Optimal Portfolio under Fast Mean-reverting Fractional Stochastic Environment

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Abstract

Empirical studies indicate the existence of long range dependence in the volatility of the underlying asset. This feature can be captured by modeling its return and volatility using functions of a stationary fractional Ornstein–Uhlenbeck (fOU) process with Hurst index $H \in (\frac{1}{2}, 1)$. In this paper, we analyze the nonlinear optimal portfolio allocation problem under this model and in the regime where the fOU process is fast mean-reverting. We first consider the case of power utility, and rigorously give first order approximations of the value and the optimal strategy by a martingale distortion transformation. We also establish the asymptotic optimality in all admissible controls of a zeroth order trading strategy. Then, we consider the case with general utility functions using the epsilon-martingale decomposition technique, and we obtain similar asymptotic optimality results within a specific family of admissible strategies.

Keywords: Optimal portfolio, fractional Ornstein–Uhlenbeck process, long range dependence, martingale distortion, asymptotic optimality.

1 Introduction

Asset allocation problems in continuous time framework are among the most widely studied problems in the field of mathematical finance, and has a long history dating back to Merton [1969, 1971]. In his original work, explicit solutions are provided on how to trade stocks and/or to consume so that one's expected utility is maximized, when the underlying assets follow the Black–Scholes–Merton model, and where the utility functions are of specific type. Since these pioneering works, a large volume of research has been done for allowing financial market imperfections, for instance, see Magill and Constantinides [1976], Guasoni and Muhle-Karbe [2013] for transaction costs, Grossman and Zhou [1993], Cvitanić and Karatzas [1995], Elie and Touzi [2008] for investment under drawdown constraint, and Cuoco and Cvitanić [1998] for trading with price impact.

Particularly, in the direction of asset modeling, the U-shaped pattern of Black–Scholes implied volatility from market option prices is widely observed when plotted against different strike prices, leading to the study of Merton problem when the volatility is stochastic, see Zariphopoulou [1999], Chacko and Viceira [2005], Fouque et al. [2015] and Lorig and Sircar [2016], to name a few. Moreover, empirical studies show that non-Markovian (dependence) structure models seem to better describe the data. Especially, in long-term investment which is related to daily data, long range dependence exhibits in both return and volatility: Breidt et al. [1998], Chronopoulou and Viens [2012a,b], Cont [2001, 2005], Engle and Patton [2001].

Our aim is to study the optimal portfolio problem when both return and volatility are driven by a long-range dependence process, denoted by $Y_t^{\epsilon,H}$, which is *fast-varying*. Specifically, we model $Y_t^{\epsilon,H}$ by a stationary fractional Ornstein–Uhlenbeck process (fOU), which follows

$$\mathrm{d}Y_t^{\epsilon,H} = -\frac{a}{\epsilon} Y_t^{\epsilon,H} \,\mathrm{d}t + \frac{1}{\epsilon^H} \,\mathrm{d}W_t^{(H)}.$$

Here, ϵ is a small parameter to make the process $Y_t^{\epsilon,H}$ fast-varying and its natural time scale to be of order ϵ (that is, its mean-reversion time scale proportional to ϵ), and $W_t^{(H)}$ is a fractional Brownian motion

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(fBm) with Hurst index $H \in (\frac{1}{2}, 1)$ to give a fOU process that is of long-range dependence. A brief review regarding fBm and fOU is given in Section 2.2, and the ϵ-scaled fOU process $Y_t^{\epsilon,H}$ is discussed in more details in Section 3.1. For further references, we refer to Mandelbrot and Van Ness [1968], Cheridito et al. [2003], Coutin [2007], Biagini et al. [2008], Kaarakka and Salminen [2011].

The reason to consider such an asset modeling is threefold.

Firstly, the stationary fOU process is Gaussian which makes its spectral decomposition (see Fouque et al. [2011]) available in explicit form when analyzing the properties of the Sharpe-ratio $\lambda(Y_t^{\epsilon,H})$ introduced in Section 2. The fOU process can be expressed as an integral of a well-studied kernel function with respect to a fBm process, which simplifies the derivation of needed estimates. Moreover, in addition to long-range correlation, it also satisfies other empirical "stylized facts", such as heavy tails and volatility clustering of returns, and persistence and mean-reversion of volatility as mentioned in Cont [2001, 2005], Engle and Patton [2001].

Secondly, when the process $Y_t^{\epsilon,H}$ is slowly varying (that is ϵ large), which is particularly important in long-term investments, the asset allocation problem has been studied in Fouque and Hu [2017b] by a *martingale distortion transformation* and regular perturbation techniques. So, it is natural to study the fast-varying regime as well.

Thirdly, although it is natural to consider multiscale factor models for risky assets, with a slow factor and a fast factor as in Fouque et al. [2015] in a Markovian framework, the analysis requires more technical details, as the martingale distortion transformation is not available. This will be presented in another paper in preparation (Hu [2017]).

In this paper, we focus on one-factor models and we study the effect of a fast time-scale on the optimal allocation problem. The analysis of long-memory models is quite challenging. This is mainly due to the fact that the process $Y_t^{\epsilon,H}$ is neither a semimartingale nor a Markov process. Consequently, the Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) is not available, to which the singular perturbation technique is usually applied. Nevertheless, when the utility is of power type, a martingale distortion transformation is available and gives a representation of the value process as well as the optimal strategy. This result was originally discovered by Zariphopoulou [1999] in the Markovian case and proved by applying a linearizing transformation to the HJB PDE. The general (non-Markovian) case was proved by Tehranchi [2004] via a conditional Hölder inequality, and by Frei and Schweizer [2008] via a BSDE approach in the case of exponential utility. Recently, it has been revisited in Fouque and Hu [2017b] under the setup (2.1) with a short proof based on a verification argument. For general utilities, the problem can be investigated using the "epsilon-martingale decomposition" method. This approach was introduced in Fouque et al. [2000] and Fouque et al. [2001], and recently developed in Garnier and Sølna [2017] for linear pricing problems where corrections to the Black–Scholes formula and implied volatility are derived when the fractional stochastic volatility is slowly varying (or has small fluctuations), and in the fast mean-reverting regime in Garnier and Sølna [2016].

Main results. In this paper, we study the nonlinear portfolio optimization problem under the fast-varying fractional stochastic environment described above. In the power utility case:

- The value function and the optimal portfolio are obtained via the martingale distortion transformation stated in Fouque and Hu [2017b]. Using the "ergodic property" of $Y_t^{\epsilon,H}$, we expand these expression in a probabilistic way, and deduce the first order approximations of both quantities. These approximations consist of a leading order term, which is related to the solution to the Merton problem with constant coefficients, and correction terms of order ϵ^{1-H}.

- The asymptotics shares remarkable similarities with the slowly-varying case (see Fouque and Hu [2017b]): we find that without using the correction term $\pi^{(1)}$ of the optimal strategy, the leading order term $\pi^{(0)}$ by itself, generates the value process up to corrections of order ϵ^{1-H}; and both $\pi^{(0)}$ and $\pi^{(1)}$ are explicit in terms of the state processes (the wealth process X_t defined below, $Y_t^{\epsilon,H}$ and the time variable t).

- The later similarity, that is, $\pi^{(0)}$ and $\pi^{(1)}$ are explicit in terms of $(X_t, Y_t^{\epsilon,H}, t)$, however, leads to a non-trivial implementation of $\pi^{(0)}$, as the fast-varying $Y_t^{\epsilon,H}$ needs to be tracked. We address this issue by suggesting a sub-optimal practical (or lazy) strategy that sacrifices some accuracy in the value process.

For general utility functions, using the epsilon-martingale decomposition method and the properties of the risk tolerance function for the Merton problem with constant coefficients, we obtain an approximation for the portfolio value corresponding to a given strategy. As in Fouque and Hu [2017a] in the Markovian case, we show that this strategy is asymptotically optimal in a specific class of admissible strategies.

The context of this paper is long-range correlation characterized by Hurst index $H \in (\frac{1}{2}, 1)$. As for the linear pricing problem in Garnier and Sølna [2016], our results hold only in the range $H \in (\frac{1}{2}, 1)$. The singular perturbation as $\epsilon \to 0$ does not commute with the limit $H \downarrow \frac{1}{2}$ (see Section 3.7). Therefore, the results in Fouque et al. [2015] in the Markovian case can not be recovered by taking $H \downarrow \frac{1}{2}$. The case $H \in (0, \frac{1}{2})$ corresponding to rough fractional stochastic volatility and short-term dependence is not addressed in this paper. Surprisingly, when $H < 1/2$, the first order corrections to the value process appear to be of order $\sqrt{\epsilon}$, and $Y_t^{\epsilon,H}$ is not visible to the leading order nor in the corrections. These findings will be presented in another paper in preparation (Fouque and Hu [2018]). In fact, the proofs of crucial lemmas in Appendix A break down when $H < \frac{1}{2}$, which is translated into divergent integrals in that case.

Organization of the paper. The rest of the paper is organized as follows. In Section 2, we restate the martingale distortion transformation under general stochastic volatility models. This is derived in the Markovian case in Zariphopoulou [1999], and in non-Markovian settings in Tehranchi [2004], Frei and Schweizer [2008], Fouque and Hu [2017b]. We also review the fractional Brownian motion and fractional Ornstein–Uhlenbeck processes. In Section 3, we introduce the fast-varying long-range dependence stochastic factor $Y_t^{\epsilon,H}$ modeled by the ϵ-scaled fOU process. Then, the asymptotic results under this modeling are derived and given in Section 3.2 and 3.3 for the value process and optimal portfolio respectively. Asymptotic optimality of the leading order strategy $\pi^{(0)}$ in the full class of admissible strategies up to ϵ^{1-H} is discussed, and the implementation difficulties are also addressed with numerical illustrations. We also compare the results with the Markovian case and comment on the influence of long-range dependence models. The problem with general utility functions is discussed and similar asymptotic optimality results are presented in Section 4. We make conclusive remarks in Section 5.

2 Merton problem under one factor stochastic environment and power utility

Let S_t be the price of the underlying asset at time t, whose return and volatility are driven by a stochastic factor Y_t,

$$\mathrm{d}S_t = S_t \left[\mu(Y_t)\,\mathrm{d}t + \sigma(Y_t)\,\mathrm{d}W_t\right]. \tag{2.1}$$

Here Y_t is a general stochastic process adapted to the natural filtration $\mathcal{G}_t$ generated by a Brownian motion W_t^Y correlated with the Brownian motion W_t which drives the asset price S_t:

$$\mathrm{d}\left\langle W_t, W_t^Y \right\rangle = \rho\,\mathrm{d}t, \quad |\rho| < 1. \tag{2.2}$$

We also define $\mathcal{F}_t$ as the natural filtration generated by the two Brownian motions (W_t, W_t^Y).

Let π_t be the amount of money invested in the underlying asset S_t at time t, while the rest earns a constant interest rate r. We require π_t to be $\mathcal{F}_t$-adapted and self-financing. Denote by X_t^π the wealth process associated to the strategy π, and, without loss of generality, assume that the interest rate r is zero, then the dynamics of X_t^π is given by:

$$\mathrm{d}X_t^\pi = \pi_t\mu(Y_t)\,\mathrm{d}t + \pi_t\sigma(Y_t)\,\mathrm{d}W_t. \tag{2.3}$$

The investor aims at finding the optimal strategy in order to maximize her expected utility of terminal wealth X_T^π. Mathematically, it consists in identifying the value process V_t defined by

$$V_t := \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_t} \mathbb{E}\left[U(X_T^\pi)|\mathcal{F}_t\right], \tag{2.4}$$

and the corresponding optimal strategy π^*, given the investor's utility function $U(\cdot)$. The form of $U(\cdot)$ varies from section to section. Specifically, in the rest of this section and Section 3, we will work with power utilities under Assumption 2.1, namely

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad \gamma \neq 1, \tag{2.5}$$

while in Section 4, the utility function is in general form satisfying Assumption 4.1. The set $\mathcal{A}_t$ contains all admissible strategies:

$$\mathcal{A}_t := \{\pi \text{ is } (\mathcal{F}_t)\text{-adapted} : X_s^\pi \text{ in (2.3) stays nonnegative } \forall s \geq t, \text{ given } \mathcal{F}_t\}, \tag{2.6}$$

with zero being an absorbing state for X^π (bankruptcy). Additionally, for the power utility case, we require that for all $\pi \in \mathcal{A}_t$, the following integrability conditions are satisfied:

$$\sup_{t\in[0,T]} \mathbb{E}\left[(X_t^\pi)^{2p(1-\gamma)}\right] < +\infty, \textit{ for some } p > 1, \quad \textit{and} \quad \mathbb{E}\left[\int_0^T (X_t^\pi)^{-2\gamma}\,\pi_t^2\sigma^2(Y_t)\,\mathrm{d}t\right] < \infty. \tag{2.7}$$

In Fouque and Hu [2017b], the value process (2.4) is studied when $U(x)$ is of power type and is represented via a *martingale distortion transformation*. For readers' convenience, we first briefly review this representation. Then, as a preparation for working under a specific fractional stochastic environment, we review the fractional Brownian motion (fBm) and fractional Ornstein-Uhlenbeck (fOU) processes.

2.1 Martingale distortion transformation

The martingale distortion transformation was derived in Tehranchi [2004] with a slightly different utility function, and recently stated in Fouque and Hu [2017b] under the same setup as in this paper.

Denote by $\widetilde{\mathbb{P}}$ the probability measure defined by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left\{-\int_0^T a_s\,\mathrm{d}W_s^Y - \frac{1}{2}\int_0^T a_s^2\,\mathrm{d}s\right\}, \tag{2.8}$$

where

$$a_t = -\rho\left(\frac{1-\gamma}{\gamma}\right)\lambda(Y_t), \tag{2.9}$$

is bounded and $\mathcal{G}_t$-adapted. Therefore, $\widetilde{W}_t^Y := W_t^Y + \int_0^t a_s\,\mathrm{d}s$ is a $\widetilde{\mathbb{P}}$-Brownian motion.

Assumption 2.1.

(i) The SDE (2.1) *for S_t has a unique strong solution, in other words,*

$$S_t = S_0 e^{\int_0^t(\mu(Y_s)-\frac{1}{2}\sigma^2(Y_s))\,\mathrm{d}s + \int_0^t \sigma(Y_s)\,\mathrm{d}W_s}$$

exists for all $t \in [0, T]$.

(ii) Assume the filtration generated by $(Y_s)_{s\leq t}$ is also $\mathcal{G}_t$, and the volatility function $\sigma(\cdot)$ is injective.

(iii) The Sharpe ratio $\lambda(\cdot) := \mu(\cdot)/\sigma(\cdot)$ is assumed to be bounded and $C^2(\mathbb{R})$. Also, the derivatives λ' and λ'' are assumed bounded.

(iv) Define the $\widetilde{\mathbb{P}}$-martingale

$$M_t = \widetilde{\mathbb{E}}\left[e^{\frac{1-\gamma}{2q\gamma}\int_0^T \lambda^2(Y_s)\,\mathrm{d}s}\,\middle|\,\mathcal{G}_t\right], \tag{2.10}$$

and write its representation

$$\mathrm{d}M_t = M_t\xi_t\,\mathrm{d}\widetilde{W}_t^Y. \tag{2.11}$$

We assume

$$\mathbb{E}\left[e^{c_\xi\int_0^T \xi_t^2\,\mathrm{d}t}\right] < \infty,$$

where the constant c_ξ is given by $c_\xi = \frac{16(1-\gamma)^2\rho^2p^2q^2}{\gamma^2}$ for $\gamma < 1$, and $c_\xi = \frac{16(1-\gamma)^2\rho^2p^2q^2}{\gamma^2} - \frac{4p(1-\gamma)}{\gamma^2}$ for $\gamma > 1$. The parameter p is introduced in (2.7) *and q is defined in terms of γ and ρ by*

$$q = \frac{\gamma}{\gamma + (1-\gamma)\rho^2}. \tag{2.12}$$

Note that q is the usual "distortion" exponent firstly introduced in Zariphopoulou [1999].

