Taylor expansion given in [23] (Formula (33)) of $\lambda_n^\pm(\kappa)$ for fixed n:

$$(6.20)\quad \lambda_n^\pm(\kappa) = \lambda_n^\pm + e^{\pm i\frac{\pi}{6}}\frac{1}{a_n'}\kappa + \mathcal{O}_n(\kappa^2)\,.$$

Since $|\lambda_n|$ behaves as $n^{\frac{2}{3}}$ (see Appendix A), the guess is that $\lambda_{n+1}^\pm(\kappa) - \lambda_n^\pm(\kappa)$ behaves as $n^{-\frac{1}{3}}$.
To justify this guess, one needs to control the derivative in a suitable neighborhood of λ_n.

PROPOSITION 30. *There exist $\eta > 0$ and $h_\infty > 0$, such that, for all $n \in \mathbb{N}^*$, for any κ such that $|\kappa| \leq h_\infty$ there exists a unique solution of (6.17) in $B(\lambda_n, \eta|\lambda_n|^{-1})$ with $\lambda_n = \lambda_n^\pm$.*

Proof of the proposition
Using the previous arguments, it is enough to establish the proposition for n large enough. Hence it remains to establish a local inversion theorem uniform with respect to n for $n \geq N$. For this purpose, we consider the holomorphic function

$$B(0,\eta) \ni t \mapsto \phi_n(t) = f(\lambda_n + t\lambda_n^{-1})\,.$$

To have a local inversion theorem uniform with respect to n, we need to control $|\phi_n'(t)|$ from below.

LEMMA 31. *For any $\eta > 0$, there exists N such that, $\forall n \geq N$,*

$$|\phi_n'(t)| \geq \frac{1}{2}\,, \quad \forall t \in B(0,\eta)\,. \tag{6.21}$$

Proof of the lemma
We have

$$\phi_n'(t) = \lambda_n^{-1} f'(\lambda_n + t\lambda_n^{-1})\,,$$

and

$$\phi_n'(0) = -i\,.$$

Hence it remains to control $\phi_n'(t) - \phi_n'(0)$ in $B(0,\eta)$. We treat the case $\lambda_n = \lambda_n^+$.
We recall that

$$\begin{aligned} f'(\lambda) &= 2\pi e^{-i\alpha}\mathrm{Ai}''(e^{-i\alpha}\lambda)\mathrm{Ai}'(e^{i\alpha}\lambda) + 2\pi e^{i\alpha}\mathrm{Ai}'(e^{-i\alpha}\lambda)\mathrm{Ai}''(e^{i\alpha}\lambda) \\ &= 2\pi\lambda\left(e^{-2i\alpha}\mathrm{Ai}(e^{-i\alpha}\lambda)\mathrm{Ai}'(e^{i\alpha}\lambda) + e^{2i\alpha}\mathrm{Ai}'(e^{-i\alpha}\lambda)\mathrm{Ai}(e^{i\alpha}\lambda)\right) \\ &= -i\lambda + 4\pi\lambda e^{2i\alpha}\mathrm{Ai}'(e^{-i\alpha}\lambda)\mathrm{Ai}(e^{i\alpha}\lambda). \end{aligned} \tag{6.22}$$

Hence we have

$$\phi_n'(t) - \phi_n'(0) = 4\pi\lambda\lambda_n^{-1}e^{2i\alpha}\,\mathrm{Ai}'(e^{-i\alpha}\lambda)\mathrm{Ai}(e^{i\alpha}\lambda) - it\lambda_n^{-2}\,, \tag{6.23}$$

with $\lambda = \lambda_n + t\lambda_n^{-1}$.

The last term in (6.22) tends to zero. It remains to control $\mathrm{Ai}'(e^{-i\alpha}\lambda)\mathrm{Ai}(e^{i\alpha}\lambda)$ in $B(\lambda_n, \eta|\lambda_n|^{-1})$ and to show that this expression tends to zero as $n \to +\infty$.
We have

$$\mathrm{Ai}'(e^{-i\alpha}\lambda) = e^{-i\alpha}(\lambda - \lambda_n)\mathrm{Ai}''(e^{-i\alpha}\tilde{\lambda}) = e^{-2i\alpha}(\lambda - \lambda_n)\,\tilde{\lambda}\,\mathrm{Ai}(e^{-i\alpha}\tilde{\lambda})\,,$$

with $\tilde{\lambda} \in B(\lambda_n, \eta|\lambda_n|^{-1})$.

Hence it remains to show that the product $|\mathrm{Ai}(e^{-i\alpha}\tilde{\lambda})\mathrm{Ai}(e^{i\alpha}\lambda)|$ for λ and $\tilde{\lambda}$ in $B(\lambda_n, \eta|\lambda_n|^{-1})$ tends to 0. For this purpose, we will use the known expansion for the Airy function (recalled in Appendix A) in the balls $B(e^{-i\alpha}\lambda_n, \eta|\lambda_n|^{-1})$ and $B(e^{i\alpha}\lambda_n, \eta|\lambda_n|^{-1})$.

(i) For the factor $|\mathrm{Ai}(e^{-i\alpha}\tilde{\lambda})|$, we need the expansion of $\mathrm{Ai}(z)$ for z in a neighborhood of a_n' of size $C|\lambda_n|^{-1}$. Using the asymptotic relation (A.7), we observe that

$$\exp\left(\pm i\frac{2}{3}z^{\frac{3}{2}}\right) = \exp\left(\pm i\left(\frac{2}{3}(-a_n')^{\frac{3}{2}}(1 + \mathcal{O}(1/|a_n'|^2))\right)\right) = \mathcal{O}(1)\,.$$

Hence we get

$$|\mathrm{Ai}(e^{-i\tilde{\alpha}}\lambda)| \leq C\,|a_n'|^{-\frac{1}{4}} \quad \forall\, \tilde{\lambda} \in B(\lambda_n, \eta|\lambda_n|^{-1})\,.$$

(ii) For the factor $|\mathrm{Ai}(e^{i\alpha}\lambda)|$, we use (A.5) to observe that

$$\exp\left(-\frac{2}{3}(e^{i\alpha}\lambda)^{\frac{3}{2}}\right) = \exp\left(-i\frac{2}{3}(-a_n')^{\frac{3}{2}}(1+\mathcal{O}((-a_n')^{-2}))\right)\,,$$

and we get, for any $\lambda \in B(\lambda_n, \eta|\lambda_n|^{-1})$

$$|\mathrm{Ai}(e^{i\alpha}\lambda)| \leq C\,|a_n'|^{-\frac{1}{4}}\,. \tag{6.24}$$

This completes the proof of the lemma and of the proposition.
Actually, we have proved on the way the more precise

Proposition 32. *For all $\eta > 0$ and $0 \leq \kappa < \frac{\eta}{2}$, there exists N such that, for all $n \geq N$, there exists a unique solution of (6.17) in $B(\lambda_n, \eta\,|\lambda_n|^{-1})$.*

Figure 1 illustrates Proposition 30. Solving Eq. (6.17) numerically, we find the first 100 zeros $\lambda_n(\kappa)$ with $\mathrm{Im}\,\lambda_n(\kappa) > 0$. According to Proposition 30, these zeros are within distance $1/|\lambda_n|$ from the zeros $\lambda_n = \lambda_n(0) = e^{i\alpha}a_n'$ which are given explicitly through the zeros a_n'. Moreover, the second order term in (6.20) that was computed in [23], suggests that the rescaled distance

$$\delta_n(\kappa) = |\lambda_n(\kappa) - \lambda_n||\lambda_n|/\kappa, \tag{6.25}$$

behaves as

$$\delta_n(\kappa) = 1 - c\kappa n^{-\frac{1}{3}} + o(n^{-\frac{1}{3}}), \tag{6.26}$$

with a nonzero constant c. Figure 1(top) shows that the distance $\delta_n(\kappa)$ remains below 1 for three values of κ: 0.1, 1, and 10. The expected asymptotic behavior given in (6.26) is confirmed by Figure 1(bottom), from which the constant c is estimated to be around 0.31.

Remark 33. *The local inversion theorem with control with respect to n permits to have the asymptotic behavior of the $\lambda_n(\kappa)$ uniformly for κ small:*

$$\lambda_n^{\pm}(\kappa) = \lambda_n^{\pm} + e^{\pm i\frac{\pi}{6}}\frac{1}{a_n'}\kappa + \frac{1}{a_n'}\mathcal{O}(\kappa^2)\,. \tag{6.27}$$

An improvement of (6.27) (as formulated by (6.26)) results from a good estimate on $\phi_n''(t)$. Observing that $|\phi_n''(t)| \leq C|a_n'|^{-\frac{1}{2}}$ in the ball $B(0,\eta)$, we obtain

$$\lambda_n^{\pm}(\kappa) = \lambda_n^{\pm} + e^{\pm i\frac{\pi}{6}}\frac{1}{a_n'}\kappa + \frac{1}{|a_n'|^{\frac{3}{2}}}\mathcal{O}(\kappa^2)\,. \tag{6.28}$$

If one needs finer estimates, one can compute $\phi_n''(0)$ and estimate ϕ_n''', and so on.

It would also be interesting to analyze the case $\kappa \to +\infty$. The limiting problem in this case is the realization of the complex Airy operator on the line which has empty spectrum. See [23] for a preliminary non rigorous analysis.

In the remaining part of this section, we describe the distribution kernel of the projector $\Pi_n^{\pm}$ associated with $\lambda_n^{\pm}(\kappa)$.

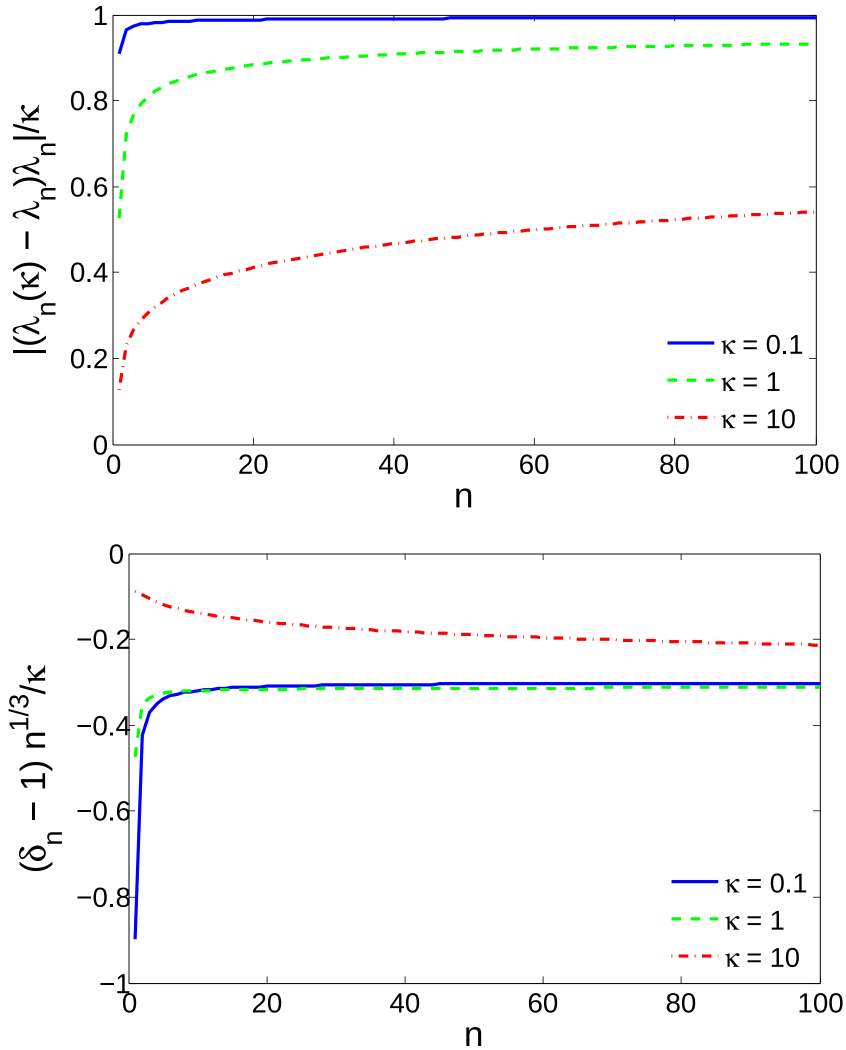

FIG. 1. *Illustration of Proposition 30 by the numerical computation of the first 100 zeros $\lambda_n^+(\kappa)$ of (6.17). At the top, the rescaled distance $\delta_n(\kappa)$ from (6.25) between $\lambda_n^+(\kappa)$ and $\lambda_n^+ = \lambda_n^+(0)$. At the bottom, the asymptotic behavior of this distance.*

PROPOSITION 34. *There exists $\kappa_0 > 0$ such that, for any $\kappa \in [0, \kappa_0]$ and any $n \in \mathbb{N}^*$, the rank of $\Pi_n^\pm$ is equal to one. Moreover, if $\psi_n^\pm$ is an eigenfunction, then*

$$\int_{-\infty}^{+\infty} \psi_n^\pm(x)^2\, dx \neq 0\,. \tag{6.29}$$

Proof
To write the projector $\Pi_n^\pm$ associated with an eigenvalue $\lambda_n^\pm$ we integrate the resolvent along a small contour $\gamma_n^\pm$ around $\lambda_n^\pm$:

$$\Pi_n^\pm = \frac{1}{2i\pi} \int_{\gamma_n^\pm} (\mathcal{A}_1^\pm - \lambda)^{-1} d\lambda\,. \tag{6.30}$$

