GROUND STATES OF A TWO PHASE MODEL WITH CROSS AND SELF ATTRACTIVE INTERACTIONS

M. CICALESE, L. DE LUCA, M. NOVAGA, AND M. PONSIGLIONE

ABSTRACT. We consider a variational model for two interacting species (or phases), subject to cross and self attractive forces. We show existence and several qualitative properties of minimizers. Depending on the strengths of the forces, different behaviors are possible: phase mixing or phase separation with nested or disjoint phases. In the case of Coulomb interaction forces, we characterize the ground state configurations.

CONTENTS

INTRODUCTION

Models of two or more interacting species find applications in several fields of science, such as physics, chemistry and biology. To cite a few examples one may think about the formation of bacterial colonies in biology [23], the self-assemble of nano-particles in physical chemistry [24], the problem of two species group consensus [13] as well as that of pedestrian dynamics [10]. The basic feature of all these models is the presence of competing forces aiming to drive two phases towards different shapes.

An interesting example of this phenomenon has been recently reported in [24]. There, it has been observed that, during the assembly process of two nano-scaled polyprotic macroions in a dilute aqueous solution, the system may be driven towards phase segregation as opposite to phase mixtures via a complex self-recognition mechanism involving multiple scales optimization.

Far from thinking to propose realistic models for these complex mechanisms, we aim at reproducing such limit behaviors while keeping the number of parameters as small as possible. We propose and study a toy model for two interacting phases subject to self and cross attractive forces depending only on the distance between particles. Such a model may be introduced as follows. Two phases, represented by two subsets of $\mathbb{R}^N$, say E_1 and E_2, with masses m_1 and m_2 respectively, interact both with themselves and with the other phase trying to minimize an energy of the form

$$(0.1) \qquad \mathcal{F}(E_1, E_2) = \sum_{i,j=1}^{2} J_{K_{ij}}(E_i, E_j).$$

Here

$$(0.2) \qquad J_{K_{ij}}(E_i, E_j) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \chi_{E_i}(x) \, \chi_{E_j}(y) \, K_{ij}(x - y) \, \mathrm{d}x \, \mathrm{d}y$$

is a nonlocal interaction energy with interaction potential $K_{ij} : \mathbb{R}^N \to \mathbb{R}$. Energy functionals of this type have been considered by many authors in the context of nonlinear aggregation-diffusion problems, modeling biological swarming and crowd congestion (see [32, 6, 8, 12, 26, 28] and the references therein).

In the present paper we initiate the analysis of the ground states of the energy functional $\mathcal{F}$ assuming that for $i, j \in \{1, 2\}$ the interaction forces, still having different intensities, obey the same nonlocal law. More precisely, we consider $K \in L^1_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R})$ a non-increasing radially symmetric interaction potential and restrict our analysis to those $K_{ij} = c_{ij} K$. Moreover, we assume that the interactions are attractive, i.e., $c_{i,j} \leq 0$. Without this assumption, different phenomena may appear, related to loss of mass at infinity. As a consequence, the minimization problem is in general ill-posed, and requires specific cares. One possibility would consist in adding some confinement conditions. In [3], the authors propose a different kind of problem: they focus on the case $c_{11} = c_{22} = 1$, $c_{12} + c_{21} = -2$, fix E_1 and study the minimization of (0.1) as a function of E_2. They prove that such a problem admits a solution if and only if $m_2 \leq m_1$. Similar threshold phenomena appear in energetic models for di-block copolymers, where a confining perimeter term and a repulsive force compete [2, 11, 14, 16, 18, 19, 25] as well as in attractive/repulsive Lennard-Jones-type models (see e.g., [4, 7, 9, 20, 21, 31] and the references therein).

Let us go back to the case of attractive interactions $c_{ij} \leq 0$ considered in this paper. We will see that, also in this case, the minimization problem above is actually ill-posed. Indeed, in Proposition 2.9 and Theorem 3.9, we will show that if $|c_{11}|, |c_{22}|$ are small enough, any minimizing sequence wants to mix the two phases. We are then led to consider a relaxed version of the problem above where the notion of phase is weakened to allow local mixing. Now the phases are described in terms of their densities $f_1, f_2 \in L^1(\mathbb{R}^N; [0, 1])$, so that $\int_{\mathbb{R}^N} f_i(x) \, \mathrm{d}x = m_i$ and the functional becomes

$$(0.3) \qquad \mathcal{E}_K(f_1, f_2) = c_{11} \, J_K(f_1, f_1) + c_{22} \, J_K(f_2, f_2) + (c_{12} + c_{21}) \, J_K(f_1, f_2),$$

where $J_K(f_i, f_j)$ has the same form of (0.2) with K and f_i in place of K_{ij} and χ_{E_i}, respectively.

For all masses $m_i > 0$ and all $c_{ij} \le 0$, we prove existence of the minimizers of $\mathcal{E}_K$ under the constraint $f_1 + f_2 \le 1$ (Theorem 1.8). Such a constraint is inherited by the original problem, naturally arising from the relaxation procedure, but has also a clear physical meaning. Indeed, if we interpret the densities f_i as proportional to the number of particles per unit volume on a certain mesoscopic ball of a lattice gas model, the condition reflects the fact that two particles are not allowed to occupy the same elementary cell. Note that for a slightly different problem in the one dimensional case, a similar existence result has appeared in [17].

In the case $c_{12} = c_{21} = 0$, our problem reduces to two independent one-phase problems given by

$$\min_{\substack{f_i \in L^1(\mathbb{R}^N;[0,1]) \\ \int_{\mathbb{R}^N} f_i(x)\,\mathrm{d}x = m_i}} c_{ii}\, J_K(f_i, f_i) \quad \text{for } i = 1, 2.$$

If $c_{ii} < 0$, it is well known that the minimizer above is (the characteristic funciton of) a ball having mass equal to m_i (see [30, 15] or Lemma 2.1). Therefore, we focus on the case $c_{12} + c_{21} < 0$. Clearly, by the scaling and symmetry properties of the energy, it is not restrictive to assume $c_{12} = c_{21} = -1$. With this interaction term in the energy the geometry of the phases becomes a more delicate issue and it drastically depends on the strength of the interaction constants c_{11} and c_{22}. On one hand, if the cross interaction forces prevail, *phase mixing* occurs, that is, a new phase appears which is a combination of the two pure phases. On the other hand, if one of the two self interaction forces is sufficiently strong, *phase segregation* occurs, with the presence of two pure phases which can be nested or adjacent, depending on the strength of the other force. The latter behavior is in a certain sense reminiscent of clusters of two phases in an infinite ambient phase, minimizing an inhomogeneous perimeter functional with surface tension depending on the two touching phases [1]. In this case the mixing of phases is impossible but, depending on the strength of the surfaces tensions, minimizers may exhibit disjoint or nested phases [27].

Our analysis focuses also on qualitative properties of solutions. In some cases, we have determined the explicit geometry of the phases of the minimizers. Such an analysis is almost complete for the Coulomb interaction kernel.

We first describe the case of general kernels (see Figure 1). First, consider the case $c_{11} + c_{22} > -2$, that we will call the *weakly attractive case*. In this case, the shape of minimizers is not explicit, except for $(c_{11} + 1)\, m_1 = (c_{22} + 1)\, m_2$ and K positive definite. If this occurs, the unique minimizer is given by $(f_1, f_2) = (\frac{m_1}{m_1 + m_2} \chi_B, \frac{m_2}{m_1 + m_2} \chi_B)$, where B is a ball with $|B| = m_1 + m_2$ (Proposition 2.9).

The *strongly attractive case* $c_{11} + c_{22} \le -2$ (Theorem 2.4) needs to be classified into the four subcases listed below. If $c_{11} = c_{22} = -1$, the problem is extremely degenerate, i.e., the minimizers are given by all the pairs (f_1, f_2), with $f_1 + f_2 = \chi_B$. If $c_{11} = -1$ and $c_{22} < -1$, then the minimizers of the problem are the pairs (f_1, f_2), where $f_1 + f_2 = \chi_B$ and f_2 is (the characteristic function of) a ball contained in B (not necessarily concentric). If $c_{22} < -1 < c_{11}$, then the minimizer is unique and it is given by a ball and a concentric annulus around it. Finally, for $c_{11}, c_{22} < -1$, the minimizer is fully characterized only in the one dimensional case and it is given by the two tangent balls (namely segments).
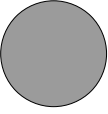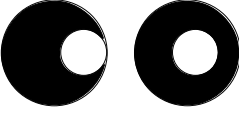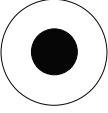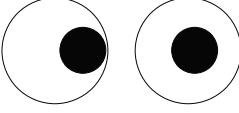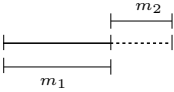
general K	$-1 < c_{22} \leq 0$	$c_{22} = -1$	$c_{22} < -1$
$-1 < c_{11} \leq 0$	if $(c_{11}+1)m_1 = (c_{22}+1)m_2$ and K positive definite	?	if $c_{11} + c_{22} \leq -2$
$c_{11} = -1$	?		
$c_{11} < -1$	if $c_{11} + c_{22} \leq -2$		m_2 m_1 if $N = 1$

Figure 1. The phase f_1 is the black one, whereas the phase f_2 is white. The grey region represents the mixing of the two phases. The gradational shaded ball in the central box represents the extremely degenerate character of minimizers for $c_{11} = c_{22} = -1$.

As for the Coulomb interactions (see Figure 2), we have fully characterized the minimizers also in the weakly attractive case.
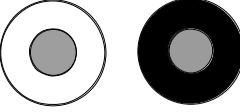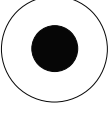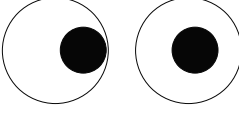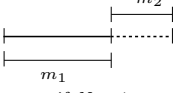
K Coulomb	$-1 < c_{22} \leq 0$	$c_{22} = -1$	$c_{22} < -1$
$-1 < c_{11} \leq 0$			
$c_{11} = -1$			
$c_{11} < -1$			m_2 m_1 if $N = 1$

Figure 2. The phase f_1 is the black one, the phase f_2 is white. The grey region represents the mixing of the two phases.

We have proven (Theorem 3.9, Corollaries 3.5 and 3.12) that if $-1 < c_{11}, c_{22} < 0$ the minimizer is given by an interior ball in which f_1 and f_2 mix each other with specific volume fractions, according with their self attraction coefficients, and a concentric annulus where only the remaining homogeneous phase is present. If $c_{22} \leq -1 < c_{11}$, then the minimizer is unique and it is given by a ball and a concentric annulus around it. In this respect, for $c_{22} \leq -1 < c_{11}$ the solution is the same in the weakly and in the strongly attractive cases.

Clearly, in the strongly attractive case the analysis done for general kernels applies in particular to the case of Coulomb interactions. The shape of minimizers for $c_{11}, c_{22} < -1$ is still open, but we can exclude the presence of a mixing phase (Proposition 3.2). The determination of the shape of the two phases in this case seems to be a challenging problem, that could be explored through numerical methods. Switching the roles of c_{11}, c_{22}, f_1, f_2 in the discussion above, the description of minimizers extend to all the other cases not explicitly mentioned.

We remark that the analysis for the Coulomb interaction kernel is much richer, since we can exploit methods and tools of potential theory such as maximum principles. The characterization of minimizers in the weakly attractive case reduces to the case $c_{11} = c_{22} = 0$, considered in Theorem 3.9. Even if the two phases interact only through a cross attractive force,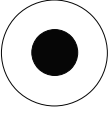 this case turns out to be non trivial. The strategy to tackle this problem is based on a rearrangement argument that resembles the Talenti inequality. This is the content of Lemma 3.8, which establishes that, given a charge configuration f which generates a potential V, one can rearrange the masses on every superlevel of V, so that the new potential turns out to be greater than the radially symmetric rearrangement V^* of V.
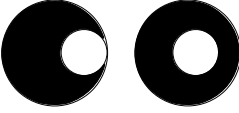
The plan of the paper is the following. In Section 1 we introduce the nonlocal model and we prove existence and compactness of minimizers. In Section 2 we show some qualitative properties of minimizers and we characterize them explicitly in some strongly attractive cases. Eventually, in Section 3 we study in detail the case of Coulomb interactions.

1. The variational problem

In this section we state our variational problem, proving existence and some qualitative properties of the minimizers.

1.1. **Description of the model.** We first introduce a functional modeling the interaction between two non-self-repulsive and mutually attractive species.

Let $N \in \mathbb{N}$ and let $K : \mathbb{R}^N \to \mathbb{R}$ be a non-increasing radially symmetric interaction potential, with $K \in L^1_{\text{loc}}(\mathbb{R}^N)$. For any pair of measurable sets (A, B) with finite measure, we set

$$(1.1) \qquad J_K(A, B) := \int_A \int_B K(x - y) \, \mathrm{d}x \, \mathrm{d}y$$

and we notice that, by the assumptions on K, the functional J_K is well defined and takes values in $\mathbb{R} \cup \{-\infty\}$.

Given $c_{ij} \leq 0$ for $i, j = 1, 2$ and $m_1, m_2 > 0$, we are interested in finding the minimizers of the functional

$$(1.2) \quad \mathcal{F}_K(E_1, E_2) := c_{11} J_K(E_1, E_1) + c_{22} J_K(E_2, E_2) + (c_{12} + c_{21}) J_K(E_1, E_2)$$

among all the pairs of measurable sets (E_1, E_2) with $|E_1| = m_1$ and $|E_2| = m_2$. Here E_1 and E_2 represent two species with masses m_1 and m_2 respectively, c_{11}, c_{22} the autointeraction and $c_{12} + c_{21}$ the cross-interaction coefficients.

As mentioned in the Introduction, for $c_{12} + c_{21} = 0$ the problem decouples into two independent minimization problems, one for each phase. These are of the form

$$\min \{-J_K(E, E) : |E| = m\}.$$

By the Riesz inequality [30] (see Lemma 2.1), such a one-phase problem is well known to be solved by a ball [15]. As a consequence we focus on the case $c_{12} + c_{21} < 0$ and furthermore, without loss of generality, we set $c_{12} + c_{21} = -2$. From Proposition 2.9 and Theorem 3.9, it will follow that if $|c_{11}|, |c_{22}|$ are small enough, the minimum problem above does not admit in general a minimizer. Roughly speaking, the reason is that, in some cases, any minimizing sequence wants to mix the two phases. As a result, we are led to consider a *relaxed problem.* More precisely, according with (1.1), for any $f_1, f_2 \in L^1(\mathbb{R}^N)$ we set

$$J_K(f_1, f_2) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_1(x) f_2(y) K(x - y) \, \mathrm{d}x \, \mathrm{d}y.$$

Then, we consider the functional $\mathcal{E}_K^{c_{11}, c_{22}} : L^1(\mathbb{R}^N; \mathbb{R}^+) \times L^1(\mathbb{R}^N; \mathbb{R}^+) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$(1.3) \qquad \mathcal{E}_K^{c_{11}, c_{22}}(f_1, f_2) = c_{11} J_K(f_1, f_1) + c_{22} J_K(f_2, f_2) - 2 J_K(f_1, f_2).$$

We introduce the class of admissible densities $\mathcal{A}_{m_1, m_2}$ defined by

$$(1.4) \quad \mathcal{A}_{m_1, m_2} := \Big\{ (f_1, f_2) \in L^1(\mathbb{R}^N; \mathbb{R}^+) \times L^1(\mathbb{R}^N; \mathbb{R}^+) :$$

$$\int_{\mathbb{R}^N} f_i(x) \, \mathrm{d}x = m_i \text{ for } i = 1, 2, \ f_1(x) + f_2(x) \le 1 \text{ for a.e. } x \in \mathbb{R}^N \Big\}.$$

It is easy to see that for any $(f_1, f_2) \in \mathcal{A}_{m_1, m_2}$

$$\mathcal{E}_K^{c_{11}, c_{22}}(f_1, f_2) = \inf \liminf_{n \to \infty} \mathcal{F}_K(E_1^n, E_2^n),$$

where the infimum is taken among all sequences $\{E_i^n\}$ ($i = 1, 2$) with $|E_i^n| = m_i$ and such that $\chi_{E_i^n}$ converge tightly to f_i. We also observe that, if the kernel K is bounded at infinity, then the energy is continuous with respect to tight convergence: if $f_i^n \overset{*}{\rightharpoonup} f_i$ and $\|f_i^n\|_1 \to \|f_i\|_1$ for $i = 1, 2$, then $\mathcal{E}_K^{c_{11}, c_{22}}(f_1^n, f_2^n) \to \mathcal{E}_K^{c_{11}, c_{22}}(f_1, f_2)$.