Remark 2.2. *In the above assumptions, by part* (ii), *$\mathcal{F}_t$ is also the filtration generated by (W, Y). Since $\sigma(\cdot)$ is one-to-one, it is also the one generated by S_t. This assumption is important, since in reality S_t is what we observed. In part* (iii), *the smoothness of λ is needed when Taylor expansions are performed to prove Theorem 3.4 and 3.5; while the boundedness of λ' and λ'' are convenient assumptions for the estimation of the error terms. Part* (iv) *on $(\xi)_{t\in[0,T]}$ seems to be a strong assumption. However, in Section 3, we will see that it is satisfied for our proposed model for $Y_t^{\epsilon,H}$.*

The assumption that $r = 0$ can be viewed as a change of numéraire. In fact, the following Proposition 2.3 could be extended to the case $r = r(Y_t)$ with only minor modifications.

Proposition 2.3 (Martingale Distortion Transformation)**.** *Let S_t follow the dynamics* (2.1), *and suppose the objective is* (2.4) *with the power utility function* (2.5). *Under Assumptions 2.1, the value process V_t is given by*

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma}\left[\widetilde{\mathbb{E}}\left(e^{\frac{1-\gamma}{2q\gamma}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s}\,\middle|\,\mathcal{G}_t\right)\right]^q. \tag{2.13}$$

The expectation $\widetilde{\mathbb{E}}[\cdot]$ is computed with respect to $\widetilde{\mathbb{P}}$ introduced in (2.8). *The parameter q is given by* (2.12).

The optimal strategy π^* is

$$\pi_t^* = \left[\frac{\lambda(Y_t)}{\gamma\sigma(Y_t)} + \frac{\rho q \xi_t}{\gamma\sigma(Y_t)}\right] X_t, \tag{2.14}$$

where ξ_t is given by the Martingale Representation Theorem in (2.11).

Proof. See [Fouque and Hu, 2017b, Proposition 2.2] for a detailed proof. ∎

Remark 2.4. *The separation of variable form* (2.13), *that is, the utility of the current wealth $U(X_t)$ multiplied by a process related to the stochastic factor Y_t, is motivated by the Markovian case firstly developed in Zariphopoulou [1999].*

When Y_t is Markovian, results in Zariphopoulou [1999] is recovered by rewriting the value process:

$$V_t = U(X_t)\left[\widetilde{\mathbb{E}}\left(e^{\frac{1-\gamma}{2q\gamma}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s}\,\middle|\,Y_t\right)\right]^q = U(X_t)v(Y_t)^q,$$

and applying the Feyman-Kac formula to $v(\cdot)$.

When the Sharpe-ratio is degenerate $\lambda(y) = \lambda_0$, the value process and the optimal strategy are reduced to

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma}e^{\frac{1-\gamma}{2\gamma}\lambda_0^2(T-t)}, \quad \pi_t^* = \frac{\lambda_0}{\gamma\sigma(Y_t)}X_t.$$

When the two Brownian motions W_t and W_t^Y are uncorrelated $\rho = 0$, the problem is already "linear" since $q = 1$. In that case, the value process and the optimal strategy are simplified as:

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma}\mathbb{E}\left[e^{\frac{1-\gamma}{2q\gamma}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s}\,\middle|\,\mathcal{G}_t\right], \quad \pi_t^* = \frac{\lambda(Y_t)}{\gamma\sigma(Y_t)}X_t.$$

The results in Proposition 2.3 can also be generalized to the case of log utility and/or with multiple assets, see Fouque and Hu [2017b] for further discussion.

2.2 Fractional Brownian motion and fractional Ornstein-Uhlenbeck processes

A standard fractional Brownian motion (fBm) is a continuous Gaussian process $\left(W_t^{(H)}\right)_{t\in\mathbb{R}}$ with zero mean and covariance structure:

$$\mathbb{E}\left[W_t^{(H)}W_s^{(H)}\right] = \frac{\sigma_H^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),$$

where σ_H is a positive constant

$$\sigma_H^2 = \frac{1}{\Gamma(2H+1)\sin(\pi H)}, \tag{2.15}$$

and $H \in (0,1)$ is the Hurst index. According to Mandelbrot and Van Ness [1968], $W_t^{(H)}$ can be represented by the following moving-average integral:

$$W_t^{(H)} = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}}\left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right)\mathrm{d}W_s^Y, \tag{2.16}$$

where $(W_t^Y)_{t\in\mathbb{R}^+}$ is the standard Brownian motion that is correlated with W_t as given in (2.2), and $(W_t^Y)_{t\in\mathbb{R}^-} := (B_{-t})_{t\in\mathbb{R}^-}$ is another Brownian motion independent of $(W_t^Y)_{t\in\mathbb{R}^+}$ and (W_t).

We then introduce the stationary fractional Ornstein-Uhlenbeck (fOU) process as

$$Y_t^H := \int_{-\infty}^t e^{-a(t-s)}\,\mathrm{d}W_s^{(H)} \tag{2.17}$$

which is the unique (in distribution) stationary solution to the Langevin equation driven by fBm (see Cheridito et al. [2003])

$$\mathrm{d}Y_t^H = -aY_t^H\,\mathrm{d}t + \mathrm{d}W_t^{(H)}, \tag{2.18}$$

where $a > 0$ is a strictly positive parameter. It has zero mean and (co)variance structure:

$$\sigma_{ou}^2 := \mathbb{E}\left[\left(Y_t^H\right)^2\right] = \frac{1}{2}a^{-2H}\Gamma(2H+1)\sigma_H^2, \tag{2.19}$$

$$\mathbb{E}\left[Y_t^H Y_{t+s}^H\right] = \sigma_{ou}^2\frac{2\sin(\pi H)}{\pi}\int_0^\infty \cos(asx)\frac{x^{1-2H}}{1+x^2}\,\mathrm{d}x := \sigma_{ou}^2\mathcal{C}_Y(s). \tag{2.20}$$

By the moving-average representation (2.16) for $W_t^{(H)}$, the stationary solution (2.17) is expressed as:

$$Y_t^H = \int_{-\infty}^t \mathcal{K}(t-s)\,\mathrm{d}W_s^Y, \tag{2.21}$$

where $\left(W_t^Y\right)_{t\in\mathbb{R}}$ is the standard Brownian motion on $\mathbb{R}$ as described after equation (2.16). The non-negative kernel $\mathcal{K}$ takes the form

$$\mathcal{K}(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\left[t^{H-\frac{1}{2}} - a\int_0^t (t-s)^{H-\frac{1}{2}}e^{-as}\,\mathrm{d}s\right], \tag{2.22}$$

and $\int_0^\infty \mathcal{K}^2(u)\,\mathrm{d}u = \sigma_{ou}^2$. For asymptotic properties of $\mathcal{K}(t)$ when $t \ll 1$ and $t \gg 1$, we refer to [Garnier and Sølna, 2017, Section 2.2]. They also provide short-range correlation properties when $H \in (0,\frac{1}{2})$, and long-range correlation properties when $H \in (\frac{1}{2},1)$. In Section 3, we will mainly focus on the case of $H > \frac{1}{2}$, as explained in the introduction. Specifically, we will study the Merton problem (2.4) when Y_t follows a rescaled version of (2.17), such that it is fast-varying.

3 Application to fast-varying fractional stochastic environment

In this section, we first introduce the ϵ-scaled stationary fOU process denote by $Y_t^{\epsilon,H}$, of which we mention several properties with proofs delayed to the Appendix. Then, we study the Merton problem (2.4) under such fractional stochastic factor $Y_t^{\epsilon,H}$. To be specific, we will give approximations of both the value process, denoted by V_t^ϵ and the corresponding optimal strategy π^*. This is done by applying Proposition 2.3 with $Y_t = Y_t^{\epsilon,H}$, then by expanding the expressions (3.11) based on the properties mentioned in Section 3.1. We also show that the "leading order" strategy alone can produce the given approximation of V_t^ϵ. However, the implementation needs to track the fast factor $Y_t^{\epsilon,H}$ using high-frequency data and this is not an easy task. To address this issue, we propose a practical strategy which does not require tracking $Y_t^{\epsilon,H}$, with numerical illustration. Finally, we compare the results with the Markovian case, and we comment on the effects of taking into account the long-range dependence.

3.1 The fast mean-reverting fOU process

The ϵ-scaled fractional Ornstein–Uhlenbeck process $Y_t^{\epsilon,H}$ is defined by

$$Y_t^{\epsilon,H} := \epsilon^{-H} \int_{-\infty}^t e^{-\frac{a(t-s)}{\epsilon}} \, \mathrm{d}W_s^{(H)} \tag{3.1}$$

where $\epsilon \ll 1$ is a small parameter and $H \in (\frac{1}{2}, 1)$. In the following moving-average integral representation

$$Y_t^{\epsilon,H} = \int_{-\infty}^t \mathcal{K}^\epsilon(t-s)\, \mathrm{d}W_s^Y, \quad \mathcal{K}^\epsilon(t) = \frac{1}{\sqrt{\epsilon}} \mathcal{K}\left(\frac{t}{\epsilon}\right), \tag{3.2}$$

W^Y is the Brownian motion that drives the process $Y_t^{\epsilon,H}$ as in (2.21), and is correlated with W_t as in (2.2).

It is a zero-mean, stationary Gaussian process with variance σ_{ou}^2 and covariance

$$\mathbb{E}\left[Y_t^{\epsilon,H} Y_{t+s}^{\epsilon,H}\right] = \sigma_{ou}^2 \mathcal{C}_Y\left(\frac{s}{\epsilon}\right) = \sigma_{ou}^2 \frac{2\sin(\pi H)}{\pi} \int_0^\infty \cos\left(\frac{asx}{\epsilon}\right) \frac{x^{1-2H}}{1+x^2}\, \mathrm{d}x, \tag{3.3}$$

which shows the natural scale of $Y_t^{\epsilon,H}$ is ϵ as desired. Moreover, the correlation function $C_Y(s)$ is not integrable at infinity and the long-range correlation exhibits the behavior:

$$\mathcal{C}_Y(s) = \frac{(as)^{2H-2}}{\Gamma(2H-1)} + o(s^{2H-2}), \quad s \gg 1.$$

The Sharpe-ratio process $\lambda(Y_t^{\epsilon,H})$ inherits this long-range correlation, namely,

$$Cov(\lambda(Y_t^{\epsilon,H}), \lambda(Y_{t+s}^{\epsilon,H})) = Var(\lambda^2(Y_t^{\epsilon,H})) C_\lambda(\frac{s}{\epsilon}), \text{ and } C_\lambda(s) \sim \mathcal{O}(s^{2H-2}), \text{ for } s \gg 1.$$

This follows from a straightforward modification of proofs in [Garnier and Sølna, 2016, Lemma 3.1].

Now, we check that Assumption 2.1(iv) is satisfied by $Y_t^{\epsilon,H}$.

Lemma 3.1. *Under Assumption 2.1*(i)-(iii)*, the fast mean-reverting stationary fractional Ornstein–Uhlenbeck process $Y_t^{\epsilon,H}$ defined in* (3.1) *satisfies Assumption 2.1*(iv).

Proof. This is a slightly different version of Lemma 3.1 in Fouque and Hu [2017b]. Using the property that $A^\epsilon(T) \equiv \int_0^t \mathcal{K}^\epsilon(s)\, \mathrm{d}s$ is of order ϵ^{1-H}, essentially the same proof applies. Thus, we omit the details here. □

We now introduce the bracket notation $\langle \cdot \rangle$ for averaging with respect to the invariant distribution of fOU process:

$$\langle g \rangle := \int_{\mathbb{R}} g(z) \frac{1}{\sqrt{2\pi}\sigma_{ou}} e^{-\frac{z^2}{2\sigma_{ou}^2}}\, \mathrm{d}z = \int_{\mathbb{R}} g(\sigma_{ou} z) p(z)\, \mathrm{d}z,$$

where $p(z)$ is the density of the standard normal distribution, as well as $\overline{\lambda}$ and $\widetilde{\lambda}$ which will be used throughout the rest of the paper:

$$\overline{\lambda} := \sqrt{\langle \lambda^2 \rangle}, \quad \widetilde{\lambda} := \langle \lambda \rangle. \tag{3.4}$$

Accordingly, we define several important quantities, which are differences between time averages and spacial averages:

$$I_t^\epsilon := \int_0^t \left(\lambda^2(Y_s^{\epsilon,H}) - \overline{\lambda}^2\right) \mathrm{d}s, \tag{3.5}$$

$$\eta_t^\epsilon := \int_0^t \left(\lambda(Y_s^{\epsilon,H}) - \widetilde{\lambda}\right) \mathrm{d}s, \tag{3.6}$$

$$\kappa_t^\epsilon := \int_0^t \left(\lambda(Y_s^{\epsilon,H})\lambda'(Y_s^{\epsilon,H}) - \langle\lambda\lambda'\rangle\right) \mathrm{d}s. \tag{3.7}$$

It is proved in Appendix A that, by the ergodicity of $Y_t^{\epsilon,H}$, these differences are small and of order ϵ^{1-H}. More properties and estimates regarding $Y_t^{\epsilon,H}$ are also stated therein.

Let C_k be the "probabilists" Hermite coefficients of the function $\lambda^2(\cdot)$:

$$C_k := \int_{\mathbb{R}} H_k(z)\lambda^2(\sigma_{ou}z)p(z)\,\mathrm{d}z, \quad H_k(z) = (-1)^k e^{z^2/2}\frac{\mathrm{d}^k\left(e^{-z^2/2}\right)}{\mathrm{d}z^k}.$$

The Hermite polynomials are naturally associated with OU processes. Now, we state a further assumption on $\lambda(\cdot)$ which is required in Lemma A.2.

Assumption 3.2. *There exists $\alpha > 4$ such that*

$$\sum_{k=0}^\infty \frac{\alpha^k C_k^2}{k!} < \infty,$$

where C_k's are the Hermite coefficients defined above.

Remark 3.3. *A sufficient condition to Assumption 3.2 given in Garnier and Sølna [2016] is stated as follows. If $\lambda^2(x)$ is of the form*

$$\lambda^2(x) = \int_{-\infty}^{x/\sigma_{ou}} f(y)\,\mathrm{d}y,$$

where the Fourier transform of the function f satisfies $\left|\hat{f}(\nu)\right| \le C\exp(-\nu^2)$ for some $C > 0$, then Assumption 3.2 is fulfilled. The proof relies on Parseval identity, and we refer to [Garnier and Sølna, 2016, Lemma A.2] for details.

In the rest of this section, we study the Merton problem (2.4), when the stochastic environment is modeled by $Y_t^{\epsilon,H}$ with H restricted to $H > \frac{1}{2}$, and when the investor's utility is of power type. Note that under such circumstance (in fact, as long as $H \neq \frac{1}{2}$), $Y_t^{\epsilon,H}$ is neither a semi-martingale nor a Markov process, thus the usual Hamilton-Jacobi-Bellman partial differential equation is not available. However, we have Proposition 2.3 which can be applied directly, and this will be the starting point of our derivation of the approximations.

