ON THE ERROR ANALYSIS OF STABILIZED FINITE ELEMENT METHODS FOR THE STOKES PROBLEM

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Abstract. For a family of stabilized mixed finite element methods for the Stokes equations a complete a priori and a posteriori error analysis is given.

Key words. Stabilized finite element methods, Galerkin least squares methods, Stokes problem, incompressible elasticity, a priori error estimates, a posteriori error estimates

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1. Introduction. Stabilization of mixed finite element methods for saddle point problems [10, 9, 4, 3] is by now a well-established technique to design stable methods with finite element spaces which do not have to satisfy the so-called Babuška–Brezzi condition. The idea is to add a properly weighted residual of the momentum balance equation to the variational bilinear form. This resembles the least squares method and hence the formulation has often been called the Galerkin least squares method. This is, however, a somewhat misleading name since the formulation does not lead to a minimization problem.

In the paper by Franca and Stenberg [6] a unified stability and error analysis for this class of methods was given. The error estimates were obtained under the assumption that the solution is regular enough, and so far a general analysis has been missing.

The purpose of this paper is to address this question. We will show that using a technique proposed recently by Gudi [7] it is possible to derive quasi-optimal a priori estimates. This technique uses estimates known from a posteriori error analysis.

In addition to the a priori analysis, we discuss a posteriori estimates. Since the added stabilization term is exactly a weighted residual, the a posteriori analysis is very straightforward. Similar a posteriori estimates were given in [18]. Our analysis seems, however, more natural.

The plan of the paper is as follows. In the next sections we recall the continuous Stokes problem and its discretization by stabilized finite element methods. Then we present the new a priori analysis. We end by deriving the a posteriori error estimates. We will use well established notation. In addition, we will use the shorthand notation $A \lesssim B$ for: there exist a positive constant C, independent of the mesh parameter h, such that $A \leq CB$.

2. The Stokes problem. We consider the Stokes equations for slow (or very viscous) steady fluid flow or equivalently, the equations of incompressible elasticity, which we normalize in such a way that $2\mu = 1$, where μ is the dynamic viscosity or first Lamé parameter, respectively. Let **div** denote the vector valued divergence applied to tensors and denote the symmetric velocity gradient/strain tensor by

$$(2.1) \qquad \boldsymbol{D}(\boldsymbol{v}) = \frac{1}{2}\big(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T\big).$$

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Introducing the second order differential operator

$$(2.2) \qquad \boldsymbol{A}\boldsymbol{v} = \mathbf{div}\boldsymbol{D}(\boldsymbol{v}),$$

the problem is: given $(\boldsymbol{f}, \boldsymbol{t}, g)$, find $(\boldsymbol{u}, p)$ such that

$$(2.3) \qquad -\boldsymbol{A}\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega,$$
$$(2.4) \qquad \operatorname{div} \boldsymbol{u} = g \quad \text{in } \Omega,$$
$$(2.5) \qquad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D,$$
$$(2.6) \qquad \big(\boldsymbol{D}(\boldsymbol{u}) - p\boldsymbol{I}\big)\boldsymbol{n} = \boldsymbol{t} \quad \text{on } \Gamma_N.$$

The domain $\Omega \subset I\!\!R^d$, $d = 2, 3$, is assumed bounded and with a polygonal or polyhedral boundary. With the bilinear form

$$(2.7) \qquad \mathcal{B}(\boldsymbol{w}, r; \boldsymbol{v}, q) = (\boldsymbol{D}(\boldsymbol{w}), \boldsymbol{D}(\boldsymbol{v})) - (\operatorname{div} \boldsymbol{v}, r) - (\operatorname{div} \boldsymbol{w}, q),$$

and the linear form

$$(2.8) \qquad \mathcal{F}(\boldsymbol{v}, q) = (\boldsymbol{f}, \boldsymbol{v}) + \langle \boldsymbol{t}, \boldsymbol{v} \rangle_{\Gamma_N} - (g, q),$$

we define the variational formulation.