For $i = 1, 2$, set $V_i := f_i * K$, so that we can write

$$
\begin{aligned}
\mathcal{E}_K^{c_{11}, c_{22}}(f_1, f_2) \ &= \ c_{11} \int_{\mathbb{R}^N} f_1(x) V_1(x) \, \mathrm{d}x + c_{22} \int_{\mathbb{R}^N} f_2(x) V_2(x) \, \mathrm{d}x \\
&\quad -2 \int_{\mathbb{R}^N} f_1(x) V_2(x) \, \mathrm{d}x \\
(1.5) \qquad &= \ c_{11} \int_{\mathbb{R}^N} f_1(x) V_1(x) \, \mathrm{d}x + c_{22} \int_{\mathbb{R}^N} f_2(x) V_2(x) \, \mathrm{d}x \\
&\quad -2 \int_{\mathbb{R}^N} f_2(x) V_1(x) \, \mathrm{d}x.
\end{aligned}
$$

We now recall the definitions of the main classes of kernels we will focus on. We say that the kernel K is *positive definite* if

$$(1.6) \qquad J_K(\varphi, \varphi) \ge 0 \ \ \forall \varphi \in L^1(\mathbb{R}^N) \ \text{ and}$$

$$J_K(\varphi, \varphi) = 0 \text{ if and only if } \varphi = 0 \text{ a.e. in } \mathbb{R}^N.$$

We denote by K_{C_N} the Coulomb kernel in $\mathbb{R}^N$, defined by

$$
(1.7) \qquad K_{C_N}(x) := \begin{cases} -\dfrac{1}{2}\,|x| & \text{for } N = 1, \\[2mm] -\dfrac{1}{2\pi}\log|x| & \text{for } N = 2, \\[2mm] \dfrac{1}{(N-2)\,\omega_N}\,\dfrac{1}{|x|^{N-2}} & \text{for } N \geq 3, \end{cases}
$$

where ω_N is the N-dimensional measure of the unitary ball in $\mathbb{R}^N$. By definition, $-\Delta K_{C_N} = \delta_0$ for any N so that $-\Delta V_i(x) = f_i(x)$. In the following Remark we list some properties of the Coulomb kernels that will be useful in the following.

Remark 1.1. By [22, Theorem 1.15] K_{C_N} is positive definite for $N \geq 3$ but not for $N = 1, 2$. Nevertheless, by [22, Theorem 1.16], for any $\varphi \in L^1(\mathbb{R}^2)$ with compact support and $\int_{\mathbb{R}^2} \varphi(x)\, dx = 0$, we have

$$
J_{K_{C_2}}(\varphi, \varphi) \geq 0,
$$

where equality holds true if and only if $\varphi(x) = 0$ for a.e. $x \in \mathbb{R}^2$. Moreover, it is easy to see that the same result holds true also for K_{C_1}.

1.2. **First and second variations.** For any given $(f_1, f_2) \in \mathcal{A}_{m_1, m_2}$ set

$$
(1.8) \quad G_i := \{x \in \mathbb{R}^N : 0 < f_i(x) < 1\}, \quad F_i := \{x \in \mathbb{R}^N : f_i(x) = 1\}, \quad i = 1, 2.
$$

Moreover, we set

$$
(1.9) \qquad\qquad S := \{x \in \mathbb{R}^N : f_1(x) + f_2(x) = 1\}.
$$

Lemma 1.2 (**First variation**). *Let (f_1, f_2) be a minimizer of $\mathcal{E}_K^{c_{11}, c_{22}}$ in $\mathcal{A}_{m_1, m_2}$. Let $i, j \in \{1, 2\}$ with $i \neq j$. For any $\varphi_i, \psi \in L^1(\mathbb{R}^N; \mathbb{R}^+)$ with $\varphi_i = 0$ a.e. in $\mathbb{R}^N \setminus (G_i \cup F_i)$, $\psi = 0$ a.e. in S, and $\int_{\mathbb{R}^N} \varphi_i(x)\, dx = \int_{\mathbb{R}^N} \psi(x)\, dx$, we have*

$$
(1.10) \qquad \int_{\mathbb{R}^N} (\psi(x) - \varphi_i(x))(c_{ii} V_i(x) - V_j(x))\, dx \geq 0.
$$

As a consequence,

$$
(1.11) \qquad\qquad c_{ii} V_i - V_j = \gamma_i \text{ a.e. in } G_i \setminus S.
$$

for some constant $\gamma_i \in \mathbb{R}$.

Proof. To simplify notation we prove the claim for $i = 1$ and $j = 2$. The proof of the other case can be obtained by switching f_1 with f_2 and c_{11} with c_{22}. Without loss of generality, we assume $\varphi_1, \psi \in L^\infty(\mathbb{R}^N; \mathbb{R}^+)$. For any $\varepsilon > 0$, we set

$$
A^\varepsilon := \{x \in G_1 \cup F_1 : \varepsilon < f_1(x) \leq 1\}, \quad B^\varepsilon := \{x \in \mathbb{R}^N : f_1(x) + f_2(x) < 1 - \varepsilon\}.
$$

It is easy to see that $A^\varepsilon \nearrow (G_1 \cup F_1)$, $B^\varepsilon \nearrow (\mathbb{R}^N \setminus S)$. Set

$$
\varphi_1^\varepsilon := \frac{\int_{\mathbb{R}^N} \varphi_1(x)\, dx}{\int_{A^\varepsilon} \varphi_1(x)\, dx} \cdot \varphi_1 \llcorner A^\varepsilon, \quad \psi^\varepsilon := \frac{\int_{\mathbb{R}^N} \psi(x)\, dx}{\int_{B^\varepsilon} \psi(x)\, dx} \cdot \psi \llcorner B^\varepsilon;
$$

then $\|\varphi_1^\varepsilon - \varphi_1\|_{L^1} \to 0$ and $\|\psi^\varepsilon - \psi\|_{L^1} \to 0$. For $t > 0$ small enough, $(f_1 + t(\psi^\varepsilon - \varphi_1^\varepsilon), f_2) \in \mathcal{A}_{m_1,m_2}$ and, since (f_1, f_2) is a minimizer for $\mathcal{E}_K^{c_{11},c_{22}}$, we have:

$$0 \le \lim_{t \to 0} \frac{\mathcal{E}_K^{c_{11},c_{22}}(f_1 + t(\psi^\varepsilon - \varphi_1^\varepsilon), f_2) - \mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2)}{t}$$

$$= \int_{\mathbb{R}^N} 2(\psi^\varepsilon(x) - \varphi_1^\varepsilon(x))\,(c_{11}V_1(x) - V_2(x))\,\mathrm{d}x.$$

As $\varepsilon \to 0$, we get the claim.

Finally, taking $\varphi_1 = \psi \equiv 0$ in $S \setminus G_1$ we are allowed to switch the roles of ψ and φ_1 in (1.10), obtaining (1.11). $\qquad\square$

From now on, given any subset E of $\mathbb{R}^N$, we will always assume that E coincides with the set of the Lebesgue points of its characteristic function. In this way, ∂E will be well defined and will always refer to this precise representative of E.

Corollary 1.3. *Let (f_1, f_2) be a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$. Then, for any $\varphi_1, \varphi_2 \in L^1(\mathbb{R}^N; \mathbb{R}^+)$ with $\varphi_i = 0$ a.e. in $\mathbb{R}^N \setminus (G_i \cup F_i)$ for $i = 1, 2$, and $\int_{\mathbb{R}^N} \varphi_1(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \varphi_2(x)\,\mathrm{d}x$, we have*

$$(1.12) \qquad \int_{\mathbb{R}^N} (\varphi_2(x) - \varphi_1(x))((c_{11} + 1)V_1(x) - (c_{22} + 1)V_2(x))\,\mathrm{d}x \ge 0.$$

In particular, for any $x_1 \in \overline{G_1 \cup F_1}$ and $x_2 \in \overline{G_2 \cup F_2}$, we have

$$(1.13) \quad (c_{11} + 1)V_1(x_1) - (c_{22} + 1)V_2(x_1) \le (c_{11} + 1)V_1(x_2) - (c_{22} + 1)V_2(x_2).$$

Moreover,

$$(1.14) \qquad (c_{11} + 1)V_1 - (c_{22} + 1)V_2 = \gamma \qquad a.e.\ in\ G_1 \cap G_2,$$

for some constant $\gamma \in \mathbb{R}$.

Proof. Notice that if $\varphi_i = 0$ a.e. in $\mathbb{R}^N \setminus G_i$ (for $i = 1, 2$), (1.12) is obtained by summing (1.10) for $i = 1, j = 2$ with $\psi = \varphi_2$ and for $i = 2, j = 1$ with $\psi = \varphi_1$. To treat the general case, it is enough to consider the variation $(f_1 + t(\varphi_2 - \varphi_1), f_2 - t(\varphi_2 - \varphi_1)) \in \mathcal{A}_{m_1,m_2}$ for t small enough. Then (1.12) and (1.13) are equivalent to the fact that the first variation of the energy is nonnegative.

Finally, taking $\varphi_1, \varphi_2 \in L_c^\infty(\mathbb{R}^N; \mathbb{R}^+)$, with $\varphi_1 = \varphi_2 = 0$ a.e. in $\mathbb{R}^N \setminus (G_1 \cap G_2)$ and $\int_{\mathbb{R}^N} \varphi_1(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \varphi_2(x)\,\mathrm{d}x$, we have that (1.12) holds true also switching φ_1 with φ_2, whence we get (1.14). $\qquad\square$

Arguing as in the proof of Corollary 1.3 (or using (1.10) and exploiting the continuity of V_i), one can easily prove the following stationarity equations for the boundaries of the two phases (see also [29, Eqs. (1.2)–(1.4)] for similar conditions in a related model for triblock copolymers).

Corollary 1.4. *Let (f_1, f_2) be a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ and assume that $f_i = \chi_{E_i}$ for some sets $E_i \subset \mathbb{R}^2$. Then, the following equalities hold:*

$$(1.15) \qquad\qquad c_{11}V_1 - V_2 \;=\; c_1 \qquad\qquad on\ \partial E_1 \setminus \partial E_2$$

$$(1.16) \qquad\qquad c_{22}V_2 - V_1 \;=\; c_2 \qquad\qquad on\ \partial E_2 \setminus \partial E_1$$

$$(1.17) \quad (c_{11} + 1)V_1 - (c_{22} + 1)V_2 \;=\; c_1 - c_2 \qquad on\ \partial E_1 \cap \partial E_1,$$

for some $c_1, c_2 \in \mathbb{R}$.

Lemma 1.5 (**Second Variation**). *Let (f_1, f_2) be a minimizer of $\mathcal{E}_K^{c_{11}, c_{22}}$ in $\mathcal{A}_{m_1, m_2}$. Then for any $\varphi \in L^1(\mathbb{R}^N; \mathbb{R})$ with $\varphi = 0$ in $\mathbb{R}^N \setminus (G_1 \cap G_2)$ and $\int_{\mathbb{R}^N} \varphi = 0$, we have*

$$(1.18) \qquad (c_{11} + c_{22} + 2) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x - y) \, \varphi(x) \, \varphi(y) \, \mathrm{d}x \, \mathrm{d}y \geq 0.$$

Proof. Without loss of generality assume that $\varphi \in L^\infty(\mathbb{R}^N; \mathbb{R})$. Since $(f_1 + t\varphi, f_2 - t\varphi) \in \mathcal{A}_{m_1, m_2}$ for t small enough, the claim follows by the positiveness of the second variation at (f_1, f_2), which is assumed to be a minimizer. The computations are left to the reader. $\square$

1.3. **Existence of minimizers.** Here we prove that for every $c_{11}, c_{22} \leq 0$, the functional $\mathcal{E}_K^{c_{11}, c_{22}}$ defined in (1.3) admits a minimizer in $\mathcal{A}_{m_1, m_2}$.

For any $m_1, m_2 > 0$, we set

$$I_{m_1, m_2}^{c_{11}, c_{22}} := \inf_{(f_1, f_2) \in \mathcal{A}_{m_1, m_2}} \mathcal{E}_K^{c_{11}, c_{22}}(f_1, f_2)$$

and we extend this definition to the case of possibly null masses, by setting

$$I_{m_1, m_2}^{c_{11}, c_{22}} := \begin{cases} \min_{\substack{f_i \in L^1(\mathbb{R}^N; [0,1]) \\ \int_{\mathbb{R}^N} f_i(x) \, \mathrm{d}x = m_i}} c_{ii} \, J_K(f_i, f_i) & \text{if } m_i > 0 \text{ and } m_j = 0, \\ 0 & \text{if } m_1 = m_2 = 0. \end{cases}$$

The following two lemmas state monotonicity and sub-additivity properties of the energy with respect to the masses m_1, m_2, for nonnegative kernels. Their proofs can be easily obtained exploiting the fact that the two phases attract each other: adding masses or moving back masses going to infinity decreases the energy. The details of the proofs are left to the reader.

Lemma 1.6. *Assume that $K(x) \geq 0$ for all $x \in \mathbb{R}^N$. For any $m_1 \geq \tilde{m}_1 \geq 0$ and $m_2 \geq \tilde{m}_2 \geq 0$ we have*

$$I_{m_1, m_2}^{c_{11}, c_{22}} \leq I_{\tilde{m}_1, \tilde{m}_2}^{c_{11}, c_{22}}.$$

Moreover, if $m_1, m_2 > 0$, equality holds true if and only if $m_i = \tilde{m}_i$ for $i = 1, 2$.

Lemma 1.7. *Assume that $K(x) \geq 0$ for all $x \in \mathbb{R}^N$. Let $\{m_1^l\}, \{m_2^l\}$ be two nonnegative sequences such that $0 \leq \tilde{m}_i := \sum_{l \in \mathbb{N}} m_i^l < +\infty$ for $i = 1, 2$. Then*

$$(1.19) \qquad \sum_{l \in \mathbb{N}} I_{m_1^l, m_2^l}^{c_{11}, c_{22}} \geq I_{\tilde{m}_1, \tilde{m}_2}^{c_{11}, c_{22}}.$$

Moreover, if $\tilde{m}_1, \tilde{m}_2 > 0$, then equality holds true if and only if $\tilde{m}_i^l \equiv 0$ for any $l \neq \bar{l}$, for some $\bar{l} \in \mathbb{N}$ and for $i = 1, 2$.