If we consider the associated kernels, we get, using (6.14) and the fact that $\mathcal{G}_0^-$ is holomorphic in λ:

$$\Pi_n^\pm(x,y\,;\kappa) = \frac{1}{2i\pi}\int_{\gamma_n^\pm} \mathcal{G}_1(x,y\,;\lambda,\kappa)\,d\lambda\,. \tag{6.31}$$

The projector is given by the following expression (with $w_x^{\pm,n} = ix + \lambda_n^\pm$) for $y > 0$

$$\Pi_n^\pm(x,y\,;\kappa) = \begin{cases} -4\pi^2 \dfrac{e^{2i\alpha}[\mathrm{Ai}'(e^{i\alpha}\lambda_n^\pm)]^2}{f'(\lambda_n^\pm)} \mathrm{Ai}(e^{-i\alpha}w_x^{\pm,n})\mathrm{Ai}(e^{-i\alpha}w_y^{\pm,n}) & (x>0)\,, \\ 2\pi \dfrac{\kappa}{f'(\lambda_n^\pm)} \mathrm{Ai}(e^{i\alpha}w_x^{\pm,n})\mathrm{Ai}(e^{-i\alpha}w_y^{\pm,n}) & (x<0)\,, \end{cases} \tag{6.32}$$

and for $y < 0$

$$\Pi_n^\pm(x,y\,;\kappa) = \begin{cases} 2\pi \dfrac{\kappa}{f'(\lambda_n^\pm)} \mathrm{Ai}(e^{-i\alpha}w_x^{\pm,n})\mathrm{Ai}(e^{i\alpha}w_y^{\pm,n}) & (x>0)\,, \\ -4\pi^2 \dfrac{e^{-2i\alpha}[\mathrm{Ai}'(e^{-i\alpha}\lambda_n^\pm)]^2}{f'(\lambda_n^\pm)} \mathrm{Ai}(e^{i\alpha}w_x^{\pm,n})\mathrm{Ai}(e^{i\alpha}w_y^{\pm,n}) & (x<0)\,. \end{cases} \tag{6.33}$$

Here we recall that we have established that for $|\kappa|$ small enough $f'(\lambda_n^\pm) \neq 0$. It remains to show that the rank of $\Pi_n^\pm$ is one that will yield an expression for the eigenfunction. It is clear from (6.32) and (6.33) that the rank of $\Pi_n^\pm$ is at most two and that every function in the range of $\Pi_n^\pm$ has the form $(c_-\mathrm{Ai}(e^{i\alpha}w_x^{\pm,n})\,,\, c_+\mathrm{Ai}(e^{-i\alpha}w_x^{\pm,n}))\,,$ where $c_-, c_+ \in \mathbb{R}\,.$ It remains to establish the existence of a relation between c_- and c_+. This is directly obtained by using the first part of the transmission condition. If $\kappa \neq 0$, the functions in the range of $\Pi_n^\pm$ have the form

$$c_n\left(\mathrm{Ai}'(e^{-i\alpha}\lambda_n^\pm)\mathrm{Ai}(e^{i\alpha}w_x^{\pm,n}), e^{2i\alpha}\mathrm{Ai}'(e^{i\alpha}\lambda_n^\pm)\mathrm{Ai}(e^{-i\alpha}w_x^{\pm,n})\right)\,,$$

with $c_n \in \mathbb{C}$. Inequality (6.29) results from an abstract lemma in [7] once we have proved that the rank of the projector is one. We have indeed

$$||\Pi_n^\pm|| = \frac{1}{|\int_{-\infty}^{+\infty} \psi_n^\pm(x)^2\,dx|}\,. \tag{6.34}$$

More generally, what we have proven can be formulated in this way:

Proposition 35. *If $f(\lambda) + \kappa = 0$ and $f'(\lambda) \neq 0\,,$ then the associated projector has rank 1 (no Jordan block).*

The condition of κ being small in Proposition 34 is only used for proving the property $f'(\lambda) \neq 0$. For the case of the Dirichlet or Neumann realization of the complex Airy operator in $\mathbb{R}_+$, we refer to Section 3. The nonemptiness was obtained directly by using the properties of the Airy function. Note that our numerical solutions did not reveal projectors of rank higher than 1. We conjecture that the rank of these projectors is 1 for any $0 \le \kappa < +\infty$ but we could only prove the weaker

Proposition 36. *For any $\kappa \ge 0$, there is at most a finite number of eigenvalues with nontrivial Jordan blocks.*

Proof

We start from

$$f(\lambda) := 2\pi\,\mathrm{Ai}'(e^{i\alpha}\lambda)\,\mathrm{Ai}'(e^{-i\alpha}\lambda)\,,$$

and get by derivation

$$\frac{1}{2\pi} f'(\lambda) = e^{i\alpha}\mathrm{Ai}''(e^{i\alpha}\lambda)\mathrm{Ai}'(e^{-i\alpha}\lambda) + e^{-i\alpha}\mathrm{Ai}'(e^{i\alpha}\lambda)\mathrm{Ai}''(e^{-i\alpha}\lambda)\,. \tag{6.35}$$

What we have to prove is that $f'(\lambda)$ is different from 0 for a large solution λ of $f(\lambda) = -\kappa$. We know already that $\mathrm{Re}\,\lambda \geq 0$. We note that $f(0) > 0$. Hence 0 is not a pole for $\kappa \geq 0$. More generally f is real and strictly positive on the real axis. Hence $f(\lambda) + \kappa > 0$ on the real axis.
We can assume that $\mathrm{Im}\,\lambda > 0$ (the other case can be treated similarly). Using the equation satisfied by the Airy function, we get

$$\frac{1}{2\pi\lambda} f'(\lambda) = e^{-i\alpha}\mathrm{Ai}(e^{i\alpha}\lambda)\mathrm{Ai}'(e^{-i\alpha}\lambda) + e^{i\alpha}\mathrm{Ai}'(e^{i\alpha}\lambda)\mathrm{Ai}(e^{-i\alpha}\lambda)\,, \tag{6.36}$$

and by the Wronskian relation (A.3):

$$e^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}\lambda)\mathrm{Ai}(e^{i\alpha}\lambda) - e^{i\alpha}\mathrm{Ai}'(e^{i\alpha}\lambda)\mathrm{Ai}(e^{-i\alpha}\lambda) = \frac{i}{2\pi}\,. \tag{6.37}$$

Suppose that $f(\lambda) = -\kappa$ and that $f'(\lambda) = 0$.
We have

$$-e^{i\alpha}\mathrm{Ai}'(e^{i\alpha}\lambda)\mathrm{Ai}(e^{-i\alpha}\lambda) = e^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}\lambda)\mathrm{Ai}(e^{i\alpha}\lambda) = \frac{i}{4\pi}\,.$$

and get

$$\kappa = -\frac{ie^{i\alpha}}{2}\,\frac{\mathrm{Ai}'(e^{i\alpha}\lambda)}{\mathrm{Ai}(e^{i\alpha}\lambda)} = \frac{ie^{-i\alpha}}{2}\,\frac{\mathrm{Ai}'(e^{-i\alpha}\lambda)}{\mathrm{Ai}(e^{-i\alpha}\lambda)}\,.$$

Using the last equality and the asymptotics (A.5), (A.6) for Ai and Ai$'$, we get as $|\lambda| \to +\infty$ satisfying the previous condition

$$\kappa \sim \frac{1}{2}|\lambda|^{\frac{1}{2}}\,,$$

which cannot be true for λ large. This completes the proof of the proposition.

7. Resolvent estimates as $|\mathrm{Im}\,\lambda| \to +\infty$. The resolvent estimates have been already proved in Section 5 and were used in the analysis of the decay of the associated semigroup. We propose here another approach which leads to more precise results. We keep in mind (6.14) and the discussion in Section 5.
For $\lambda = \lambda_0 + i\eta$, we have

$$\|\mathcal{G}_0^-(\,\cdot\,,\,\cdot\,;\lambda)\|_{L^2(\mathbb{R}^2)} = \|\mathcal{G}_0^-(\,\cdot\,,\,\cdot\,;\lambda_0)\|_{L^2(\mathbb{R}^2)}\,.$$

Hence the Hilbert-Schmidt norm of the resolvent $(\mathcal{A}^+ - \lambda)^{-1}$ does not depend on the imaginary part of λ.
As a consequence, to recover Lemma 27 by this approach, it only remains to check the following lemma

LEMMA 37. *For any λ_0, there exist $C > 0$ and $\eta_0 > 0$ such that*

$$\sup_{|\eta|>\eta_0} \|\mathcal{G}_1(\cdot\,,\,\cdot;\lambda_0 + i\eta)\|_{L^2(\mathbb{R}^2)} \leq C\,. \tag{7.1}$$

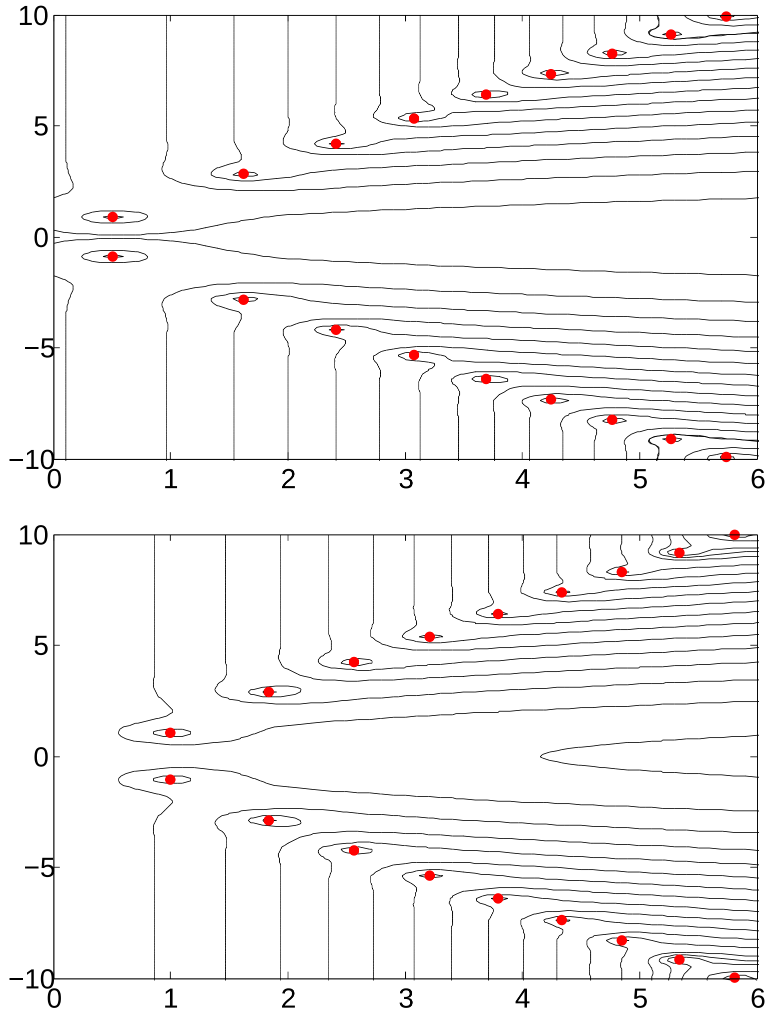

Fig. 2. *Numerically computed pseudospectrum in the complex plane of the complex Airy operator with Neumann boundary conditions (top) and with the transmission boundary condition at the origin with $\kappa = 1$ (bottom). The red points show the poles $\lambda_n^\pm(\kappa)$ found by solving numerically Eq. (6.17) that corresponds to the original problem on $\mathbb{R}$. The presented picture corresponds to a zoom (eliminating numerical artefacts) in a computation done for a large interval $[-L, +L]$ with the transmission condition at the origin and Dirichlet boundary conditions at $\pm L$. The pseudospectrum was computed for $L^3 = 10^4$ by projecting the complex Airy operator onto the orthogonal basis of eigenfunctions of the corresponding Laplace operator and then diagonalizing the obtained truncated matrix representation (see Appendix E for details). We only keep a few lines of pseudospectra for the clarity of the picture. As predicted by the theory, the vertical lines are related to the pseudospectrum of the free complex Airy operator on the line.*

The proof is included in the proof of the following improvement which is the main result of this section and is confirmed by the numerical computations. One indeed observes that the lines of the pseudospectrum are asymptotically vertical as $\mathrm{Im}\,\lambda \to \pm\infty$ when $\mathrm{Re}\,\lambda > 0$, see Figure 2.

PROPOSITION 38. *For any $\lambda_0 > 0$,*

$$\lim_{\eta\to\pm\infty} \|\mathcal{G}_1(\cdot\,,\,\cdot;\lambda_0+i\eta)\|_{L^2(\mathbb{R}^2)} = 0\,.$$

Moreover, this convergence is uniform for λ_0 in a compact set.

Proof

We have

$$e^{i\alpha}\lambda = e^{i\alpha}\lambda_0 - e^{i\pi/6}\eta$$

and

$$e^{-i\alpha}\lambda = e^{-i\alpha}\lambda_0 + e^{-i\pi/6}\eta\,.$$

Then according to (A.6), one can easily check that the term $\mathrm{Ai}'(e^{\pm i\alpha}\lambda)$ decays exponentially as $\eta\to\mp\infty$ and grows exponentially as $\eta\to\pm\infty$. On the other hand, the term $\mathrm{Ai}'(e^{i\alpha}\lambda)$ decays exponentially as $\eta\to\pm\infty$.