THE CONTINUOUS PROBLEM. *Find* $(\boldsymbol{u}, p) \in \boldsymbol{H}_D^1(\Omega) \times L^2(\Omega)$ *such that*

$$(2.9) \qquad \mathcal{B}(\boldsymbol{u}, p; \boldsymbol{v}, q) = \mathcal{F}(\boldsymbol{v}, q) \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{H}_D^1(\Omega) \times L^2(\Omega).$$

Here, $\boldsymbol{H}_D^1(\Omega) = \boldsymbol{H}^1(\Omega) \cap \{ \boldsymbol{v} \mid \boldsymbol{v}_{|\Gamma_D} = \boldsymbol{0} \}$.

The stability of this is a consequence of Korn's inequality

$$(2.10) \qquad C\|\boldsymbol{v}\|_1 \le \|\boldsymbol{D}(\boldsymbol{v})\|_0 \le \|\boldsymbol{v}\|_1$$

and the condition

$$(2.11) \qquad \sup_{\boldsymbol{v} \in \boldsymbol{H}_D^1(\Omega)} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \gtrsim \|q\|_0 \quad \forall q \in L^2(\Omega).$$

Together they imply the stability:

THEOREM 2.1. *It holds*

$$(2.12) \qquad \sup_{(\boldsymbol{v},q) \in \boldsymbol{H}_D^1(\Omega) \times L^2(\Omega)} \frac{\mathcal{B}(\boldsymbol{w}, r; \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1 + \|q\|_0} \gtrsim (\|\boldsymbol{w}\|_1 + \|r\|_0)$$
$$\forall (\boldsymbol{w}, r) \in \boldsymbol{H}_D^1(\Omega) \times L^2(\Omega).$$

Classical mixed finite element methods are based on the variational formulation above posed in the finite element subspaces. By analogy with the continuous problem, the discrete spaces have to satisfy the Babuška–Brezzi condition, i.e. the discrete counterpart of (2.11). The recent monograph [1] contains the state of the art information on stable velocity–pressure pairs.

3. Stabilized methods. We denote the piecewise polynomial finite element subspaces for the velocity and pressure by $\boldsymbol{V}_h \subset \boldsymbol{H}_D^1(\Omega)$ and $P_h \subset L^2(\Omega)$, respectively. The underlying mesh is denoted by $\mathcal{C}_h$. As usual, the diameter of an element $K \in \mathcal{C}_h$, is denoted by h_K. Next, we define the bilinear and linear forms

$$(3.1) \qquad \mathcal{S}_h(\boldsymbol{w}, r; \boldsymbol{v}, q) = \sum_{K \in \mathcal{C}_h} h_K^2(-\boldsymbol{A}\boldsymbol{w} + \nabla r, -\boldsymbol{A}\boldsymbol{v} + \nabla q)_K,$$

$$(3.2) \qquad \mathcal{L}_h(\boldsymbol{v}, q) = \sum_{K \in \mathcal{C}_h} h_K^2(\boldsymbol{f}, -\boldsymbol{A}\boldsymbol{v} + \nabla q)_K.$$

From the differential equation (2.3) it follows.

LEMMA 3.1. *If $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$ it holds*

$$(3.3) \qquad \mathcal{S}_h(\boldsymbol{u}, p; \boldsymbol{v}, q) = \mathcal{L}_h(\boldsymbol{v}, q) \quad \forall(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times P_h.$$

Proof. The differential equation (2.3) has to be interpreted in the sense of distributions. However, with the assumption $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$ the sum $-\boldsymbol{A}\boldsymbol{v} + \nabla q$ is in $\boldsymbol{L}_2(\Omega)$ and hence both $\mathcal{S}_h(\boldsymbol{u}, p; \boldsymbol{v}, q)$ and $\mathcal{L}_h(\boldsymbol{v}, q)$ are well defined and equal. $\square$