Theorem 1.8. *Let $c_{11}, c_{22} \leq 0$. Then, the functional $\mathcal{E}_K^{c_{11}, c_{22}}$ defined in (1.3) admits a minimizer in $\mathcal{A}_{m_1, m_2}$. More precisely, let $\{(f_{1,n}, f_{2,n})\}$ be a minimizing sequence. Then, there exists a sequence of translations $\{\tau_n\} \subset \mathbb{R}^N$ such that (up to a subsequence) $f_{i,n}(\cdot - \tau_n) \to f_i$ tightly for some $(f_1, f_2) \in \mathcal{A}_{m_1, m_2}$ which minimizes $\mathcal{E}_K^{c_{11}, c_{22}}$.*

Proof. We distinguish between two cases.

First case: $\lim_{|x| \to +\infty} K(x) = -\infty$. For every $\varepsilon > 0$ and for every pair of sets $A_{1,n}, A_{2,n} \subset \mathbb{R}^N$ such that

$$\int_{A_{i,n}} f_{i,n}(x) \, \mathrm{d}x \geq \varepsilon,$$

we have $\mathrm{dist}(A_{1,n}, A_{2,n}) \leq C$ for some C independent of n; otherwise, we would clearly have $-J_K(f_{1,n}, f_{2,n}) \to +\infty$. As a consequence, by the triangular inequality we deduce that for every pair of sets $A_{i,n}, B_{i,n} \subset \mathbb{R}^N$ such that

$$\int_{A_{i,n}} f_{i,n}(x) \, \mathrm{d}x \geq \varepsilon, \qquad \int_{B_{i,n}} f_{i,n}(x) \, \mathrm{d}x \geq \varepsilon,$$

we have $\mathrm{dist}(A_{i,n}, B_{i,n}) \leq C$ for some C independent of n. As a result there exists $\{\tau_n\} \subset \mathbb{R}$ such that, up to a subsequence, $f_{i,n}(\cdot - \tau_n)$ tightly converge to some f_i in L^1. By the lower semicontinuity of $\mathcal{E}_K^{c_{11}, c_{22}}$ with respect to the tight convergence, we conclude that (f_1, f_2) is a minimizer of $\mathcal{E}_K^{c_{11}, c_{22}}$ in $\mathcal{A}_{m_1, m_2}$.

Second case: $\lim_{|x| \to +\infty} K(x) + C = 0$ *for some* $C \in \mathbb{R}$. For simplicity, we assume that $C = 0$, since additive constants in the kernel bring only an additive constant in the total energy. Set $Q_0 := [0,1]^N$, and for every $z \in \mathbb{Z}^N$, let $Q^z := z + Q_0$ and $m_{i,n}^z := \int_{Q^z} f_{i,n}(x) \, \mathrm{d}x$. For any given $\varepsilon > 0$, we set

$$\mathcal{I}_{\varepsilon,n} := \{z \in \mathbb{Z}^N : m_{i,n}^z \leq \varepsilon, \, i = 1, 2\}, \qquad \mathcal{J}_{\varepsilon,n} := \{z \in \mathbb{Z}^N : \max_i m_{i,n}^z > \varepsilon\},$$

$$A_{\varepsilon,n} := \bigcup_{z \in \mathcal{I}_{\varepsilon,n}} Q^z, \qquad g_{i,n}^\varepsilon := f_{i,n} \chi_{A_{\varepsilon,n}},$$

$$E_{\varepsilon,n} := \bigcup_{z \in \mathcal{J}_{\varepsilon,n}} Q^z, \qquad f_{i,n}^\varepsilon := f_{i,n} \chi_{E_{\varepsilon,n}}$$

We first prove that

$$(1.20) \quad J_K(g_{1,n}^\varepsilon, f_{1,n}) + J_K(g_{2,n}^\varepsilon, f_{2,n}) + J_K(g_{1,n}^\varepsilon, f_{2,n}) + J_K(f_{1,n}, g_{2,n}^\varepsilon) \leq r(\varepsilon),$$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$. We show only that $J_K(g_{1,n}^\varepsilon, f_{2,n}) < r(\varepsilon)$ (the other cases being analogous). For every fixed $R \in \mathbb{N}$ we have

$$(1.21) \quad J_K(g_{1,n}^\varepsilon, f_{2,n}) = \sum_{z \in \mathcal{I}_{\varepsilon,n}} \sum_{w \in \mathbb{Z}^N} J_K(f_{1,n} \chi_{Q^z}, f_{2,n} \chi_{Q^w})$$

$$= \sum_{z \in \mathcal{I}_{\varepsilon,n}, w \in \mathbb{Z}^N : |z-w| \leq R} J_K(f_{1,n} \chi_{Q^z}, f_{2,n} \chi_{Q^w})$$

$$+ \sum_{z \in \mathcal{I}_{\varepsilon,n}, w \in \mathbb{Z}^N : |z-w| > R} J_K(f_{1,n} \chi_{Q^z}, f_{2,n} \chi_{Q^w}).$$

Using that K is integrable and by Riesz inequality (see Lemma 2.1), it is easy to see that

$$J_K(f_{1,n} \chi_{Q^z}, f_{2,n} \chi_{Q^w}) \leq h(m_{1,n}^z) \, m_{2,n}^w,$$

where $h(t) := \int_{B^t} K(x) \, dx$ with B^t the ball centered at the origin and with mass t, so that $\lim_{t \to 0} h(t) = 0$. We deduce that the first addendum in (1.21) tends to zero as $\varepsilon \to 0$ (for R fixed). Moreover, the second addendum is bounded (uniformly with respect to ε) from above by a function $\omega(R)$, such that $\omega(R) \to 0$ as $R \to \infty$. This completes the proof of (1.20).

By the mass constraints on f_i we have that $\sharp \mathcal{J}_{\varepsilon,n} \leq \frac{m_1 + m_2}{\varepsilon}$. Therefore, up to a subsequence, we can always write $\mathcal{J}_{\varepsilon,n} = \cup_{l=1}^{H_\varepsilon} J_{\varepsilon,n}^l$ for some $H_\varepsilon \leq \frac{m_1 + m_2}{\varepsilon}$, where $J_{\varepsilon,n}^l$ are pairwise disjoint and satisfy:

  (1) for every l, $\mathrm{diam}(J_{\varepsilon,n}^l) \leq M_\varepsilon$ for some $M_\varepsilon \in \mathbb{R}$ independent of n;

(2) for every $l_1 \neq l_2$, $\mathrm{dist}(J_{\varepsilon,n}^{l_1}, J_{\varepsilon,n}^{l_2}) \to \infty$ as $n \to \infty$.

Notice that by (1.20) we deduce that, for ε small enough, $\mathcal{J}_{\varepsilon,n} \neq \emptyset$ and $H_\varepsilon \geq 1$ (otherwise $I_{m_1,m_2}^{c_{11},c_{22}}$ would be zero). Set $f_{i,n}^{\varepsilon,l} := f_{i,n}^{\varepsilon} \llcorner \bigcup_{z \in J_{\varepsilon,n}^l} Q^z$ for $i = 1, 2$ and for every $l = 1, \ldots, H_\varepsilon$. There exists a translation $\tau_{l,n}$ such that, up to a subsequence, $f_{i,n}^{\varepsilon,l}(\cdot - \tau_{l,n})$ converge tightly to some $f_i^{\varepsilon,l}$. By (1.20), recalling that $\lim_{|x| \to +\infty} K(x) = 0$ and using the continuity of the energy with respect to the tight convergence, we have

$$(1.22) \quad \lim_n \mathcal{E}_K^{c_{11},c_{22}}(f_{1,n}, f_{2,n}) \geq \limsup_n \mathcal{E}_K^{c_{11},c_{22}}(f_{1,n}^{\varepsilon}, f_{2,n}^{\varepsilon}) - r(\varepsilon)$$

$$= \limsup_n \sum_{l=1}^{H_\varepsilon} \mathcal{E}_K^{c_{11},c_{22}}(f_{1,n}^{\varepsilon,l}, f_{2,n}^{\varepsilon,l}) - r(\varepsilon) \geq \sum_{l=1}^{H_\varepsilon} \mathcal{E}_K^{c_{11},c_{22}}(f_1^{\varepsilon,l}, f_2^{\varepsilon,l}) - r(\varepsilon).$$

Let now $\{\varepsilon_k\}$ be a decreasing sequence converging to zero as $k \to \infty$. We notice that H_{ε_k} is nondecreasing with respect to k and then $H_{\varepsilon_k} \to H \in \mathbb{N} \cup \infty$. We can always choose the labels in such a way that $\{f_{i,n}^{\varepsilon_k,l}\}$, and so also their limits $f_i^{\varepsilon_k,l}$, are monotone with respect to k. As a consequence, it is not restrictive to assume that the translation vectors $\tau_{l,n}$ are independent of ε. By monotonicity, $f_i^{\varepsilon_k,l}$ converge strongly in L^1 to some f_i^l for any $1 \leq l \leq H$ and $i = 1, 2$. By (1.22) and the continuity of $\mathcal{E}_K^{c_{11},c_{22}}$ with respect to the tight convergence, it follows that

$$(1.23) \quad I_{m_1,m_2}^{c_{11},c_{22}} = \lim_n \mathcal{E}_K^{c_{11},c_{22}}(f_{1,n}, f_{2,n}) \geq \sum_{l=1}^{H} \mathcal{E}_K^{c_{11},c_{22}}(f_1^l, f_2^l).$$

Let $m_i^l := \int_{\mathbb{R}^N} f_i^l(x)\, \mathrm{d}x$, then $\tilde{m}_i := \sum_{l=1}^{H} m_i^l \leq m_i$ for $i = 1, 2$.
By (1.23) and lemmas 1.7 and 1.6, we get

$$I_{m_1,m_2}^{c_{11},c_{22}} \geq \sum_{l=1}^{H} \mathcal{E}_K^{c_{11},c_{22}}(f_1^l, f_2^l) \geq \sum_{l=1}^{H} I_{m_1^l,m_2^l}^{c_{11},c_{22}} \geq I_{\tilde{m}_1,\tilde{m}_2}^{c_{11},c_{22}} \geq I_{m_1,m_2}^{c_{11},c_{22}};$$

it follows that all the inequalities above are in fact equalities, $H = 1$ and $\tilde{m}_i = m_i$, which concludes the proof. $\qquad\square$

Remark 1.9. The problem considered in this paper could be generalized to the case of more than two phases, with mutual and self attractive interactions. We notice that, with minor changes, the existence of a solution for this generalized problem would follow along the lines of the proof of Theorem 1.8.

Remark 1.10. Notice that in the case of $c_{11}, c_{22} > 0$ the functional $\mathcal{E}_K^{c_{11},c_{22}}$ does not admit in general a minimizer in $\mathcal{A}_{m_1,m_2}$. For instance, if $c_{11} > 0$, then it is easy to see that, for m_1 large enough, any minimizing sequence $f_{1,n}$ for the first phase tends to lose mass at infinity. As a consequence, $\mathcal{E}_K^{c_{11},c_{22}}$ does not admit a minimizer in $\mathcal{A}_{m_1,m_2}$ for m_1 large enough.

Moreover, assume that K be a positive definite kernel as in (1.6), and let $c_{11}, c_{22} \geq 1$. Then, for any $(f_1, f_2) \in \mathcal{A}_{m_1,m_2}$, we have

$$\mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2) = (c_{11}-1)\, J_K(f_1, f_1) + (c_{22}-1)\, J_K(f_2, f_2) + J_K(f_1 - f_2, f_1 - f_2) \geq 0.$$

It is easy to see that the infimum of $\mathcal{E}_K^{c_{11},c_{22}}$ is zero. It follows that (f_1, f_2) is a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if $m_1 = m_2$, $c_{11} = c_{22} = 1$ and $f_i = f \in L^1(\mathbb{R}^N; [0, \frac{1}{2}])$ with $\int_{\mathbb{R}^N} f(x)\, \mathrm{d}x = m_1 = m_2$.

Finally, if $m_1 = m_2$ and $\max\{c_{11}, c_{22}\} > 1$, then still the energy does not admit a minimizer. Indeed any minimizer (f_1, f_2) should satisfy $f_1 = f_2$ a.e. and the energy becomes

$$(c_{11} + c_{22} - 2)\, J_K(f, f),$$

which does not admit a minimizer in the class of functions $f \in L^1(\mathbb{R}^N; [0, \frac{1}{2}])$ with $\int_{\mathbb{R}^N} f(x)\, \mathrm{d}x = m_1 = m_2$.

1.4. Compactness of minimizers.

Here we prove the compactness property of minimizers.

Proposition 1.11. *Every minimizer (f_1, f_2) of $\mathcal{E}_K^{c_{11}, c_{22}}$ in $\mathcal{A}_{m_1, m_2}$ has compact support.*

Proof. Assume by contradiction that f_1 has not compact support. Recalling the definition of S in (1.9), we set $r := (2\frac{m_1 + m_2}{\omega_N})^{1/N}$ so that $|B_r \setminus S| > 0$. For $R > 0$ we now set $\varphi_1^R := f_1 \chi_{(\mathbb{R}^N \setminus B_R)}$ and observe that for R large enough we can find $\psi^R \in L^1(\mathbb{R}^N; \mathbb{R}^+)$ such that $\psi^R \equiv 0$ in $S \cup (\mathbb{R}^N \setminus B_r)$ and at the same time $\int_{\mathbb{R}^N} \varphi_1^R(x)\, \mathrm{d}x = \int_{\mathbb{R}^N} \psi^R(x)\, \mathrm{d}x > 0$. Hence by (1.10) we have

$$\int_{B_r} \psi^R(x)(c_{11} V_1(x) - V_2(x))\, \mathrm{d}x \geq \int_{\mathbb{R}^N \setminus B_R} \varphi_1^R(x)(c_{11} V_1(x) - V_2(x))\, \mathrm{d}x,$$

or, equivalently,

$$\int_{B_r} \psi^R(x)(|c_{11}| V_1(x) + V_2(x))\ \mathrm{d}x \ \leq\ \int_{\mathbb{R}^N \setminus B_R} \varphi_1^R(x)(|c_{11}| V_1(x) + V_2(x))\ \mathrm{d}x.$$

Since $\int_{\mathbb{R}^N \setminus B_R} \varphi_1^R(x)\, \mathrm{d}x = \int_{B_r} \psi^R(x)\, \mathrm{d}x$, the previous inequality implies that

$$\inf_{B_r}(|c_{11}| V_1 + V_2) \leq \sup_{\mathbb{R}^N \setminus B_R}(|c_{11}| V_1 + V_2)$$

which gives a contradiction for R large enough. $\qquad\square$

2. Qualitative properties of minimizers and some explicit solutions

In this Section we discuss some qualitative properties of the minimizers of $\mathcal{E}_K^{c_{11}, c_{22}}$, and for some specific choice of the coefficients c_{11}, c_{22} we determine the explicit solution.

2.1. Some preliminary results.

First, we recall the classical Riesz inequality [30]. To this purpose, for any $m > 0$ and $x_0 \in \mathbb{R}^N$, we denote by $B^m(x_0)$ the ball centered in x_0 with $|B^m(x_0)| = m$ (B^m if $x_0 = 0$). With a little abuse of notation, for any $x_0 \in \mathbb{R}^N$ and for any $f \in L^1(\mathbb{R}^N)$, we set $B^f(x_0) := B^{\|f\|_{L^1}}(x_0)$ ($B^f := B^{\|f\|_{L^1}}$ if $x_0 = 0$). Moreover, for every function $u \in L^1(\mathbb{R}^N; \mathbb{R}^+)$ we denote by u^* the spherical symmetric nonincreasing rearrangement of u, satisfying

(2.1) $\qquad \{u^* > t\} = B^{m_t}$ where $m_t := |\{u > t\}|$ $\qquad$ for all $t > 0$.