More precisely, we have

$$\begin{aligned}
|\mathrm{Ai}'(e^{i\alpha}(\lambda_0+i\eta))|^2 &\sim |c|^2\eta^{\frac12}\exp\left(\tfrac{2\sqrt2}{3}\eta^{\frac32}\right)\,,\ \text{ as } \eta\to+\infty\,;\\
&\sim |c|^2(-\eta)^{\frac12}\exp\left(-\tfrac{2\sqrt2}{3}\eta^{\frac32}\right)\,,\ \text{ as } \eta\to-\infty\,;\\
|\mathrm{Ai}'(e^{-i\alpha}(\lambda_0+i\eta))|^2 &\sim |c|^2\eta^{\frac12}\exp\left(-\tfrac{2\sqrt2}{3}\eta^{\frac32}\right)\,,\ \text{ as } \eta\to+\infty\,;\\
&\sim |c|^2(-\eta)^{\frac12}\exp\left(\tfrac{2\sqrt2}{3}\eta^{\frac32}\right)\,,\ \text{ as } \eta\to-\infty\,.
\end{aligned}\tag{7.2}$$

As a consequence, the function $f(\lambda)$, which was defined in (6.10) by

$$f(\lambda) := 2\pi\mathrm{Ai}'(e^{-i\alpha}\lambda)\mathrm{Ai}'(e^{i\alpha}\lambda)\,,$$

has the following asymptotic behavior as $\eta\to\mp\infty$:

$$f(\lambda_0+i\eta) = 2\pi|c|^2\,|\eta|^{\frac12}\big(1+o(1)\big)\,.\tag{7.3}$$

We treat the case $\eta > 0$ (the other case can be deduced by considering the complex conjugate).

Coming back to the two formulas giving $\mathcal{G}_1$ in (6.15) and (6.16) and starting with the first one, we have to analyze the L^2 norm over $\mathbb{R}_+\times\mathbb{R}_+$ of

$$(x,y)\mapsto -4\pi^2\frac{e^{2i\alpha}[\mathrm{Ai}'(e^{i\alpha}\lambda)]^2}{f(\lambda)+\kappa}\mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_y)\,.$$

This norm N_1 is given by

$$N_1 := 4\pi^2\ |\mathrm{Ai}'(e^{i\alpha}\lambda)|^2\ |f(\lambda)+\kappa|^{-1}\ ||\mathrm{Ai}(e^{-i\alpha}w_x)||^2_{L^2(\mathbb{R}_+)}\,.$$

Hence we have to estimate $\int_0^{+\infty}|\mathrm{Ai}(e^{-i\alpha}w_x)|^2dx$. We observe that

$$e^{-i\alpha}w_x = e^{-i\frac{\pi}{6}}(x+\eta) + e^{-i\alpha}\lambda_0\,,$$

and that the argument of $e^{-i\alpha}w_x$ is very close to $-\frac{\pi}{6}$ as $\eta\to+\infty$ (uniformly for $x>0$). This is rather simple for $\eta>0$ because x and η have the same sign. We can use the asymptotics (A.5) (with $z = e^{-i\alpha}w_x$) in order to get

$$\int_0^{+\infty}|\mathrm{Ai}(e^{-i\alpha}w_x)|^2dx \le C\,(|\eta|^2+1)^{-\frac12}\,\exp\left(-\frac{2\sqrt2}{3}|\eta|^{\frac32}\right)\,.\tag{7.4}$$

Here we have used that, for $\beta > 0$,

$$\int_{\eta}^{+\infty} \exp\left(-\beta y^{\frac{3}{2}}\right) dy = \frac{2}{3\beta} \exp\left(-\beta \eta^{\frac{3}{2}} (1 + \mathcal{O}(|\eta|^{-\frac{1}{2}}))\right) .$$

The control of $|\mathrm{Ai}'(e^{i\alpha}(\lambda_0 + i\eta))|^2$ given in (7.2) and (7.3) finally yields

$$N_1 \lesssim (|\eta|^2 + 1)^{-\frac{1}{2}} . \tag{7.5}$$

By the notation $\lesssim$, we mean that there exists a constant C such that

$$N_1 \leq C(|\eta|^2 + 1)^{-\frac{1}{2}} .$$

For the L^2-norm of the second term (see (6.16)),

$$N_2 := \left|\left| -2\pi \frac{f(\lambda)}{f(\lambda) + \kappa} \mathrm{Ai}(e^{i\alpha} w_x) \mathrm{Ai}(e^{-i\alpha} w_y) \right|\right|_{L^2(\mathbb{R}_x^- \times \mathbb{R}_y^+)} ,$$

we observe that

$$N_2 \lesssim ||\mathrm{Ai}(e^{i\alpha} w_x)||_{L^2(\mathbb{R}_-)} \; ||\mathrm{Ai}(e^{-i\alpha} w_x)||_{L^2(\mathbb{R}_+)} ,$$

and having in mind (7.4), we have only to bound $\int_{-\infty}^{0} |\mathrm{Ai}(e^{i\alpha} w_x)|^2 dx$. We can no more use the asymptotic for the Airy function as $(x + \eta)$ is small. We have indeed

$$e^{i\alpha} w_x = -e^{i\frac{\pi}{6}} (x + \eta) + e^{i\alpha} \lambda_0 .$$

We rewrite the integral as the sum

$$\begin{aligned} \int_{-\infty}^{0} |\mathrm{Ai}(e^{i\alpha} w_x)|^2 dx &= \int_{-\infty}^{-\eta - C} |\mathrm{Ai}(e^{i\alpha} w_x)|^2 dx \\ &\quad + \int_{-\eta - C}^{-\eta + C} |\mathrm{Ai}(e^{i\alpha} w_x)|^2 dx + \int_{-\eta + C}^{0} |\mathrm{Ai}(e^{i\alpha} w_x)|^2 dx . \end{aligned}$$

The integral in the middle of the r.h.s. is bounded. The first one is also bounded according to the behavior of the Airy function. So the dominant term is the third one

$$\begin{aligned} \int_{-\eta + C}^{0} |\mathrm{Ai}(e^{i\alpha} w_x)|^2 dx &= \int_{C}^{\eta} |\mathrm{Ai}(-e^{i\frac{\pi}{6}} x + e^{i\alpha} \lambda_0)|^2 dx \\ &\leq \tilde{C}(|\eta|^2 + 1)^{\frac{1}{4}} \; \exp\left(+ \frac{2\sqrt{2}}{3} |\eta|^{\frac{3}{2}} \right) . \end{aligned}$$

Combining with (7.4), the L^2-norm of the second term decays as $\eta \to +\infty$:

$$N_2 \lesssim (|\eta|^2 + 1)^{-\frac{1}{8}} . \tag{7.6}$$

This achieves the proof of the proposition, the uniformity for λ_0 in a compact being controlled at each step of the proof.

8. Proof of the completeness. We have already recalled or established in Section 3 (Propositions 10, 14, and 17) the results for the Dirichlet, Neumann or Robin realization of the complex Airy operator in $\mathbb{R}_+$. The aim of this section is to establish the same result in the case with transmission. The new difficulty is that the operator is no longer sectorial.

8.1. Reduction to the case $\kappa = 0$. We first reduce the analysis to the case $\kappa = 0$ by comparison of the two kernels. We have indeed

$$\begin{aligned}\mathcal{G}^-(x,y\,;\lambda,\kappa) - \mathcal{G}^-(x,y\,;\lambda,0) &= \mathcal{G}_1(x,y\,;\lambda,\kappa) - \mathcal{G}_1(x,y\,;\lambda,0)\\ &= -\kappa(f(\lambda)+\kappa)^{-1}\mathcal{G}_1(x,y\,;\lambda,0)\,,\end{aligned} \tag{8.1}$$

where $\mathcal{G}^-(x,y\,;\lambda,\kappa)$ denotes the kernel of the resolvent for the transmission problem associated to $\kappa \geq 0$ and $D_x^2 - ix$.
We will also use the alternative equivalent relation:

$$\mathcal{G}^-(x,y\,;\lambda,\kappa) = \mathcal{G}^-(x,y\,;\lambda,0)f(\lambda)\,(f(\lambda)+\kappa)^{-1} + \kappa(f(\lambda)+\kappa)^{-1}\,\mathcal{G}_0^-(x,y\,;\lambda,0)\,. \tag{8.2}$$

REMARK 39. *This formula gives another way for proving that the operator with kernel $\mathcal{G}^\pm(x,y\,;\lambda,\kappa)$ is in a suitable Schatten class (see Proposition 21). It is indeed enough to have the result for $\kappa = 0$, that is to treat the Neumann case on the half line.*

Another application of this formula is

PROPOSITION 40. *There exists $M > 0$ such that for all $\lambda > 0$,*

$$\|(\mathcal{A}_1^\pm - \lambda)^{-1}\|_{HS} \leq M(1+\lambda)^{-\frac{1}{4}}(\log\lambda)^{\frac{1}{2}}\,. \tag{8.3}$$

Proof
Proposition 40 is a consequence of Proposition 15, and Formula (8.1).

REMARK 41. *Similar estimates are obtained in the case without boundary (typically for a model like the Davies operator $D_x^2 + ix^2$) by Dencker-Sjöstrand-Zworski [14] or more recently by Sjöstrand [35].*

8.2. Estimate for $f(\lambda)$. We recall that $f(\lambda)$ was defined in (6.10) by

$$f(\lambda) := 2\pi \mathrm{Ai}'(e^{-i\alpha}\lambda)\mathrm{Ai}'(e^{i\alpha}\lambda)\,.$$

Recalling the asymptotic expansions (A.6) and (A.8) of Ai', it is immediate to get

LEMMA 42. *The function $\lambda \mapsto f(\lambda)$ is an entire function of type $\frac{3}{2}$, i.e. there exists $D > 0$ such that*

$$|f(\lambda)| \leq D\exp\big(D|\lambda|^{\frac{3}{2}}\big)\,, \quad \forall \lambda \in \mathbb{C}\,. \tag{8.4}$$

Focusing now on the main purpose of this section, we get from (A.6) that for any $\epsilon > 0$ there exists $\lambda_1 > 0$ such that, for $\lambda \geq \lambda_1$,

$$|\mathrm{Ai}'(e^{i\alpha}\lambda)|^2 = |\mathrm{Ai}'(e^{-i\alpha}\lambda)|^2 \geq \frac{1-\epsilon}{4\pi}\lambda^{\frac{1}{2}}\exp\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right)\,. \tag{8.5}$$

Here we have also used that

$$\mathrm{Ai}(z) = \overline{\mathrm{Ai}(\bar{z})} \quad \text{and} \quad \mathrm{Ai}'(z) = \overline{\mathrm{Ai}'(\bar{z})} \tag{8.6}$$

(note that $\mathrm{Ai}(x)$ is real for x real). Thus there exists $C_1 > 0$ such that, for $\lambda \geq 1$,

$$\frac{1}{|f(\lambda)|} \leq \frac{C_1}{\lambda^{\frac{1}{2}}}\exp\left(-\frac{4}{3}\lambda^{\frac{3}{2}}\right)\,. \tag{8.7}$$

8.3. Estimate of the L^2 norm of $\mathcal{G}_1(\cdot\,,\,\cdot\,;\lambda,0)$. Having in mind (6.15)-(6.16) and noting that, for $\lambda > 0$,

$$\frac{|\mathrm{Ai}'(e^{i\alpha}\lambda)|^2}{|f(\lambda)|} = \frac{1}{2\pi}\,,$$

it is enough to estimate

$$\int_0^{+\infty} |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2 dx = I_0(\lambda) = \int_{-\infty}^0 |\mathrm{Ai}(e^{i\alpha}(ix+\lambda))|^2 dx\,. \tag{8.8}$$

It is enough to observe from (3.10), (8.6) and the comparison of the domain of integration in $\mathbb{R}^2$, that

$$2I_0(\lambda)^2 \leq ||\mathcal{G}_0^-(\cdot\,,\,\cdot\,;\lambda)||^2\,. \tag{8.9}$$

Applying (3.7), we get

$$I_0(\lambda) \lesssim \lambda^{-\frac{1}{4}} \exp\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right)\,. \tag{8.10}$$

Hence, coming back to (8.1), we have obtained

Proposition 43. *There exist κ_0, C and $\lambda_0 > 0$ such that, for all $\kappa \in [0,\kappa_0]$, for all $\lambda \geq \lambda_0$,*

$$||\mathcal{G}^-(\cdot\,,\,\cdot\,;\lambda,\kappa)\, - \,\mathcal{G}^-(\cdot\,,\,\cdot\,;\lambda,0)||_{L^2(\mathbb{R}^2)} \leq C\kappa\,|\lambda|^{-\frac{3}{4}}\,. \tag{8.11}$$

Hence we are reduced to the case $\kappa = 0$ which can be decoupled (see Remark 29) in two Neumann problems on $\mathbb{R}_-$ and $\mathbb{R}_+$.

Using (8.2) and the estimates established for $\mathcal{G}_0^-(\cdot\,,\,\cdot\,;\lambda,0)$ (which depends only on $\mathrm{Re}\,\lambda$) (see (3.7) or (3.5)), we have

Proposition 44. *For all κ_0, there exist C and $\lambda_0 > 0$ such that, for all $\kappa \in [0,\kappa_0]$, for all real $\lambda \geq \lambda_0$, one has*

$$||\mathcal{G}^-(\cdot\,,\,\cdot\,;\lambda,\kappa)\, - \,(f(\lambda)(f(\lambda)+\kappa)^{-1})\mathcal{G}^-(\cdot\,,\,\cdot\,;\lambda,0)||_{L^2(\mathbb{R}^2)} \leq C\kappa\,|\lambda|^{-\frac{3}{4}}\,. \tag{8.12}$$

This immediately implies

Proposition 45. *For any $g = (g_-,g_+)$, $h = (h_-,h_+)$ in $L^2_- \times L^2_+$, we have*

$$|\langle \mathcal{G}^-(\lambda,\kappa)g\,,\,h\rangle\, - \,(f(\lambda)(f(\lambda)+\kappa)^{-1})\langle \mathcal{G}^-(\lambda,0)g\,,\,h\rangle| \leq C(g,h)\kappa\,|\lambda|^{-\frac{3}{4}}\,, \tag{8.13}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in the Hilbert space $L^2_- \times L^2_+$.