3.2 First order approximation to the value process

Let S_t follow the dynamics

$$\mathrm{d}S_t = S_t\left[\mu(Y_t^{\epsilon,H})\,\mathrm{d}t + \sigma(Y_t^{\epsilon,H})\,\mathrm{d}W_t\right], \tag{3.8}$$

where $Y_t^{\epsilon,H}$ is the ϵ-scaled stationary fOU process (3.1) described above with $H > \frac{1}{2}$. Then, the wealth process X_t^π becomes

$$\mathrm{d}X_t^\pi = \pi_t\mu(Y_t^{\epsilon,H})\,\mathrm{d}t + \pi_t\sigma(Y_t^{\epsilon,H})\,\mathrm{d}W_t. \tag{3.9}$$

Denote by V_t^ϵ the value process at time t under the current setup:

$$V_t^\epsilon := \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_t^\epsilon} \mathbb{E}\left[U(X_T^\pi)|\mathcal{F}_t\right], \tag{3.10}$$

where the superscript ϵ emphasizes the dependence on ϵ brought by $Y_t^{\epsilon,H}$, and the notation of admissible set is also changed from $\mathcal{A}_t$ to $\mathcal{A}_t^\epsilon$ accordingly. Directly applying Proposition 2.3 with $Y_t = Y_t^{\epsilon,H}$ gives the following expression for V_t^ϵ:

$$V_t^\epsilon = \frac{X_t^{1-\gamma}}{1-\gamma}\left[\widetilde{\mathbb{E}}\left(e^{\frac{1-\gamma}{2q\gamma}\int_t^T \lambda^2(Y_s^{\epsilon,H})\,\mathrm{d}s}\Big|\mathcal{G}_t\right)\right]^q. \tag{3.11}$$

Theorem 3.4. *In the regime of ϵ small, under Assumptions 2.1 and 3.2, for fixed $t \in [0,T)$, V_t^ϵ takes the form*

$$V_t^\epsilon = Q_t^\epsilon(X_t) + o(\epsilon^{1-H}), \tag{3.12}$$

where

$$Q_t^\epsilon(x) = \frac{x^{1-\gamma}}{1-\gamma}e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)}\left[1 + \frac{1-\gamma}{\gamma}\left(\phi_t^\epsilon + \epsilon^{1-H}\rho\overline{\lambda}\left(\frac{1-\gamma}{\gamma}\right)\frac{\langle\lambda\lambda'\rangle\,(T-t)^{H+\frac{1}{2}}}{a\Gamma(H+\frac{3}{2})}\right)\right]. \tag{3.13}$$

Here ϕ_t^ϵ is the random process defined as

$$\phi_t^\epsilon = \mathbb{E}\left[\frac{1}{2}\int_t^T\left(\lambda^2(Y_s^{\epsilon,H}) - \overline{\lambda}^2\right)\mathrm{d}s\Big|\,\mathcal{G}_t\right], \tag{3.14}$$

which is of order ϵ^{1-H} as proved in Lemma A.1(ii)*. The notation $o(\epsilon^{1-H})$ denotes a $\mathcal{F}_t$-adapted random variable whose order is higher than ϵ^{1-H} in L^1.*

Proof. In order to obtain (3.12)-(3.13), we start by expanding

$$\Psi_t^\epsilon := \widetilde{\mathbb{E}}\left[e^{\frac{1-\gamma}{2q\gamma}\int_t^T\left(\lambda^2(Y_s^{\epsilon,H})-\overline{\lambda}^2\right)\mathrm{d}s}\Big|\mathcal{G}_t\right], \tag{3.15}$$

then, we apply Taylor formula to the function x^q.

Using the fact that I_t^ϵ is "small" and Taylor expansion of e^x in x, one deduces

$$\begin{aligned}\Psi_t^\epsilon &= \widetilde{\mathbb{E}}\left[1 + \frac{1-\gamma}{2q\gamma}\int_t^T\left(\lambda^2(Y_s^{\epsilon,H}) - \overline{\lambda}^2\right)\mathrm{d}s + R_{[t,T]}\Big|\mathcal{G}_t\right] \\ &= 1 + \frac{1-\gamma}{q\gamma}\widetilde{\mathbb{E}}\left[\frac{1}{2}\int_t^T\left(\lambda^2(Y_s^{\epsilon,H}) - \overline{\lambda}^2\right)\mathrm{d}s\Big|\mathcal{G}_t\right] + \widetilde{\mathbb{E}}\left[R_{[t,T]}|\mathcal{G}_t\right],\end{aligned} \tag{3.16}$$

where $R_{[t,T]} = e^\chi\left[\frac{1-\gamma}{2q\gamma}\int_t^T\left(\lambda^2(Y_s^{\epsilon,H}) - \overline{\lambda}^2\right)\mathrm{d}s\right]^2$ with χ being the bounded Lagrange remainder. Thus the term $\widetilde{\mathbb{E}}\left[R_{[t,T]}|\mathcal{G}_t\right]$ is of order ϵ^{2-2H} in L^1 by Lemma A.2(i).

Define the $\widetilde{\mathbb{P}}$-martingale $\widetilde{\psi}_t^\epsilon$ by

$$\widetilde{\psi}_t^\epsilon = \widetilde{\mathbb{E}}\left[\int_0^T G(Y_s^{\epsilon,H})\,\mathrm{d}s\Big|\mathcal{G}_t\right], \quad G(y) = \frac{1}{2}(\lambda^2(y) - \overline{\lambda}^2).$$

Taylor expanding $G(Y_s^{\epsilon,H})$ at $y = \widetilde{Y}_s^{\epsilon,H} := \int_{-\infty}^s \mathcal{K}^\epsilon(s-u)\,\mathrm{d}\widetilde{W}_u^Y$, together with $Y_s^{\epsilon,H} - \widetilde{Y}_s^{\epsilon,H} \sim O(\epsilon^{1-H})$ (see Lemma A.3(i)) yields

$$\begin{aligned}\widetilde{\psi}_t^\epsilon &= \widetilde{\mathbb{E}}\left[\int_0^T G(\widetilde{Y}_s^{\epsilon,H})\,\mathrm{d}s\Big|\mathcal{G}_t\right] + \widetilde{\mathbb{E}}\left[\int_0^T G'(\widetilde{Y}_s^{\epsilon,H})\left(Y_s^{\epsilon,H} - \widetilde{Y}_s^{\epsilon,H}\right)\mathrm{d}s\Big|\mathcal{G}_t\right] \\ &\quad + \widetilde{\mathbb{E}}\left[\int_0^T G''(\chi_s)\left(Y_s^{\epsilon,H} - \widetilde{Y}_s^{\epsilon,H}\right)^2\mathrm{d}s\Big|\mathcal{G}_t\right] \\ &= \widetilde{\mathbb{E}}\left[\int_0^T G(\widetilde{Y}_s^{\epsilon,H})\,\mathrm{d}s\Big|\mathcal{G}_t\right] + \widetilde{\mathbb{E}}\left[\int_0^T G'(\widetilde{Y}_s^{\epsilon,H})\int_0^s \rho\left(\frac{1-\gamma}{\gamma}\right)\lambda(Y_u^{\epsilon,H})\mathcal{K}^\epsilon(s-u)\,\mathrm{d}u\,\mathrm{d}s\Big|\mathcal{G}_t\right] + \mathcal{O}(\epsilon^{2-2H}) \\ &:= \widetilde{\psi}_t^{\epsilon,1} + \widetilde{\psi}_t^{\epsilon,2} + \mathcal{O}(\epsilon^{2-2H}).\end{aligned}$$

Now it remains to find approximations for $\widetilde{\psi}_t^{\epsilon,j}$, $j = 1, 2$, up to order ϵ^{1-H}. To this end, we need the following estimates in L^1:

$$R_t^{(1)} := \epsilon^{1-H} \int_0^t (T-u)^{H-\frac{1}{2}} \left(\lambda(Y_u^{\epsilon,H}) - \widetilde{\lambda} \right) \mathrm{d}u \sim o(\epsilon^{1-H}), \tag{3.17}$$

$$R_t^{(2)} := \widetilde{\mathbb{E}} \left[\int_0^T \left(G'(\widetilde{Y}_s^{\epsilon,H}) - \langle \lambda\lambda' \rangle \right) \int_0^s \rho \left(\frac{1-\gamma}{\gamma} \right) \lambda(Y_u^{\epsilon,H}) \mathcal{K}^\epsilon(s-u) \, \mathrm{d}u \, \mathrm{d}s \Big| \mathcal{G}_t \right] \sim o(\epsilon^{1-H}), \tag{3.18}$$

$$R_t^{(3)} := \widetilde{\mathbb{E}} \left[\int_0^T \int_0^s \left(\lambda(Y_u^{\epsilon,H}) - \widetilde{\lambda} \right) \mathcal{K}^\epsilon(s-u) \, \mathrm{d}u \, \mathrm{d}s \Big| \mathcal{G}_t \right] \sim o(\epsilon^{1-H}). \tag{3.19}$$

The proofs are technical and lengthy, thus deferred to Lemma A.4. To condense the notation, we define

$$\psi_t^\epsilon = \mathbb{E} \left[\frac{1}{2} \int_0^T \left(\lambda^2(Y_s^{\epsilon,H}) - \overline{\lambda}^2 \right) \mathrm{d}s \Big| \mathcal{G}_t \right], \tag{3.20}$$

$$\vartheta_t^\epsilon := \int_t^T \mathbb{E} \left[G'(Y_s^{\epsilon,H}) | \mathcal{G}_t \right] \mathcal{K}^\epsilon(s-t) \, \mathrm{d}s, \tag{3.21}$$

$$\widetilde{\vartheta}_t^\epsilon := \int_t^T \widetilde{\mathbb{E}}[G'(\widetilde{Y}_s^{\epsilon,H}) | \mathcal{G}_t] \mathcal{K}^\epsilon(s-t) \, \mathrm{d}s, \tag{3.22}$$

where ψ_t^ϵ is a $\mathbb{P}$-martingale satisfying $\mathrm{d}\psi_t^\epsilon = \vartheta_t^\epsilon \, \mathrm{d}W_t^Y$ (see details in Lemma A.1(i)). Similarly we have $\mathrm{d}\widetilde{\psi}_t^\epsilon = \widetilde{\vartheta}_t^\epsilon \, \mathrm{d}\widetilde{W}_t^Y$, and the difference between ϑ_t^ϵ and $\widetilde{\vartheta}_t^\epsilon$ is discussed in Lemma A.3(ii)).

Next, the terms $\widetilde{\psi}_t^{\epsilon,1}$ and $\widetilde{\psi}_t^{\epsilon,2}$ are computed as follows:

$$\begin{aligned}
\widetilde{\psi}_t^{\epsilon,1} &= \widetilde{\mathbb{E}} \left[\int_0^T G(\widetilde{Y}_s^{\epsilon,H}) \, \mathrm{d}s \Big| \mathcal{G}_t \right] = \widetilde{\mathbb{E}} \left[\int_0^T G(\widetilde{Y}_s^{\epsilon,H}) \, \mathrm{d}s \Big| \mathcal{G}_0 \right] + \int_0^t \widetilde{\vartheta}_u^\epsilon \, \mathrm{d}\widetilde{W}_u^Y \qquad (\widetilde{Y}_s^{\epsilon,H} | \mathcal{G}_0 \overset{\mathcal{D}}{=} Y_s^{\epsilon,H} | \mathcal{G}_0) \\
&= \mathbb{E} \left[\int_0^T G(Y_s^{\epsilon,H}) \, \mathrm{d}s \Big| \mathcal{G}_0 \right] + \int_0^t \vartheta_u^\epsilon \, \mathrm{d}W_u^Y + \int_0^t \left(\widetilde{\vartheta}_u^\epsilon - \vartheta_u^\epsilon \right) \mathrm{d}W_u^Y - \int_0^t \widetilde{\vartheta}_u^\epsilon \rho \left(\frac{1-\gamma}{\gamma} \right) \lambda(Y_u^{\epsilon,H}) \, \mathrm{d}u \\
&\qquad \qquad (\text{expression of } \psi_t^\epsilon \text{ and } \widetilde{\vartheta}_u^\epsilon - \vartheta_u^\epsilon \sim \mathcal{O}(\epsilon^{2-2H})) \\
&= \psi_t^\epsilon - \rho \left(\frac{1-\gamma}{\gamma} \right) \int_0^t \vartheta_u^\epsilon \lambda(Y_u^{\epsilon,H}) \, \mathrm{d}u + o(\epsilon^{1-H}) \qquad (\vartheta_u^\epsilon = \epsilon^{1-H} \theta_u + \widetilde{\theta}_u^\epsilon) \\
&= \psi_t^\epsilon - \epsilon^{1-H} \rho \left(\frac{1-\gamma}{\gamma} \right) \int_0^t \theta_u \lambda(Y_u^{\epsilon,H}) \, \mathrm{d}u - \rho \left(\frac{1-\gamma}{\gamma} \right) \int_0^t \widetilde{\theta}_u^\epsilon \lambda(Y_u^{\epsilon,H}) \, \mathrm{d}u + o(\epsilon^{1-H}) \\
&\qquad \qquad (\widetilde{\theta}_u^\epsilon \sim o(\epsilon^{1-H})) \\
&= \psi_t^\epsilon - \epsilon^{1-H} \rho \left(\frac{1-\gamma}{\gamma} \right) \widetilde{\lambda} \int_0^t \theta_u \, \mathrm{d}u - \epsilon^{1-H} \rho \left(\frac{1-\gamma}{\gamma} \right) \int_0^t \theta_u \left(\lambda(Y_u^{\epsilon,H}) - \widetilde{\lambda} \right) \mathrm{d}u + o(\epsilon^{1-H}) \\
&\qquad \qquad (\text{definition of } \theta_u \text{ and estimate of } R_t^{(1)}) \\
&= \psi_t^\epsilon - \epsilon^{1-H} \rho \left(\frac{1-\gamma}{\gamma} \right) \widetilde{\lambda} \frac{\langle \lambda\lambda' \rangle}{a\Gamma(H+\frac{3}{2})} \left(T^{H+\frac{1}{2}} - (T-t)^{H+\frac{1}{2}} \right) + o(\epsilon^{1-H}),
\end{aligned}$$

and

$$
\begin{aligned}
\widetilde{\psi}_t^{\epsilon,2} &= \widetilde{\mathbb{E}}\left[\int_0^T G'(\widetilde{Y}_s^{\epsilon,H})\int_0^s \rho\left(\frac{1-\gamma}{\gamma}\right)\lambda(Y_u^{\epsilon,H})\mathcal{K}^\epsilon(s-u)\,\mathrm{d}u\,\mathrm{d}s\Big|\mathcal{G}_t\right] \\
&= \langle\lambda\lambda'\rangle\,\widetilde{\mathbb{E}}\left[\int_0^T\int_0^s \rho\left(\frac{1-\gamma}{\gamma}\right)\lambda(Y_u^{\epsilon,H})\mathcal{K}^\epsilon(s-u)\,\mathrm{d}u\,\mathrm{d}s\Big|\mathcal{G}_t\right] + R_t^{(2)} \\
&= \langle\lambda\lambda'\rangle\,\rho\left(\frac{1-\gamma}{\gamma}\right)\widetilde{\lambda}\int_0^T\int_0^s \mathcal{K}^\epsilon(s-u)\,\mathrm{d}u\,\mathrm{d}s + R_t^{(2)} + R_t^{(3)} \qquad \text{(estimates of } R_t^{(2)} \text{ and } R_t^{(3)}) \\
&= \epsilon^{1-H}\,\langle\lambda\lambda'\rangle\,\rho\left(\frac{1-\gamma}{\gamma}\right)\widetilde{\lambda}\frac{T^{H+\frac{1}{2}}}{a\Gamma(H+\frac{3}{2})} + o(\epsilon^{1-H}).
\end{aligned}
$$

All reasonings are mentioned in the parentheses from line to line and proofs can be found in Lemmas A.1(i), A.3 and A.4. Combining the expansions of $\widetilde{\psi}_t^{\epsilon,1}$ and $\widetilde{\psi}_t^{\epsilon,2}$ together yields,

$$
\widetilde{\psi}_t^{\epsilon} = \psi_t^\epsilon + \epsilon^{1-H}\rho\left(\frac{1-\gamma}{\gamma}\right)\widetilde{\lambda}\frac{\langle\lambda\lambda'\rangle}{a\Gamma(H+\frac{3}{2})}(T-t)^{H+\frac{1}{2}} + o(\epsilon^{1-H}). \tag{3.23}
$$

Subtracting $\int_0^t G(Y_u^{\epsilon,H})\,\mathrm{d}u$ from both sides of (3.23), together with (3.16), (3.14) and (3.20), brings

$$
\Psi_t^\epsilon = 1 + \frac{1-\gamma}{q\gamma}\left(\phi_t^\epsilon + \epsilon^{1-H}\rho\left(\frac{1-\gamma}{\gamma}\right)\widetilde{\lambda}\frac{\langle\lambda\lambda'\rangle}{a\Gamma(H+\frac{3}{2})}(T-t)^{H+\frac{1}{2}}\right) + o(\epsilon^{1-H}). \tag{3.24}
$$

Taylor expanding x^q produces the desired result

$$
\begin{aligned}
V_t^\epsilon &= \frac{X_t^{1-\gamma}}{1-\gamma}e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)}\left(\Psi_t^\epsilon\right)^q \\
&= \frac{X_t^{1-\gamma}}{1-\gamma}e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)}\left\{1 + \frac{1-\gamma}{\gamma}\left(\phi_t^\epsilon + \epsilon^{1-H}\rho\left(\frac{1-\gamma}{\gamma}\right)\widetilde{\lambda}\frac{\langle\lambda\lambda'\rangle}{a\Gamma(H+\frac{3}{2})}(T-t)^{H+\frac{1}{2}}\right)\right\} + o(\epsilon^{1-H}).
\end{aligned}
$$

Observe that there are two corrections to the leading term: a random component ϕ_t^ϵ, and a deterministic function of t, X_t and the spatial average with respect to $Y_t^{\epsilon,H}$, both being of order ϵ^{1-H}. □

3.3 First order expansion of the optimal strategy

We now turn to the optimal portfolio π^* that leads to V_t^ϵ. Under the fractional stochastic environment $Y_t^{\epsilon,H}$, the form of the optimal strategy (2.14) in Proposition 2.3 becomes

$$
\pi_t^* = \left[\frac{\lambda(Y_t^{\epsilon,H})}{\gamma\sigma(Y_t^{\epsilon,H})} + \frac{\rho q\xi_t}{\gamma\sigma(Y_t^{\epsilon,H})}\right]X_t. \tag{3.25}
$$

It is not fully explicit due to the presence of ξ_t given by the martingale representation theorem (2.11). In the regime of ϵ small, we use (3.24) derived above to obtain the following expansion for π_t^*.