Next, we define the forms

$$(3.4) \qquad \mathcal{B}_h(\boldsymbol{w}, r; \boldsymbol{v}, q) = \mathcal{B}(\boldsymbol{w}, r; \boldsymbol{v}, q) - \alpha \mathcal{S}_h(\boldsymbol{w}, r; \boldsymbol{v}, q)$$

and

$$(3.5) \qquad \mathcal{F}_h(\boldsymbol{v}, q) = \mathcal{F}(\boldsymbol{v}, q) - \alpha \mathcal{L}_h(\boldsymbol{v}, q),$$

where α is a positive constant less than the constant C_I in the following inverse inequality, which is valid in piecewise polynomial spaces with shape regular elements:

$$(3.6) \qquad C_I \sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{A}\boldsymbol{v}\|_{0,K}^2 \leq \|\boldsymbol{D}(\boldsymbol{v})\|_0^2.$$

The stabilized formulation is then the following.

THE FINITE ELEMENT METHOD. *Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times P_h$ such that*

$$(3.7) \qquad \mathcal{B}_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}, q) = \mathcal{F}_h(\boldsymbol{v}, q) \quad \forall(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times P_h.$$

The consistency follows from Lemma 3.1.

THEOREM 3.2. *Suppose that $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$. Then finite element method is consistent, in the sense that the exact solution $(\boldsymbol{u}, p) \in \boldsymbol{H}_D^1(\Omega) \times L^2(\Omega)$ to (2.9) satisfies the discrete variational form*

$$(3.8) \qquad \mathcal{B}_h(\boldsymbol{u}, p; \boldsymbol{v}, q) = \mathcal{F}_h(\boldsymbol{v}, q) \quad \forall(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times P_h.$$

Next, we outline the main steps for analyzing the stability of the formulation. For $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times P_h$, the inverse inequality (3.6) and the assumption $0 < \alpha < C_I$ give

$$
\begin{aligned}
\mathcal{B}_h(\boldsymbol{w}, r; \boldsymbol{w}, -r) &= \|\boldsymbol{D}(\boldsymbol{w})\|_0^2 - \alpha \sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{A}\boldsymbol{w}\|_{0,K}^2 + \alpha \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla r\|_{0,K}^2 \\
(3.9) \qquad &\geq \left(1 - \frac{\alpha}{C_I}\right) \|\boldsymbol{D}(\boldsymbol{w})\|_0^2 + \alpha \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla r\|_{0,K}^2 \\
&\gtrsim \left(\|\boldsymbol{D}(\boldsymbol{w})\|_0^2 + \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla r\|_{0,K}^2\right).
\end{aligned}
$$

As for the continuous problem the stability for the velocity follows from Korn's inequality (2.10), whereas the stability of the pressure is in the mesh dependent seminorm

$$(3.10) \qquad \Big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla r\|_{0,K}^2 \Big)^{1/2}$$

as a consequence of the added stabilization term. This gives stability for all pressures except the piecewise constants. In the case of continuous pressure approximations, all modes except the globally constant function are stabilized. For discontinuous pressures the stable subspace is that of pressures orthogonal to the space of piecewise constants denoted by

$$(3.11) \qquad P_h^0 = \{\, q \in L^2(\Omega) \,|\, q|_K \in P_0(K) \ \forall K \in \mathcal{C}_h \,\}.$$

The stabilization term has no influence on P_h^0. Hence the stability has to be based on the original bilinear form $\mathcal{B}$, i.e. we have to assume that the following discrete stability inequality is valid:

$$(3.12) \qquad \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \gtrsim \|q\|_0 \quad \forall q \in P_h^0.$$

The final stability estimate with the L_2-norm for the pressure is then proved using (3.9) and (3.12) and a "trick", first introduced by Pitkäranta [13], and later applied for the Stokes problem by Verfürth [15]. Our stability theorem is formulated as follows.