We now adapt the proof of the completeness from [2].
If we denote by E the closed space generated by the generalized eigenfunctions of $\mathcal{A}_1^-$, the proof of [2] in the presentation of [28] consists in introducing

$$F(\lambda) = \langle \mathcal{G}^-(\lambda,\kappa)g\,,\,h\rangle,$$

where

$$h \in E^\perp \quad \text{and} \quad g \in L^2_- \times L^2_+. \tag{8.14}$$

As a consequence of the assumption on h, one observes that $F(\lambda)$ is an entire function and the problem is to show that F is identically 0. The completeness is obtained if we prove this statement for any g and h satisfying the condition (8.14).
Outside the numerical range of $\mathcal{A}_1^-$, i.e. in the negative half-plane, it is immediate to see that $F(\lambda)$ tends to zero as $\mathrm{Re}\,\lambda \to -\infty$. If we show that $|F(\lambda)| \leq C(1+|\lambda|)^M$ for some $M > 0$ in the whole complex plane, we will get by Liouville's theorem that F is a polynomial and, with the control in the left half-plane, we should get that F is identically 0.
Hence it remains to control $F(\lambda)$ in a neighborhood of the positive half-plane $\{\lambda \in \mathbb{C}\,,\, \mathrm{Re}\,\lambda \geq 0\}$.

As in [2], we apply Phragmen-Lindelöf principle (see Appendix D). The natural idea (suggested by the numerical picture) is to control the resolvent on the positive real axis. We first recall some additional material present in Chapter 16 in [2].

THEOREM 46. *Let $\phi(\lambda)$ be an entire complex-valued function of finite order ρ (and $\phi(\lambda)$ is not identically equal to 0). Then for any $\epsilon > 0$ there exists an infinite increasing sequence $(r_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^+$ and tending to $+\infty$ such that*

$$\min_{|\lambda|=r_k} |\phi(\lambda)| > \exp(-r_k^{\rho+\epsilon})\,.$$

For this theorem (Theorem 6.2 in [2]), S. Agmon refers to the book of E.C. Titchmarsh [38] (p. 273).

This theorem is used for proving an inequality of the type ρ with $\rho = 2$ in the Hilbert-Schmidt case. We avoid an abstract lemma [2] (Lemma 16.3) but follow the scheme of its proof for controlling directly the Hilbert-Schmidt norm of the resolvent along an increasing sequence of circles.

PROPOSITION 47. *For $\epsilon > 0$, there exists an infinite increasing sequence $(r_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^+$ tending to $+\infty$ such that*

$$\max_{|\lambda|=r_k} ||\mathcal{G}^{\pm}(\cdot\,,\,\cdot\,;\lambda,\kappa)||_{HS} \leq \exp\big(r_k^{\frac{3}{2}+\epsilon}\big)\,.$$

Proof
We start from

$$\mathcal{G}^-(x,y\,;\lambda,\kappa) = \mathcal{G}^-(x,y\,;\lambda,0)f(\lambda)\,(f(\lambda)+\kappa)^{-1} + \kappa(f(\lambda)+\kappa)^{-1}\,\mathcal{G}_0^-(x,y\,;\lambda,0)\,. \tag{8.15}$$

We apply Theorem 46 with $\phi(\lambda) = f(\lambda) + \kappa$. It is proven in Lemma 42 that f is of type $\frac{3}{2}$. Hence we get for $\epsilon > 0$ (arbitrary small) the existence of a sequence $r_1 < r_2 < \cdots < r_k < \cdots$ such that

$$\max_{|\lambda|=r_k} \left|\frac{1}{f(\lambda)+\kappa}\right| \leq \exp\big(r_k^{\frac{3}{2}+\epsilon}\big)\,.$$

In view of (8.15), it remains to control the Hilbert-Schmidt norm of

$$\mathcal{G}^-(x,y\,;\lambda,0)f(\lambda)\; + \kappa\,\mathcal{G}_0^-(x,y\,;\lambda,0)\,.$$

Hence the remaining needed estimates only concern the case $\kappa = 0$. The estimate on the Hilbert-Schmidt norm of $\mathcal{G}_0^-$ is recalled in (3.7). It remains to get an estimate for the entire function $\mathcal{G}^-(x,y\,;\lambda,0)f(\lambda)$.

Because $\kappa = 0$, this is reduced to the Neumann problem on the half-line for the complex Airy operator $D_x^2 - ix$. For $y > 0$ and $x > 0$, $f(\lambda)\,\mathcal{G}_1^N(x, y\,;\lambda)$ is given by the following expression

$$f(\lambda)\,\mathcal{G}_1^N(x, y\,;\lambda) = -4\pi^2[e^{2i\alpha}\mathrm{Ai}'(e^{i\alpha}\lambda)]^2\mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_y)\,. \tag{8.16}$$

We only need the estimate for λ in a sector containing $\mathbb{R}_+ \times \mathbb{R}_+$. This is done in [28] but we will give a direct proof below. In the other region, we can first control the resolvent in $\mathcal{L}(L^2)$ and then use the resolvent identity

$$\mathcal{G}^{\pm,N}(\lambda) - \mathcal{G}^{\pm,N}(\lambda_0) = (\lambda - \lambda_0)\mathcal{G}^{\pm,N}(\lambda)\mathcal{G}^{\pm,N}(\lambda_0)\,.$$

This shows that in order to control the Hilbert-Schmidt norm of $\mathcal{G}^{\pm,N}(\lambda)$ for any λ, it is enough to control the Hilbert-Schmidt norm of $\mathcal{G}^{\pm,N}(\lambda_0)$ for some λ_0, as well as the $\mathcal{L}(L^2)$ norm of $\mathcal{G}^{\pm,N}(\lambda)$, the latter being easier to estimate.

More directly the control of the Hilbert-Schmidt norm is reduced to the existence of a constant $C > 0$ such that

$$\int_0^{+\infty} |\mathrm{Ai}(e^{-i\alpha}(ix + \lambda))|^2 dx \leq C\,\exp(C|\lambda|^{\frac{3}{2}})\,.$$

In this case, we have to control the resolvent in a neighborhood of the sector $\mathrm{Im}\,\lambda \leq 0$, $\mathrm{Re}\,\lambda \geq 0$, which corresponds to the numerical range of the operator. As $x \to +\infty$, the dominant term in the argument of the Airy function is $e^{i(-\alpha + \frac{\pi}{2})}x = e^{-i\frac{\pi}{6}}x$. As expected we arrive in a zone of the complex plane where the Airy function is exponentially decreasing. It remains to estimate for which x we enter in this zone. We claim that there exists $C > 0$ such that if $x \geq C|\lambda|$ and $|\lambda| \geq 1$, then

$$|\mathrm{Ai}(e^{-i\alpha}(ix + \lambda))| \leq C\exp(-C(x + |\lambda|)^{\frac{3}{2}})\,.$$

In the remaining zone, we obtain an upper bound of the integral by $\mathcal{O}\big(\exp(C|\lambda|^{\frac{3}{2}})\big)$.

We will then use the Phragmen-Lindelöf principle (Theorem 54). For this purpose, it remains to control the resolvent on the positive real line. It is enough to prove the theorem for $g^+ = (0, g_+)$ and $g^- = (g_-, 0)$. In other words, it is enough to consider F_+ (resp. F_-) associated with g^+ (resp. g^-).
Let us treat the case of F_+ and use Formula (8.2) and Proposition 45:

$$|\langle\mathcal{G}^-(\lambda, \kappa)g^+\,,\,h\rangle\,-\,(f(\lambda)(f(\lambda) + \kappa)^{-1})\langle\mathcal{G}^-(\lambda, 0)g_+\,,\,h_+\rangle| \leq C(g, h)\kappa\,|\lambda|^{-\frac{3}{4}}\,. \tag{8.17}$$

This estimate is true on the positive real axis. It remains to control the term $|\langle\mathcal{G}^-(\lambda, 0)g^+\,,\,h\rangle|$. Along this positive real axis, we have by Proposition 15 the decay of $F_+(\lambda)$. Using Phragmen-Lindelöf principle completes the proof.

Note that for $F_-(\lambda)$, we have to use the symmetric (with respect to the real axis) curve in $\mathrm{Im}\,\lambda > 0$.

In summary, we have obtained the following

PROPOSITION 48. *For any $\kappa \geq 0$, the space generated by the generalized eigenfunctions of the complex Airy operator on the line with transmission is dense in $L^2_- \times L^2_+$.*

Appendices.

Appendix A. Basic properties of the Airy function.

In this Appendix, we summarize the basic properties of the Airy function $\mathrm{Ai}(z)$ and its derivative $\mathrm{Ai}'(z)$ that we used (see [1] for details).
We recall that the Airy function is the unique solution of

$$(D_x^2 + x)u = 0$$

on the line such that $u(x)$ tends to 0 as $x \to +\infty$ and

$$\mathrm{Ai}(0) = \frac{1}{3^{\frac{2}{3}}\,\Gamma\left(\frac{2}{3}\right)}\,.$$

This Airy function extends into a holomorphic function in $\mathbb{C}$.

The Airy function is positive decreasing on $\mathbb{R}_+$ but has an infinite number of zeros in $\mathbb{R}_-$. We denote by a_n ($n \in \mathbb{N}$) the decreasing sequence of zeros of Ai. Similarly we denote by a_n' the sequence of zeros of Ai'. They have the following asymptotics (see for example [1]), as $n \to +\infty$,

$$a_n \underset{n\to+\infty}{\sim} -\left(\frac{3\pi}{2}(n-1/4)\right)^{\frac{2}{3}}, \tag{A.1}$$

and

$$a_n' \underset{n\to+\infty}{\sim} -\left(\frac{3\pi}{2}(n-3/4)\right)^{\frac{2}{3}}. \tag{A.2}$$

The functions $\mathrm{Ai}(e^{i\alpha}z)$ and $\mathrm{Ai}(e^{-i\alpha}z)$ (with $\alpha = 2\pi/3$) are two independent solutions of the differential equation

$$\left(-\frac{d^2}{dz^2} + z\right) w(z) = 0\,.$$

Their Wronskian reads

$$e^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}z)\mathrm{Ai}(e^{i\alpha}z) - e^{i\alpha}\mathrm{Ai}'(e^{i\alpha}z)\mathrm{Ai}(e^{-i\alpha}z) = \frac{i}{2\pi} \quad \forall\, z \in \mathbb{C}\,. \tag{A.3}$$

Note that these two functions are related to $\mathrm{Ai}(z)$ by the identity

$$\mathrm{Ai}(z) + e^{-i\alpha}\mathrm{Ai}(e^{-i\alpha}z) + e^{i\alpha}\mathrm{Ai}(e^{i\alpha}z) = 0 \quad \forall\, z \in \mathbb{C}\,. \tag{A.4}$$

The Airy function and its derivative satisfy different asymptotic expansions depending on their argument:
(i) For $|\arg z| < \pi$,

$$\mathrm{Ai}(z) = \frac{1}{2\sqrt{\pi}}\, z^{-\frac{1}{4}}\, \exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right)\left(1 + \mathcal{O}(|z|^{-\frac{3}{2}})\right), \tag{A.5}$$

$$\mathrm{Ai}'(z) = -\frac{1}{2\sqrt{\pi}}\, z^{\frac{1}{4}}\, \exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right)\left(1 + \mathcal{O}(|z|^{-\frac{3}{2}})\right) \tag{A.6}$$

(moreover $\mathcal{O}$ is, for any $\epsilon > 0$, uniform when $|\arg z| \le \pi - \epsilon$).

(ii) For $|\arg z| < \frac{2}{3}\pi$,

$$\text{(A.7)}\qquad \begin{aligned}\mathrm{Ai}(-z) = \frac{1}{\sqrt{\pi}}\, z^{-\frac{1}{4}} \Bigg(&\sin\left(\frac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4}\right)(1+\mathcal{O}(|z|^{-\frac{3}{2}})\\ &-\frac{5}{72}\left(\frac{2}{3}z^{\frac{3}{2}}\right)^{-1}\cos\left(\frac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4}\right)(1+\mathcal{O}(|z|^{-\frac{3}{2}})\Bigg)\end{aligned}$$

$$\text{(A.8)}\qquad \begin{aligned}\mathrm{Ai}'(-z) = -\frac{1}{\sqrt{\pi}}\, z^{\frac{1}{4}} \Bigg(&\cos\left(\frac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4}\right)(1+\mathcal{O}(|z|^{-\frac{3}{2}}))\\ &+\frac{7}{72}\left(\frac{2}{3}z^{\frac{3}{2}}\right)^{-1}\sin\left(\frac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4}\right)(1+\mathcal{O}(|z|^{-\frac{3}{2}}))\Bigg)\end{aligned}$$

(moreover $\mathcal{O}$ is for any $\epsilon > 0$, uniform in the sector $\{|\arg z| \leq \frac{2\pi}{3} - \epsilon\}$).

Appendix B. Analysis of the resolvent of $\mathcal{A}^+$ on the line for $\lambda > 0$ (after [32]).

On the line $\mathbb{R}$, $\mathcal{A}^+$ is the closure of the operator $\mathcal{A}_0^+$ defined on $C_0^\infty(\mathbb{R})$ by $\mathcal{A}_0^+ = D_x^2 + ix$. A detailed description of its properties can be found in [26]. In this appendix, we give the asymptotic control of the resolvent $(\mathcal{A}^+ - \lambda)^{-1}$ as $\lambda \to +\infty$. We successively discuss the control in $\mathcal{L}(L^2(\mathbb{R}))$ and in the Hilbert-Schmidt space $\mathcal{C}^2(L^2(\mathbb{R}))$. These two spaces are equipped with their canonical norms.