Lemma 2.1 (Riesz inequality). *Let $f, g \in L^1(\mathbb{R}^N; [0,1])$ with $\|f\|_{L^1}, \|g\|_{L^1} > 0$. Then,*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x)\, g(y)\, K(x - y)\, \mathrm{d}x\, \mathrm{d}y \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f^*(x)\, g^*(y)\, K(x - y)\, \mathrm{d}x\, \mathrm{d}y$$

$$\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \chi_{B^f}(x)\, \chi_{B^g}(y)\, K(x - y)\, \mathrm{d}x\, \mathrm{d}y,$$

where the first inequality is in fact an equality if and only if $f(\cdot) = f^*(\cdot - x_0)$ and $g(\cdot) = g^*(\cdot - x_0)$ for some $x_0 \in \mathbb{R}^N$, whereas the second inequality holds with the equality if and only if $f^* = \chi_{B^f}$ and $g^* = \chi_{B^g}$.

The following lemma states that, for $c_{11} = 0$, there exists a minimizer $(\tilde{f}_1, f_2)$ such that $\tilde{f}_1 + f_2 = 1$ on the support of $\tilde{f}_1$. Its proof can be easily obtained by (1.5); the details are left to the reader.

Lemma 2.2 (**Superlevels**). *Let (f_1, f_2) be a minimizer of $\mathcal{E}_K^{0,c_{22}}$ in $\mathcal{A}_{m_1,m_2}$. Then, there exists a unique $t > 0$ such that the pair $(\tilde{f}_1, f_2)$ is still a minimizer of $\mathcal{E}_K^{0,c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if $\tilde{f}_1 \in L^1(\mathbb{R}^N; \mathbb{R}^+)$ satisfies (i), (ii) and (iii) below:*

- (i) $\int_{\mathbb{R}^N} \tilde{f}_1(x) \, \mathrm{d}x = m_1$;
- (ii) $\tilde{f}_1(x) = 1 - f_2(x)$ *if* $V_2(x) > t$;
- (iii) $\tilde{f}_1(x) = 0$ *if* $V_2(x) < t$.

Moreover, if $|\{V_2 = t\}| = 0$, then $\tilde{f}_1$ is uniquely determined, and clearly $f_1 = \tilde{f}_1$.
A similar statement holds true for the case $c_{22} = 0$.

We recall that the sets G_i are defined in (1.8).

Corollary 2.3. *Let (f_1, f_2) be a minimizer for $\mathcal{E}_K^{0,c_{22}}$ in $\mathcal{A}_{m_1,m_2}$. Then, for any measurable set $E_1 \subset G_1 \setminus G_2$ with $|E_1| = \int_{G_1 \setminus G_2} f_1(x) \, \mathrm{d}x$, the function*

$$\tilde{f}_1(x) := \begin{cases} \chi_{E_1} & \text{if } x \in G_1 \setminus G_2 \\ f_1(x) & \text{otherwise in } \mathbb{R}^N, \end{cases}$$

satisfies

$$\mathcal{E}_K^{0,c_{22}}(\tilde{f}_1, f_2) \leq \mathcal{E}_K^{0,c_{22}}(f_1, f_2).$$

A similar statement holds true in the case $c_{22} = 0$.

2.2. The strongly attractive case $c_{11} + c_{22} \leq -2$. In the following theorem we characterize the minimizers for every c_{11}, c_{22} such that $c_{11} + c_{22} \leq -2$ and $\max\{c_{11}, c_{22}\} \geq -1$.

Theorem 2.4. *Let $c_{11} + c_{22} \leq -2$. The following statements hold true.*

- (i) *if $c_{11} = c_{22} = -1$, then (f_1, f_2) is a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if $f_1 + f_2 = \chi_{B^{m_1+m_2}(x_0)}$, for some $x_0 \in \mathbb{R}^N$;*
- (ii) *if $c_{11} = -1$ and $c_{22} < -1$, then $(f_1, f_2) \in \mathcal{A}_{m_1,m_2}$ is a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if $f_1 + f_2 = \chi_{B^{m_1+m_2}(x_0)}$ for some $x_0 \in \mathbb{R}^N$, and $f_2 = \chi_{B^{m_2}(y_0)}$ for some $y_0 \in \mathbb{R}^N$ with $B^{m_2}(y_0) \subset B^{m_1+m_2}(x_0)$;*
- (ii') *if $c_{22} = -1$ and $c_{11} < -1$, then $(f_1, f_2) \in \mathcal{A}_{m_1,m_2}$ is a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if $f_1 + f_2 = \chi_{B^{m_1+m_2}(x_0)}$ for some $x_0 \in \mathbb{R}^N$ and $f_1 = \chi_{B^{m_1}(y_0)}$ for some $y_0 \in \mathbb{R}^N$ with $B^{m_1}(y_0) \subset B^{m_1+m_2}(x_0)$;*
- (iii) *if $c_{22} < -1$ and $-1 < c_{11} \leq 0$, then $(f_1, f_2) \in \mathcal{A}_{m_1,m_2}$ is a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if $f_1 + f_2 = \chi_{B^{m_1+m_2}(x_0)}$ and $f_2 = \chi_{B^{m_2}(x_0)}$ for some $x_0 \in \mathbb{R}^N$;*
- (iii') *if $c_{11} < -1$ and $-1 < c_{22} \leq 0$, then $(f_1, f_2) \in \mathcal{A}_{m_1,m_2}$ is a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if $f_1 + f_2 = \chi_{B^{m_1+m_2}(x_0)}$ and $f_1 = \chi_{B^{m_1}(x_0)}$ for some $x_0 \in \mathbb{R}^N$.*
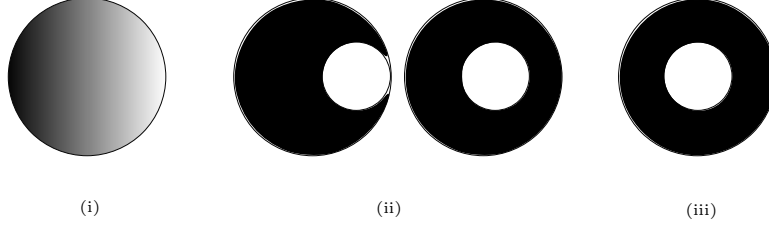
<div align="center">

(i)   (ii)   (iii)

</div>

Figure 3. The phase f_1 is the black one, whereas the phase f_2 is white. The first cartoon represents the minimizers in (i). In this case, all the configurations (f_1, f_2) such that $f_1 + f_2 = \chi_{B^{m_1 + m_2}}$ are minimizers of the energy. The second and the third cartoons are two examples of minimizers in the case (ii). The last cartoon is the unique minimizer in the case (iii). Minimizers in cases (ii') and (iii') can be obtained by the balls above switching the balck parts with the white ones.

Proof. We prove only (i), (ii), and (iii), being the proofs of (ii') and (iii') the same as to the ones of (ii) and (iii), respectively.

It is easy to see that

$$\mathcal{E}_K^{c_{11}, c_{22}}(f_1, f_2) = c_{11}\, J_K(f_1 + f_2, f_1 + f_2) - 2(c_{11} + 1)\, J_K(f_2, f_1 + f_2)$$
$$+ (c_{11} + c_{22} + 2)\, J_K(f_2, f_2).$$

The claim (iii) follows immediately by applying Lemma 2.1 to each of the three addenda above. Moreover,

$$\mathcal{E}_K^{-1,-1}(f_1, f_2) = -J_K(f_1 + f_2, f_1 + f_2),$$
$$\mathcal{E}_K^{-1,c_{22}}(f_1, f_2) = -J_K(f_1 + f_2, f_1 + f_2) + (c_{22} + 1)\, J_K(f_2, f_2)$$

and hence (i) and (ii) easily follow by applying once again Lemma 2.1. $\qquad\square$

The next proposition gives a characterization for $N = 1$ of the minimizer of $\mathcal{E}_K^{c_{11}, c_{22}}$ in the case $c_{11}, c_{22} < -1$ which is left open in Theorem 2.4.

Proposition 2.5. *Let $N = 1$ and $c_{11}, c_{22} < -1$. Then*

$$(f_1, f_2) = (\chi_{[-m_1, 0]}, \chi_{[0, m_2]}) \ and \ (f_1, f_2) = (\chi_{[0, m_1]}, \chi_{[-m_2, 0]}),$$

are (up to a translation) the unique minimizers of $\mathcal{E}_K^{c_{11}, c_{22}}$ in $\mathcal{A}_{m_1, m_2}$.

Proof. It is easy to see that for any (f_1, f_2)

$$\mathcal{E}_K^{c_{11}, c_{22}}(f_1, f_2) = \mathcal{E}_K^{c_{11}, -1}(f_1, f_2) + (c_{22} + 1)J_K(f_2, f_2);$$

since the second addendum is minimized when f_2 is the characteristic function of an interval, the claim follows by Theorem 2.4 (ii'). $\qquad\square$

Remark 2.6. In the general multi-dimensional case, we do not know the explicit form of the minimizers if $c_{11}, c_{22} < -1$. One could guess that f_i are characteristic functions as in the Coulomb case considered in Proposition 3.2 where, by means of first variation techniques, we can exclude that the solution is given by two tangent balls as well as by a ball and a concentric annulus around it. A natural issue to consider is then the asymptotic behaviour of minimizers for c_{11}, c_{22} which tend to

the boundary (and at infinity) of the region $\{c_{11}, c_{22} < -1\}$. In fact, there are many interesting limits that one could study:

(1) $c_{11} < -1$, $c_{22n} \nearrow -1$. Let $(f_1^n, f_2^n) \in \mathcal{A}_{m_1, m_2}$ ba a minimizer of $\mathcal{E}^{c_{11}, c_{22n}}$ in $\mathcal{A}_{m_1, m_2}$. Notice that the limit problem does not admit a unique solution. Nevertheless, we expect that, up to a unique translation, f_1^n and $f_1^n + f_2^n$ converge strongly in L^1 to characteristic functions of two innerly tangent balls. Indeed, this is the minimizer for $c_{11}, c_{22} < 1$, among the family of pairs of nested balls.

(2) $c_{11}, c_{22} < -1$, $c_{11}, c_{22} \nearrow -1$. In this case the limit problem is the most degenerate one for which it seems difficult to have a clear guess.

(3) $c_{11} < -1$, $c_{22n} \to -\infty$. In this case we expect that the second phase tends to a ball, while the first phase tends to the characteristic function of a set which is not a ball.

(4) $c_{11}, c_{22} \to -\infty$: In this case we have that the two phases converge to two tangent balls. This is precisely the content of Proposition 2.7 below.

Proposition 2.7. *Let $\{c_{11n}\}, \{c_{22n}\} \subset \mathbb{R}$ be such that $c_{11n}, c_{22n} \to -\infty$. For any $n \in \mathbb{N}$, let $(f_1^n, f_2^n) \in \mathcal{A}_{m_1, m_2}$ be a minimizer of $\mathcal{E}_K^{c_{11n}, c_{22n}}$ in $\mathcal{A}_{m_1, m_2}$. Then, up to a unique translation, f_1^n, f_2^n converge strongly in L^1 to characteristic functions of two tangent balls, i.e., there exists a family of translations $\{\tau_n\}$ and a unitary vector $\nu \in \mathbb{R}^N$, such that*

$$f_1^n(\cdot - \tau_n) \to \chi_{B^{m_1}}, \quad f_2^n(\cdot - \tau_n) \to \chi_{B^{m_2}(r\,\nu)}, \quad \text{with } r := \left(\frac{m_1}{\omega_N}\right)^{\frac{1}{N}}.$$

Proof. First, notice that there exists a constant C such that

$$-2J_K(f_1^n, f_2^n) \geq C, \quad c_{11n} J_K(f_1^n, f_1^n) \geq c_{11n} I_{m_1, 0}^{-1, 0}, \quad c_{22n} J_K(f_2^n, f_2^n) \geq c_{22n} I_{0, m_2}^{0, -1},$$

so that

$$(2.2) \qquad I_{m_1, m_2}^{c_{11n}, c_{22n}} = \mathcal{E}_K^{c_{11n}, c_{22n}}(f_1^n, f_2^n) \geq c_{11n} I_{m_1, 0}^{-1, 0} + c_{22n} I_{0, m_2}^{0, -1} + C.$$

On the other hand,

$$(2.3) \qquad I_{m_1, m_2}^{c_{11n}, c_{22n}} \leq \mathcal{E}_K^{c_{11n}, c_{22n}}(\chi_{B^{m_1}}, \chi_{B^{m_2}(r\,\nu)}) = c_{11n} I_{m_1, 0}^{-1, 0} + c_{22n} I_{0, m_2}^{0, -1} + C,$$

which, togehter with (2.2), yields

$$J_K(f_1^n, f_1^n) \to I_{m_1, 0}^{-1, 0}, \qquad J_K(f_2^n, f_2^n) \to I_{0, m_2}^{0, -1}.$$

Therefore, by Theorem 1.8 applied to $I_{m_1, 0}^{-1, 0}$ and $I_{0, m_2}^{0, -1}$, there exist two sequences of translations $\{\tau_i^n\}$ (for $i = 1, 2$) such that

$$f_1^n(\cdot - \tau_1^n) \to \chi_{B^{m_1}}, \qquad f_2^n(\cdot - \tau_2^n) \to \chi_{B^{m_2}} \qquad \text{strongly in } L^1.$$

It remains to prove that $|\tau_1^n - \tau_2^n| \to r$ as $n \to \infty$. Set

$$\lambda_n := \frac{|\tau_1^n - \tau_2^n|}{r}.$$

Notice that $\liminf_{n \to \infty} \lambda_n \geq 1$ (otherwise, for n large, f_1^n and f_2^n would be close in L^1 to characteristic functions of two intersecting balls, so that (f_1^n, f_2^n) would not be admissible). Up to a subsequence, we can assume that $\limsup_{n \to \infty} \lambda_n = \lim_{n \to \infty} \lambda_n =: \lambda$, with $\lambda \geq 1$. Then, set

$$\tilde{f}_1^n := \chi_{B^{m_1}(\tau_1^n)}, \quad \tilde{f}_2^n := \chi_{B^{m_2}(\tau_2^n)};$$

notice that $\|\tilde{f}_i^n - f_i^n\|_{L^1} \to 0$ as $n \to \infty$ for $i = 1, 2$. Then, by the lower semicontinuity property of J_K with respect to the strong L^1 convergence, we get

$$\liminf_n J_K(f_1^n, f_2^n) - J_K(\tilde{f}_1^n, \tilde{f}_2^n) \geq 0$$

We conclude

$$I_{m_1,m_2}^{c_{11}n,c_{22}n} \geq \mathcal{E}_K^{c_{11}n,c_{22}n}(\tilde{f}_1^n, \tilde{f}_2^n) + \rho(n) \geq \mathcal{E}_K^{c_{11}n,c_{22}n}(\chi_{B^{m_1}}, \chi_{B^{m_2}(r\nu)}) + \rho(n) + \omega(\lambda_n),$$

where $\rho(n) \to 0$ as $n \to \infty$ and $\omega : [1, +\infty) \to [0, +\infty)$ is an increasing function vanishing at 1. By minimality it easily follows that $\lambda = 1$ and hence the claim. $\square$

2.3. **The weakly attractive case** $c_{11} + c_{22} > -2$. Here we will consider the case $c_{11} + c_{22} > -2$, and we will characterize the solution only for the purely weakly attractive case $0 \geq c_{11}, c_{22} > -1$ with $(c_{11} + 1)m_1 = (c_{22} + 1)m_2$. Moreover, we will assume that K is positive definite, according to definition (1.1). Notice that this implies that the functional $J_K(\varphi, \varphi)$ is strictly convex.