Theorem 3.5. *Under Assumption 2.1 and 3.2, we have the following approximation of the optimal strategy π_t^*:*

$$
\begin{aligned}
\pi_t^* &= \left[\frac{\lambda(Y_t^{\epsilon,H})}{\gamma\sigma(Y_t^{\epsilon,H})} + \epsilon^{1-H}\frac{\rho(1-\gamma)}{\gamma^2\sigma(Y_t^{\epsilon,H})}\frac{\langle\lambda\lambda'\rangle}{a\Gamma(H+\frac{1}{2})}(T-t)^{H-\frac{1}{2}}\right]X_t + o(\epsilon^{1-H}) \\
&:= \pi_t^{(0)} + \epsilon^{1-H}\pi_t^{(1)} + o(\epsilon^{1-H}).
\end{aligned} \tag{3.26}
$$

Proof. This is done by deriving the expansion of ξ_t from its definition (2.11). We rewrite M_t in terms of Ψ_t^ϵ by comparing (2.10) to (3.15),

$$
M_t = \Psi_t^\epsilon\, e^{\frac{1-\gamma}{2q\gamma}\int_0^t \lambda^2(Y_s^{\epsilon,H})\,\mathrm{d}s}\, e^{\frac{1-\gamma}{2q\gamma}\overline{\lambda}^2(T-t)},
$$

and then use the approximation (3.24) of Ψ_t^ϵ.

Since, by definition, M_t is a $\widetilde{\mathbb{P}}$-martingale, in the following calculation where Itô's formula is applied to M_t, we will only concentrate on the diffusion part. More precisely, the drift terms will not be computed explicitly and are replaced by "$\mathrm{d}t$ terms", in other words, calculations are omitted as long as they do not contribute to the diffusion part:

$$\begin{aligned}
\mathrm{d}M_t &= M_t(\Psi_t^\epsilon)^{-1}\,\mathrm{d}\Psi_t^\epsilon + \mathrm{d}t \text{ terms } = M_t(\Psi_t^\epsilon)^{-1}\frac{1-\gamma}{q\gamma}\,\mathrm{d}\phi_t^\epsilon + \mathrm{d}t \text{ terms} \\
&= M_t(\Psi_t^\epsilon)^{-1}\frac{1-\gamma}{q\gamma}\,\mathrm{d}\psi_t^\epsilon + \mathrm{d}t \text{ terms } = M_t(\Psi_t^\epsilon)^{-1}\frac{1-\gamma}{q\gamma}\vartheta_t^\epsilon\,\mathrm{d}W_t^Y + \mathrm{d}t \text{ terms} \\
&= M_t(\Psi_t^\epsilon)^{-1}\frac{1-\gamma}{q\gamma}\vartheta_t^\epsilon\,\mathrm{d}\widetilde{W}_t^Y.
\end{aligned}$$

In the above derivation, we have successively used (3.24), $\mathrm{d}\psi_t^\epsilon = \mathrm{d}\phi_t^\epsilon + \mathrm{d}t$ terms , and $\mathrm{d}\psi_t^\epsilon = \vartheta_t^\epsilon\,\mathrm{d}W_t^Y$.

Then ξ_t is easily identified and the approximation is deduced

$$\begin{aligned}
\xi_t &= (\Psi_t^\epsilon)^{-1}\frac{1-\gamma}{q\gamma}\vartheta_t^\epsilon = \epsilon^{1-H}\frac{1-\gamma}{q\gamma}\theta_t + o(\epsilon^{1-H}) \\
&= \epsilon^{1-H}\frac{1-\gamma}{q\gamma}\frac{\langle\lambda\lambda'\rangle}{a\Gamma(H+\frac{1}{2})}(T-t)^{H-\frac{1}{2}} + o(\epsilon^{1-H})
\end{aligned}$$

using $\vartheta_t^\epsilon = \epsilon^{1-H}\theta_t + \widetilde{\theta}_t^\epsilon$ (see Lemma A.1(i) for details). Plugging the above expression into (3.25) yields the desired result (3.26). □

Note that, in the above approximation, both the leading order strategy $\pi_t^{(0)}$ and the first order correction term $\pi_t^{(1)}$ are in feedback forms in terms of the state processes. Therefore, if one decides to track the fast-varying process $Y_t^{\epsilon,H}$ to implement $\pi_t^{(0)}$, no further computational cost is required when $\pi_t^{(1)}$ is also included in order to incorporate the inter-temporal hedging. On the other hand, tracking $Y_t^{\epsilon,H}$ is not easy and requires sophisticated econometric techniques. This issue will be addressed in Section 3.5. Before that, we discuss how good the strategy $\pi_t^{(0)}$ is.

3.4 Asymptotic optimality of $\pi_t^{(0)}$

In this subsection, we investigate the relation between V_t^ϵ and the value function obtained by following the zeroth-order strategy given in (3.26):

$$\pi_t^{(0)} = \frac{\lambda(Y_t^{\epsilon,H})}{\gamma\sigma(Y_t^{\epsilon,H})}X_t.$$

Let $X_t^{\pi^{(0)}}$ be the wealth process associated to $\pi_t^{(0)}$:

$$\begin{aligned}
\mathrm{d}X_t^{\pi^{(0)}} &= \mu(Y_t^{\epsilon,H})\pi_t^{(0)}\,\mathrm{d}t + \sigma(Y_t^{\epsilon,H})\pi_t^{(0)}\,\mathrm{d}W_t \\
&= \frac{\lambda^2(Y_t^{\epsilon,H})}{\gamma}X_t^{\pi^{(0)}}\,\mathrm{d}t + \frac{\lambda(Y_t^{\epsilon,H})}{\gamma}X_t^{\pi^{(0)}}\,\mathrm{d}W_t,
\end{aligned}$$

and denote by $V_\cdot^{\pi^{(0)},\epsilon}$ the corresponding value process

$$V_t^{\pi^{(0)},\epsilon} := \mathbb{E}\left[U\left(X_T^{\pi^{(0)}}\right)\middle|\mathcal{F}_t\right],$$

then, the following result holds:

Corollary 3.6. *Under Assumptions 2.1 and 3.2, for fixed $t\in[0,T)$ and the observed value X_t, $V_t^{\pi^{(0)},\epsilon}$ is approximated by*

$$V_t^{\pi^{(0)},\epsilon} = Q_t^\epsilon(X_t) + o(\epsilon^{1-H}), \tag{3.27}$$

where Q_t^ϵ is given in (3.13).

Proof. In Section 4 Proposition 4.3, such approximation result is given under a more general setup, that is, $U(\cdot)$ is in general form that includes the power utility case (2.5). Therefore, the proof here is a straightforward application by adapting the notation $v^{(0)}$, $v^{(1)}$ ... in Proposition 4.3 to the power utility case, and (3.27) is easily verified. $\square$

Now, combining Theorem 3.4 with Corollary 3.6 gives that $V_t^{\pi^{(0)},\epsilon} - V_t^\epsilon$ is of order $o(\epsilon^{1-H})$, which indicates that $\pi_t^{(0)}$ already generates the leading order term plus two corrections of order ϵ^{1-H} given by (3.13). Therefore, we state that:

$\pi_t^{(0)}$ *is asymptotically optimal within all admissible strategy* $\mathcal{A}_t^\epsilon$ *up to order* ϵ^{1-H}.

3.5 A practical strategy

The analysis above relies on the assumption that $Y_t^{\epsilon,H}$ is observable or trackable. In other words, to implement the principal term $\pi_t^{(0)}$, one needs to track the fast-varying factor $Y_t^{\epsilon,H}$ for any $t \in [0,T]$. This is usually not practical and long-term investors will not tackle this issue, since it usually requires high-frequency data and to deal with microstructure issues, as mentioned in Fouque et al. [2015]. Instead, they would prefer to look for a practical strategy which does not depend on the factor $Y_t^{\epsilon,H}$. To this end, we propose such a strategy and quantify its loss in terms of utility.

In the regime of ϵ small, the optimal Y-independent strategy proportional to the current wealth level is:

$$\bar\pi_t^{(0)} = \frac{\overline{\mu}}{\gamma\overline{\sigma}^2} X_t, \tag{3.28}$$

where the coefficients are

$$\overline{\mu} = \langle \mu \rangle, \quad \overline{\sigma}^2 = \langle \sigma^2 \rangle.$$

This is obtained by making the ansatz $\bar\pi_t^{(0)} = cX_t$, and then determining c by optimizing the leading order term of the corresponding problem value. Under self-financing, the wealth process (3.9) following the ansatz becomes:

$$X_t^{\bar\pi^{(0)}} = X_0 e^{\int_0^t \left(c\mu(Y_s^{\epsilon,H}) - \frac{1}{2}c^2\sigma^2(Y_s^{\epsilon,H})\right)\,\mathrm{d}t + \int_0^t c\sigma(Y_s^{\epsilon,H})\,\mathrm{d}W_s},$$

and the value to the problem is computed as

$$\begin{aligned}
V_t^{\bar\pi^{(0)},\epsilon} &= \mathbb{E}[U(X_T^{\bar\pi^{(0)}})|\mathcal{F}_t] \\
&= \frac{X_t^{1-\gamma}}{1-\gamma}\mathbb{E}\left(e^{(1-\gamma)\int_t^T \left(c\mu(Y_s^{\epsilon,H}) - \frac{1}{2}c^2\sigma^2(Y_s^{\epsilon,H})\right)\mathrm{d}t + (1-\gamma)\int_t^T c\sigma(Y_s^{\epsilon,H})\,\mathrm{d}W_s}\middle|\mathcal{F}_t\right) \\
&= \frac{X_t^{1-\gamma}}{1-\gamma}\widehat{\mathbb{E}}\left(e^{\int_t^T \left((1-\gamma)c\mu(Y_s^{\epsilon,H}) - \frac{\gamma-\gamma^2}{2}c^2\sigma^2(Y_s^{\epsilon,H})\right)\mathrm{d}t}\middle|\mathcal{G}_t\right),
\end{aligned}$$

where $W_t - (1-\gamma)c\int_0^t \sigma(Y_s^{\epsilon,H})\,\mathrm{d}s$ is a standard Brownian motion under $\widehat{\mathbb{P}}$. Using ergodic property of $Y_t^{\epsilon,H}$:

$$\int_t^T \left(\mu(Y_s^{\epsilon,H}) - \overline{\mu}\right)\,\mathrm{d}s \sim o(1), \quad \text{and} \quad \int_t^T \left(\sigma^2(Y_s^{\epsilon,H}) - \overline{\sigma}^2\right)\,\mathrm{d}s \sim o(1),$$

and Taylor expanding the function e^x at $x = 0$ (a similar derivation as in Theorem 3.4) one deduces:

$$\begin{aligned}
V_t^{\bar\pi^{(0)},\epsilon} &= \frac{X_t^{1-\gamma}}{1-\gamma}e^{[c(1-\gamma)\overline{\mu} - \frac{\gamma-\gamma^2}{2}c^2\overline{\sigma}^2](T-t)}\widehat{\mathbb{E}}\left(e^{\int_t^T \left((1-\gamma)c(\mu(Y_s^{\epsilon,H})-\overline{\mu}) - \frac{\gamma-\gamma^2}{2}c^2(\sigma^2(Y_s^{\epsilon,H})-\overline{\sigma}^2)\right)\mathrm{d}t}\middle|\mathcal{G}_t\right) \\
&= \frac{X_t^{1-\gamma}}{1-\gamma}e^{[c(1-\gamma)\overline{\mu} - \frac{\gamma-\gamma^2}{2}c^2\overline{\sigma}^2](T-t)} + o(1).
\end{aligned}$$

The leading order is optimized at $c^* = \frac{\overline{\mu}}{\gamma\overline{\sigma}^2}$, which leads to (3.28), and gives the optimal leading order term

$$\frac{X_t^{1-\gamma}}{1-\gamma}e^{\frac{1-\gamma}{2\gamma}\frac{\overline{\mu}^2}{\overline{\sigma}^2}(T-t)}.$$

This can be interpreted as the optimal value with Sharpe ratio $\overline{\mu}/\overline{\sigma}$.

The loss in utility of using $\bar{\pi}_t^{(0)}$ is quantified by comparing the above term with the leading order term of V_t^ϵ given in (3.12)-(3.13):

$$\frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2 (T-t)},$$

and is measured by the Cauchy-Schwarz gap

$$\overline{\lambda}^2 = \left\langle \frac{\mu^2}{\sigma^2} \right\rangle \geq \frac{\langle \mu^2 \rangle}{\langle \sigma^2 \rangle} \geq \frac{\overline{\mu}^2}{\overline{\sigma}^2},$$

as in the Markovian setup in Fouque et al. [2015]. Note that $\overline{\sigma}^2 = \langle \sigma^2 \rangle$ is the average which arises in the linear problem of option pricing as observed in Garnier and Sølna [2016] in the long-range memory case.