B.1. Control in $\mathcal{L}(L^2(\mathbb{R}))$. Here we follow an idea present in the book of E. B. Davies [13] and used in J. Martinet's PHD [32] (see also [26]).

Proposition 49.
For all $\lambda > \lambda_0$,

$$\text{(B.1)}\qquad \|(\mathcal{A}^+ - \lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \sqrt{2\pi}\ \lambda^{-\frac{1}{4}} \exp\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right)\big(1+o(1)\big)\,.$$

Proof
The proof is obtained by considering $\mathcal{A}^+$ in the Fourier space, i.e.

$$\text{(B.2)}\qquad \widehat{\mathcal{A}}^+ = \xi^2 - \frac{d}{d\xi}\,.$$

The associated semi-group $T_t := \exp(-\widehat{\mathcal{A}}^+ t)$ is given by

$$\text{(B.3)}\qquad T_t u(\xi) = \exp\left(-\xi^2 t - \xi t^2 - \frac{t^3}{3}\right) u(\xi - t)\,,\quad \forall\, u \in \mathcal{S}(\mathbb{R})\,.$$

T_t appears as the composition of a multiplication by $\exp(-\xi^2 t - \xi t^2 - \frac{t^3}{3})$ and a translation by t. Computing $\sup_\xi\{\exp(-\xi^2 t - \xi t^2 - \frac{t^3}{3})\}$ leads to

$$\text{(B.4)}\qquad \|T_t\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \exp\left(-\frac{t^3}{12}\right)\,.$$

It is then easy to get an upper bound for the resolvent. For $\lambda > 0$, we have

$$\text{(B.5)}\qquad \begin{aligned}\|(\mathcal{A}^+ - \lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} &= \|(\widehat{\mathcal{A}}^+ - \lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))}\\ &\leq \int_0^{+\infty} \exp(t\lambda)\, \|T_t\|_{\mathcal{L}(L^2(\mathbb{R}))} dt\\ &\leq \int_0^{+\infty} \exp\left(t\lambda - \frac{t^3}{12}\right) dt\,.\end{aligned}$$

The right hand side can be estimated by using the Laplace integral method. Setting $t=\lambda^{\frac12}s$, we have

$$\int_0^{+\infty}\exp\left(t\lambda-\frac{t^3}{12}\right)dt=\lambda^{\frac12}\int_0^{+\infty}\exp\left(\lambda^{\frac32}\left(s-\frac{s^3}{12}\right)\right)ds\,. \tag{B.6}$$

We observe that $\hat\phi(s)=s-\frac{s^3}{12}$ admits a global non-degenerate maximum on $[0,+\infty)$ at $s=2$ with $\hat\phi(2)=\frac43$ and $\hat\phi''(2)=-1$. The Laplace integral method gives the following equivalence as $\lambda\to+\infty$:

$$\int_0^{+\infty}\exp\left(\lambda^{\frac32}\left(s-\frac{s^3}{12}\right)\right)ds\sim\sqrt{2\pi}\;\lambda^{-\frac34}\exp\left(\frac43\lambda^{\frac32}\right)\,. \tag{B.7}$$

This completes the proof of the proposition. We note that this upper bound is not optimal in comparison with Bordeaux-Montrieux's formula (3.5).

B.2. Control in Hilbert-Schmidt norm. In this part, we give a proof of Proposition 7. As in the previous subsection, we use the Fourier representation and analyze $\widehat{\mathcal{A}}^+$. Note that

$$\|(\widehat{\mathcal{A}}^+-\lambda)^{-1}\|_{HS}^2=\|(\mathcal{A}^+-\lambda)^{-1}\|_{HS}^2\,. \tag{B.8}$$

We have then an explicit description of the resolvent by

$$(\widehat{\mathcal{A}}^+-\lambda)^{-1}u(\xi)=\int_{-\infty}^{\xi}u(\eta)\exp\left(\frac13(\eta^3-\xi^3)+\lambda(\xi-\eta)\right)d\eta\,.$$

Hence, we have to compute

$$\|(\widehat{\mathcal{A}}^+-\lambda)^{-1}\|_{HS}^2=\int\int_{\eta<\xi}\exp\left(\frac23(\eta^3-\xi^3)+2\lambda(\xi-\eta)\right)d\eta d\xi\,.$$

After the change of variable $(\xi_1,\eta_1)=(\lambda^{-\frac12}\xi,\lambda^{-\frac12}\eta)$, we get

$$\|(\widehat{\mathcal{A}}^+-\lambda)^{-1}\|_{HS}^2=\lambda\int\int_{\eta_1<\xi_1}\exp\left(\lambda^{\frac32}\left[\frac23(\eta_1^3-\xi_1^3)+2(\xi_1-\eta_1)\right]\right)\,d\xi_1d\eta_1\,.$$

With

$$h=\lambda^{-\frac32}\,, \tag{B.9}$$

we can write

$$\|(\widehat{\mathcal{A}}^+-\lambda)^{-1}\|_{HS}^2=h^{-\frac23}\Phi(h), \tag{B.10}$$

where

$$\Phi(h)=\int\int_{y<x}\exp\left(\frac2h[\phi(x)-\phi(y)]\right)dxdy\,, \tag{B.11}$$

with

$$\phi(x)=x-\frac{x^3}{3}\,. \tag{B.12}$$

$\Phi(h)$ can now be split in three terms

$$\Phi(h) = I_1(h) + I_2(h) + I_3(h)\,, \tag{B.13}$$

with

$$I_1(h) = \int\int_{\substack{y<x\\ y>0}} \exp\left(\frac{2}{h}[\phi(x)-\phi(y)]\right) dxdy\,,$$

$$I_2(h) = \int\int_{\substack{y<x\\ x<0}} \exp\left(\frac{2}{h}[\phi(x)-\phi(y)]\right) dxdy\,,$$

$$I_3(h) = \int\int_{\substack{x\in\mathbb{R}^+\\ y\in\mathbb{R}^-}} \exp\left(\frac{2}{h}[\phi(x)-\phi(y)]\right) dxdy\,.$$

We observe now that by the change of variable $(x,y)\mapsto(-y,-x)$, we get

$$I_1(h) = I_2(h)\,,$$

and that

$$I_3(h) = I_4(h)^2\,,$$

with

$$I_4(h) = \int_{\mathbb{R}^+} \exp\left(\frac{2}{h}\phi(x)\right) dx\,.$$

Hence, it remains to estimate, as $h\to 0$, the integrals $I_1(h)$ and $I_4(h)$.

B.2.1. Control of $I_1(h)$.
The function $\phi(x)$ is positive on $(0,\sqrt{3})$ and negative decreasing on $(\sqrt{3},+\infty)$, with $\phi(0)=\phi(\sqrt{3})=0$. It admits a unique non-degenerate maximum at $x=1$ with $\phi(1)=\frac{2}{3}$.
We first observe the trivial estimates

$$\exp\left(-\frac{2}{h}\phi(y)\right) \le 1\,,\quad \forall\, y\in[0,\sqrt{3}]\,,$$

and

$$\exp\left(\frac{2}{h}\phi(x)\right) \le 1\,,\quad \forall\, x\in[\sqrt{3},+\infty[\,.$$

We will also have to estimate, for $x\ge\sqrt{3}$,

$$J(h,x) := \int_{\sqrt{3}}^{x} \exp\left(-\frac{2}{h}\phi(y)\right) dy\,.$$

For this purpose, we integrate by parts, observing that

$$\exp\left(-\frac{2}{h}\phi(y)\right) = -\frac{h}{2}\frac{1}{\phi'(y)}\frac{d}{dy}\exp\left(-\frac{2}{h}\phi(y)\right)\,.$$

We get

$$J(h,x) = -\frac{h}{2}\frac{1}{\phi'(x)}\exp\left(-\frac{2}{h}\phi(x)\right) - \frac{h}{2}\frac{1}{\phi'(\sqrt{3})} + \frac{h}{2}\int_{\sqrt{3}}^{x}\left(\frac{1}{\phi'}\right)'(y)\,\exp\left(-\frac{2}{h}\phi(y)\right)\,dy\,.$$

This implies

$$J(h,x) \le \frac{h}{2}\frac{1}{x^2-1}\exp\left(-\frac{2}{h}\phi(x)\right) - \frac{h}{2}\frac{1}{\phi'(\sqrt{3})} + ChJ(h,x)\,,$$

and finally, for h small enough and another constant $C > 0$

$$J(h,x) \le \frac{h}{2}\,\frac{1+Ch}{x^2-1}\exp\left(-\frac{2}{h}\phi(x)\right) + Ch\,, \quad \forall x \in [\sqrt{3},+\infty)\,. \tag{B.14}$$

Similarly, one can show that

$$\int_{\sqrt{3}}^{+\infty}\exp\left(\frac{2}{h}\phi(x)\right)dx \le \frac{h}{4}\,. \tag{B.15}$$

With these estimates, we can bound $I_1(h)$ from above in the following way

$$\begin{aligned}
I_1(h) = &\int_0^{\sqrt{3}}\exp\left(\frac{2}{h}\phi(x)\right)\left(\int_0^{x}\exp\left(-\frac{2}{h}\phi(y)\right)dy\right)dx\\
&+\int_{\sqrt{3}}^{+\infty}\exp\left(\frac{2}{h}\phi(x)\right)\left(\int_0^{\sqrt{3}}\exp\left(-\frac{2}{h}\phi(y)\right)dy\right)dx\\
&+\int_{\sqrt{3}}^{+\infty}\exp\left(\frac{2}{h}\phi(x)\right)\,J(h,x)\\
\le &\int_0^{\sqrt{3}}\exp\left(\frac{2}{h}\phi(x)\right)\left(\int_0^{\sqrt{3}}\exp\left(-\frac{2}{h}\phi(y)\right)dy\right)dx\\
&+(\sqrt{3}+Ch)\int_{\sqrt{3}}^{+\infty}\exp\left(\frac{2}{h}\phi(x)\right)dx\\
&+(1+Ch)\frac{h}{2}\int_{\sqrt{3}}^{+\infty}\frac{1}{x^2-1}\,dx\\
\le &\,3\sup_{[0,\sqrt{3}]}\left\{\exp\left(\frac{2}{h}\phi(x)\right)\right\}\\
&+\frac{(\sqrt{3}+Ch)h}{4}\\
&+(1+Ch)\frac{h}{2}\int_{\sqrt{3}}^{+\infty}\frac{1}{x^2-1}\,dx\\
\le &\,3\exp\left(\frac{4}{3h}\right) + \frac{\sqrt{3}h(1+Ch)}{4} + \frac{h}{2}(1+Ch)\int_{\sqrt{3}}^{+\infty}\frac{1}{x^2-1}dx\,.
\end{aligned}$$

Hence we have shown the existence of $\hat{C} > 0$ and $h_0 > 0$ such that, for $h \in (0,h_0)$,

$$I_1(h) \le \hat{C}\exp\left(\frac{4}{3h}\right)\,. \tag{B.16}$$

Consequently, $I_1(h)$ and $I_2(h)$ appear as remainder terms.

B.2.2. Asymptotic of $I_4(h)$. Here, using the properties of ϕ, we get by the standard Laplace integral method

$$I_4(h) \sim \sqrt{\pi/2}\,\sqrt{h}\,\exp\left(\frac{4}{3h}\right)\,. \tag{B.17}$$

Hence, putting altogether the estimates, we get, as $h \to 0$,

$$\Phi(h) \sim \frac{\pi h}{2}\exp\left(\frac{8}{3h}\right) \tag{B.18}$$

Coming back to (B.8), (B.9) and (B.10), this achieves the proof of Proposition 7.

Appendix C. Analysis of the resolvent for the Dirichlet realization in the half-line.

C.1. Main statement. The aim of this appendix is to give the proof of Proposition 11. Although it is not used in our main text, it is interesting to get the main asymptotic for the Hilbert-Schmidt norm of the resolvent in Proposition 11.