Lemma 2.8. *Let K be positive definite. For any $-1 < c < 1$ and for any $m > 0$, the (unique up to a translation) minimizer of $\mathcal{E}_K^{c,c}$ in $\mathcal{A}_{m,m}$ is given by the pair $(f_1^0, f_2^0) = \left(\frac{1}{2}\chi_{B^{2m}}, \frac{1}{2}\chi_{B^{2m}}\right)$.*

Proof. Let $(f_1, f_2) \in \mathcal{A}_{m,m}$. We first notice that the convexity of the functional $J_K(f, f)$ immediately implies that

(2.4) $J_K(f_1, f_2) = 2J_K(\frac{f_1+f_2}{2}, \frac{f_1+f_2}{2}) - \frac{J_K(f_1,f_1)}{2} - \frac{J_K(f_2,f_2)}{2} \leq J_K(\frac{f_1+f_2}{2}, \frac{f_1+f_2}{2}).$

Moreover

$$\mathcal{E}_K^{c,c}(f_1, f_2) = c\, J_K(f_1 + f_2, f_1 + f_2) - 2(1+c)\, J_K(f_1, f_2),$$

which, together with (2.4), yields

$$\mathcal{E}_K^{c,c}(f_1, f_2) \geq c\, J_K\left(\frac{f_1+f_2}{2} + \frac{f_1+f_2}{2}, \frac{f_1+f_2}{2} + \frac{f_1+f_2}{2}\right)$$
$$- 2(1 + c)\, J_K\left(\frac{f_1+f_2}{2}, \frac{f_1+f_2}{2}\right) = \mathcal{E}_K^{c,c}\left(\frac{f_1+f_2}{2}, \frac{f_1+f_2}{2}\right),$$

where in the inequality we have also used that $c + 1 > 0$. By the strict convexity of $J_K(f, f)$, the inequality is strict whenever $f_1 \neq f_2$. We deduce that $f_1 = f_2 = \frac{f_1+f_2}{2} =: f$. Since $\mathcal{E}_K^{c,c}(f, f) = 2(c - 1)\, J_K(f, f)$, by Lemma 2.1, we conclude that $\mathcal{E}_K^{c,c}(f_1, f_2)$ attains its unique minimum when $f_1 = f_2 = \frac{1}{2}\chi_{B^{2m}}$. $\square$

Let us introduce the coefficients a_i (depending on c_{11} and c_{22}) which represent the volume fractions of the two phases where they mix:

(2.5) $$a_1 := := \frac{c_{22} + 1}{c_{11} + c_{22} + 2}, \qquad a_2 := \frac{c_{11} + 1}{c_{11} + c_{22} + 2}.$$

Notice that, if $(c_{11} + 1)m_1 = (c_{22} + 1)m_2$, then

$$a_1 = \frac{m_1}{m_1 + m_2}, \qquad a_2 = \frac{m_2}{m_1 + m_2}.$$

Proposition 2.9. *Let $-1 < c_{11}, c_{22} \leq 0$. If $(c_{11} + 1)m_1 = (c_{22} + 1)m_2$, then the (unique up to a translation) minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ is given by the pair*

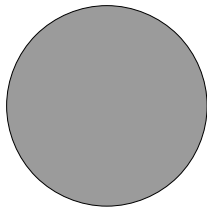$$(f_1, f_2) = (a_1\chi_{B^{m_1+m_2}}, a_2\chi_{B^{m_1+m_2}}).$$

Figure 4. The phase f_1 is black and the phase f_2 is white. Under the assumptions of Proposition 2.9, the minimzer is given by a ball where the two phases mix each other. The mixing is represented by the grey color.

Proof. By Lemma 2.8 we get directly the claim in the case $c_{11} = c_{22}$, since by assumption this implies $m_1 = m_2$.

We now prove the result in the general case. For any $(f_1, f_2) \in \mathcal{A}_{m_1, m_2}$, we set

$$(2.6) \qquad h_1 := (1 + \tfrac{c_{11}}{2})f_1 - \tfrac{c_{22}}{2}f_2, \quad h_2 := -\tfrac{c_{11}}{2}f_1 + (1 + \tfrac{c_{22}}{2})f_2.$$

It is easy to see that $h_1, h_2 \geq 0$, $h_1 + h_2 = f_1 + f_2 \leq 1$ and, by assumption,

$$\int_{\mathbb{R}^N} h_1(x) \, \mathrm{d}x = \int_{\mathbb{R}^N} h_2(x) \, \mathrm{d}x = \frac{m_1 + m_2}{2} =: m.$$

By straightforward computations it follows that, setting $c := \frac{c_{11}c_{22}}{2 - c_{11}c_{22}}$,

$$\mathcal{E}_K^{c_{11}, c_{22}}(f_1, f_2) = \frac{2 - c_{11}c_{22}}{2 + c_{11} + c_{22}} \, \mathcal{E}_K^{c,c}(h_1, h_2).$$

Notice that, since $-1 < c_{11}, c_{22} < 0$, we have that $0 < c < 1$ and $\frac{2 - c_{11}c_{22}}{2 + c_{11} + c_{22}} > 0$; therefore, (f_1, f_2) minimizes $\mathcal{E}_K^{c_{11}, c_{22}}$ (in $\mathcal{A}_{m_1, m_2}$) if and only if (h_1, h_2) minimizes $\mathcal{E}_K^{c,c}$ in $\mathcal{A}_{m,m}$. By Lemma 2.8, the unique minimizer of $\mathcal{E}_K^{c,c}$ in $\mathcal{A}_{m,m}$ is given by $(h_1, h_2) = (\tfrac{1}{2}\chi_{B^{2m}}, \tfrac{1}{2}\chi_{B^{2m}})$. Hence the claim for $c_{11} \neq c_{22}$ follows by (2.6). $\qquad \square$

Remark 2.10. Proposition 2.9 establishes that, for very special coefficients c_{11} and c_{22} depending on the masses m_1, m_2, the minimizer is given by a homogenous density that mixes the two phases with specific volume fractions. The proof is based on the convexity of J_K. One may wonder whether, under this assumption, the result still holds for generic c_{11} and c_{22}. We will see that this is not the case even for the Coulomb kernel (see Corollary 3.5 and Theorem 3.9).

3. THE COULOMB KERNEL

Through this section we will assume that $K = K_{C_N}$ is the Coulomb kernel defined in (1.7). We will provide the explicit form of the solutions for all the choices of the (nonpositive) parameters c_{11}, c_{22}, except when they are both strictly less than -1 in which case we will only be able to say that f_i are characteristic functions of sets.

3.1. **Consequences of the first variation.** We specialize the results of section 1.2 to the case of Coulomb kernels. We recall that the sets G_i, F_i and S are defined in (1.8), (1.9).

Proposition 3.1. *Let (f_1, f_2) be a minimizer of $\mathcal{E}_{K_{C_N}}^{c_{11}, c_{22}}$ in $\mathcal{A}_{m_1, m_2}$. The following facts hold true.*

(i) $(c_{11} + 1)f_1 - (c_{22} + 1)f_2 = 0$ *a.e. in* $G_1 \cap G_2$. *In particular, if either* $(c_{11} + 1)(c_{22} + 1) < 0$, *or* $c_{11} = -1 \neq c_{22}$, *or* $c_{22} = -1 \neq c_{11}$, *then* $|G_1 \cap G_2| = 0$.

(ii) *if* $c_{11} \neq 0$, *then* $|G_1 \setminus G_2| = 0$, *while if* $c_{22} \neq 0$, *then* $|G_2 \setminus G_1| = 0$.

(iii) $G_1 \cap G_2 \subset S$.

(iv) *If* $c_{11} \neq -1$ *or* $c_{22} \neq -1$, *then*

$$(3.1) \qquad\qquad f_1 = a_1, \quad f_2 = a_2 \qquad \text{a.e. in } G_1 \cap G_2,$$

where a_i *are defined in* (2.5).

Proof. (i) is a consequence of (1.14). To prove (ii) notice that $G_1 \setminus G_2 \subset G_1 \setminus S$, which implies by (1.11) that $c_{11}f_1 = f_2$ in $G_1 \setminus G_2$. Furthermore, in this region $f_2 = 0$, so that (since $c_{11} \neq 0$) also $f_1 = 0$. The case $c_{22} \neq 0$ is proved in the same way.

The proof of (iii) follows recalling that by (1.11) we have $0 > c_{11}f_1 - f_2 = 0$ in $(G_1 \cap G_2) \setminus S$ and hence $|(G_1 \cap G_2) \setminus S| = 0$. The claim in (iv) follows by (1.14) recalling that, in view of (iii), $f_1 + f_2 = 1$. □

3.2. The strongly attractive case $c_{11} + c_{22} \leq -2$. In Theorem 2.4 we have characterized the minimizers for every c_{11}, c_{22} such that $c_{11} + c_{22} \leq -2$ and $\max\{c_{11}, c_{22}\} \geq -1$. Clearly such result applies also to Coulomb kernels. The (general N dimensional) case $c_{11}, c_{22} < -1$ was left open. In the following proposition, we show that for Coulomb kernels the minimizers f_i are characteristic functions of sets E_i whose shape is unknown (see Remark 2.6 for some further comments in this direction).

Proposition 3.2. *Let* $c_{11} + c_{22} \leq -2$ *with* $(c_{11}, c_{22}) \neq (-1, -1)$. *If* (f_1, f_2) *is a minimizer of* $\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}$ *in* $\mathcal{A}_{m_1,m_2}$, *then* $f_1 = \chi_{F_1}$ *and* $f_2 = \chi_{F_2}$ *for some* F_1, $F_2 \subset \mathbb{R}^N$.

Proof. By Theorem 2.4 and Proposition 2.5 the claim holds true in the one dimensional case and in the general N dimensional case for $\max\{c_{11}, c_{22}\} \geq -1$, so that it is enough to prove the claim in the case $N \geq 2$ and $c_{11}, c_{22} < -1$. Since $c_{11} + c_{22} + 2 < 0$, by applying Lemma 1.5 with $\varphi \in L^1(\mathbb{R}^N; \mathbb{R})$, $\varphi = 0$ a.e. in $\mathbb{R}^N \setminus (G_1 \cap G_2)$ and $\int_{\mathbb{R}^N} \varphi \, dx = 0$, we get

$$(3.2) \qquad \int_{G_1 \cap G_2} \int_{G_1 \cap G_2} K_{C_N}(x - y) \, \varphi(x) \, \varphi(y) \, dx \, dy \leq 0.$$

By Remark 1.1 we deduce that the above inequality is actually an equality and that $\varphi = 0$ in $G_1 \cap G_2$. By the arbitrariness of φ, it follows that $|G_1 \cap G_2| = 0$. Finally, by Proposition 3.1(ii), we have that $|G_1 \setminus G_2| = |G_2 \setminus G_1| = 0$, so we conclude that $|G_1| = |G_2| = 0$.

□

3.3. The weakly attractive case $c_{11} + c_{22} > -2$ **(preliminary results).** For any measurable set $E \subset \mathbb{R}^N$, we set $V_E := \chi_E * K$.

Lemma 3.3. *Let* $-1 \leq c_{11}, c_{22} \leq 0$ *with* $c_{11} \neq -1$ *or* $c_{22} \neq -1$. *Then, there exists a minimizer* (f_1, f_2) *of* $\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}$ *in* $\mathcal{A}_{m_1,m_2}$, *such that* $|G_1 \setminus G_2| = |G_2 \setminus G_1| = 0$ *and either* $|F_1| = 0$ *or* $|F_2| = 0$.

Moreover, any minimizer (f_1, f_2) *of* $\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}$ *in* $\mathcal{A}_{m_1,m_2}$ *is such that either* $|G_1 \setminus G_2| + |F_1| = 0$ *or* $|G_2 \setminus G_1| + |F_2| = 0$.

Proof. Let (f_1, f_2) be a minimizer of $\mathcal{E}_{K_{C_N}}^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$. By Proposition 3.1(ii) and Corollary 2.3 we can always assume

(3.3) $\{f_1 \neq 0\} = (G_1 \cap G_2) \cup F_1, \qquad \{f_2 \neq 0\} = (G_1 \cap G_2) \cup F_2$ a.e.,

so that $|G_1 \setminus G_2| = |G_2 \setminus G_1| = 0$.

Now, let us prove that either $|F_1| = 0$ or $|F_2| = 0$. We first focus on the case $N \geq 3$. By (3.3) and (3.1) we have

(3.4) $f_1 = a_1 \chi_{G_1 \cap G_2} + \chi_{F_1}, \quad f_2 = a_2 \chi_{G_1 \cap G_2} + \chi_{F_2}.$

It follows that

$$V_1 = a_1 V_{G_1 \cap G_2} + V_{F_1}, \quad V_2 = a_2 V_{G_1 \cap G_2} + V_{F_2}$$

which together with (1.13) easily yields

(3.5) $(c_{11} + 1)V_{F_1}(x_2) - (c_{22} + 1)V_{F_2}(x_2) \geq (c_{11} + 1)V_{F_1}(x_1) - (c_{22} + 1)V_{F_2}(x_1)$

for any $x_1 \in \bar{F}_1$ and any $x_2 \in \bar{F}_2$. Set $U(x) := (c_{11} + 1)V_{F_1}(x) - (c_{22} + 1)V_{F_2}(x)$. Then U solves

(3.6) $\begin{cases} -\Delta U = (c_{11} + 1)\chi_{F_1} - (c_{22} + 1)\chi_{F_2} & \text{in } \mathbb{R}^N \\ U(x) \to 0 & \text{if } |x| \to \infty. \end{cases}$

So, U is subharmonic in $\mathbb{R}^N \setminus \overline{F}_1$ and hence either $U \leq 0$ or U reaches its maximum on $\overline{F}_1$. Analogously, since U is superharmonic in $\mathbb{R}^N \setminus \overline{F}_2$, either $U \geq 0$ or U reaches its minimum on $\overline{F}_2$. Now, if $U \equiv 0$, then $|F_1| = |F_2| = 0$; otherwise, assume, for instance, that U reaches its maximum on $\overline{F}_1$. By (3.5) and by (3.6), it follows that U is constant in F_2, and hence $|F_2| = 0$. Analogously, if U reaches its minimum on $\overline{F}_2$, we get that $|F_1| = 0$.

The proofs for the cases $N = 1, 2$ are analogous, the only care being that, for $N = 2$, the boundary condition in (3.6) should be replaced either by $U(x) \to 0$ or $U(x) \to \pm\infty$, according with the sign of $(c_{11} + 1)|F_1| - (c_{22} + 1)|F_2|$. For $N = 1$ a direct proof shows that U reaches its maximum on $\overline{F}_1$ and its minimum on $\overline{F}_2$.

We pass to the proof of the last claim of the lemma. Assume by contradiction that $G_1 \setminus G_2 + |F_1| > 0$ and $|G_2 \setminus G_1| + |F_2| > 0$. By Proposition 3.1(ii) and Corollary 2.3 we deduce that there exists a minimizer satisfying (3.3) with both F_1 and F_2 with positive measure. Following the lines of the proof of the first claim of the lemma, this provides a contradiction. $\qquad\square$

The remaining part of this section is devoted to the uniqueness and characterization of the minimizer. In particular, we will see that the unique minimizer in the purely weakly attractive case, corresponding to $-1 < c_{11}, c_{22} \leq 0$, is given by a ball where the two phases are mixed proportionally to their self attraction coefficents and by an annulus around this ball (see Corollary 3.5 for the case $N = 1$ and Theorem 3.9 and Corollary 3.12 for the case $N \geq 2$). Moreover, we will see that also in the reamainig cases, i.e., $c_{11} \leq -1 \leq c_{22} \leq 0$ and $c_{22} \leq -1 \leq c_{11} \leq 0$, with $c_{11} + c_{22} > -2$, the unique minimizer is given once again by a ball and an annulus around it, where the internal ball corresponds to the phase having the stronger self-attraction coefficient (see Proposition 3.4 for the case $N = 1$ and Corollary 3.10 for the case $N \geq 2$).