THEOREM 3.3. *Suppose that one of the following conditions is valid:*

(i) $P_h \subset \mathcal{C}^0(\Omega)$,

(ii)   *the stability inequality* (3.12) *is valid.*

For $0 < \alpha < C_I$ it then holds

$$(3.13) \qquad \sup_{(\boldsymbol{v},q) \in \boldsymbol{V}_h \times P_h} \frac{\mathcal{B}_h(\boldsymbol{w}, r; \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1 + \|q\|_0} \gtrsim (\|\boldsymbol{w}\|_1 + \|r\|_0) \quad \forall (\boldsymbol{w}, r) \in \boldsymbol{V}_h \times P_h.$$

We emphasize the generality of the formulation. For continuous pressures all elements, triangles, quadrilaterals, tetrahedrons, prisms, hexahedrons and pyramids can be used, and also mixing them is allowed provided the mesh is conforming. For discontinuous elements the only condition is that the stability estimate (3.12) is valid. In two dimensions this is true if the local element are $[P_2(K)]^2$ and $[Q_2(K)]^2$ for triangles and quadrilaterals, respectively. In three dimensions the choices $[P_3(K)]^3$ and $[Q_2(K)]^3$ are sufficient for tetrahedrons and hexahedrons.

The following error estimate presented in the papers [10, 9, 4, 6, 5] is a direct consequence of the stability and consistency

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\|_0 \lesssim \inf_{\boldsymbol{v} \in \boldsymbol{V}_h} \Big(\|\boldsymbol{u} - \boldsymbol{v}\|_1 + \big(\sum_{K \in \mathcal{C}_h} h_K^2 |\boldsymbol{u} - \boldsymbol{v}|_{2,K}^2 \big)^{1/2} \Big)$$
$$+ \inf_{q \in P_h} \Big(\|p - q\|_0 + \big(\sum_{K \in \mathcal{C}_h} h_K^2 |p - q|_{1,K}^2 \big)^{1/2} \Big).$$

The drawback of this estimate is that it requires that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ and $p \in H^1(\Omega)$. For less regular solutions the convergence was left open in the papers cited above. In the following we will amend this situation by using arguments introduced by Gudi [7].

4. A refined a priori error analysis. First, we recall results from a posteriori error analysis [16, 17]. For an edge or face E in the mesh, we denote by ω_E the union of all elements in $\mathcal{C}_h$ having E as an edge or a face. We define $\mathrm{osc}_K(\boldsymbol{f})$ by

$$(4.1) \qquad \mathrm{osc}_K(\boldsymbol{f}) = h_K \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,K},$$

where $\boldsymbol{f}_h \in \boldsymbol{V}_h$ is the L_2-projection of $\boldsymbol{f}$. Similarly, we define

$$(4.2) \qquad \mathrm{osc}_E(\boldsymbol{t}) = h_E^{1/2} \|\boldsymbol{t} - \boldsymbol{t}_h\|_{0,E},$$

with $\boldsymbol{t}_h \in \boldsymbol{V}_h|_{\Gamma_N}$ being the L_2-projection. The global oscillation terms are defined through

$$(4.3) \qquad \mathrm{osc}(\boldsymbol{f})^2 = \sum_{K \in \mathcal{C}_h} \mathrm{osc}_K(\boldsymbol{f})^2 \quad \text{and} \quad \mathrm{osc}(\boldsymbol{t})^2 = \sum_{E \subset \Gamma_N} \mathrm{osc}_E(\boldsymbol{t})^2.$$

For an edge or face $E = \partial K \cap \partial K'$ the jump in the normal traction is

$$(4.4) \qquad [\![(\boldsymbol{D}(\boldsymbol{v}) - q\boldsymbol{I})\boldsymbol{n}]\!]|_E = (\boldsymbol{D}(\boldsymbol{v}) - q\boldsymbol{I})\big|_K \boldsymbol{n}_K - (\boldsymbol{D}(\boldsymbol{v}) - q\boldsymbol{I})\big|_{K'} \boldsymbol{n}_{K'}.$$

The following lower bounds are proved in [16, 17].