PROPOSITION 50. *As $\lambda \to +\infty$, we have*

$$||\mathcal{G}^{-,D}(\lambda)||_{HS} \sim \frac{\sqrt{3}}{2\sqrt{2}}\,\lambda^{-\frac{1}{4}}(\log\lambda)^{\frac{1}{2}}\,. \tag{C.1}$$

C.2. The Hilbert-Schmidt norm of the resolvent for real λ . The Hilbert-Schmidt norm of the resolvent can be written as

$$||\mathcal{G}^{-,D}||^2_{HS} = \int_{\mathbb{R}^2_+} |\mathcal{G}^{-,D}(x,y\,;\lambda)|^2 dxdy = 8\pi^2\int_0^\infty Q(x;\lambda)dx, \tag{C.2}$$

where

$$\begin{aligned} Q(x;\lambda) = {} & \frac{|\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2}{|\mathrm{Ai}(e^{-i\alpha}\lambda)|^2}\times \\ & \times\int_0^x \left|\mathrm{Ai}(e^{i\alpha}(iy+\lambda))\mathrm{Ai}(e^{-i\alpha}\lambda) - \mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\mathrm{Ai}(e^{i\alpha}\lambda)\right|^2 dy. \end{aligned} \tag{C.3}$$

Using the identity (A.4), we observe that

$$\begin{aligned} & \mathrm{Ai}(e^{i\alpha}(iy+\lambda))\mathrm{Ai}(e^{-i\alpha}\lambda) - \mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\mathrm{Ai}(e^{i\alpha}\lambda) \\ & \quad = e^{-i\alpha}\left(\mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\mathrm{Ai}(\lambda) - \mathrm{Ai}(iy+\lambda)\mathrm{Ai}(e^{-i\alpha}\lambda)\right)\,. \end{aligned} \tag{C.4}$$

Hence we get

$$Q(x;\lambda) = |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2\int_0^x \left|\mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\frac{\mathrm{Ai}(\lambda)}{\mathrm{Ai}(e^{-i\alpha}\lambda)} - \mathrm{Ai}(iy+\lambda)\right|^2 dy\,. \tag{C.5}$$

C.3. More facts on Airy expansions. As a consequence of (A.5), we can write for $\lambda > 0$ and $x > 0$

$$|\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))| = \frac{\exp\big(-\frac{2}{3}\lambda^{\frac{3}{2}} u(x/\lambda)\big)}{2\sqrt{\pi}(\lambda^2+x^2)^{\frac{1}{8}}}\big(1+\mathcal{O}(\lambda^{-\frac{3}{2}})\big), \tag{C.6}$$

where

$$u(s) = -(1+s^2)^{\frac{3}{4}}\cos\left(\frac{3}{2}\tan^{-1}(s)\right) = \frac{\sqrt{\sqrt{1+s^2}+1}\,(\sqrt{1+s^2}-2)}{\sqrt{2}}. \tag{C.7}$$

We note indeed that $|e^{-i\alpha}(ix+\lambda)| = \sqrt{x^2+\lambda^2} \geq \lambda \geq \lambda_0$ and that we have a control of the argument $\arg(e^{-i\alpha}(ix+\lambda)) \in [-\frac{2\pi}{3}, -\frac{\pi}{6}]$ which permits to apply (A.5).
Similarly, we obtain

$$|\mathrm{Ai}(ix+\lambda)| = \frac{\exp\big(\frac{2}{3}\lambda^{\frac{3}{2}} u(x/\lambda)\big)}{2\sqrt{\pi}(\lambda^2+x^2)^{\frac{1}{8}}}\big(1+\mathcal{O}(\lambda^{-\frac{3}{2}})\big)\,. \tag{C.8}$$

We note indeed that $|ix+\lambda| = \sqrt{x^2+\lambda^2}$ and $\arg((ix+\lambda)) \in [0, +\frac{\pi}{2}]$ so that one can then again apply (A.5). In particular the function $|\mathrm{Ai}(ix+\lambda)|$ grows super-exponentially as $x \to +\infty$.
Figure 3 illustrates that, for large λ, both equations (C.6) and (C.8) are very accurate approximations for $|\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|$ and $|\mathrm{Ai}(ix+\lambda)|$, respectively.
The control of the next order term (as given in (A.5)) implies that there exist $C > 0$ and $\epsilon_0 > 0$, such that, for any $\epsilon \in (0, \epsilon_0]$, any $\lambda > \varepsilon^{-\frac{2}{3}}$ and any $x \geq 0$, one has

$$(1-C\epsilon)\frac{\exp\big(-\frac{2}{3}\lambda^{\frac{3}{2}} u(x/\lambda)\big)}{2\sqrt{\pi}\,(\lambda^2+x^2)^{\frac{1}{8}}} \leq |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))| \leq (1+C\epsilon)\frac{\exp\big(-\frac{2}{3}\lambda^{\frac{3}{2}} u(x/\lambda)\big)}{2\sqrt{\pi}\,(\lambda^2+x^2)^{\frac{1}{8}}} \tag{C.9}$$

and

$$(1-C\epsilon)\frac{\exp\big(\frac{2}{3}\lambda^{\frac{3}{2}} u(x/\lambda)\big)}{2\sqrt{\pi}\,(\lambda^2+x^2)^{\frac{1}{8}}} \leq |\mathrm{Ai}(ix+\lambda)| \leq (1+C\epsilon)\frac{\exp\big(\frac{2}{3}\lambda^{\frac{3}{2}} u(x/\lambda)\big)}{2\sqrt{\pi}\,(\lambda^2+x^2)^{\frac{1}{8}}}\,, \tag{C.10}$$

where the function u is explicitly defined in Eq. (C.7).

Basic properties of u.
Note that

$$u'(s) = \frac{3}{2\sqrt{2}}\;\frac{s}{\sqrt{1+\sqrt{1+s^2}}} \geq 0 \qquad (s \geq 0), \tag{C.11}$$

and u has the following expansion at the origin

$$u(s) = -1 + \frac{3}{8}s^2 + \mathcal{O}(s^4)\,. \tag{C.12}$$

For large s, one has

$$u(s) \sim \frac{s^{\frac{3}{2}}}{\sqrt{2}}\,, \qquad u'(s) \sim \frac{3s^{\frac{1}{2}}}{2\sqrt{2}}\,. \tag{C.13}$$

One concludes that the function u is monotonously increasing on $[0, +\infty)$ with $u(0) = -1$ and $\lim_{s\to+\infty} u(s) = +\infty\,.$

C.4. Upper bound. We start from the simple upper bound (for any $\epsilon > 0$)

$$Q(x,\lambda) \leq \left(1 + \frac{1}{\epsilon}\right) Q_1(x,\lambda) + (1+\epsilon) Q_2(x,\lambda)\,, \tag{C.14}$$

with

$$Q_1(x,\lambda) := |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2 \, \frac{|\mathrm{Ai}(\lambda)|^2}{|\mathrm{Ai}(e^{-i\alpha}\lambda)|^2} \int\limits_0^x |\mathrm{Ai}(e^{-i\alpha}(iy+\lambda))|^2 \, dy$$

and

$$Q_2(x,\lambda) := |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2 \int\limits_0^x |\mathrm{Ai}(iy+\lambda)|^2 \, dy\,.$$

We then write

$$Q_1(x,\lambda) \leq |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2 \, \frac{|\mathrm{Ai}(\lambda)|^2}{|\mathrm{Ai}(e^{-i\alpha}\lambda)|^2} \int\limits_0^{+\infty} |\mathrm{Ai}(e^{-i\alpha}(iy+\lambda))|^2 \, dy$$

and integrating over x

$$\int_0^{+\infty} Q_1(x,\lambda) dx \leq I_0(\lambda)^2 \, \frac{|\mathrm{Ai}(\lambda)|^2}{|\mathrm{Ai}(e^{-i\alpha}\lambda)|^2},$$

where $I_0(\lambda)$ is given by (8.8).

Using (8.10) and (A.5), we obtain

$$\int_0^{+\infty} Q_1(x,\lambda) dx \leq C \lambda^{-\frac{1}{2}}\,. \tag{C.15}$$

Hence at this stage, we have proven the existence of $C > 0$, $\epsilon_0 > 0$ and λ_0 such that for any $\epsilon \in (0, \epsilon_0]$ and any $\lambda \geq \lambda_0$

$$||\mathcal{G}^{-,D}||_{HS}^2 \leq (1+\epsilon) \left(8\pi^2 \int\limits_0^{\infty} Q_2(x;\lambda) dx\right) + C \lambda^{-\frac{1}{2}} \epsilon^{-1}\,. \tag{C.16}$$

It remains to estimate

$$\int_0^{+\infty} Q_2(x,\lambda) dx = \int_0^{+\infty} dx \int\limits_0^x |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda)) \mathrm{Ai}(iy+\lambda)|^2 \, dy\,. \tag{C.17}$$

Using the estimates (C.6) and (C.8), we obtain

LEMMA 51. *There exist C and ϵ_0, such that, for any $\epsilon \in (0, \epsilon_0)$, for $\lambda > \epsilon^{-\frac{2}{3}}$, the integral of $Q_2(x;\lambda)$ can be bounded as*

$$\frac{1}{2}(1 - C\epsilon)\, I(\lambda) \;\leq 8\pi^2 \int_0^{+\infty} Q_2(x,\lambda) dx \leq \frac{1}{2}(1 + C\epsilon)\, I(\lambda)\,, \tag{C.18}$$

where

$$I(\lambda)=\int_0^\infty dx\ \frac{\exp\big(-\frac{4}{3}\lambda^{\frac{3}{2}}u(x/\lambda)\big)}{(\lambda^2+x^2)^{\frac{1}{4}}}\int_0^x dy\ \frac{\exp\big(\frac{4}{3}\lambda^{\frac{3}{2}}u(y/\lambda)\big)}{(\lambda^2+y^2)^{\frac{1}{4}}}. \tag{C.19}$$

Control of $I(\lambda)$.
It remains to control $I(\lambda)$ as $\lambda\to+\infty$. Using a change of variables, we get

$$I(\lambda)=\lambda\int_0^\infty dx\ \frac{\exp\big(-\frac{4}{3}\lambda^{\frac{3}{2}}u(x)\big)}{(1+x^2)^{\frac{1}{4}}}\int_0^x dy\ \frac{\exp\big(\frac{4}{3}\lambda^{\frac{3}{2}}u(y)\big)}{(1+y^2)^{\frac{1}{4}}}\,. \tag{C.20}$$

Hence, introducing

$$t=\frac{4}{3}\lambda^{\frac{3}{2}}\,, \tag{C.21}$$

we reduce the analysis to $\hat I(t)$ defined for $t\ge t_0$ by

$$\hat I(t):=\int_0^\infty dx\ \frac{1}{(1+x^2)^{\frac{1}{4}}}\int_0^x dy\ \frac{\exp\big(t(u(y)-u(x))\big)}{(1+y^2)^{\frac{1}{4}}}\,, \tag{C.22}$$

with

$$I(\lambda)=\lambda\,\widehat I\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right)\,. \tag{C.23}$$

The following analysis is close to that of the asymptotic behavior of a Laplace integral.

Asymptotic upper bound of $\hat I(t)$.
Although $u(y)\le u(x)$ in the domain of integration in Eq. (C.22), a direct use of this upper bound will lead to an upper bound by $+\infty$.

Let us start by a heuristic discussion. The maximum of $u(y)-u(x)$ should be on $x=y$. For $x-y$ small, we have $u(y)-u(x)\sim(y-x)u'(x)$. This suggests a concentration near $x=y=0$, whereas a contribution for large x is of smaller order. More rigorously, we write

$$\widehat I(t)=\widehat I_1(t,\epsilon)+\widehat I_2(t,\epsilon,\xi)+\widehat I_3(t,\epsilon), \tag{C.24}$$

with, for $0<\epsilon<\xi$,

$$\begin{aligned}
\widehat I_1(t,\epsilon)&=\int_0^\epsilon dx\ \frac{1}{(1+x^2)^{\frac{1}{4}}}\int_0^x dy\ \frac{\exp\big(t(u(y)-u(x))\big)}{(1+y^2)^{\frac{1}{4}}},\\
\widehat I_2(t,\epsilon,\xi)&=\int_\epsilon^\xi dx\ \frac{1}{(1+x^2)^{\frac{1}{4}}}\int_0^x dy\ \frac{\exp\big(t(u(y)-u(x))\big)}{(1+y^2)^{\frac{1}{4}}},\\
\widehat I_3(t,\xi)&=\int_\xi^{+\infty} dx\ \frac{1}{(1+x^2)^{\frac{1}{4}}}\int_0^x dy\ \frac{\exp\big(t(u(y)-u(x))\big)}{(1+y^2)^{\frac{1}{4}}}\,.
\end{aligned} \tag{C.25}$$

We now observe that $u(s)$ has the form $u(s) = v(s^2)$ where $v' > 0$, so that

(C.26)
$$\forall x, y \text{ s.t. } 0 \leq y \leq x \leq \tau_0\,, \quad \left(\sup_{\tau\in[0,\tau_0]} v'(\tau)\right)(x^2-y^2) \geq u(x)-u(y) \geq \left(\inf_{\tau\in[0,\tau_0]} v'(\tau)\right)(x^2-y^2)\,.$$

Analysis of $\widehat{I}_1(t,\epsilon)$.
Using the right hand side of inequality (C.26) with $\tau_0 = \epsilon$, we show the existence of constants C and $\epsilon_0 > 0$, such that, $\forall \epsilon \in (0, \epsilon_0)$

$$(1 - C\epsilon) J_\epsilon \left((1 + C\epsilon)\frac{3}{8}t\right) \leq \widehat{I}_1(t,\epsilon) \leq (1 + C\epsilon) J_\epsilon \left((1 - C\epsilon)\frac{3}{8}t\right)\,, \tag{C.27}$$

with

$$J_\epsilon(\sigma) := \int_0^\epsilon dx \int_0^x \exp\big(\sigma(y^2 - x^2)\big)dy\,,$$

which has now to be estimated for large σ.
For $\frac{1}{\sqrt{\epsilon\sigma}} \leq \epsilon$, we write

$$J_\epsilon(\sigma) = J_\epsilon^1(\sigma) + J_\epsilon^2(\sigma)\,,$$

with

$$J_\epsilon^1(\sigma) := \int_0^{\frac{1}{\sqrt{\epsilon\sigma}}} dx \int_0^x \exp\big(\sigma(y^2 - x^2)\big)dy\,,$$
$$J_\epsilon^2(\sigma) := \int_{\frac{1}{\sqrt{\epsilon\sigma}}}^\epsilon dx \int_0^x \exp\big(\sigma(y^2 - x^2)\big)dy\,.$$

Using the trivial estimate

$$\int_0^x \exp\big(\sigma(y^2 - x^2)\big)dy \leq x,$$

we get

$$J_\epsilon^1(\sigma) \leq \frac{1}{2\epsilon\sigma}\,. \tag{C.28}$$

We have now to analyze $J_\epsilon^2(\sigma)$.
The formula giving $J_\epsilon^2(\sigma)$ can be expressed by using the Dawson function (cf [1], p. 295 and 319)

$$s \mapsto D(s) := \int_0^s \exp(y^2 - s^2)\,dy$$

and its asymptotics as $s \to +\infty$,

$$D(s) = \frac{1}{2s}(1 + \delta(s))\,, \tag{C.29}$$

where the function $\delta(s)$ satisfies $\delta(s) = \mathcal{O}(s^{-1})$.
We get indeed

$$J_\epsilon^2(\sigma) = \frac{1}{\sigma}\int_{\epsilon^{-\frac{1}{2}}}^{\epsilon\sigma^{\frac{1}{2}}} D(s)ds\,.$$