3.4. **The weakly attractive case $c_{11} + c_{22} > -2$ (in dimension $N = 1$).** In the following proposition we study the minimizer of $\mathcal{E}_{K_{C_1}}^{c_{11},c_{22}}$ when $c_{11} \leq -1 \leq c_{22} \leq 0$ and $c_{11} + c_{22} > -2$. In the subsequent corollary we take advantage of this result via a re-parameterization of the energies to study the case $-1 < c_{11}, c_{22} \leq 0$.

Proposition 3.4. *Let $c_{11} \leq -1$ and $-1 \leq c_{22} \leq 0$ (resp. $c_{22} \leq -1$ and $-1 \leq c_{11} \leq 0$) with $c_{11} + c_{22} > -2$. Then the (unique up to a translation) minimizer of $\mathcal{E}_{K_{C_1}}^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ is given by the pair*

$$(f_1, f_2) = (\chi_{B^{m_1}}, \chi_{B^{m_1+m_2} \setminus B^{m_1}}) \quad (resp.\ (f_1, f_2) = (\chi_{B^{m_1+m_2} \setminus B^{m_2}}, \chi_{B^{m_2}})).$$

Proof. We prove the claim only for $c_{11} \leq -1$ and $-1 \leq c_{22} \leq 0$ with $c_{11} + c_{22} > -2$, being the proof of the other case analogous. Let (f_1, f_2) be a minimizer of $\mathcal{E}_{K_{C_1}}^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$. By (i) and (ii) of Proposition 3.1, we have that $f_1 = \chi_{F_1}$ and $f_2 = \chi_{F_2} + f_2 \llcorner G_2$. We can assume without loss of generality that $F_1 \cup F_2 \cup G_2$ is an interval, since reducing the distances decreases the energy. For the same reason, it is easy to see that $|G_2| = 0$. Notice that

$$\mathcal{E}_{K_{C_1}}^{c_{11},c_{22}}(f_1, f_2) = \mathcal{E}_{K_{C_1}}^{-1,c_{22}}(f_1, f_2) + (c_{11} + 1)J_{K_{C_1}}(f_1, f_1),$$

so it is enough to prove the claim for $c_{11} = -1$. We now prove that $V_2' = 0$ in F_1. By (1.13), we have

$$V_2(x_1) \geq V_2(x_2) \quad \text{for any } x_1 \in \overline{F_1} \text{ and } x_2 \in \overline{F_2},$$

and, by the maximum principle, V_2 attains its maximum in $\overline{F_2}$ (notice that $V_2 \to -\infty$ as $|x| \to +\infty$). It follows that for any $x \in F_1$, $V_2(x) = \max V_2$. We have

$$0 = V_2'(x) = \frac{1}{2}\left(|F_2 \cap (-\infty, x]| - |F_2 \cap [x, \infty)|\right) \quad \text{for any } x \in F_1,$$

and hence F_1 is connected and centered in $F_1 \cup F_2$. $\qquad\square$

Corollary 3.5. *Let $-1 < c_{11}, c_{22} \leq 0$. Then, the following results hold true (recall that a_i are defined in (2.5)).*

(i) *If $(c_{22} + 1)m_2 \geq (c_{11} + 1)m_1$, then (up to a translation)*

$$(3.7) \qquad (f_1, f_2) = \left(a_1 \chi_{B^{\frac{m_1}{a_1}}}, \chi_{B^{m_2+m_1}} - a_1 \chi_{B^{\frac{m_1}{a_1}}}\right)$$

is the (unique) minimizer of $\mathcal{E}_{K_{a_2}}^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$.

(ii) *If $(c_{11} + 1)m_1 > (c_{22} + 1)m_2$, then (up to a translation)*

$$(f_1, f_2) = \left(\chi_{B^{m_2+m_1}} - a_2 \chi_{B^{\frac{m_2}{a_2}}}, a_2 \chi_{B^{\frac{m_2}{a_2}}}\right)$$

is the (unique) minimizer of $\mathcal{E}_{K_{a_2}}^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$.

Proof. We prove only (i) since the proof of (ii) is analogous.

Let (f_1, f_2) be a minimizer of $\mathcal{E}_{K_{C_1}}^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$. Arguing as in the proof of Proposition 3.4, one can show that $|G_2 \setminus G_1| + |G_1 \setminus G_2| = 0$, and hence

$$(3.8) \qquad f_1 = a_1 \chi_{G_1 \cap G_2} \text{ and } f_2 = a_2 \chi_{G_1 \cap G_2} + \chi_{F_2}.$$

Set $A := G_1 = G_2$, $B := F_2$, $\tilde{m}_1 := \frac{m_1}{a_1}$ and $\tilde{m}_2 := m_2 - \frac{c_{11}+1}{c_{22}+1}m_1 > \tilde{m}_1$; then, by easy computations, it follows that

$$
\begin{aligned}
\mathcal{E}_{K_{C_1}}^{c_{11},c_{22}}(f_1,f_2) &= \frac{1-c_{11}c_{22}}{c_{11}+c_{22}+2}\left[-J_{K_{C_1}}(A,A) + c_{22}\frac{c_{11}+c_{22}+2}{1-c_{11}c_{22}}J_{K_{C_1}}(B,B)\right.\\
&\quad \left. -2J_{K_{C_1}}(A,B)\right]\\
&= \frac{1-c_{11}c_{22}}{c_{11}+c_{22}+2}\mathcal{E}_{K_{C_1}}^{-1,\tilde{c}_{22}}(\chi_A,\chi_B),
\end{aligned}
$$

with $\tilde{c}_{22} := c_{22}\frac{c_{11}+c_{22}+2}{1-c_{11}c_{22}} \in (-1,0)$. Since $\frac{1-c_{11}c_{22}}{c_{11}+c_{22}+2} > 0$, it follows that (f_1,f_2) is a minimizer of $\mathcal{E}_{K_{C_1}}^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if (χ_A,χ_B) minimizes $\mathcal{E}_{K_{C_1}}^{-1,\tilde{c}_{22}}$ in $\mathcal{A}_{\tilde{m}_1,\tilde{m}_2}$. By Proposition 3.4, the unique minimizer of $\mathcal{E}_{K_{C_1}}^{-1,\tilde{c}_{22}}(\chi_A,\chi_B)$ among the pairs (A,B) with $|A| = \tilde{m}_1$ and $|B| = \tilde{m}_2$ is given by $(B^{\tilde{m}_1}, B^{\tilde{m}_1+\tilde{m}_2} \setminus B^{\tilde{m}_1})$. The claim follows thanks to formula (3.8).   □

One might wonder whether the assumption that $K = K_{C_1}$ is crucial in order to prove Proposition 3.4 and Corollary 3.5. In the following Remark, we exhibit an example of kernel for which the pair (f_1,f_2) in (3.7) is not the minimizer of $\mathcal{E}_K^{0,0}$ in $\mathcal{A}_{m_1,m_2}$, for suitably chosen $m_1,m_2 > 0$.

Remark 3.6. Let $\rho > 0$ and let $m_1,m_2 > 0$ be such that $m_1 > 2\rho$, $m_2 > m_1+4\rho$. Consider the kernel $K := \chi_{[-\rho,\rho]}$ and set $A := (-m_1,m_1)$, $B := (-\frac{m_1+m_2}{2}, \frac{m_1+m_2}{2})$, $(f_1,f_2) = (\frac{1}{2}\chi_A, \chi_B - \frac{1}{2}\chi_A)$. Then,

$$
\mathcal{E}^{0,0}(f_1,f_2) = -[-\tfrac{1}{2}J_K(A,A) + J_K(A,B)].
$$

One can easily check that $J_K(A,A) = 4\rho m_1 - \rho^2$ and $J_K(A,B) = 4\rho m_1$; it follows that

$$
\mathcal{E}_K^{0,0}(f_1,f_2) = -\rho(2m_1 + \tfrac{\rho}{2}).
$$

Let now split A into two intervals $A_1 := (-c_{11}\rho-m_1, -c_{11}\rho)$ and $A_2 := (c_{11}\rho, c_{11}\rho+m_1)$, with $\frac{1}{2} < c_{11} < 1$, and consider the energy of the admissible pair

$$
(g_1,g_2) := (\tfrac{1}{2}\chi_{A_1} + \tfrac{1}{2}\chi_{A_2}, \chi_B - \tfrac{1}{2}\chi_{A_1} - \tfrac{1}{2}\chi_{A_2}).
$$

By symmetry $J_K(A_2,A_2) = J_K(A_1,A_1)$ and $J_K(A_2,B) = J_K(A_1,B)$. Hence

$$
\mathcal{E}_K^{0,0}(g_1,g_2) = -[-J_K(A_1,A_1) - J_K(A_1,A_2) + 2J_K(A_1,B)],
$$

where $J_K(A_1,A_1) = 2\rho m_1 - \rho^2$, $J_K(A_1,A_2) = 0$ (since $c_{11} > \frac{1}{2}$) and $J_K(A_1,B) = 2\rho m_1$. It follows that $\mathcal{E}_K^{0,0}(g_1,g_2) = -\rho(2m_1 + \rho) < \mathcal{E}_K^{0,0}(f_1,f_2)$ and therefore (f_1,f_2) is not the minimizer of $\mathcal{E}_K^{0,0}$ in $\mathcal{A}_{m_1,m_2}$. One can easily check that the above result holds true also taking $K(x) := \chi_{[-\rho,\rho]}(x)\,(\rho - |x|)$ and m_1,m_2 as above.

3.5. **The weakly attractive case $c_{11} + c_{22} > -2$ (the case $N \geq 2$).** Now we focus on the case $N \geq 2$, considering first the case $c_{11} = c_{22} = 0$ (Theorem 3.9) and then the remaining cases (Corollaries 3.10 and 3.12).

We first introduce some preliminary notation and recall some well known results we will use in this section. For any $g \in L^2(\mathbb{R}^N;\mathbb{R}^+)$, we set $V := K_{C_N} *g$. Moreover, we recall that for every function $u \in L^1(\mathbb{R}^N;\mathbb{R}^+)$, u^* is the spherical symmetric nonincreasing rearrangement of u defined in (2.1). Clearly, the notion of spherical symmetric nonincreasing rearrangement can be extended in the obvious way to functions $u \in L^1_{loc}(\mathbb{R}^N;\mathbb{R})$ tending to $-\infty$ for $x \to +\infty$.

Lemma 3.7. *Let $g \in L^2(\mathbb{R}^N; \mathbb{R}^+)$, let $m := \int_{\mathbb{R}^N} g(x) \, \mathrm{d}x$, and let $V := K_{C_N} * g$. Moreover, for $N = 2$ assume that g has compact support. Then,*

$$(3.9) \qquad V(x) \to 0 \text{ as } |x| \to +\infty \qquad\qquad \text{for } N \geq 3;$$

$$(3.10) \qquad V(x) = -\frac{m}{2\pi} \log |x| + r(x) \qquad\qquad \text{for } N = 2;$$

where $r(x) \to 0$ as $|x| \to +\infty$. As a consequence, $V - V^* \to 0$ as $|x| \to +\infty$.

Let now $f \in L^1(\mathbb{R}^N; \mathbb{R}^+)$. For any $r > 0$ we denote by $t(r)$ the unique $t \in \mathbb{R}$ such that $|\{V > t\}| \leq \omega_N r^N \leq |\{V \geq t\}|$. Let $\tilde{f} : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$(3.11) \qquad \tilde{f}(x) := \frac{1}{N\omega_N |x|^{N-1}} \frac{\mathrm{d}t}{\mathrm{d}r}\Big|_{t=t(|x|)} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\{V>t\}} f(y) \, \mathrm{d}y\right)\Big|_{t=t(|x|)}.$$

We notice that $B_r = \{V^* > t(r)\}$ and that

$$(3.12) \qquad \int_{\{V^*>t\}} \tilde{f}(x) \, \mathrm{d}x = \int_{\{V>t\}} f(x) \, \mathrm{d}x \qquad \text{for every } t \in \mathbb{R}.$$

Moreover, one can easily check that also $\tilde{f}$ takes values in $\mathbb{R}^+$, and

$$(3.13) \qquad \|\tilde{f}\|_1 = \|f\|_1, \qquad \|\tilde{f}\|_p \leq \|f\|_p \quad \text{for all } 1 < p \leq +\infty.$$

Lemma 3.8. *Let $f \in L^2(\mathbb{R}^N; \mathbb{R}^+)$, with $N \geq 2$, and let $V := K_{C_N} * f$. Moreover, let $\tilde{f} \in L^2(\mathbb{R}^N; \mathbb{R}^+)$ be defined as in (3.11), and let $\tilde{V} := K_{C_N} * \tilde{f}$. Then, $\tilde{V} \geq V^*$, and*

$$\tilde{V}(x) > V^*(x) \qquad \text{for a.e. } x \in B_{r(t_{max})},$$

where t_{max} is the maximal level such that $\{V > t\}$ is a ball for every $t \leq t_{max}$.

Proof. By the coarea formula and the isoperimetric inequality, for almost every $t \in \mathbb{R}$ we have

$$\int_{\partial\{V>t\}} |\nabla V(x)| \, \mathrm{d}\mathcal{H}^{N-1} \geq \int_{\partial\{V^*>t\}} |\nabla V^*(x)| \, \mathrm{d}\mathcal{H}^{N-1},$$

with strict inequality whenever $\{V > t\}$ is not a ball. Therefore, by (3.12)

$$(3.14) \quad \int_{\partial\{V^*>t\}} |\nabla \tilde{V}(x)| \, \mathrm{d}\mathcal{H}^{N-1} \geq -\int_{\{V^*>t\}} \Delta\tilde{V}(x) \, \mathrm{d}x = -\int_{\{V>t\}} \Delta V(x) \, \mathrm{d}x$$

$$= \int_{\partial\{V>t\}} |\nabla V(x)| \, \mathrm{d}\mathcal{H}^{N-1} \geq \int_{\partial\{V^*>t\}} |\nabla V^*(x)| \, \mathrm{d}\mathcal{H}^{N-1}$$

with strict inequalities whenever $\{V > t\}$ is not a ball. Since $\tilde{V} - V^*$ is radial and in view of Lemma 3.7 it vanishes at infinity, we have

$$(3.15) \quad \tilde{V}(r) - V^*(r) = \int_r^{+\infty} \frac{\mathrm{d}}{\mathrm{d}s}\left(V^*(s) - \tilde{V}(s)\right) \, \mathrm{d}s =$$

$$\int_r^{+\infty} \frac{1}{N\omega_N s^{N-1}} \, \mathrm{d}s \int_{\partial B_s} -|\nabla V^*(x)| + |\nabla \tilde{V}(x)| \, \mathrm{d}x.$$

The claim follows since the integrand is nonnegative, and it is strictly positive in a subset of positive measure of $(r, +\infty)$, for all $r < r_{t_{max}}$.

$\square$

Lemma 3.8 establishes that we can rearrange the mass of f_1 in order to obtain a new radial charge configuration $\tilde{f}_1$, increasing the corresponding potential. Exploiting such a result, we deduce that the minimizer of $\mathcal{E}_K^{0,0}$ has radial symmetry. This is done in the next theorem.