3.6 Numerical illustration

Next, we illustrate numerically the asymptotic optimality property of $\pi_t^{(0)}$ and the sub-optimality of $\bar{\pi}_t^{(0)}$, that is, we compute V_t^ϵ, $V_t^{\pi^{(0)},\epsilon}$, and $V_t^{\bar{\pi}^{(0)},\epsilon}$ at time $t = 0$ using Monte Carlo simulations, and compare their differences. Using equation (3.11) and changing the measure from $\widetilde{\mathbb{P}}$ to $\mathbb{P}$, one deduces

$$V_0^\epsilon = \frac{X_0^{1-\gamma}}{1-\gamma} \left[\mathbb{E}\left(e^{\left(\frac{1-\gamma}{2\gamma}\right) \int_0^T \lambda^2(Y_s^{\epsilon,H})\, \mathrm{d}s + \rho\left(\frac{1-\gamma}{\gamma}\right) \int_0^T \lambda(Y_s^{\epsilon,H})\, \mathrm{d}W_s^Y} \middle| \mathcal{G}_0 \right) \right]^q .$$

Solving the SDE for $X_t^{\pi^{(0)}}$ and plugging the solution into the definition of $V_t^{\pi^{(0)},\epsilon}$ bring

$$V_0^{\pi^{(0)},\epsilon} = \frac{X_0^{1-\gamma}}{1-\gamma} \mathbb{E}\left(e^{\left(\frac{-2\gamma^2+3\gamma-1}{2\gamma^2}\right) \int_0^T \lambda^2(Y_s^{\epsilon,H})\, \mathrm{d}s + \left(\frac{1-\gamma}{\gamma}\right) \int_0^T \lambda(Y_s^{\epsilon,H})\, \mathrm{d}W_s} \middle| \mathcal{F}_0 \right).$$

Similarly, the value process following the Y-independent strategy $\bar{\pi}_t^{(0)}$ is given by

$$V_0^{\bar{\pi}^{(0)},\epsilon} = \frac{X_0^{1-\gamma}}{1-\gamma} \mathbb{E}\left(e^{\left(\frac{1-\gamma}{\gamma}\right) \frac{\overline{\mu}}{\overline{\sigma}^2} \int_0^T \mu(Y_s^{\epsilon,H})\, \mathrm{d}s - \left(\frac{1-\gamma}{2\gamma^2}\right) \frac{\overline{\mu}^2}{\overline{\sigma}^4} \int_0^T \sigma^2(Y_s^{\epsilon,H})\, \mathrm{d}s + \left(\frac{1-\gamma}{\gamma}\right) \frac{\overline{\mu}}{\overline{\sigma}^2} \int_0^T \sigma(Y_s^{\epsilon,H})\, \mathrm{d}W_s} \middle| \mathcal{F}_0 \right).$$

The model parameters are chosen as:

$$T = 1, \quad H = 0.6, \quad a = 1, \quad \gamma = 0.4, \quad \rho = -0.5, \quad \mu(y) = \frac{0.1 \times \lambda(y)}{0.1 + \lambda(y)}, \quad \lambda^2(y) = \frac{1}{2} \int_{-\infty}^{y/\sigma_{ou}} p(z/2)\, \mathrm{d}z,$$

where we recall that $p(z)$ is the $\mathcal{N}(0,1)$-density. Note that the choice of $\lambda(y)$ above satisfies Assumption 2.1(i) and 3.2 (see [Garnier and Sølna, 2016, Lemma A.2]) and $\overline{\lambda} = \sqrt{\langle \lambda^2 \rangle} = 0.7$. Note also that with our choice for $\mu(y)$, both $\mu(y)$ and $\sigma^2(y) = \mu^2(y)/\lambda^2(y)$ are integrable with respect to the invariant distribution of $Y^{\epsilon,H}$, so that $\overline{\mu}$ and $\overline{\sigma}^2$ are finite and equal to .087 and .0176 respectively.

Due to the natural non-Markovian structure, we first generate a "historical" path W_t^Y between $-M$ and 0, and then evaluate each conditional expectation by the average of 500,000 paths. The fast-varying factor $(Y_t^{\epsilon,H})_{t\in[0,T]}$ (3.2) is generated using Euler scheme with mesh size $\Delta t = 10^{-3}$, and $M = (T/\Delta t)^{1.5}$ (cf. Bardet et al. [2003]).

The numerical results presented in Table 1 are only for a purpose of illustration as we computed the values for only a few "omegas" denoted by #1, #2, and #3.

Table 1: The value processes V_0^ϵ *vs.* $V_0^{\pi^{(0)},\epsilon}$ *vs.* $V_0^{\bar{\pi}^{(0)},\epsilon}$ for the power utility case.

		#1	#2	#3
$\epsilon = 1$	V_0^ϵ	1.5772	1.5644	1.3016
	$V_0^\epsilon - V_0^{\pi^{(0)},\epsilon}$	0.0018	0.0019	0.0024
	$V_0^\epsilon - V_0^{\bar{\pi}^{(0)},\epsilon}$	0.0689	0.0643	0.0820
$\epsilon = 0.5$	V_0^ϵ	1.5567	1.4965	1.3183
	$V_0^\epsilon - V_0^{\pi^{(0)},\epsilon}$	0.0025	0.0028	0.0028
	$V_0^\epsilon - V_0^{\bar{\pi}^{(0)},\epsilon}$	0.0760	0.0593	0.0999
$\epsilon = 0.1$	V_0^ϵ	1.4514	1.4417	1.3976
	$V_0^\epsilon - V_0^{\pi^{(0)},\epsilon}$	0.0026	0.0026	0.0025
	$V_0^\epsilon - V_0^{\bar{\pi}^{(0)},\epsilon}$	0.0761	0.0756	0.0823
$\epsilon = 0.05$	V_0^ϵ	1.4376	1.4375	1.4105
	$V_0^\epsilon - V_0^{\pi^{(0)},\epsilon}$	0.0022	0.0022	0.0021
	$V_0^\epsilon - V_0^{\bar{\pi}^{(0)},\epsilon}$	0.0750	0.0762	0.0806
$\epsilon = 0.01$	V_0^ϵ	1.4417	1.4416	1.4276
	$V_0^\epsilon - V_0^{\pi^{(0)},\epsilon}$	0.0015	0.0015	0.0015
	$V_0^\epsilon - V_0^{\bar{\pi}^{(0)},\epsilon}$	0.0724	0.0727	0.0748

As expected, the strategy $\pi_t^{(0)}$ performs well for ϵ small, the relative difference $(V_0^\epsilon - V_0^{\pi^{(0)},\epsilon})/V_0^\epsilon$ being about 0.1%. What is more surprising is that it also performs well even for not so small values of ϵ. Again, as expected, the sub-optimal "lazy" strategy $\bar{\pi}_t^{(0)}$ underperforms $\pi_t^{(0)}$ but it performs relatively well since $(V_0^\epsilon - V_0^{\bar{\pi}^{(0)},\epsilon})/V_0^\epsilon$ is about 5%.

3.7 Comparison with the Markovian case

In the Markovian case, which corresponds to $H = \frac{1}{2}$ in the modeling of $Y_t^{\epsilon,H}$ (3.1), approximations to the value function and the optimal portfolio have been derived in Fouque et al. [2015]. They are given by:

$$V^\epsilon(t, X_t) = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)} \left[1 - \sqrt{\epsilon}\rho\left(\frac{1-\gamma}{\gamma}\right)^2 \frac{\langle\lambda\theta'\rangle}{2}(T-t)\right] + \mathcal{O}(\epsilon) \tag{3.29}$$

$$\pi^*(t, X_t, Y_t^{\epsilon,H}) = \left[\frac{\lambda(Y_t^{\epsilon,H})}{\gamma\sigma(Y_t^{\epsilon,H})} + \sqrt{\epsilon}\frac{\rho(1-\gamma)}{\gamma^2\sigma(Y_t^{\epsilon,H})}\frac{\theta'(Y_t^{\epsilon,H})}{2}\right] X_t + \mathcal{O}(\epsilon) \tag{3.30}$$

where $\theta(y)$ solves the Poisson equation $\frac{1}{2}\theta''(y) - ay\theta'(y) = \lambda^2(y) - \overline{\lambda}^2$. These can be viewed as the limits $\lim_{\epsilon\to 0}\lim_{H\downarrow\frac{1}{2}}$ of our current setup.

However, these limits do not commute. For instance, if we consider the small ϵ expansion of π^* from (3.26) and formally let $H = \frac{1}{2}$, we obtain

$$\left[\frac{\lambda(Y_t^{\epsilon,H})}{\gamma\sigma(Y_t^{\epsilon,H})} + \sqrt{\epsilon}\frac{\rho(1-\gamma)}{\gamma^2\sigma(Y_t^{\epsilon,H})}\frac{\langle\lambda\lambda'\rangle}{a}\right] X_t + o(\sqrt{\epsilon}), \tag{3.31}$$

which corresponds to the other order of limits $\lim_{H\downarrow\frac{1}{2}}\lim_{\epsilon\to 0}$. The two expansions (3.30) and (3.31) are different and in particular they track the first order correction in different ways.

Regarding the value process V_t^ϵ, one first observes that the path-dependent component ϕ_t^ϵ disappears in (3.13) in the limit $H \downarrow \frac{1}{2}$. To be precise,

$$\lim_{H\downarrow\frac{1}{2}}\lim_{\epsilon\to 0} \epsilon^{H-1}\phi_t^\epsilon = 0, \tag{3.32}$$

by Lemma A.1(ii). This is because $\epsilon^{H-1}\phi_t^\epsilon$ converges in distribution to $\mathcal{N}(0,\sigma_\phi^2(T-t)^{2H})$, where σ_ϕ^2 is given by

$$\sigma_\phi^2 = \sigma_{ou}^2 \langle\lambda\lambda'\rangle^2 \left(\frac{1}{\Gamma(2H+1)\sin(\pi H)} - \frac{1}{2H\Gamma^2(H+\frac{1}{2})}\right).$$

Then, the claim (3.32) is obtained by setting $H=\frac{1}{2}$ in σ_ϕ^2. Now, we conclude that V_t^ϵ only exhibits a feedback-type correction when taking a formal limit $H\downarrow\frac{1}{2}$ in (3.13):

$$\frac{X_t^{1-\gamma}}{1-\gamma}e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)}\left[1+\sqrt{\epsilon}\rho\left(\frac{1-\gamma}{\gamma}\right)^2\frac{\widetilde{\lambda}\langle\lambda\lambda'\rangle}{a}(T-t)\right]+o(\sqrt{\epsilon}).$$

However the first order correction is in general not the same as in (3.29).

We remark that although the two sets of expansions ($H=\frac{1}{2}$ *vs.* $H\in(\frac{1}{2},1)$) share the same form, the coefficients are not identical. This is because our derivations in Theorem 3.4 and 3.5 are only valid for $H\in(\frac{1}{2},1)$, and the singular perturbation is "singular" at $H=\frac{1}{2}$. Consequently, the order of limits $H\downarrow\frac{1}{2}$ and $\epsilon\to 0$ is not interchangeable, and this leads to different expansion results.

Finally, note that in the Markovian case $H=\frac{1}{2}$, the first order correction to V_t^ϵ is "deterministic", while in the case $H>\frac{1}{2}$, the stochastic correction ϕ_t^ϵ of the same order also appears, as a consequence of having long-range dependence in the stochastic environment $Y_t^{\epsilon,H}$.

4 General utilities and fractional stochastic environment

In this section, we analyze the nonlinear asset allocation problem using asymptotic methods where the utility function $U(x)$ is general, and when, as in (3.8), the log-return μ and volatility σ of the risky asset S_t are driven by the fast-varying fractional stochastic factor $Y_t^{\epsilon,H}$ defined in (3.1) and discussed in Section 3.1. For the linear pricing problem when the volatility is modeled by $Y_t^{\epsilon,H}$, approximation results have been developed in Garnier and Sølna [2016] using the same technique.

Here, unlike in the power utility case, the representations (2.13) and (2.14) for the value process and the corresponding optimal strategy are not available, therefore asymptotic expansions can not be done directly. However, we are able to follow the idea developed in [Fouque and Hu, 2017a, Section 4] and [Fouque and Hu, 2017b, Section 4] and partially solve this problem. We first study the value process following a specific strategy called $\pi^{(0)}$ introduced in (4.8), and then show that $\pi^{(0)}$ is the best up to order ϵ^{1-H} among the following subset $\widetilde{\mathcal{A}}_t^\epsilon$ of admissible strategies $\mathcal{A}_t^\epsilon$,

$$\widetilde{\mathcal{A}}_t^\epsilon[\widetilde{\pi}^0,\widetilde{\pi}^1,\alpha] := \left\{\pi=\widetilde{\pi}^0+\epsilon^\alpha\widetilde{\pi}^1 : \pi\in\mathcal{A}_t^\epsilon, \alpha>0, 0<\epsilon\leq 1\right\}, \tag{4.1}$$

The detailed definition of $\widetilde{\mathcal{A}}_t^\epsilon$ will be given in Section 4.2. Note that the full optimality of $\pi^{(0)}$ in the whole class $\mathcal{A}_t^\epsilon$ remains an open problem.

In the rest of this section, we briefly review the classical Merton problem, where μ and σ are constants in (2.1) . Denote by $M(t,x;\lambda)$ the corresponding value function, if the utility $U(x)$ is $C^2(0,\infty)$, strictly increasing, strictly concave, and satisfies the Inada and Asymptotic Elasticity conditions (see Kramkov and Schachermayer [2003] for details)

$$U'(0+)=\infty, \quad U'(\infty)=0, \quad \mathrm{AE}[U] := \lim_{x\to\infty} x\frac{U'(x)}{U(x)} < 1,$$

then, the Merton value function $M(t,x;\lambda)$ is strictly increasing, strictly concave in the wealth variable x, and decreasing in the time variable t. It is $C^{1,2}([0,T]\times\mathbb{R}^+)$ and solves the HJB equation

$$M_t + \sup_\pi\left\{\frac{1}{2}\sigma^2\pi^2 M_{xx}+\mu\pi M_x\right\} = M_t - \frac{1}{2}\lambda^2\frac{M_x^2}{M_{xx}} = 0, \quad M(T,x;\lambda)=U(x), \tag{4.2}$$

where $\lambda=\mu/\sigma$ is the constant Sharpe ratio, and appears as a parameter in (4.2).

Based on the Merton value function $M(t,x;\lambda)$, one defines the risk-tolerance function by

$$R(t,x;\lambda) := -\frac{M_x(t,x;\lambda)}{M_{xx}(t,x;\lambda)}. \tag{4.3}$$

It is clear that $R(t,x;\lambda)$ is continuous and strictly positive due to the regularity, concavity and monotonicity of $M(t,x;\lambda)$. Further properties regarding $R(t,x;\lambda)$ are also discussed in Källblad and Zariphopoulou [2014], and Fouque and Hu [2017a] under general utility with additional assumptions. Some of them are repeatedly used in the derivations and will be mentioned during the proofs.

4.1 Portfolio performance of a given strategy

Denote by $v^{(0)}(t,x)$ the value function at "averaged" Sharpe-ratio $\overline{\lambda}$

$$v^{(0)}(t,x) := M(t,x;\overline{\lambda}), \tag{4.4}$$

with $\overline{\lambda}$ given in (3.4). Using the notations from Fouque et al. [2015]:

$$D_k := R(t,x;\overline{\lambda})^k \partial_x^k, \qquad k = 1,2,\cdots, \tag{4.5}$$

$$\mathcal{L}_{t,x}(\lambda) := \partial_t + \frac{1}{2}\lambda^2 D_2 + \lambda^2 D_1, \tag{4.6}$$

and the Merton PDE (4.2), $v^{(0)}$ also satisfies

$$\mathcal{L}_{t,x}(\overline{\lambda}) v^{(0)}(t,x) = 0. \tag{4.7}$$

The strategy $\pi^{(0)}$ is defined as

$$\pi^{(0)}(t,x,y) := -\frac{\lambda(y)}{\sigma(y)}\frac{v_x^{(0)}(t,x)}{v_{xx}^{(0)}(t,x)} = \frac{\lambda(y)}{\sigma(y)} R(t,x;\overline{\lambda}), \tag{4.8}$$

and our aim is to compute the following quantity:

$$V_t^{\pi^{(0)},\epsilon} := \mathbb{E}\left[U(X_T^{\pi^{(0)}})|\mathcal{F}_t\right], \tag{4.9}$$

where $X_t^{\pi^{(0)}}$ is the wealth process following the feedback-form strategy $\pi^{(0)}$

$$\begin{aligned} \mathrm{d}X_t^{\pi^{(0)}} &= \mu(Y_t^{\epsilon,H})\pi^{(0)}(t,X_t^{\pi^{(0)}},Y_t^{\epsilon,H})\,\mathrm{d}t + \sigma(Y_t^{\epsilon,H})\pi^{(0)}(t,X_t^{\pi^{(0)}},Y_t^{\epsilon,H})\,\mathrm{d}W_t \\ &= \lambda^2(Y_t^{\epsilon,H})R(t,X_t^{\pi^{(0)}};\overline{\lambda})\,\mathrm{d}t + \lambda(Y_t^{\epsilon,H})R(t,X_t^{\pi^{(0)}};\overline{\lambda})\,\mathrm{d}W_t. \end{aligned} \tag{4.10}$$

The technique used to study $V_t^{\pi^{(0)},\epsilon}$ is called "epsilon-martingale decomposition", which was firstly introduced in Fouque et al. [2000] to solve the linear pricing problem, and later developed in Fouque et al. [2001], Garnier and Sølna [2017, 2016], Fouque and Hu [2017b]. The idea is to make an ansatz $Q_t^{\pi^{(0)},\epsilon}$ for $V_t^{\pi^{(0)},\epsilon}$ in the form of a martingale plus something small (non-martingale part) with the right terminal condition. Then this ansatz is indeed the approximation to $V_t^{\pi^{(0)},\epsilon}$ with an error that is of order of the non-martingale part. Detailed discussion can be found in the references we just mentioned.