By taking ϵ small enough to use the asymptotics of $D(\cdot)$, we get

$$J_\epsilon^2(\sigma) = \frac{1}{2\sigma}\left(\int_{\epsilon^{-\frac{1}{2}}}^{\sigma^{\frac{1}{2}}\epsilon} \frac{1}{s} ds + \int_{\epsilon^{-\frac{1}{2}}}^{\sigma^{\frac{1}{2}}\epsilon} \frac{\delta(s)}{s} ds\right) = \frac{1}{4}\,\frac{\log\sigma}{\sigma} + \frac{C}{\sigma}(\log\epsilon + \mathcal{O}(1))\,. \tag{C.30}$$

Hence we have shown the existence of constants $C > 0$ and ϵ_0 such that if $t \geq C\epsilon^{-3}$ and $\epsilon \in (0, \epsilon_0)$, then

$$\widehat{I}_1(t,\epsilon) \leq \frac{2}{3}\frac{\log t}{t} + C\left(\epsilon\frac{\log t}{t} + \frac{1}{\epsilon}\frac{1}{t}\right)\,. \tag{C.31}$$

Analysis of $\widehat{I}_3(t,\xi)$

We start from

$$\widehat{I}_3(t,\xi) = \int_\xi^{+\infty} dx\, \frac{1}{(1+x^2)^{\frac{1}{4}}} \int_0^x dy\, \frac{\exp\big(t(u(y)-u(x))\big)}{(1+y^2)^{\frac{1}{4}}}\,.$$

Having in mind the properties of u, we can choose ξ large enough in order to have for some $c_\xi > 0$ the property that for $x \geq \xi$ and $\frac{x}{2} \leq y \leq x$,

$$\begin{aligned} u(x) &\geq c_\xi x^{\frac{3}{2}} \\ u(x) - u(x/2) &\geq c_\xi x^{\frac{3}{2}}\,, \\ u(x) - u(y) &\geq c_\xi x^{\frac{1}{2}}\,(x-y)\,. \end{aligned} \tag{C.32}$$

This determines our choice of ξ. Using these inequalities, we rewrite $\widehat{I}_3(t,\xi)$ as the sum

$$\widehat{I}_3(t,\xi) = \widehat{I}_{31}(t) + \widehat{I}_{32}(t)\,,$$

with

$$\begin{aligned} \widehat{I}_{31}(t) &= \int_\xi^{+\infty} dx\, \frac{1}{(1+x^2)^{\frac{1}{4}}} \int_0^{\frac{x}{2}} dy\, \frac{\exp\big(t(u(y)-u(x))\big)}{(1+y^2)^{\frac{1}{4}}}, \\ \widehat{I}_{32}(t) &= \int_\xi^{+\infty} dx\, \frac{1}{(1+x^2)^{\frac{1}{4}}} \int_{\frac{x}{2}}^{x} dy\, \frac{\exp\big(t(u(y)-u(x))\big)}{(1+y^2)^{\frac{1}{4}}}. \end{aligned}$$

Using the monotonicity of u, we obtain the upper bound

$$
\begin{aligned}
\widehat{I}_{31}(t) &\leq \int_{\xi}^{+\infty} dx \, \frac{1}{(1+x^2)^{\frac{1}{4}}} \int_0^{\frac{x}{2}} dy \, \exp\big(t(u(y)-u(x))\big) \\
&\leq \frac{1}{2} \int_{\xi}^{+\infty} x^{\frac{1}{2}} \, \exp\big(t(u(x/2)-u(x))\big) \, dx \\
&\leq \frac{1}{2} \int_{\xi}^{+\infty} x^{\frac{1}{2}} \, \exp\big(-c_{\xi} \, t \, x^{\frac{3}{2}}\big) \, dx \\
&\leq \frac{1}{3} \int_{\xi^{\frac{3}{2}}}^{+\infty} \exp\big(-c_{\xi} t s\big) \, ds \\
&\leq \frac{1}{3c_{\xi} t} \, \exp\big(-c_{\xi} \xi^{\frac{3}{2}} t\big) \, .
\end{aligned}
$$

Hence, there exists $\epsilon_{\xi} > 0$ such that as $t \to +\infty$,

(C.33) $$\widehat{I}_{31}(t) = \mathcal{O}\big(\exp(-\epsilon_{\xi} t)\big) \, .$$

The last term to control is $\widehat{I}_{32}(t)$. Using (C.32) and

$$(1+y^2)^{-\frac{1}{4}} \leq (1+(x/2)^2)^{-\frac{1}{4}} \leq \sqrt{2}\,(4+x^2)^{-\frac{1}{4}} \leq \sqrt{2}\,(1+x^2)^{-\frac{1}{4}}$$

for $x/2 \leq y \leq x$, we get

(C.34)
$$
\begin{aligned}
\widehat{I}_{32}(t) &\leq \sqrt{2} \int_{\xi}^{+\infty} dx \, \frac{1}{(1+x^2)^{\frac{1}{2}}} \int_{\frac{x}{2}}^{x} dy \, \exp(t(u(y)-u(x)) \\
&\leq \sqrt{2} \int_{\xi}^{+\infty} dx \, \frac{1}{(1+x^2)^{\frac{1}{2}}} \int_{\frac{x}{2}}^{x} dy \, \exp\big(-c_{\xi} t x^{\frac{1}{2}} (x-y)\big) \\
&\leq \frac{\sqrt{2}}{c_{\xi} t} \int_{\xi}^{+\infty} x^{-\frac{3}{2}} \, dx \; = \frac{2\sqrt{2}}{\sqrt{\xi} c_{\xi} t} \, .
\end{aligned}
$$

Hence putting together (C.33) and (C.34) we have, for this choice of ξ, the existence of $\hat{C}_{\xi} > 0$ and $t_{\xi} > 0$ such that

(C.35) $$\forall t \geq t_{\xi} \, , \quad \widehat{I}_3(t) \leq \hat{C}_{\xi}/t \, .$$

Analysis of $\widehat{I}_2(t,\epsilon,\xi)$.
We recall that

$$\widehat{I}_2(t,\epsilon,\xi) = \int_{\epsilon}^{\xi} dx \, \frac{1}{(1+x^2)^{\frac{1}{4}}} \int_0^{x} dy \, \frac{\exp\big(t(u(y)-u(x))\big)}{(1+y^2)^{\frac{1}{4}}} \, .$$

We first observe that

$$\widehat{I}_2(t,\epsilon,\xi) \leq \int_{\epsilon}^{\xi} dx \int_0^{x} dy \exp\big(t(u(y)-u(x))\big) \leq \int_{\epsilon}^{\xi} dx \int_0^{x} dy \exp\big(c_{\xi} t (y^2 - x^2)\big) \, ,$$

with

$$c_\xi = \inf_{[0,\xi]} v' > 0\,.$$

Using now

$$\int_0^x \exp\big(c_\xi t(y^2-x^2)\big)dy \le \int_0^x \exp\big(c_\xi tx(y-x)\big)dy = \frac{1}{c_\xi tx}\left(1-\exp(-c_\xi tx^2)\right) \le \frac{1}{c_\xi tx}\,,$$

we get

$$\widehat{I}_2(t,\epsilon,\xi) \le \frac{1}{c_\xi t}\left(\log\xi - \log\epsilon\right)\,. \tag{C.36}$$

Putting together (C.24), (C.31), (C.35) and (C.36), we have shown the existence of $C>0$ and ϵ_0 such that if $t \ge C\epsilon^{-3}$ and $\epsilon \in (0,\epsilon_0)$, then

$$\widehat{I}(t) \le \frac{2}{3}\frac{\log t}{t} + C\left(\epsilon\frac{\log t}{t} + \frac{1}{\epsilon}\frac{1}{t}\right)\,. \tag{C.37}$$

Coming back to (C.23) and using (C.16), we show the existence of $C>0$ and ϵ_0 such that if $\lambda \ge C\epsilon^{-2}$, then

$$||\mathcal{G}^{-,D}(\lambda)||_{HS}^2 \le \frac{3}{8}\lambda^{-\frac12}\log\lambda + C\left(\epsilon\lambda^{-\frac12}\log\lambda + \frac{1}{\epsilon}\lambda^{-\frac12}\right)\,.$$

Taking $\epsilon = (\log\lambda)^{-\frac12}$, we obtain

Lemma 52. *There exist $C>0$ and λ_0 such that for $\lambda \ge \lambda_0$*

$$||\mathcal{G}^{-,D}(\lambda)||_{HS}^2 \le \frac{3}{8}\lambda^{-\frac12}(1 + C\,(\log\lambda)^{-\frac12})\log\lambda\,.$$

C.5. Lower bound. Once the upper bounds are established, the proof of the lower bound is easy. We start from the simple lower bound (for any $\epsilon > 0$)

$$Q(x,\lambda) \ge -\frac{1}{\epsilon}Q_1(x,\lambda) + (1-\epsilon)Q_2(x,\lambda)\,, \tag{C.38}$$

and consequently

$$\int_0^{+\infty} Q(x,\lambda)\,dx \ge (1-\epsilon)\int_0^{+\infty} Q_2(x,\lambda)\,dx - \frac{1}{\epsilon}\int_0^{+\infty} Q_1(x,\lambda)\,dx\,. \tag{C.39}$$

Taking $\epsilon = (\log\lambda)^{-\frac12}$ and using the upper bound (C.15), it remains to find a lower bound for $\int_0^{+\infty} Q_2(x,\lambda)dx$, which can be worked out in the same way as for the upper bound. We can use (C.18), (C.27), (C.30) and

$$\widehat{I}(t) \ge \widehat{I}_1(t,\epsilon) \ge \frac{2}{3}\frac{\log t}{t} - C\left(\epsilon\frac{\log t}{t} + \frac{1}{\epsilon}\frac{1}{t}\right)\,. \tag{C.40}$$

This gives the proof of

Lemma 53. *There exist $C>0$ and λ_0 such that for $\lambda \ge \lambda_0$*

$$||\mathcal{G}^{-,D}(\lambda)||_{HS}^2 \ge \frac{3}{8}\lambda^{-\frac12}\,(1 - C\,(\log\lambda)^{-\frac12})\log\lambda\,.$$

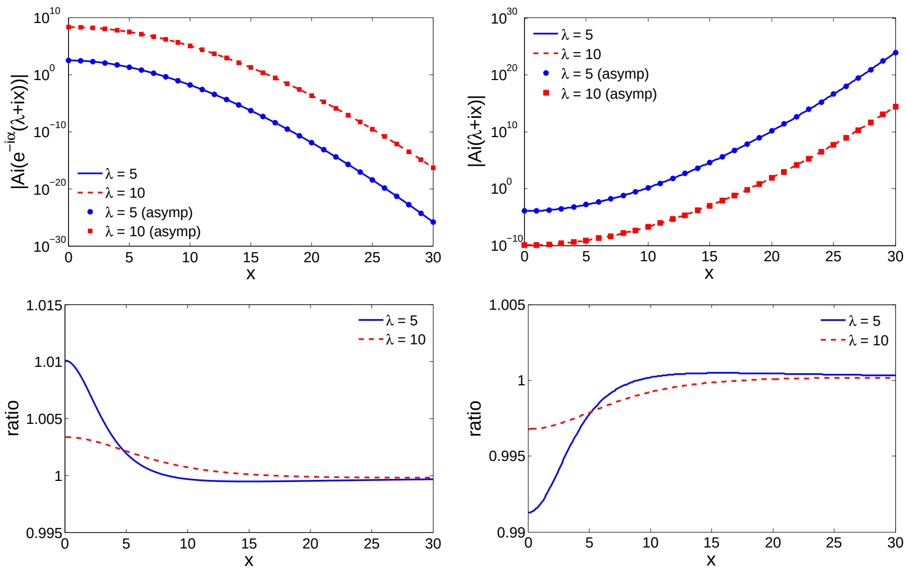

Fig. 3. *(Top) Asymptotic behavior of* $|\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|$ *(left) and* $|\mathrm{Ai}(ix+\lambda)|$ *(right) for large* λ. *(Bottom) The ratio between these functions and their asymptotics given by* (C.6) *and* (C.8).

Appendix D. Phragmen-Lindelöf theorem.

The Phragmen-Lindelöf Theorem (see Theorem 16.1 in [2]) reads

Theorem 54. *(Phragmen-Lindelöf) Let us assume that there exist two rays*

$$\mathcal{R}_1 = \{re^{i\theta_1} : r \geq 0\} \text{ and } \mathcal{R}_2 = \{re^{i\theta_2} : r \geq 0\}$$

with (θ_1, θ_2) *such that* $|\theta_1 - \theta_2| = \frac{\pi}{\alpha}$, *and a continuous function* F *in the closed sector delimited by the two rays, holomorphic in the open sector, satisfying the properties*

(i)

$$\exists C > 0, \quad \exists N \in \mathbb{R}, \ s. \ t. \ \forall \lambda \in \mathcal{R}_1 \cup \mathcal{R}_2, \quad |F(\lambda)| \leq C\big(|\lambda|^2 + 1\big)^{N/2}.$$

(ii) There exist an increasing sequence (r_k) *tending to* $+\infty$, *and* C *such that*

$$\forall k, \quad \max_{|\lambda| = r_k} |F(\lambda)| \leq C \, \exp(r_k^{\beta}) \,, \tag{D.1}$$

with $\beta < \alpha$.