Theorem 3.9. *For $m_2 \geq m_1$, the (unique up to a translation) minimizer of $\mathcal{E}_{K_{C_N}}^{0,0}$ in $\mathcal{A}_{m_1,m_2}$ is given by the pair (f_1, f_2) where*

$$f_1 := \tfrac{1}{2}\chi_{B^{2m_1}}, \qquad f_2 := \chi_{B^{m_1+m_2}} - \tfrac{1}{2}\chi_{B^{2m_1}}.$$

Proof. Let (f_1, f_2) be a minimizer of $\mathcal{E}_{K_{C_N}}^{0,0}$ in $\mathcal{A}_{m_1,m_2}$. Let V_1 be the potential generated by f_1 and let $\tilde{f}_1$ and $\tilde{V}_1$ be defined according to Lemma 3.8. Notice that $0 \leq \tilde{f}_1 \leq 1$, and that $\|\tilde{f}_1\|_{L^1(\mathbb{R}^N)} = m_1$. Let us observe that by standard regularity theory, $\tilde{V}_1$ attains a maximum. We denote it by $\tilde{M}_1$. We first show that there exists $\tilde{t} < \tilde{M}_1$ such that

$$\text{(3.16)} \qquad \int_{\{\tilde{V}_1 > \tilde{t}\}} (1 - \tilde{f}_1(x))\, \mathrm{d}x = m_2.$$

Suppose by contradiction that there does not exist $\tilde{t}$ such that (3.16) holds true. Notice that $-\Delta \tilde{V} = \tilde{f}$ and that $\tilde{f}$ and $\tilde{V}$ are radially symmetric. Therefore, $\tilde{V}$ may have a flat region only in a ball centered at the origin, whereas it is strictly decreasing with respect to $|x|$ elsewhere. We deduce that $\int_{\{\tilde{V}_1 > t\}} (1 - \tilde{f}_1(x))\, \mathrm{d}x > m_2$ for any $t < \tilde{M}_1$, and in particular that $\int_{\{\tilde{V}_1 = \tilde{M}_1\}} (1 - \tilde{f}_1(x))\, \mathrm{d}x \geq m_2$. It follows that $|\{\tilde{V}_1 = \tilde{M}_1\}| \geq m_2$ and, since $\tilde{V}_1$ is radially symmetric, $\{\tilde{V}_1 = \tilde{M}_1\}$ is a ball centered at the origin containing B^{m_2}. Set $\hat{f}_2 := \chi_{B^{m_2}}$; we have that $(\tilde{f}_1, \hat{f}_2) \in \mathcal{A}_{m_1,m_2}$, and by Lemma 2.2 $(\tilde{f}_1, \hat{f}_2)$ is a minimizer. Notice that the supports of $\tilde{f}_1$ and $\hat{f}_2$ are disjoint, but this is in contradiction with Proposition 3.3. We conclude that there exists $\tilde{t}$ satisfying (3.16). Set

$$\tilde{f}_2(x) := \begin{cases} 1 - \tilde{f}_1(x) & \text{for } x \in \{\tilde{V}_1 > \tilde{t}\} \\ 0 & \text{otherwise.} \end{cases};$$

by construction $(\tilde{f}_1, \tilde{f}_2) \in \mathcal{A}_{m_1,m_2}$ ($\int_{\mathbb{R}^N} \tilde{f}_2(x)\, \mathrm{d}x = m_2$ by (3.16)).

Let now $\hat{t} \leq \tilde{t}$ be such that

$$\{V_1^* > \hat{t}\} \subseteq \{\tilde{V}_1 > \tilde{t}\} \subseteq \{V_1^* \geq \hat{t}\}.$$

This is possible since the superlevel set $\{\tilde{V}_1 > \tilde{t}\}$ is a ball centered at the origin. Let $A := \{\tilde{V}_1 > \tilde{t}\} \setminus \{V_1^* > \hat{t}\}$. Since $A \subseteq \{V_1^* = \hat{t}\}$, we have $\tilde{f}_1 = 0$ a.e. on A, and hence $\tilde{f}_2 = 1$ a.e. on A. Moreover, by Corollary 2.3 we can always assume that

$$\text{(3.17)} \qquad \mathrm{supp}\, f_2 = \{V_1 \geq \hat{t}\} \cup A', \quad f_2 = 1 - f_1 \text{ on } \{V_1 > \hat{t}\}, \quad f_2 \equiv 1 \text{ on } A',$$

for some set $A' = \{V_1 \geq \hat{t}\}$ with $|A'| = |A|$. By the Coarea Formula and Lemma 3.8 we have

$$
\begin{aligned}
\mathcal{E}_{K_{C_N}}^{0,0}(\tilde{f}_1, \tilde{f}_2) &= -2 \int_{\mathbb{R}^N} \tilde{f}_2(x) \tilde{V}_1(x) \, \mathrm{d}x \\
(3.18) \qquad &\leq -2 \int_{\mathbb{R}^N} \tilde{f}_2(x) V_1^*(x) \, \mathrm{d}x \\
&= -2 \hat{t} |A| - 2 \int_{\hat{t}}^{+\infty} t \int_{\{V_1^* > t\}} (1 - \tilde{f}_1(x)) \, \mathrm{d}x \, \mathrm{d}t \\
&= -2 \hat{t} |A| - 2 \int_{\hat{t}}^{+\infty} t \int_{\{V_1 > t\}} (1 - f_1(x)) \, \mathrm{d}x \, \mathrm{d}t \\
&= -2 \hat{t} |A| - 2 \int_{\{V_1 > \hat{t}\}} (1 - f_1(x)) V_1(x) \, \mathrm{d}x \\
(3.19) \qquad &= -2 \int_{A'} f_2(x) \hat{t} \, \mathrm{d}x - 2 \int_{\{V_1 > \hat{t}\}} (1 - f_1(x)) V_1(x) \, \mathrm{d}x \\
(3.20) \qquad &= -2 \int_{\mathbb{R}^N} f_2(x) V_1(x) \, \mathrm{d}x = \mathcal{E}_{K_{C_N}}^{0,0}(f_1, f_2),
\end{aligned}
$$

where the equality in (3.19) follows from (3.17). By minimality, the inequality in (3.18) is actually an equality, and hence $V_1^* \equiv \tilde{V}_1$. It follows that all the superlevels of V_1 are balls. By Proposition 3.1, $f_1 = \frac{1}{2}$ in $G_1 \cap G_2$, whereas by Lemma 3.3 $G_1 \cup F_1 = G_1 \cap G_2$, so that $f_1 := \frac{1}{2}\chi_E$ for some set E. Since all the superlevel sets of V_1 are balls, we conclude that, up to a translation, $f_1 := \frac{1}{2}\chi_{B^{2m_1}}$. By Corollary 2.2 we also deduce that $f_2 := \chi_{B^{m_1+m_2}} - \frac{1}{2}\chi_{B^{2m_1}}$ and this concludes the proof. $\qquad \square$

Corollary 3.10. *Let $c_{11} \leq -1$ and $-1 \leq c_{22} \leq 0$ (resp. $c_{22} \leq -1$ and $-1 \leq c_{11} \leq 0$) with $c_{11} + c_{22} > -2$. Then, the (unique up to a translation) minimizer of $\mathcal{E}_{K_{C_N}}^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ is given by the pair*

$$
(f_1, f_2) = (\chi_{B^{m_1}}, \chi_{B^{m_1+m_2} \setminus B^{m_1}}) \quad (resp. \ (f_1, f_2) = (\chi_{B^{m_1+m_2} \setminus B^{m_2}}, \chi_{B^{m_2}}).
$$

Figure 5. The phase f_1 is black and the phase f_2 is white. The minimizer in the case $c_{11} \leq -1 \leq c_{22} \leq 0$ is represented on the left, whereas on the right there is the minimizer in the case $c_{22} \leq -1 \leq c_{11} \leq 0$.

Proof. We prove the claim only for $c_{11} \leq -1$ and $-1 \leq c_{22} \leq 0$ with $c_{11} + c_{22} > -2$, the proof of the other case being fully analogous. Let (f_1, f_2) be a minimizer of $\mathcal{E}_{K_{C_N}}^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$.

Set $\tilde{m}_1 := \frac{m_1}{2}$, $\tilde{m}_2 := \frac{m_1}{2} + m_2 > \tilde{m}_1$,

$$(3.21) \qquad g_1 := \frac{f_1}{2}, \qquad g_2 := \frac{f_1}{2} + f_2.$$

It is easy to see that $g_i \geq 0$, $\int_{\mathbb{R}^N} g_i(x)\,dx = \tilde{m}_i$ (for $i = 1, 2$) and $g_1 + g_2 = f_1 + f_2 \leq 1$, so that $(g_1, g_2) \in \mathcal{A}_{\tilde{m}_1, \tilde{m}_2}$. A straightforward computation yields

$$
\begin{aligned}
\mathcal{E}_{K_{C_N}}^{c_{11},c_{22}}(f_1, f_2) &= (c_{11} + 1)J_{K_{C_N}}(f_1, f_1) + \mathcal{E}_{K_{C_N}}^{-1,c_{22}}(f_1, f_2) \\
&= (c_{11} + 1)J_{K_{C_N}}(f_1, f_1) + c_{22}J_{K_{C_N}}(f_1 + f_2, f_1 + f_2) \\
&\quad + (1 + c_{22})(-J_{K_{C_N}}(f_1, f_1) - 2J_{K_{C_N}}(f_1, f_2)) \\
(3.22) \qquad &= 4\,(c_{11} + 1)J_{K_{C_N}}(g_1, g_1) + c_{22}J_{K_{C_N}}(g_1 + g_2, g_1 + g_2) \\
&\quad + 2(1 + c_{22})\,\mathcal{E}_{K_{C_N}}^{0,0}(g_1, g_2),
\end{aligned}
$$

and hence (f_1, f_2) is a minimizer of $\mathcal{E}_{K_{C_N}}^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if (g_1, g_2) minimizes the energy

$$(3.23) \quad 4\,(c_{11}+1)J_{K_{C_N}}(g_1, g_1) + c_{22}J_{K_{C_N}}(g_1 + g_2, g_1 + g_2) + 2(1 + c_{22})\,\mathcal{E}_{K_{C_N}}^{0,0}(g_1, g_2)$$

in $\mathcal{A}_{\tilde{m}_1, \tilde{m}_2}$. By Theorem 3.9, the third addendum in (3.23) is minimized (in $\mathcal{A}_{\tilde{m}_1, \tilde{m}_2}$) if and only if

$$(g_1, g_2) = (\tfrac{1}{2}\chi_{B^{2\tilde{m}_1}}, \tfrac{1}{2}\chi_{B^{2\tilde{m}_1}} + \chi_{B^{\tilde{m}_1 + \tilde{m}_2} \setminus B^{2\tilde{m}_1}}).$$

We notice that such configuration minimizes also the first and the second addendum. The claim follows directly by (3.21). $\square$

Quantitative Riesz inequalities have been recently studied in [5, Theorem 1.5]. For any measurable set $E \subset \mathbb{R}^N$ with finite measure, let $E^* := B^{|E|}$ be the ball centered at the origin such that $|E^*| = |E|$. From Corollary 3.10 with $c_{11} = -1$ and $c_{22} = 0$ we immediately get the following improved Riesz inequality.

Corollary 3.11. *For any measurable sets $E_1 \subseteq E_2 \subset \mathbb{R}^N$ with finite measure, there holds*

$$(3.24) \quad J_{K_{C_N}}(E_1^*, E_2^*) - J_{K_{C_N}}(E_1, E_2) \geq \frac{1}{2}\left(J_{K_{C_N}}(E_1^*, E_1^*) - J_{K_{C_N}}(E_1, E_1)\right).$$

Moreover, for any measurable sets $A_1 \subseteq A_2 \subset \mathbb{R}^N$ with finite measure, there holds

$$(3.25) \quad J_{K_{C_N}}(A_2, A_2) - J_{K_{C_N}}(A_1, A_1) \leq J_{K_{C_N}}(B^{|A_2|}, B^{|A_2|})$$
$$- J_{K_{C_N}}(B^{|A_2|} \setminus B^{|A_2| - |A_1|}, B^{|A_2|} \setminus B^{|A_2| - |A_1|}).$$

Proof. We only prove (3.24), since (3.25) is indeed equivalent to (3.24) replacing E_1 with $A_2 \setminus A_1$ and E_2 with A_2.

Let $f_1 := \chi_{E_1}$, $f_2 := \chi_{E_2 \setminus E_1}$. By Corollary 3.10 we have

$$
\begin{aligned}
J_{K_{C_N}}(E_1, E_1) - 2J_{K_{C_N}}(E_1, E_2) &= J_{K_{C_N}}(f_1, f_1) - 2J_{K_{C_N}}(f_1, f_1 + f_2) \\
&= \mathcal{E}_{K_{C_N}}^{-1,0}(f_1, f_2) \geq \mathcal{E}_{K_{C_N}}^{-1,0}(\chi_{E_1^*}, \chi_{E_2^* \setminus E_1^*}) = J_{K_{C_N}}(E_1^*, E_1^*) - 2J_{K_{C_N}}(E_1^*, E_2^*).
\end{aligned}
$$

$\square$

In the next corollary we will consider the case $-1 < c_{11} \leq 0$, $-1 < c_{22} \leq 0$, completing the analysis of the weakly attractive case for the Coulomb interaction kernel. Recall the coefficients a_i defined in (2.5).

Corollary 3.12. *Let $-1 < c_{11} \leq 0$, $-1 < c_{22} \leq 0$. The following results hold true.*

(i) If $(c_{22}+1)m_2 > (c_{11}+1)m_1$, then the (unique up to a translation) minimizer of $\mathcal{E}_{K_{C_N}}^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ is given by the pair

$$(f_1, f_2) = \left(a_1\, \chi_{B^{\frac{m_1}{a_1}}},\, \chi_{B^{m_2+m_1}} - a_1\, \chi_{B^{\frac{m_1}{a_1}}} \right).$$

(ii) If $(c_{11}+1)m_1 > (c_{22}+1)m_2$, then the (unique up to a translation) minimizer of $\mathcal{E}_{K_{C_N}}^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ is given by the pair

$$(f_1, f_2) = \left(\chi_{B^{m_2+m_1}} - a_2\, \chi_{B^{\frac{m_2}{a_2}}},\, a_2\, \chi_{B^{\frac{m_2}{a_2}}} \right).$$
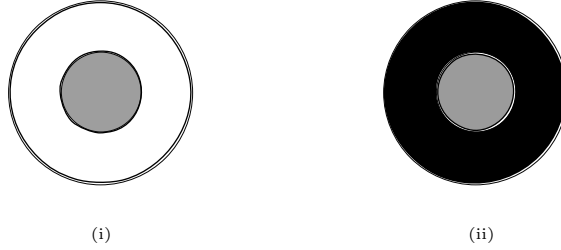

(i)   (ii)

Figure 6. The phase f_1 is black and the phase f_2 is white. The mixing of the two phases is represented by the grey color. The cartoon on the left represents the unique minimizer in (i). In this case, the two phases mix each other in the inner ball, and the remainig mass of f_2 is arranged in an annulus around such ball. In the case (ii), the minimizer has the same form, but replacing f_2 (white) with f_1 (black).

Proof. We prove only (i) since the proof of (ii) is analogous. Let (f_1, f_2) be a minimizer of $\mathcal{E}_{K_{C_N}}^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$. We first notice that, in the case $c_{22} < 0$, by i) and ii) of Proposition 3.1 we have $f_1 = a_1\chi_A$, $f_2 = a_2\chi_A + \chi_B$ for some measurable sets $A, B \subset \mathbb{R}^N$. Then, one can argue as in the proof of Corollary 3.5 (applying Corollary 3.10 instead of Proposition 3.4). The details are left to the reader.