To prove that the ansatz $Q_t^{\pi^{(0)},\epsilon}$ is indeed a martingale plus the non-martingale part of the desired order, we further require Assumption 4.1 for the utility function and Assumption 4.2 for the value function $v^{(0)}(t,x)$. Basically, we work under the same setup of $U(\cdot)$ as in Fouque and Hu [2017a], and we restate these requirements here for convenience. Detailed discussions about general utility functions can be found there in Section 2.3.

Assumption 4.1. *Throughout this section, we make the following assumptions on the utility $U(x)$:*

(i) U(x) is $C^6(0,\infty)$, strictly increasing, strictly concave and satisfying the following conditions (Inada and Asymptotic Elasticity):

$$U'(0+) = \infty, \quad U'(\infty) = 0, \quad AE[U] := \lim_{x\to\infty} x\frac{U'(x)}{U(x)} < 1. \tag{4.11}$$

(ii) U(0+) is finite. Without loss of generality, we assume U(0+) = 0.

(iii) Denote by $R(x)$ the risk tolerance,

$$R(x) := -\frac{U'(x)}{U''(x)}. \tag{4.12}$$

Assume that $R(0) = 0$, R(x) is strictly increasing and $R'(x) < \infty$ on $[0,\infty)$, and there exists $K \in \mathbb{R}^+$, such that for $x \geq 0$, and $2 \leq i \leq 4$,

$$\left|\partial_x^{(i)} R^i(x)\right| \leq K. \tag{4.13}$$

(iv) Define the inverse function of the marginal utility $U'(x)$ as $I : \mathbb{R}^+ \to \mathbb{R}^+$, $I(y) = U'^{(-1)}(y)$, and assume that, for some positive α, κ, $I(y)$ satisfies the polynomial growth condition:

$$I(y) \leq \alpha + \kappa y^{-\alpha}. \tag{4.14}$$

Note that the item *(ii)* above excludes the case of power utility $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ when $\gamma > 1$. However, all results in this section still hold for the case $\gamma > 1$, with a slight modification in the proofs. Below is the additional assumption needed jointly on $v^{(0)}(t,x)$ and $X_t^{\pi^{(0)}}$, which is also considered as a hidden assumption on $U(\cdot)$.

Assumption 4.2. *The process $v^{(0)}(t, X_t^{\pi^{(0)}})$ is in L^4 uniformly in ϵ and in $t \in [0,T]$, i.e.,*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left(v^{(0)}(t, X_t^{\pi^{(0)}})\right)^4\right] \leq C_1 \tag{4.15}$$

where C_1 is independent of ϵ.

Now we state the following proposition which gives $Q_t^{\pi^{(0)},\epsilon}$.

Proposition 4.3. *Under Assumption 2.1*(i)-(iii)*, 3.2, 4.1 and 4.2, for fixed $t \in [0,T)$, the $\mathcal{F}_t$-measurable value process $V_t^{\pi^{(0)},\delta}$ defined in* (4.9) *is approximated by $Q_t^{\pi^{(0)},\epsilon}$ up to order ϵ^{1-H}:*

$$V_t^{\pi^{(0)},\epsilon} = Q_t^{\pi^{(0)},\epsilon}(X_t^{\pi^{(0)}}) + o(\epsilon^{1-H}), \tag{4.16}$$

where $Q_t^{\pi^{(0)},\epsilon}(x)$ is given by:

$$Q_t^{\pi^{(0)},\epsilon}(x) = v^{(0)}(t,x) + D_1 v^{(0)}(t,x)\phi_t^\epsilon + \epsilon^{1-H}\rho\widetilde{\lambda} v^{(1)}(t,x). \tag{4.17}$$

The function $v^{(0)}$ is defined in (4.4) *and satisfies $\mathcal{L}_{t,x}(\overline{\lambda})v^{(0)}(t,x) = 0$, D_1 and $\widetilde{\lambda}$ are from* (4.5) *and* (3.4) *respectively, $(\phi_t^\epsilon)_{t\in[0,T]}$ is the $\mathcal{F}_t$-measurable process of order ϵ^{1-H} given in* (3.14) *and $v^{(1)}(t,x)$ is defined as*

$$v^{(1)}(t,x) = D_1^2 v^{(0)}(t,x) C_{t,T}, \quad C_{t,T} = \frac{\langle \lambda\lambda' \rangle}{a\Gamma(H+\frac{3}{2})}(T-t)^{H+\frac{1}{2}}. \tag{4.18}$$

Proof. Based on the epsilon-martingale decomposition, it suffices to show that $Q_t^{\pi^{(0)},\epsilon}$ can be decomposed as $M_t^\epsilon + R_t^\epsilon$, where M_t^ϵ is a true martingale, and R_t^ϵ is of order $o(\epsilon^{1-H})$. In the sequel, we shall focus on the derivation of determining $Q_t^{\pi^{(0)},\epsilon}$, which involves finding corrections of order ϵ^{1-H} so that R_t^ϵ is pushed to a higher order, while the proofs regarding M_t^ϵ and R_t^ϵ are delayed to Appendix A.

Applying Itô formula to $v^{(0)}(t, X_t^{\pi^{(0)}})$ brings

$$\begin{aligned}
\mathrm{d}v^{(0)}(t, X_t^{\pi^{(0)}}) &= \mathcal{L}_{t,x}(\lambda(Y_t^{\epsilon,H}))v^{(0)}(t, X_t^{\pi^{(0)}})\,\mathrm{d}t + \sigma(Y_t^{\epsilon,H})\pi^{(0)}(t, X_t^{\pi^{(0)}}, Y_t^{\epsilon,H})v_x^{(0)}(t, X_t^{\pi^{(0)}})\,\mathrm{d}W_t \\
&= \frac{1}{2}\left(\lambda^2(Y_t^{\epsilon,H}) - \overline{\lambda}^2\right) D_1 v^{(0)}(t, X_t^{\pi^{(0)}})\,\mathrm{d}t + \,\mathrm{d}M_t^{(1)},
\end{aligned} \tag{4.19}$$

where $M_t^{(1)}$ is the martingale given by

$$\mathrm{d}M_t^{(1)} = \sigma(Y_t^{\epsilon,H})\pi^{(0)}(t, X_t^{\pi^{(0)}}, Y_t^{\epsilon,H})v_x^{(0)}(t, X_t^{\pi^{(0)}})\,\mathrm{d}W_t, \tag{4.20}$$

and the relations (4.7) and $D_1 v^{(0)}(t,x) = -D_2 v^{(0)}(t,x)$ have been used.

Recall ϕ_t^ϵ and ψ_t^ϵ defined in (3.14) and (3.20) respectively, then, we have $\mathrm{d}\psi_t^\epsilon - \mathrm{d}\phi_t^\epsilon = \frac{1}{2}\left(\lambda^2(Y_t^{\epsilon,H}) - \overline{\lambda}^2\right)\mathrm{d}t$, and the first term in (4.19) becomes

$$\frac{1}{2}\left(\lambda^2(Y_t^{\epsilon,H}) - \overline{\lambda}^2\right) D_1 v^{(0)}(t, X_t^{\pi^{(0)}})\,\mathrm{d}t = D_1 v^{(0)}(t, X_t^{\pi^{(0)}})\left(\,\mathrm{d}\psi_t^\epsilon - \,\mathrm{d}\phi_t^\epsilon\right).$$

To further simplify $D_1 v^{(0)}(t, X_t^{\pi^{(0)}})\,\mathrm{d}\phi_t^\epsilon$, which corresponds to finding the corrector to $v^{(0)}(t, X_t^{\pi^{(0)}})$ at order ϵ^{1-H}, we compute the total differential of $D_1 v^{(0)}(t, X_t^{\pi^{(0)}})\phi_t^\epsilon$ (the arguments of $v^{(0)}(t, X_t^{\pi^{(0)}})$ will be omitted systematically in the following):

$$\begin{aligned}
\mathrm{d}\left(D_1 v^{(0)}\phi_t^\epsilon\right) &= D_1 v^{(0)}\,\mathrm{d}\phi_t^\epsilon + \phi_t^\epsilon \mathcal{L}_{t,x}(\lambda(Y_t^{\epsilon,H}))D_1 v^{(0)}\,\mathrm{d}t + \phi_t^\epsilon \sigma(Y_t^{\epsilon,H})\pi^{(0)}(t, X_t^{\pi^{(0)}}, Y_t^{\epsilon,H})\partial_x D_1 v^{(0)}\,\mathrm{d}W_t \\
&\quad + \sigma(Y_t^{\epsilon,H})\pi^{(0)}(t, X_t^{\pi^{(0)}}, Y_t^{\epsilon,H})\partial_x D_1 v^{(0)}\,\mathrm{d}\langle W, \phi^\epsilon\rangle_t \\
&= D_1 v^{(0)}\,\mathrm{d}\phi_t^\epsilon + \phi_t^\epsilon\left[\frac{1}{2}(\lambda^2(Y_t^{\epsilon,H}) - \overline{\lambda}^2)(D_2 + 2D_1)D_1 v^{(0)}\right]\mathrm{d}t \\
&\quad + \phi_t^\epsilon \lambda(Y_t^{\epsilon,H})D_1^2 v^{(0)}\,\mathrm{d}W_t + \rho\lambda(Y_t^{\epsilon,H})D_1^2 v^{(0)}\,\mathrm{d}\left\langle W^Y, \psi^\epsilon\right\rangle_t
\end{aligned}$$

In the derivation, we have used the definition of D_1 and $R(t,x;\lambda)$ (cf. (4.5) and (4.3)), and

$$\mathcal{L}_{t,x}(\overline{\lambda})D_1 v^{(0)} = D_1 \mathcal{L}_{t,x}(\overline{\lambda})v^{(0)} = 0, \text{ and } \mathrm{d}\langle W, \phi^\epsilon\rangle_t = \rho\,\mathrm{d}\left\langle W^Y, \psi^\epsilon\right\rangle_t.$$

The results in Lemma A.1(i): $\mathrm{d}\left\langle W^Y, \psi^\epsilon\right\rangle_t = \vartheta_t^\epsilon\,\mathrm{d}t = \left(\epsilon^{1-H}\theta_t + \widetilde{\theta}_t^\epsilon\right)\mathrm{d}t$, together with the above derivation produce

$$\begin{aligned}
\mathrm{d}\left(D_1 v^{(0)}\phi_t^\epsilon\right) &= -\frac{1}{2}\left(\lambda^2(Y_t^{\epsilon,H}) - \overline{\lambda}^2\right) D_1 v^{(0)}\,\mathrm{d}t + \phi_t^\epsilon\left[\frac{1}{2}(\lambda^2(Y_t^{\epsilon,H}) - \overline{\lambda}^2)(D_2 + 2D_1)D_1 v^{(0)}\right]\mathrm{d}t \\
&\quad + \epsilon^{1-H}\rho\lambda(Y_t^{\epsilon,H})D_1^2 v^{(0)}\theta_t\,\mathrm{d}t + \rho\lambda(Y_t^{\epsilon,H})D_1^2 v^{(0)}\widetilde{\theta}_t^\epsilon\,\mathrm{d}t + \,\mathrm{d}M_t^{(2)},
\end{aligned} \tag{4.21}$$

and

$$\mathrm{d}M_t^{(2)} = D_1 v^{(0)}(t, X_t^{\pi^{(0)}})\,\mathrm{d}\psi_t^\epsilon + \phi_t^\epsilon \lambda(Y_t^{\epsilon,H})D_1^2 v^{(0)}(t, X_t^{\pi^{(0)}})\,\mathrm{d}W_t. \tag{4.22}$$

The term $\epsilon^{1-H}\rho\lambda(Y_t^{\epsilon,H})D_1^2 v^{(0)}\theta_t\,\mathrm{d}t$ is taken care of by adding the term $\epsilon^{1-H}\rho\widetilde{\lambda}v^{(1)}$ to $Q_t^{\pi^{(0)},\epsilon}$. By using the relation $\theta_t = -\partial_t C_{t,T}$, one has

$$\begin{aligned}
\mathrm{d}v^{(1)}(t, X_t^{\pi^{(0)}}) &= \mathcal{L}_{t,x}(\lambda(Y_t^{\epsilon,H}))v^{(1)}(t, X_t^{\pi^{(0)}})\,\mathrm{d}t + \sigma(Y_t^{\epsilon,H})\pi^{(0)}(t, X_t^{\pi^{(0)}}, Y_t^{\epsilon,H})v_x^{(1)}(t, X_t^{\pi^{(0)}})\,\mathrm{d}W_t \\
&= \frac{1}{2}(\lambda^2(Y_t^{\epsilon,H}) - \overline{\lambda}^2)(D_2 + 2D_1)v^{(1)}(t, X_t^{\pi^{(0)}})\,\mathrm{d}t - D_1^2 v^{(0)}(t, X_t^{\pi^{(0)}})\theta_t\,\mathrm{d}t + \,\mathrm{d}M_t^{(3)},
\end{aligned} \tag{4.23}$$

where $M_t^{(3)}$ is the martingale defined by

$$\mathrm{d}M_t^{(3)} = \sigma(Y_t^{\epsilon,H})\pi^{(0)}(t, X_t^{\pi^{(0)}}, Y_t^{\epsilon,H})v_x^{(1)}(t, X_t^{\pi^{(0)}})\,\mathrm{d}W_t. \tag{4.24}$$

Combining equation (4.19), (4.21) and (4.23) yields

$$\begin{aligned}
\mathrm{d}Q_t^{\pi^{(0)},\epsilon}(X_t^{\pi^{(0)}}) =&\ \mathrm{d}\left(v^{(0)}(t,X_t^{\pi^{(0)}}) + D_1 v^{(0)}(t,X_t^{\pi^{(0)}})\phi_t^\epsilon + \epsilon^{1-H}\rho\widetilde{\lambda}v^{(1)}(t,X_t^{\pi^{(0)}})\right) \\
=&\ \phi_t^\epsilon\left[\frac{1}{2}(\lambda^2(Y_t^{\epsilon,H}) - \overline{\lambda}^2)(D_2+2D_1)D_1 v^{(0)}\right]\mathrm{d}t + \epsilon^{1-H}\rho\left(\lambda(Y_t^{\epsilon,H}) - \widetilde{\lambda}\right)D_1^2 v^{(0)}\theta_t\,\mathrm{d}t \\
&+ \rho\lambda(Y_t^{\epsilon,H})D_1^2 v^{(0)}\widetilde{\theta}_t^\epsilon\,\mathrm{d}t + \frac{1}{2}\epsilon^{1-H}\rho\widetilde{\lambda}(\lambda^2(Y_t^{\epsilon,H}) - \overline{\lambda}^2)(D_2+2D_1)v^{(1)}(t,X_t^{\pi^{(0)}})\,\mathrm{d}t \\
&+ \mathrm{d}M_t^{(1)} + \mathrm{d}M_t^{(2)} + \epsilon^{1-H}\rho\widetilde{\lambda}\,\mathrm{d}M_t^{(3)}.
\end{aligned}$$

Denote by $R_{t,T}^{(j)}$, $j=1,2,3,4$ the first four terms in the above expression

$$R_{t,T}^{(1)} := \int_t^T \phi_s^\epsilon\left[\frac{1}{2}(\lambda^2(Y_s^{\epsilon,H}) - \overline{\lambda}^2)(D_2+2D_1)D_1 v^{(0)}(s,X_s^{\pi^{(0)}})\right]\mathrm{d}s, \tag{4.25}$$

$$R_{t,T}^{(2)} := \int_t^T \epsilon^{1-H}\rho\left(\lambda(Y_s^{\epsilon,H}) - \widetilde{\lambda}\right)D_1^2 v^{(0)}(s,X_s^{\pi^{(0)}})\theta_s\,\mathrm{d}s, \tag{4.26}$$

$$R_{t,T}^{(3)} := \int_t^T \rho\lambda(Y_s^{\epsilon,H})D_1^2 v^{(0)}(s,X_s^{\pi^{(0)}})\widetilde{\theta}_s^\epsilon\,\mathrm{d}s, \tag{4.27}$$

$$R_{t,T}^{(4)} := \int_t^T \frac{1}{2}\epsilon^{1-H}\rho\widetilde{\lambda}(\lambda^2(Y_s^{\epsilon,H}) - \overline{\lambda}^2)(D_2+2D_1)v^{(1)}(s,X_s^{\pi^{(0)}})\,\mathrm{d}s, \tag{4.28}$$

and it is proved in Lemma A.6 that they are $o(\epsilon^{1-H})$ terms in L^1:

$$\lim_{\epsilon\to 0}\epsilon^{H-1}\,\mathbb{E}\left|R_{t,T}^{(j)}\right| = 0, \quad \forall j=1,2,3,4. \tag{4.29}$$

Lemma A.5 also shows that $M_t^{(j)}$, $j=1,2,3$ are indeed true $\mathbb{P}$-martingales.