Then we have

$$|F(\lambda)| \leq C \left(|\lambda|^2 + 1\right)^{N/2}$$

for all λ *between the two rays* $\mathcal{R}_1$ *and* $\mathcal{R}_2$.

Appendix E. Numerical computation of eigenvalues.

In order to compute numerically the eigenvalues of the realization $\mathcal{A}_{1,L}^{+,D}$ of the complex Airy operator $\mathcal{A}_0^+ := D_x^2 + ix = -\frac{d^2}{dx^2} + ix$ on the real line with a transmission condition, we impose auxiliary Dirichlet boundary conditions at $x = \pm L$, i.e., we search for eigenpairs $\{\lambda_L, u_L(\cdot)\}$ of the following problem:

$$\begin{aligned} &\left(-\frac{d^2}{dx^2}+ix\right)u_L(x)=\lambda_L u_L(x), \qquad (-L<x<L),\\ &u_L(\pm L)=0\,, \qquad u_L'(0_+)=u_L'(0_-)=\kappa\big(u_L(0_+)-u_L(0_-)\big)\,, \end{aligned} \tag{E.1}$$

with a positive parameter κ.

Since the interval $[-L,L]$ is bounded, the spectrum of the above differential operator is discrete. To compute its eigenvalues, one can either discretize the second derivative, or represent this operator in an appropriate basis in the form of an infinite-dimensional matrix. Following [21], we choose the second option and use the basis formed by the eigenfunctions of the Laplace operator $-\frac{d^2}{dx^2}$ with the above boundary conditions. Once the matrix representation is found, it can be truncated to compute the eigenvalues numerically. Finally, one considers the limit $L\to+\infty$ to remove the auxiliary boundary conditions at $x=\pm L$.

There are two sets of Laplacian eigenfunctions in this domain:

(i) symmetric eigenfunctions

$$v_{n,1}(x)=\sqrt{1/L}\cos(\pi(n+1/2)x/L), \quad \mu_{n,1}=\pi^2(n+1/2)^2/L^2, \tag{E.2}$$

enumerated by the index $n\in\mathbb{N}$.

(ii) antisymmetric eigenfunctions

$$v_{n,2}(x)=\begin{cases}+(\beta_n/\sqrt{L})\sin(\alpha_n(1-x/L)) & (x>0),\\ -(\beta_n/\sqrt{L})\sin(\alpha_n(1+x/L)) & (x<0),\end{cases} \tag{E.3}$$

with $\mu_{n,2}=\alpha_n^2/L^2$, where α_n $(n=0,1,2,\ldots)$ satisfy the equation

$$\alpha_n \operatorname{ctan}(\alpha_n)=-2\kappa L\,, \tag{E.4}$$

while the normalization constant β_n is

$$\beta_n=\left(1+\frac{2\kappa L}{\alpha_n^2+4\kappa^2L^2}\right)^{-\frac{1}{2}}\,. \tag{E.5}$$

The solutions α_n of Eq. (E.4) lie in the intervals $(\pi n+\pi/2, \pi n+\pi)$, with $n\in\mathbb{N}$.

In what follows, we use the double index (n,j) to distinguish symmetric and antisymmetric eigenfunctions and to enumerate eigenvalues, eigenfunctions, as well as the elements of governing matrices and vectors. We introduce two (infinite-dimensional) matrices Λ and $\mathcal{B}$ to represent the Laplace operator and the position operator in the Laplacian eigenbasis:

$$\Lambda_{n,j;n',j'}=\delta_{n,n'}\delta_{j,j'}\,\mu_{n,j}\,, \tag{E.6}$$

and

$$\mathcal{B}_{n,j;n',j'}=\int\limits_{-L}^{L} dx\; v_{n,j}(x)\; x\; v_{n',j'}(x)\,. \tag{E.7}$$

The symmetry of eigenfunctions $v_{n,j}$ implies $\mathcal{B}_{n,1;n',1}=\mathcal{B}_{n,2;n',2}=0$, while

$$\begin{aligned}\mathcal{B}_{n,1;n',2}&=\mathcal{B}_{n',2;n,1}\\ &=-2L\beta_{n'}\frac{\sin(\alpha_{n'})(\alpha_{n'}^2+\pi^2(n+1/2)^2)-(-1)^n(2n+1)\pi\alpha_{n'}}{(\alpha_{n'}^2-\pi^2(n+1/2)^2)^2}.\end{aligned} \tag{E.8}$$

	L	$\lambda_{1,L}$	$\lambda_{3,L}$	$\lambda_{5,L}$
$\kappa = 0$	4	0.5161 - 0.8918i	1.2938 - 2.1938i	3.7675 - 1.9790i
	6	0.5094 - 0.8823i	1.1755 - 3.9759i	1.6066 - 2.7134i
	8	0.5094 - 0.8823i	1.1691 - 5.9752i	1.6233 - 2.8122i
	10	0.5094 - 0.8823i	1.1691 - 7.9751i	1.6241 - 2.8130i
	∞	0.5094 - 0.8823i		1.6241 - 2.8130i
$\kappa = 1$	4	1.0516 - 1.0591i	1.3441 - 2.0460i	4.1035 - 1.7639i
	6	1.0032 - 1.0364i	1.1725 - 3.9739i	1.7783 - 2.7043i
	8	1.0029 - 1.0363i	1.1691 - 5.9751i	1.8364 - 2.8672i
	10	1.0029 - 1.0363i	1.1691 - 7.9751i	1.8390 - 2.8685i
	∞	1.0029 - 1.0363i		1.8390 - 2.8685i

TABLE 1

The convergence of the eigenvalues $\lambda_{n,L}$ computed by diagonalization of the matrix $\Lambda + i\mathcal{B}$ truncated to the size 100×100. Due to the reflection symmetry of the interval, all eigenvalues appear in complex conjugate pairs: $\lambda_{2n,L} = \bar{\lambda}_{2n-1,L}$. The last line presents the poles of the resolvent of the complex Airy operator $\mathcal{A}_1^+$ obtained by solving numerically the equation (6.17). The intermediate column shows the eigenvalue $\lambda_{3,L}$ coming from the auxiliary boundary conditions at $x = \pm L$ (as a consequence, it does not depend on the transmission coefficient κ). Since the imaginary part of these eigenvalues diverges as $L \to +\infty$, they can be easily identified and discarded.

The infinite-dimensional matrix $\Lambda + i\mathcal{B}$ represents the complex Airy operator $\mathcal{A}_{1,L}^{+,D}$ on the interval $[-L, L]$ in the Laplacian eigenbasis. As a consequence, the eigenvalues and eigenfunctions can be numerically obtained by truncating and diagonalizing this matrix. The obtained eigenvalues are ordered according to their increasing real part:

$$\mathrm{Re}\,\lambda_{1,L} \le \mathrm{Re}\,\lambda_{2,L} \le \dots$$

Table 1 illustrates the rapid convergence of these eigenvalues to the eigenvalues of the complex Airy operator $\mathcal{A}_1^+$ on the whole line with transmission, as L increases. The same matrix representation was used for plotting the pseudospectrum of $\mathcal{A}_1^+$ (Fig. 2).

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions,* Dover Publisher, New York, 1965.

[2] S. Agmon, *Elliptic boundary value problems,* D. Van Nostrand Company, 1965.

[3] Y. Almog, *The stability of the normal state of superconductors in the presence of electric currents*, SIAM J. Math. Anal. 40 (2) (2008), pp. 824-850.

[4] Y. Almog and B. Helffer, *On the spectrum of non-selfadjoint Schrödinger operators with compact resolvent*, Comm. in PDE 40 (8) (2015), pp. 1441-1466.

[5] Y. Almog and R. Henry, *Spectral analysis of a complex Schrödinger operator in the semiclassical limit*, SIAM J. Math. Anal., 48 (4) (2016), pp. 2962-2993

[6] Y. Almog, D. S. Grebenkov, and B. Helffer, *Spectral analysis of a complex Schrödinger operator in the semiclassical limit: Robin and Transmission cases.* In preparation.

[7] A. Aslanyan and E. B. Davies, *Spectral instability for some Schrödinger operators*, Numer. Math. 85 (4) (2000), pp. 525-552.

[8] C. Bardos, D. S. Grebenkov, and A. Rozanova-Pierrat, *Short-time heat diffusion in compact domains with discontinuous transmission boundary conditions*, Math. Models Methods Appl. Sci. 26 (2016), pp. 59-110.

[9] K. Beauchard, B. Helffer, R. Henry and L. Robbiano, *Degenerate parabolic operators of Kolmogorov type with a geometric control condition*, ESAIM Control Optim. Calc. Var. 21 (2) (2015), pp. 487-512.

[10] W. Bordeaux-Montrieux, *Estimation de résolvante et construction de quasimode près du bord du pseudospectre,* arXiv:1301.3102v1.

[11] H. S. Carslaw and J. C. Jaeger, *Conduction of heat in solids,* Clarendon Press, Oxford, 1986.

[12] E. B. Davies, *One-Parameter Semi-groups,* Academic Press (1980).

[13] E. B. Davies, *Linear operators and their spectra*, Cambridge Studies in Advanced Mathematics (N106) p.250-251.

[14] N. Dencker, J. Sjöstrand, and M. Zworski, *Pseudospectra of (pseudo) differential operators*, Comm. Pure Appl. Math. 57 (2004), pp. 384-415.

[15] T. M. de Swiet and P. N. Sen, *Decay of nuclear magnetization by bounded diffusion in a constant field gradient*, J. Chem. Phys. 100 (1994), pp. 5597-5604.

[16] N. Dunford and J. T. Schwartz, *Linear operators. Part 2: Spectral theory, self adjoint operators in Hilbert space*, New York, 1963.

[17] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*, Oxford University Press, Oxford, 1987.

[18] P. B. Gilkey and K. Kirsten, *Heat Content Asymptotics with Transmittal and Transmission Boundary Conditions*, J. London Math. Soc. 68 (2003), pp. 431-443.

[19] D. S. Grebenkov, *NMR Survey of Reflected Brownian Motion*, Rev. Mod. Phys. 79 (2007), pp. 1077-1137.

[20] D. S. Grebenkov, *Pulsed-gradient spin-echo monitoring of restricted diffusion in multilayered structures*, J. Magn. Reson. 205 (2010), pp. 181-195.

[21] D. S. Grebenkov, D. V. Nguyen, and J.-R. Li, *Exploring diffusion across permeable barriers at high gradients. I. Narrow pulse approximation*, J. Magn. Reson. 248 (2014), pp. 153-163.

[22] D. S. Grebenkov, *Exploring diffusion across permeable barriers at high gradients. II. Localization regime*, J. Magn. Reson. 248 (2014), pp. 164-176.

[23] D. S. Grebenkov, Supplementary materials to [22].

[24] D. S. Grebenkov and B. Helffer. *On spectral properties of the Bloch-Torrey operator in two dimensions* (submitted; available online at https://arxiv.org/abs/1608.01925).

[25] B. Helffer, *On pseudo-spectral problems related to a time dependent model in superconductivity with electric current*, Confluentes Math. 3 (2) (2011), pp. 237-251.

[26] B. Helffer, *Spectral theory and its applications*, Cambridge University Press, 2013.

[27] B. Helffer and J. Sjöstrand, *From resolvent bounds to semigroup bounds, Appendix of a course by J. Sjöstrand*, Proceedings of the Evian Conference, 2009; preprint arXiv:1001.4171

[28] R. Henry, *Etude de l'opérateur de Airy complexe*, Mémoire de Master 2 (2010).

[29] R. Henry, *Spectral instability for the complex Airy operator and even non self-adjoint anharmonic oscillators*, J. Spectral theory 4 (2014), pp. 349-364.

[30] R. Henry, *On the semi-classical analysis of Schrödinger operators with purely imaginary electric potentials in a bounded domain*, ArXiv:1405.6183.

[31] T. Kato, *Perturbation Theory for Linear operators*, Springer-Verlag, Berlin New-York,

1966.
[32] J. Martinet, *Sur les propriétés spectrales d'opérateurs non-autoadjoints provenant de la mécanique des fluides*, Thèse de doctorat Univ. Paris-Sud, Déc. 2009. Appendix B.
[33] E. G. Novikov, D. van Dusschoten, and H. Van As, *Modeling of Self-Diffusion and Relaxation Time NMR in Multi-Compartment Systems*, J. Magn. Reson. 135 (1998), pp. 522-528.
[34] J. G. Powles, M. J. D. Mallett, G. Rickayzen, and W. A. B. Evans, *Exact analytic solutions for diffusion impeded by an infinite array of partially permeable barriers*, Proc. R. Soc. London. A 436 (1992), pp. 391-403.
[35] J. Sjöstrand, *Resolvent estimates for non-selfadjoint operators via semigroups. Around the research of Vladimir Maz'ya. III.* Int. Math. Ser. 13, Springer, New York (2010), pp. 359-384.
[36] S. D. Stoller, W. Happer, and F. J. Dyson, *Transverse spin relaxation in inhomogeneous magnetic fields*, Phys. Rev. A 44 (1991), pp. 7459-7477.
[37] J. E. Tanner, *Transient diffusion in a system partitioned by permeable barriers. Application to NMR measurements with a pulsed field gradient*, J. Chem. Phys. 69 (1978), pp. 1748-1754.
[38] E. C. Titchmarsh, *The theory of functions*, 2-nd edition, Oxford University Press, 1939.
[39] H. C. Torrey, *Bloch equations with diffusion terms*, Phys. Rev. 104 (1956), pp. 563-565.
[40] O. Vallée and M. Soares, *Airy functions and applications to physics.* Imperial College Press, London, 2004.