It remains to prove the claim for $c_{22} = 0$. In this case set $\tilde{m}_1 := \frac{c_{11}+2}{2}m_1$ and $\tilde{m}_2 := -\frac{c_{11}}{2}m_1 + m_2$. By assumption $\tilde{m}_2 > \tilde{m}_1$. Set moreover

(3.26) $$g_1 := \frac{c_{11}+2}{2}f_1, \qquad g_2 := -\frac{c_{11}}{2}f_1 + f_2.$$

It is easy to see that $g_i \geq 0$, $\int_{\mathbb{R}^N} g_i(x)\,\mathrm{d}x = \tilde{m}_i$ (for $i = 1, 2$) and $g_1 + g_2 = f_1 + f_2 \leq 1$, so that $(g_1, g_2) \in \mathcal{A}_{\tilde{m}_1, \tilde{m}_2}$. Moreover, a straightforward computation yields

$$\mathcal{E}_{K_{C_N}}^{c_{11},0}(f_1, f_2) = \frac{2}{c_{11}+2}\,\mathcal{E}_{K_{C_N}}^{0,0}(g_1, g_2),$$

and hence (f_1, f_2) is a minimizer of $\mathcal{E}_{K_{C_N}}^{c_{11},0}$ in $\mathcal{A}_{m_1,m_2}$ if and only if (g_1, g_2) is a minimizer of $\mathcal{E}_{K_{C_N}}^{0,0}$ in $\mathcal{A}_{\tilde{m}_1, \tilde{m}_2}$. By Theorem 3.9, the unique (up to a translation) minimizer of $\mathcal{E}_{K_{C_N}}^{0,0}$ in $\mathcal{A}_{\tilde{m}_1, \tilde{m}_2}$ is given by $(g_1, g_2) = (\frac{1}{2}\chi_{B^{2\tilde{m}_1}}, \frac{1}{2}\chi_{B^{2\tilde{m}_1}} + \chi_{B^{\tilde{m}_1+\tilde{m}_2}\setminus B^{\tilde{m}_1}})$. This, together with (3.26), concludes the proof.   □

Conclusions and perspectives

We have studied existence and qualitative properties of minimizers of the energy

$$\mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2) = c_{11}\, J_K(f_1, f_1) + c_{22}\, J_K(f_2, f_2) - 2J_K(f_1, f_2),$$

in the class of densities $(f_1, f_2) \in L^1(\mathbb{R}^N; [0,1]) \times L^1(\mathbb{R}^N; [0,1])$ with fixed masses m_1, m_2 and satisfying the constraint $f_1 + f_2 \leq 1$. We have focused on the attractive case $c_{11}, c_{22} \leq 0$ (the checkerboard region in Figure 7), and proved the existence of a minimizer in this case for all the values of masses m_1, m_2 (see Theorem 1.8). Moreover, for $0 < c_{11} = c_{22} \leq 1$, $m_1 = m_2$ and K positive definite (the dashed segment in Figure 7), we have proved that there exists a minimizer (see Lemma 2.8 and Remark 1.10). Finally, for $c_{11}, c_{22} \geq 1$ with $\max\{c_{11}, c_{22}\} > 1$ (grey region in the Figure 7), the energy $\mathcal{E}_K^{c_{11},c_{22}}$ does not admit a minimizer for any pair of values m_1 and m_2 (see Remark 1.10).
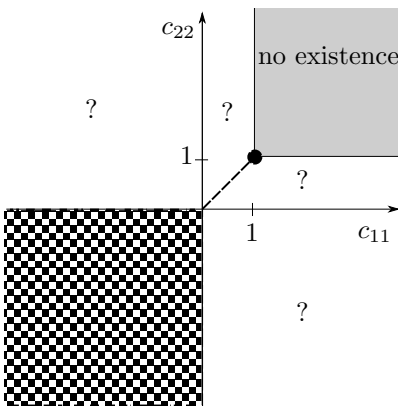

Figure 7. Existence/Non existence regions of parameters c_{11}, c_{22}

A natural question arising from these (partial) results is whether existence of minimizers can be proven in the remaining cases. A general existence result, i.e., independent of the masses, seems to be false if at least one of the coefficients is strictly positive. Indeed, the corresponding phase would loose some of its (if too large) mass. In this case, existence results depending on the masses seems to be an interesting issue.

A relevant aspect of our analysis is that, for the Coulomb interaction kernel, we have found the explicit shape of minimizers for all choices of negative coefficients, expect when they are both strictly less than -1 (see Figure 8). In this case, we can still say that f_i are characteristic functions of two pairwise disjoint sets. But their specific shape is unknown, and could be analyzed using numerical methods.

For general kernels our analysis is far from being complete. Nevertheless, there are many possible generalizations we would like to comment on.

First of all, one may study the minimum problem above for some specific kernels that are very used in the context of population dynamics (see for instance [8, 12] and the references therein) such as, Gaussian, Morse or power law kernels, or suitable combinations of these ones. Moreover, one might remove the assumption that the cross and self interaction kernels K_{ij} are all multiples of a given K. Actually, it would be interesting also to understand whether the improved Riesz inequality
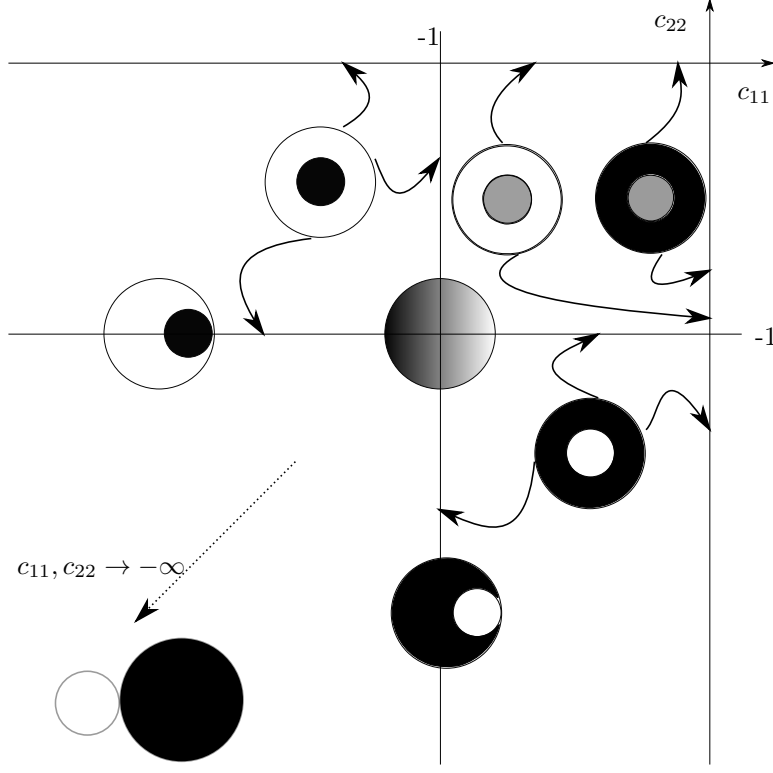
Figure 8. Minimizers for Coulomb interactions

established in Corollary 3.11 holds true for more general kernels. We notice that this Corollary is equivalent to Theorem 3.9 once one knows that there is not coexistence of two homogeneous phases, i.e., when f_i are as in Lemma 3.3.

Another interesting direction is the extension of the model to the case of n species, i.e., considering minimizers of functionals of the type

$$\mathcal{E}_K(f_1, \ldots, f_n) := \sum_{i,j=1}^{n} J_{K_{ij}}(f_i, f_j)$$

under the constraint $\sum_{i=1}^{n} f_i \leq 1$ and $\int_{\mathbb{R}^N} f_i(x) \, dx = m_i$ for $i = 1, 2, \ldots, n$. We believe that some of the techniques developed here could be slightly modified in order to prove existence and some qualitative properties of the minimizers. As already mentioned, the explicit shape of minimizers might require a specific analysis and could be subject of numerical investigation.

Finally, we point out that our analysis focuses only on the global minimizers of the functional $\mathcal{E}_K^{c_{11}, c_{22}}$. Notice that ground states play a crucial role in the long time asymptotics of nonlinear aggregation-diffusion models. Nevertheless, the analysis of stationary states (rather than minimizers) would provide a better understanding of such problems. In this respect, an interesting analysis would concern the dynamics of two phases governed by the energy proposed in this paper. A suitable notion of Wasserstein gradient flow could be considered, in the spirit of [8, 28].

References

[1] L. Ambrosio, A. Braides: Functionals defined on partitions in sets of finite perimeter. II. *J. Math. Pures Appl.* **69**, no. 3, 307–333, 1990.

[2] M. Bonacini, R. Cristoferi: Local and global minimality results for a nonlocal isoperimetric problem on $\mathbb{R}^N$. *SIAM J. Math. Anal.* **46**, no. 4, 2310–2349, 2014.

[3] M. Bonacini, H. Knüpfer, M. Röger: Optimal distribution of oppositely charged phases: perfect screening and other properties. Preprint, 2015.

[4] J. A. Cañizo, J. A. Carrillo, F. S. Patacchini: Existence of Compactly Supported Global Minimisers for the Interaction Energy. *Arch. Rat. Mech. Anal.* **217** , 1197–1217, 2015.

[5] E.A. Carlen, F. Maggi: Stability for the Brunn-Minkowski and Riesz rearrangement inequalities, with applications to Gaussian concentration and finite range non-local isoperimetry. Preprint 2015.

[6] E.A. Carlen, M. Carvalho, R. Esposito, J.L. Lebowitz, R. Marra: Free energy minimizers for a two-species model with segregation and liquid-vapor transition. *Nonlinearity* **16**, 1075–1105, 2003.

[7] J.A. Carrillo, M. Chipot, Y. Huang: On global minimizers of repulsive-attractive power-law interaction energies. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372**, no. 2028, 2014.

[8] J.A. Carrillo, M. Di Francesco, A. Figalli, T. Laurent; D. Slepčev. Global-in-time weak measure solutions and finite-time aggregation for nonlocal interaction equations. *Duke Math. J.* **156**, no. 2, 229–271, 2011.

[9] R. Choksi, R.C. Fetecau, I. Topaloglu: On minimizers of interaction functionals with competing attractive and repulsive potentials. *Annales IHP (C) Nonlinear Analysis*, to appear.

[10] E. Cristiani, B. Piccoli, A. Tosin: *Multiscale Modeling of Pedestrian Dynamics: Modeling, Simulation and Applications*, vol. 12, Springer International Publishing, 2014.

[11] A. Di Castro, M. Novaga, B. Ruffini, E. Valdinoci: Nonlocal quantitative isoperimetric inequalities. To appear on *Calc. Var. Partial Differential Equations*.

[12] M. Di Francesco, S. Fagioli: Measure solutions for non-local interaction PDEs with two species. *Nonlinearity* **26**, no. 10, 2777–2808, 2013.

[13] C. Escudero, F. Maciá, J.J.L. Velázquez: Two-species-coagulation approach to consensus by group level interactions. *Phys. Rev. E* **82**, no.1, 016113, 2010.

[14] A. Figalli, N. Fusco, F. Maggi, V. Millot, M. Morini: Isoperimetry and stability properties of balls with respect to nonlocal energies. *Comm. Math. Phys.* **336**, no. 1, 441–507, 2015.

[15] E.H. Lieb, M. Loss.: *Analysis*, vol. 14 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2001.

[16] V. Julin: Isoperimetric problem with a Coulomb repulsive term. *Indiana Univ. Math. J.* **63**, no.1, 77–89, 2014.

[17] P. Kevrevidis, A. G. Stefanov, H. Xu: Treveling waves for the mass in mass model of granular chains. Preprint 2015.

[18] H. Knüpfer, C. Muratov.: On an isoperimetric problem with a competing non-local term. I. The planar case. *Comm. Pure Appl. Math.* **66**, no.7, 1129–1162, 2013.

[19] H. Knüpfer, C. Muratov.: On an isoperimetric problem with a competing non-local term. II. The general case. *Comm. Pure Appl. Math.* **67**, no. 12, 1974–1994, 2014.

[20] T. Kolokolnikov, Y. Huang, M. Pavlovski: Singular patterns for an aggregation model with a confining potential. *Phys. D*, 260, 65–76, 2013.

[21] T. Kolokolnikov, H. Sun, D. Uminsky, A.L. Bertozzi: A theory of complex patterns arising from 2D particle interactions. *Phys. Rev. E, Rapid Communications* 84, 015203 (R), 2011.

[22] N.S. Landkof: *Foundations of modern potential theory*, translated from the Russian by A. P. Doohovskoy. Die Grundlehren der mathematischen Wissenschaften, Band 180. Springer-Verlag, New York-Heidelberg, 1972.

[23] H. Levine, E. Ben-Jacob, I. Cohen, W.-J. Rappel: Swarming patterns in Microorganisms: Some new modeling results. *Proceedings IEEE CDC*, 5073–5077, 2006.

[24] T. Liu, M.L.K. Langston, D. Li, J.M. Pigga, C. Pichon, A M. Todea, A. Müller: Self-recognition among different polyprotic macroions during assembly processes in dilute solution. *Science*, 331, no. 6024, 1590–1592, 2011.

[25] J. Lu, F. Otto: Nonexistence of minimizers for Thomas-Fermi-Dirac-von Weizsäcker model. *Comm. Pure Appl. Math.* **67**, no. 10, 1605–1617, 2014.

[26] A. Mackey, T. Kolokolnikov, A.L. Bertozzi: Two-species particle aggregation and stability of co-dimension one solutions. *Discrete Contin. Dyn. Syst. Ser. B* **19**, no. 5, 1411–1436, 2014.

[27] A. Magni, M. Novaga: A note on non lower semicontinuous perimeter functionals on partitions. Preprint, 2015.

[28] B. Maury, A. Roudneff-Chupin, F. Santambrogio, J. Venel: Handling congestion in crowd motion models. *Netw. Heterog. Media* 6, no. 3, 485–519, 2011.

[29] X. Ren, J. Wei: A double bubble assembly as a new phase of a ternary inhibitory system. *Arch. Rat. Mech. Anal.* **215**, no. 3, 967–1034, 2015.

[30] F. Riesz: Sur une inégalité intégrale. *Journ. London Math. Soc.* 5,162–168, 1930.

[31] R. Simione, D. Slepcev, I. Topaloglu: Existence of ground states of nonlocal-interaction energies. *J. Stat. Phys.* **159**, no. 4, 972–986, 2015.

[32] J.D. van der Waals, I. Verhandelingen: Kon.Akad. Wet. Amsterdam 20, 1880; II Théorie moléculaire d'une substance composée de deux matiéres différentes , *Arch. néerl.* 24, 1, 1891.

(Marco Cicalese) Zentrum Mathematik - M7, Technische Universität München, Boltzmannstrasse 3, 85748 Garching, Germany

E-mail address, M. Cicalese: `cicalese@ma.tum.de`

(Lucia De Luca) Zentrum Mathematik - M7, Technische Universität München, Boltzmannstrasse 3, 85748 Garching, Germany

E-mail address, L. De Luca: `deluca@ma.tum.de`

(Matteo Novaga) Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy

E-mail address, M. Novaga: `novaga@dm.unipi.it`

(Marcello Ponsiglione) Dipartimento di Matematica "Guido Castelnuovo", Sapienza Università di Roma, P.le Aldo Moro 5, I-00185 Roma, Italy

E-mail address, M. Ponsiglione: `ponsigli@mat.uniroma1.it`