Therefore, define the martingale M_t^ϵ and the non-martingale part R_t^ϵ respectively by

$$\begin{aligned}
M_t^\epsilon &:= \int_0^t \mathrm{d}M_s^{(1)} + \mathrm{d}M_s^{(2)} + \epsilon^{1-H}\rho\widetilde{\lambda}\,\mathrm{d}M_s^{(3)}, \\
R_T^\epsilon - R_t^\epsilon &:= R_{t,T}^{(1)} + R_{t,T}^{(2)} + R_{t,T}^{(3)} + R_{t,T}^{(4)},
\end{aligned}$$

and observe that $Q_T^{\pi^{(0)},\epsilon}(x) = v^{(0)}(T,x) = U(x)$ (since $\phi_T^\epsilon = v^{(1)}(T,x) = 0$ by definition), and then we obtain the desired result

$$\begin{aligned}
V_t^{\pi^{(0)},\epsilon} &= \mathbb{E}\left[Q_T^{\pi^{(0)},\epsilon}(X_T^{\pi^{(0)}})\middle|\mathcal{F}_t\right] = Q_t^{\pi^{(0)},\epsilon}(X_t^{\pi^{(0)}}) + \mathbb{E}[M_T^\epsilon - M_t^\epsilon|\mathcal{F}_t] + \mathbb{E}[R_T^\epsilon - R_t^\epsilon|\mathcal{F}_t] \\
&= Q_t^{\pi^{(0)},\epsilon}(X_t^{\pi^{(0)}}) + \mathbb{E}[R_{t,T}^{(1)} + R_{t,T}^{(2)} + R_{t,T}^{(3)} + R_{t,T}^{(4)}|\mathcal{F}_t] = Q_t^{\pi^{(0)},\epsilon}(X_t^{\pi^{(0)}}) + o(\epsilon^{1-H}).
\end{aligned}$$

□

When the utility $U(\cdot)$ is of power type, the functions $v^{(0)}$, $D_1 v^{(0)}$ and $v^{(1)}$ in (4.17) can be computed explicitly:

$$\begin{aligned}
v^{(0)}(t,x) &= \frac{x^{1-\gamma}}{1-\gamma}e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)}, \quad D_1 v^{(0)}(t,x) = \frac{x^{1-\gamma}}{\gamma}e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)}, \\
v^{(1)}(t,x) &= \frac{1-\gamma}{\gamma^2}x^{1-\gamma}e^{\frac{1-\gamma}{2\gamma}\overline{\lambda}^2(T-t)}\frac{\langle\lambda\lambda'\rangle}{a\Gamma(H+\frac{3}{2})}(T-t)^{H+\frac{1}{2}},
\end{aligned}$$

which leads to $Q_t^{\pi^{(0)},\epsilon} = Q_t^\epsilon$ and completes the proof of Corollary 3.6.

4.2 Asymptotic optimality of $\pi^{(0)}$

Now we study the optimality of $\pi^{(0)}$ within the smaller class of admissible strategies $\widetilde{\mathcal{A}}_t^\epsilon$, which are of the forms

$$\widetilde{\mathcal{A}}_t^\epsilon[\widetilde{\pi}^0, \widetilde{\pi}^1, \alpha] := \left\{\pi = \widetilde{\pi}^0 + \epsilon^\alpha \widetilde{\pi}^1 : \pi \in \mathcal{A}_t^\epsilon, \alpha > 0, 0 < \epsilon \leq 1\right\}.$$

Note that $\widetilde{\pi}^0$ and $\widetilde{\pi}^1$ are not required to be feedback controls (even $\pi^{(0)}$ is chosen to be in this form), but only adapted random processes , namely, $\widetilde{\pi}_t^0 \in \mathcal{F}_t$ and $\widetilde{\pi}_t^1 \in \mathcal{F}_t$. Furthermore, we require $\widetilde{\pi}^0$ and $\widetilde{\pi}^1$ to satisfy Assumption 4.4 and B.1. The parameter α is restricted to be positive since $\widetilde{\pi}^0 + \delta^0\widetilde{\pi}^1 = \widetilde{\pi}^0 + \widetilde{\pi}^1 + \delta^\alpha \cdot 0$. To show the optimality of $\pi^{(0)}$, we compare the value processes $V_t^{\pi^{(0)}}$ to $V_t^{\pi,\epsilon}$. The later one is defined by

$$V_t^{\pi,\epsilon} := \mathbb{E}\left[U\left(X_T^\pi\right)\middle|\mathcal{F}_t\right], \tag{4.30}$$

where π denotes an admissible strategy $\pi \in \widetilde{\mathcal{A}}_t^\epsilon[\widetilde{\pi}^0, \widetilde{\pi}^1, \alpha]$, and X_t^π is the corresponding wealth process:

$$\mathrm{d}X_t^\pi = \mu(Y_t^{\epsilon,H})\pi_t \,\mathrm{d}t + \sigma(Y_t^{\epsilon,H})\pi_t \,\mathrm{d}W_t. \tag{4.31}$$

To this end, we first find the approximation of $V_t^{\pi,\epsilon}$ using the epsilon-martingale decomposition technique as demonstrated in Proposition 4.3, and then asymptotically compare it with (4.17).

Assumption 4.4. *For a fixed choice of ($\widetilde{\pi}^0$, $\widetilde{\pi}^1$, $\alpha > 0$), we require:*

(i) The whole family (in ϵ) of strategies $\{\widetilde{\pi}^0 + \epsilon^\alpha\widetilde{\pi}^1\}$ is contained in $\mathcal{A}_t^\epsilon$;

(ii) The process $v^{(0)}(t, X_t^\pi)$ is in L^4 uniformly in ϵ and $t \in [0, T]$, i.e.,

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left(v^{(0)}(t, X_t^\pi)\right)^4\right] \leq C_2 \tag{4.32}$$

where C_2 is independent of ϵ, and X_t^π follows (4.31) with $\pi = \widetilde{\pi}^0 + \epsilon^\alpha\widetilde{\pi}^1$.

Theorem 4.5. *Under Assumptions 2.1*(i)-(iii)*, 3.2, 4.1, 4.2, 4.4 and B.1, for any family of trading strategies $\widetilde{\mathcal{A}}_t^\epsilon[\widetilde{\pi}^0, \widetilde{\pi}^1, \alpha]$, the following limit exists in L^1 and satisfies*

$$\ell := \lim_{\epsilon\to 0} \frac{V_t^{\pi,\epsilon} - V_t^{\pi^{(0)},\epsilon}}{\epsilon^{1-H}} \leq 0, \ \text{in } L^1, \tag{4.33}$$

where $V_t^{\pi^{(0)},\epsilon}$ and $V_t^{\pi,\epsilon}$ are defined in (4.9) and (4.30) respectively.

That is, the strategy $\pi^{(0)}$ given by (4.8) which generates $V_t^{\pi^{(0)},\epsilon}$ asymptotically outperforms any family $\widetilde{\mathcal{A}}_t^\epsilon[\widetilde{\pi}^0, \widetilde{\pi}^1, \alpha]$ producing $V_t^{\pi,\epsilon}$ at order ϵ^{1-H}. Moreover, the inequality can be written according to the following four cases:

(i) $\widetilde{\pi}^0 = \pi^{(0)}$, $\alpha > (1-H)/2$: $\ell = 0$ and $V_t^{\pi,\epsilon} = V_t^{\pi^{(0)},\epsilon} + o(\epsilon^{1-H})$;

(ii) $\widetilde{\pi}^0 = \pi^{(0)}$, $\alpha = (1-H)/2$: $-\infty < \ell < 0$ and $V_t^{\pi,\epsilon} = V_t^{\pi^{(0)},\epsilon} + \mathcal{O}(\epsilon^{1-H})$ with $\mathcal{O}(\epsilon^{1-H}) < 0$;

(iii) $\widetilde{\pi}^0 = \pi^{(0)}$, $\alpha < (1-H)/2$: $\ell = -\infty$ and $V_t^{\pi,\epsilon} = V_t^{\pi^{(0)},\epsilon} + \mathcal{O}(\epsilon^{2\alpha})$ with $\mathcal{O}(\epsilon^{2\alpha}) < 0$;

(iv) $\widetilde{\pi}^0 \neq \pi^{(0)}$: $\lim_{\epsilon\to 0} V_t^{\pi,\epsilon} < \lim_{\epsilon\to 0} V_t^{\pi^{(0)},\epsilon}$,

where all relations between $V_t^{\pi,\epsilon}$ and $V_t^{\pi^{(0)},\epsilon}$ hold under L^1 sense.

Remark 4.6. *To better understand the limit* (4.33)*, and to show that different values of α lead to different levels of accuracy in the expansion of $V_t^{\pi,\epsilon}$, the result in Theorem 4.5 has been decomposed into the four possible cases. In the cases where we got $\ell = 0$, the result means that the strategy $\pi^{(0)}$ is as good as the family of strategies $\{\widetilde{\pi}^0 + \epsilon^\alpha\widetilde{\pi}^1\}$ at order ϵ^{1-H}. In other cases where a strict inequality is obtained, $\pi^{(0)}$ outperforms other strategies, and adding the "correction" $\epsilon^\alpha\widetilde{\pi}^1$ (even if $\widetilde{\pi}^0 = \pi^{(0)}$) will not help in increasing the expected utility of terminal wealth. On the contrary, it leads to a negative effect on the value process $V_t^{\pi,\epsilon}$ at order $\epsilon^{2\alpha}$* (resp. *order one), even when one follows $\pi^{(0)}$ in the leading order (*resp. $\widetilde{\pi}^0$ deviate from $\pi^{(0)}$*). Therefore, overall we say that $\pi^{(0)}$ is "asymptotically" optimal in the class $\widetilde{\mathcal{A}}^\epsilon$ is at least at order ϵ^{1-H}, no matter what α is.*

Proof. We start with the case $\widetilde{\pi}^0 = \pi^{(0)}$. Following the same procedure as in Proposition 4.3, one deduces

$$\begin{aligned}
\mathrm{d}Q_t^{\pi^{(0)},\epsilon}(X_t^\pi) &= \mathrm{d}(v^{(0)}(t, X_t^{\pi^{(0)}}) + D_1 v^{(0)}(t, X_t^{\pi^{(0)}})\phi_t^\epsilon + \epsilon^{1-H}\rho\widetilde{\lambda}v^{(1)}(t, X_t^\pi)) \\
&= \mathrm{d}\widetilde{R}_t^\epsilon + \mathrm{d}\widetilde{M}_t^\epsilon + \epsilon^{2\alpha}\,\mathrm{d}N_t^\epsilon
\end{aligned}$$

where $\mathrm{d}\widetilde{M}_t^\epsilon$, $\mathrm{d}\widetilde{R}_t^\epsilon$ and $\mathrm{d}N_t^\epsilon$ are given by

$$\begin{aligned}
\mathrm{d}\widetilde{M}_t^\epsilon &= \sigma(Y_t^{\epsilon,H})\pi_t v_x^{(0)}(t, X_t^{\pi^{(0)}})\,\mathrm{d}W_t + D_1 v^{(0)}(t, X_t^\pi)\,\mathrm{d}\psi_t^\epsilon + \phi_t^\epsilon\sigma(Y_t^{\epsilon,H})\pi_t\partial_x D_1 v^{(0)}(t, X_t^\pi)\,\mathrm{d}W_t \\
&\quad + \sigma(Y_t^{\epsilon,H})\pi_t v_x^{(1)}(t, X_t^\pi)\,\mathrm{d}W_t, \\
\mathrm{d}N_t^\epsilon &= \frac{1}{2}\sigma^2(Y_t^{\epsilon,H})\left(\widetilde{\pi}_t^1\right)^2 v_{xx}^{(0)}(t, X_t^\pi)\,\mathrm{d}t, \\
\mathrm{d}\widetilde{R}_t^\epsilon &= \frac{1}{2}\phi_t^\epsilon(\lambda^2(Y_t^{\epsilon,H}) - \overline{\lambda}^2)(D_2 + 2D_1)D_1^2 v^{(0)}\,\mathrm{d}t + \epsilon^\alpha\phi_t^\epsilon\mu(Y_t^{\epsilon,H})\widetilde{\pi}_t^1(\partial_x + R(t, X_t^\pi; \overline{\lambda})\partial_{xx})D_1 v^{(0)}\,\mathrm{d}t \\
&\quad + \frac{1}{2}\epsilon^{2\alpha}\phi_t^\epsilon\sigma^2(Y_t^{\epsilon,H})(\widetilde{\pi}_t^1)^2\partial_{xx}D_1 v^{(0)}\,\mathrm{d}t + \rho\lambda(Y_t^{\epsilon,H})D_1^2 v^{(0)}\widetilde{\theta}_t^\epsilon\,\mathrm{d}t + \epsilon^\alpha\sigma(Y_t^{\epsilon,H})\widetilde{\pi}_t^1\partial_x D_1 v^{(0)}\vartheta_t^\epsilon\,\mathrm{d}t \\
&\quad + \epsilon^{1-H}\rho(\lambda(Y_t^{\epsilon,H}) - \widetilde{\lambda})D_1^2 v^{(0)}\theta_t\,\mathrm{d}t + \epsilon^{1-H+\alpha}\rho\widetilde{\lambda}\mu(Y_t^{\epsilon,H})\widetilde{\pi}_t^1(v_x^{(1)} + R(t, X_t^\pi; \overline{\lambda})v_{xx}^{(1)})\,\mathrm{d}t \\
&\quad + \frac{1}{2}\epsilon^{1-H}\rho\widetilde{\lambda}(\lambda^2(Y_t^{\epsilon,H}) - \overline{\lambda}^2)(D_2 + 2D_1)v^{(1)}\,\mathrm{d}t + \frac{1}{2}\epsilon^{1-H+2\alpha}\rho\widetilde{\lambda}\sigma^2(Y_t^{\epsilon,H})(\widetilde{\pi}_t^1)^2 v_{xx}^{(1)}\,\mathrm{d}t,
\end{aligned}$$

and in the expression of $\widetilde{R}_t^\epsilon$, the arguments of $v^{(0)}(t, X_t^\pi)$ and $v^{(1)}(t, X_t^\pi)$ are omitted to condense the notation.