LEMMA 4.1. *For all $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times P_h$ it holds:*

$$(4.5) \quad h_K \|\boldsymbol{A}\boldsymbol{v} - \nabla q + \boldsymbol{f}\|_{0,K} \lesssim \|\boldsymbol{D}(\boldsymbol{u} - \boldsymbol{v})\|_{0,K} + \|p - q\|_{0,K} + \mathrm{osc}_K(\boldsymbol{f}) \quad \forall K \in \mathcal{C}_h.$$

For E in the interior of Ω

$$(4.6) \quad h_E^{1/2} \big\|[\![(\boldsymbol{D}(\boldsymbol{v}) - q\boldsymbol{I})\boldsymbol{n}]\!]\big\|_{0,E} \lesssim \|\boldsymbol{D}(\boldsymbol{u} - \boldsymbol{v})\|_{0,\omega_E} + \|p - q\|_{0,\omega_E} + \sum_{K \subset \omega_E} \mathrm{osc}_K(\boldsymbol{f})$$

and for $E \subset \Gamma_N$

$$(4.7) \qquad h_E^{1/2} \big\|(\boldsymbol{D}(\boldsymbol{v}) - q\boldsymbol{I})\boldsymbol{n} - \boldsymbol{t}\big\|_{0,E} \lesssim \|\boldsymbol{D}(\boldsymbol{u} - \boldsymbol{v})\|_{0,\omega_E} + \|p - q\|_{0,\omega_E}$$

$$(4.8) \qquad \qquad \qquad + \sum_{K \subset \omega_E} \mathrm{osc}_K(\boldsymbol{f}) + \mathrm{osc}_E(\boldsymbol{t}).$$

Now we state the new error estimate. Note that $\mathrm{osc}(\boldsymbol{f})$ is a higher order term.

THEOREM 4.2. *It holds*

$$(4.9) \qquad \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\|_0 \lesssim \inf_{\boldsymbol{v} \in \boldsymbol{V}_h} \|\boldsymbol{u} - \boldsymbol{v}\|_1 + \inf_{q \in P_h} \|p - q\|_0 + \mathrm{osc}(\boldsymbol{f}).$$

Proof. Let $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times P_h$ be arbitrary. By the stability estimate (3.13) there exists $(\boldsymbol{w}, r) \in \boldsymbol{V}_h \times P_h$ with

$$(4.10) \qquad \|\boldsymbol{w}\|_1 + \|r\|_0 = 1$$

and

$$(4.11) \qquad \|\boldsymbol{u}_h - \boldsymbol{v}\|_1 + \|p_h - q\|_0 \lesssim \mathcal{B}_h(\boldsymbol{u}_h - \boldsymbol{v}, p_h - q; \boldsymbol{w}, r).$$

Using (3.7), (3.5), (2.9) and (3.4) yield

$$
\begin{aligned}
\mathcal{B}_h(\boldsymbol{u}_h - \boldsymbol{v}, p_h - q; \boldsymbol{w}, r) &= \mathcal{B}_h(\boldsymbol{u}_h, p_h, \boldsymbol{w}, r) - \mathcal{B}_h(\boldsymbol{v}, q; \boldsymbol{w}, r) \\
&= \mathcal{F}_h(\boldsymbol{w}, r) - \mathcal{B}_h(\boldsymbol{v}, q; \boldsymbol{w}, r) \\
&= \mathcal{F}(\boldsymbol{w}, r) - \alpha \mathcal{L}_h(\boldsymbol{w}, r) - \mathcal{B}_h(\boldsymbol{v}, q; \boldsymbol{w}, r) \\
&= \mathcal{B}(\boldsymbol{u}, p; \boldsymbol{w}, r) - \alpha \mathcal{L}_h(\boldsymbol{w}, r) - \mathcal{B}_h(\boldsymbol{v}, q; \boldsymbol{w}, r) \\
&= \mathcal{B}(\boldsymbol{u}, p; \boldsymbol{w}, r) - \alpha \mathcal{L}_h(\boldsymbol{w}, r) - \mathcal{B}(\boldsymbol{v}, q; \boldsymbol{w}, r) + \alpha \mathcal{S}_h(\boldsymbol{v}, q; \boldsymbol{w}, r) \\
&= \mathcal{B}(\boldsymbol{u} - \boldsymbol{v}, p - q; \boldsymbol{w}, r) + \alpha \big(\mathcal{S}_h(\boldsymbol{v}, q; \boldsymbol{w}, r) - \mathcal{L}_h(\boldsymbol{w}, r) \big).
\end{aligned}
$$
(4.12)

From the boundedness of the bilinear form $\mathcal{B}$ and the normalization (4.10), we have

$$
\mathcal{B}(\boldsymbol{u} - \boldsymbol{v}, p - q; \boldsymbol{w}, r) \lesssim \big(\|\boldsymbol{u} - \boldsymbol{v}\|_1 + \|p - q\|_0 \big). \tag{4.13}
$$

From the definitions (3.1) and (3.2) we have

$$
\mathcal{S}_h(\boldsymbol{v}, q; \boldsymbol{w}, r) - \mathcal{L}_h(\boldsymbol{w}, r) = \sum_{K \in \mathcal{C}_h} h_K^2 (-\boldsymbol{A}\boldsymbol{v} + \nabla q - \boldsymbol{f}, -\boldsymbol{A}\boldsymbol{w} + \nabla r)_K. \tag{4.14}
$$

Cauchy–Schwarz inequality then yields

$$
\begin{aligned}
&|\mathcal{S}_h(\boldsymbol{v}, q; \boldsymbol{w}, r) - \mathcal{L}_h(\boldsymbol{w}, r)| \\
&\leq \Big(\sum_{K \in \mathcal{C}_h} h_K^2 \| - \boldsymbol{A}\boldsymbol{v} + \nabla q - \boldsymbol{f}\|_{0,K}^2 \Big)^{1/2} \Big(\sum_{K \in \mathcal{C}_h} h_K^2 \| - \boldsymbol{A}\boldsymbol{w} + \nabla r\|_{0,K}^2 \Big)^{1/2}.
\end{aligned}
$$

By local inverse inequalities we have

$$
\begin{aligned}
\Big(\sum_{K \in \mathcal{C}_h} h_K^2 \| - \boldsymbol{A}\boldsymbol{w} + \nabla r\|_{0,K}^2 \Big)^{1/2} &\leq \Big(2 \sum_{K \in \mathcal{C}_h} h_K^2 \big(\|\boldsymbol{A}\boldsymbol{w}\|_{0,K}^2 + \|\nabla r\|_{0,K}^2 \big) \Big)^{1/2} \\
&\lesssim (\|\boldsymbol{w}\|_1 + \|r\|_0).
\end{aligned}
$$

Hence, (4.5) gives

$$
\begin{aligned}
|\mathcal{S}_h(\boldsymbol{v}, q; \boldsymbol{w}, r) - \mathcal{L}_h(\boldsymbol{w}, r)| &\lesssim \Big(\sum_{K \in \mathcal{C}_h} h_K^2 \| - \boldsymbol{A}\boldsymbol{v} + \nabla q - \boldsymbol{f}\|_{0,K}^2 \Big)^{1/2} \\
&\lesssim \big(\|\boldsymbol{u} - \boldsymbol{v}\|_1 + \|p - q\|_0 + \mathrm{osc}(\boldsymbol{f}) \big).
\end{aligned}
$$

The assertion now follows from (4.11), (4.12), (4.13) and (4.15). $\square$

REMARK 4.3. *The above estimates are also valid for the Douglas–Wang formulation* [3], *provided that the stabilizing term of inter-element pressure jumps is dropped.*

5. A posteriori estimates. For the a posteriori estimates we define the local estimators

$$
\eta_K^2 = h_K^2 \|\boldsymbol{A}\boldsymbol{u}_h - \nabla p_h + \boldsymbol{f}\|_{0,K}^2 + \|\mathrm{div}\,\boldsymbol{u}_h - g\|_{0,K}^2 \tag{5.1}
$$

and

$$
\eta_E^2 = \begin{cases} h_E \big\| [\![(\boldsymbol{D}(\boldsymbol{u}_h) - p_h \boldsymbol{I})\boldsymbol{n}]\!] \big\|_{0,E}^2, & \text{when } E \subset \Omega, \\ h_E \big\| (\boldsymbol{D}(\boldsymbol{u}_h) - p_h \boldsymbol{I})\boldsymbol{n} - \boldsymbol{t} \big\|_{0,E}^2, & \text{when } E \subset \Gamma_N. \end{cases} \tag{5.2}
$$

By $\mathcal{E}_h$ we denote the collection of edges/faces in Ω and on Γ_N. The global error estimator is then defined as

$$(5.3) \qquad \eta^2 = \sum_{K \in \mathcal{C}_h} \eta_K^2 + \sum_{E \in \mathcal{E}_h} \eta_E^2.$$

Taking $(\boldsymbol{v}, q) = (\boldsymbol{u}_h, p_h)$ in Lemma 4.1 yields a local lower bound for the error. Now we will prove the following upper bound.

THEOREM 5.1. *It holds*

$$(5.4) \qquad \big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\|_0\big) \lesssim \eta.$$

Proof. By the stability of the continuous problem (2.12), there exists $(\boldsymbol{v}, q) \in \boldsymbol{H}_D^1(\Omega) \times L^2(\Omega)$ with

$$(5.5) \qquad \|\boldsymbol{v}\|_1 + \|q\|_0 = 1$$

and

$$(5.6) \qquad \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\|_0 \lesssim \mathcal{B}(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h; \boldsymbol{v}, q).$$

Let $\tilde{\boldsymbol{v}} \in \boldsymbol{V}_h$ be the Clément interpolant [2] of $\boldsymbol{v}$ for which we have the estimate

$$(5.7) \qquad \Big(\sum_{K \in \mathcal{C}_h} h_K^{-2} \|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_{0,K}^2 + \sum_{E \in \mathcal{E}_h} h_E^{-1} \|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_{0,E}^2 \Big)^{1/2} \lesssim \|\boldsymbol{v}\|_1 \lesssim 1.$$

Choosing the pair $(\boldsymbol{v}, q) = (\tilde{\boldsymbol{v}}, 0)$ in the finite element formulation (3.7) and the consistency equation (3.8), we get

$$(5.8) \qquad \mathcal{B}_h(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h, \tilde{\boldsymbol{v}}, 0) = 0.$$

Subtracting this from the right hand side in (5.6), and using the definition of $\mathcal{B}_h$, we obtain

$$\mathcal{B}(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h; \boldsymbol{v}, q) = \mathcal{B}(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h; \boldsymbol{v}, q) - \mathcal{B}_h(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h, \tilde{\boldsymbol{v}}, 0)$$
$$(5.9) \qquad = \mathcal{B}(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h; \boldsymbol{v} - \tilde{\boldsymbol{v}}, q) - \alpha \mathcal{S}_h(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h; \tilde{\boldsymbol{v}}, 0).$$

The first term above is estimated exactly as in the analysis of the standard mixed method [16, 17], using element by element integration by parts and the interpolation estimate (5.7). This results in

$$(5.10) \qquad \mathcal{B}(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h; \boldsymbol{v} - \tilde{\boldsymbol{v}}, q) \lesssim \eta.$$

Recalling definition (3.1), equation (2.3), and using an inverse inequality together with estimate (5.7), we get

$$\big|\mathcal{S}_h(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h; \tilde{\boldsymbol{v}}, 0)\big| = \Big| \sum_{K \in \mathcal{C}_h} h_K^2 (\boldsymbol{f} + \boldsymbol{A}\boldsymbol{u}_h - \nabla p_h, -\boldsymbol{A}\tilde{\boldsymbol{v}})_K \Big|$$
$$\leq \Big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{f} + \boldsymbol{A}\boldsymbol{u}_h - \nabla p_h\|_{0,K}^2 \Big)^{1/2} \Big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{A}\tilde{\boldsymbol{v}}\|_{0,K}^2 \Big)^{1/2}$$
$$\lesssim \eta \|\tilde{\boldsymbol{v}}\|_1 \lesssim \eta.$$

The assertion now follows by combining the above estimates. □

REMARK 5.2. *Let us finally note that previous works on the a posteriori estimates for stabilized methods have mostly been confined to low order methods or to methods with stabilizing pressure jump terms, cf.* [8, 11, 12, 14, 19, 18]. *As mentioned in the introduction, our estimates are the same as in* [18], *but our analysis is more straightforward.*

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