The process N_t^ϵ is strictly decreasing following from the strict concavity of $v^{(0)}(t, x) = M(t, x; \overline{\lambda})$. The true martingality of $\widetilde{M}_t^\epsilon$ and the fact that $\widetilde{R}_t^\epsilon \sim o(\epsilon^{1-H})$ are guaranteed by Assumption B.1. Thus we deduce

$$\begin{aligned}
V_t^{\pi,\epsilon} &= \mathbb{E}[Q_T^{\pi^{(0)},\epsilon}(X_T^\pi)|\mathcal{F}_t] = Q_t^{\pi^{(0)},\epsilon}(X_t^\pi) + \mathbb{E}[\widetilde{R}_T^\epsilon - \widetilde{R}_t^\epsilon|\mathcal{F}_t] + \epsilon^{2\alpha}\mathbb{E}[N_T^\epsilon - N_t^\epsilon|\mathcal{F}_t] \\
&= Q_t^{\pi^{(0)},\epsilon}(X_t^\pi) + o(\epsilon^{1-H}) + \mathcal{O}(\epsilon^{2\alpha}) = V_t^{\pi^{(0)},\epsilon} + o(\epsilon^{1-H}) + \mathcal{O}(\epsilon^{2\alpha}),
\end{aligned} \tag{4.34}$$

with $\mathcal{O}(\epsilon^{2\alpha}) < 0$. This leads to the first three cases in the theorem.

In the case that $\widetilde{\pi}^0 \neq \pi^{(0)}$, similar derivation brings

$$\mathrm{d}v^{(0)}(t, X_t^\pi) = \mathrm{d}\widehat{R}_t^\epsilon + \mathrm{d}\widehat{M}_t^\epsilon + \mathrm{d}\widehat{N}_t^\epsilon,$$

where $\widehat{M}_t^\epsilon$, $\widehat{R}_t^\epsilon$ and $\widehat{N}_t^\epsilon$ are defined by

$$\begin{aligned}
\mathrm{d}\widehat{M}_t^\epsilon &= \sigma(Y_t^{\epsilon,H})\pi_t v_x^{(0)}(t, X_t^\pi)\,\mathrm{d}W_t, \\
\mathrm{d}\widehat{N}_t^\epsilon &= \frac{1}{2}\sigma^2(Y_t^{\epsilon,H})\left(\widetilde{\pi}_t^0 - \pi^{(0)}(t, X_t^\pi, Y_t^{\epsilon,H})\right)^2 v_{xx}^{(0)}(t, X_t^\pi)\,\mathrm{d}t, \\
\mathrm{d}\widehat{R}_t^\epsilon &= \frac{1}{2}(\lambda^2(Y_t^{\epsilon,H}) - \overline{\lambda}^2)D_1 v^{(0)}\,\mathrm{d}t + \epsilon^\alpha\left[\mu\widetilde{\pi}_t^1 v_x^{(0)} + \sigma^2\widetilde{\pi}_t^0\widetilde{\pi}_t^1 v_{xx}^{(0)} + \frac{1}{2}\epsilon^\alpha\sigma^2\left(\widetilde{\pi}_t^1\right)^2 v_{xx}^{(0)}\right]\mathrm{d}t,
\end{aligned}$$

and the arguments of $\mu(Y_t^{\epsilon,H})$, $\sigma(Y_t^{\epsilon,H})$ and $v^{(0)}(t, X_t^\pi)$ are omitted in the equation of $\widehat{R}_t^\epsilon$. As in the previous case, $\widehat{N}_t^\epsilon$ is strictly decreasing due to the concavity of $v^{(0)}$, and Assumption B.1 ensures that $\widehat{M}_t^\epsilon$ is a true martingale and that $\widehat{R}_t^\epsilon \sim \mathcal{O}(\epsilon^{(1-H)\wedge\alpha})$. This gives the last case in the theorem, since

$$\begin{aligned}
V_t^\epsilon &= \mathbb{E}[v^{(0)}(T, X_T^\pi)|\mathcal{F}_t] = v^{(0)}(t, X_t^\pi) + \mathbb{E}[\widehat{R}_T^\epsilon - \widehat{R}_t^\epsilon|\mathcal{F}_t] + \mathbb{E}[\widehat{N}_T^\epsilon - \widehat{N}_t^\epsilon|\mathcal{F}_t] \\
&< v^{(0)}(t, X_t^\pi) + \mathcal{O}(\epsilon^{(1-H)\wedge\alpha}),
\end{aligned} \tag{4.35}$$

and $\lim_{\epsilon\to 0} V_t^{\pi^{(0)},\delta} = v^{(0)}(t, X_t^{\pi^{(0)}})$.

□

5 Conclusion

In this paper, we study the nonlinear problem of portfolio optimization in the context of a one-factor fractional stochastic environment. This factor is modeled as a long-range memory fractional Ornstein–Uhlenbeck process with Hurst index in $(\frac{1}{2}, 1)$ and varying on a fast time-scale characterized by a small parameter ϵ. In this context, and with power utilities, the value process can be expressed explicitly thanks to a martingale distortion transformation allowing us to perform an expansion as $\epsilon \to 0$ and obtain explicit formulas for the zeroth order term and the first order corrections of order ϵ^{1-H}. Likewise, we can expand the optimal strategy and show that its zeroth order approximation is optimal up to the first order in the value process. We also extend this analysis in the case of general utility functions and we show that the asymptotic optimality of the zeroth order strategy in a specific sub-class of admissible strategies.

A Technical Lemmas

In this section, we present several lemmas used in Section 3 and Section 4. Note that the constants K, K' in all lemmas do not depend on ϵ and may vary from line to line, and we denote the function $G(y)$ as

$$G(y) = \frac{1}{2}(\lambda^2(y) - \overline{\lambda}^2),$$

and $\|X\|_p := (\mathbb{E}X^p)^{1/p}$ as the L^p-norm of X.

Lemma A.1.

(i) The martingale ψ_t^ϵ defined in (3.20):

$$\psi_t^\epsilon = \mathbb{E}\left[\int_0^T G(Y_s^{\epsilon,H})\,\mathrm{d}s \Big| \mathcal{G}_t\right],$$

satisfies

$$\mathrm{d}\psi_t^\epsilon = \vartheta_t^\epsilon\,\mathrm{d}W_t^Y, \quad \vartheta_t^\epsilon := \int_t^T \mathbb{E}\left[G'(Y_s^{\epsilon,H})|\mathcal{G}_t\right]\mathcal{K}^\epsilon(s-t)\,\mathrm{d}s.$$

Moreover, the process ϑ_t^ϵ can be written as, for all $t \in [0,T]$

$$\vartheta_t^\epsilon = \epsilon^{1-H}\theta_t + \widetilde{\theta}_t^\epsilon,$$

where θ_t is a deterministic function

$$\theta_t = \frac{\langle G'\rangle}{a\Gamma(H+\frac{1}{2})}(T-t)^{H-\frac{1}{2}},$$

and $\widetilde{\theta}_t^\epsilon$ is random and of high order than ϵ^{1-H} in L^2 sense uniformly in $t \in [0,T]$

$$\limsup_{\epsilon\to 0} \epsilon^{H-1} \sup_{t\in[0,T]} \left\|\widetilde{\theta}_t^\epsilon\right\|_2 = 0.$$

(ii) The random component ϕ_t^ϵ defined in (3.14) has the form

$$\phi_t^\epsilon = \mathbb{E}\left[\int_t^T G(Y_s^{\epsilon,H})\,\mathrm{d}s \Big| \mathcal{G}_t\right].$$

It is a random variable with mean zero and variance of order ϵ^{2-2H}:

$$Var(\phi_t^\epsilon) \le K\epsilon^{2-2H},$$

uniformly in $t \in [0,T]$. Moreover, as $\epsilon \to 0$, the random variable $\epsilon^{H-1}\phi_t^\epsilon$ converges in distribution to $\mathcal{N}(0, \sigma_\phi^2(T-t)^{2H})$, where σ_ϕ^2 is defined by

$$\sigma_\phi^2 = \sigma_{ou}^2 \langle\lambda\lambda'\rangle^2 \left(\frac{1}{\Gamma(2H+1)\sin(\pi H)} - \frac{1}{2H\Gamma^2(H+\frac{1}{2})}\right).$$

(iii) Recall the random process η_t^ϵ *defined in* (3.6)

$$\eta_t^\epsilon = \int_0^t \left(\lambda(Y_s^{\epsilon,H}) - \widetilde{\lambda}\right) \mathrm{d}s,$$

It is of order ϵ^{1-H} *in* L^2 *sense uniformly in* $t \in [0,T]$*:*

$$\sup_{t\in[0,T]} \|\eta_t^\epsilon\|_2 \leq K\epsilon^{1-H}.$$

(iv) Recall the random process κ_t^ϵ *defined in* (3.7)

$$\kappa_t^\epsilon = \int_0^t \left(\lambda(Y_s^{\epsilon,H})\lambda'(Y_s^{\epsilon,H}) - \langle\lambda\lambda'\rangle\right) \mathrm{d}s.$$

It is of order ϵ^{1-H} *in* L^2 *sense uniformly in* $t \in [0,T]$*:*

$$\sup_{t\in[0,T]} \|\kappa_t^\epsilon\|_2 \leq K\epsilon^{1-H}.$$

Lemma A.2. *Under Assumption 3.2,*

(i) The random process I_t^ϵ *defined in* (3.5)

$$I_t^\epsilon = \int_0^t \left(\lambda^2(Y_s^{\epsilon,H}) - \overline{\lambda}^2\right) \mathrm{d}s,$$

satisfies

$$\sup_{t\in[0,T]} \mathbb{E}[(I_t^\epsilon)^4] \leq Ke^{4-4H}.$$

(ii) Define the random process φ_t^ϵ *by:*

$$\varphi_t^\epsilon = \frac{1}{2}\int_0^t \left(\lambda^2(Y_s^{\epsilon,H}) - \overline{\lambda}^2\right) \phi_s^\epsilon \,\mathrm{d}s, \tag{A.1}$$

it is of order $o(\epsilon^{1-H})$ *in* L^2 *sense uniformly in* $t \in [0,T]$

$$\limsup_{\epsilon\to 0} \epsilon^{H-1} \sup_{t\in[0,T]} \|\varphi_t^\epsilon\|_2 = 0.$$

(iii) The L^4 *norm of* ϕ_t^ϵ *is of order* ϵ^{1-H}*, uniformly in* $t \in [0,T]$*:*

$$\sup_{t\in[0,T]} \|\phi_t^\epsilon\|_4 \leq K\epsilon^{1-H}.$$

Proof of Lemma A.1 and A.2. All results are slightly different versions or straightforward generalizations of lemmas in [Garnier and Sølna, 2016, Appendix A,B], thus we omit the details here. ☐

Lemma A.3.

(i) Denote by $\widetilde{Y}_t^{\epsilon,H}$ *the* $\widetilde{\mathbb{P}}$*-stationary fractional Ornstein–Ulenbeck process, whose moving average representation is of the form*

$$\widetilde{Y}_t^{\epsilon,H} := \int_{-\infty}^t \mathcal{K}^\epsilon(t-s)\,\mathrm{d}\widetilde{W}_s^Y.$$

Then, $\sup_{t\in[0,T]} \left|\widetilde{Y}_t^{\epsilon,H} - Y_t^{\epsilon,H}\right| \leq K\epsilon^{1-H}$.

(ii) Recall the stochastic process ϑ_t^ϵ defined in (3.21)*, and $\widetilde{\vartheta}_t^\epsilon$ defined in* (3.22)*:*

$$\vartheta_t^\epsilon := \int_t^T \mathbb{E}[G'(\widetilde{Y}_s^{\epsilon,H})|\mathcal{G}_t]\mathcal{K}^\epsilon(s-t)\,\mathrm{d}s, \quad \widetilde{\vartheta}_t^\epsilon := \int_t^T \widetilde{\mathbb{E}}[G'(\widetilde{Y}_s^{\epsilon,H})|\mathcal{G}_t]\mathcal{K}^\epsilon(s-t)\,\mathrm{d}s,$$

then $\sup_{t\in[0,T]}\left|\widetilde{\vartheta}_t^\epsilon - \vartheta_t^\epsilon\right| \le K\epsilon^{2-2H}$.

Proof. Part (i) follows straightforwardly by the boundedness of $\lambda(\cdot)$ and the fact that $\mathcal{K}(t) - \frac{t^{H-\frac{3}{2}}}{a\Gamma(H-\frac{1}{2})} \in L^1$.

For part (ii), we first compute the conditional distribution of $Y_s^{\epsilon,H}$ and $\widetilde{Y}_s^{\epsilon,H}$given $\mathcal{G}_t$, for $t \le s$:

$$Y_s^{\epsilon,H}|\mathcal{G}_t \overset{\mathbb{P}}{\sim} \mathcal{N}\left(\int_{-\infty}^t \mathcal{K}^\epsilon(s-u)\,\mathrm{d}W_u^Y, (\sigma_{0,s-t}^\epsilon)^2\right), \quad \widetilde{Y}_s^{\epsilon,H}|\mathcal{G}_t \overset{\widetilde{\mathbb{P}}}{\sim} \mathcal{N}\left(\int_{-\infty}^t \mathcal{K}^\epsilon(s-u)\,\mathrm{d}\widetilde{W}_u^Y, (\sigma_{0,s-t}^\epsilon)^2\right),$$

with $(\sigma_{l,r}^\epsilon)^2 = \int_l^r \mathcal{K}^\epsilon(u)^2\,\mathrm{d}u$. Therefore the difference is computed as

$$\begin{aligned}
&\widetilde{\vartheta}_t^\epsilon - \vartheta_t^\epsilon \\
&= \int_t^T \left\{\widetilde{\mathbb{E}}[G'(\widetilde{Y}_s^{\epsilon,H})|\mathcal{G}_t] - \mathbb{E}[G'(Y_s^{\epsilon,H})|\mathcal{G}_t]\right\}\mathcal{K}^\epsilon(s-t)\,\mathrm{d}s \\
&= \int_t^T \int_{\mathbb{R}} \left\{G'\left(\int_{-\infty}^t \mathcal{K}^\epsilon(s-u)\,\mathrm{d}\widetilde{W}_u^Y + \sigma_{0,s-t}^\epsilon z\right) - G'\left(\int_{-\infty}^t \mathcal{K}^\epsilon(s-u)\,\mathrm{d}W_u^Y + \sigma_{0,s-t}^\epsilon z\right)\right\} p(z)\,\mathrm{d}z\mathcal{K}^\epsilon(s-t)\,\mathrm{d}s \\
&= -\int_t^T \int_{\mathbb{R}} G''(\chi)\int_0^t \mathcal{K}^\epsilon(s-u)\rho\left(\frac{1-\gamma}{\gamma}\right)\lambda(Y_u^{\epsilon,H})\,\mathrm{d}u p(z)\,\mathrm{d}z\mathcal{K}^\epsilon(s-t)\,\mathrm{d}s,
\end{aligned}$$

where χ is a $\mathcal{G}_t$-adapted random variable determined by the remainder of Taylor expansion. Now, taking absolute value on both sides, together with fact that G'' and λ are bounded, and $\int_l^r \mathcal{K}^\epsilon(u)\,\mathrm{d}u \sim \mathcal{O}(\epsilon^{1-H})$ uniformly in $l, r \in [0,T]$ brings the desired result. □

Lemma A.4. *The quantities $R_t^{(j)}$ defined in* (3.17)-(3.19)

$$\begin{aligned}
R_t^{(1)} &:= \epsilon^{1-H}\int_0^t (T-u)^{H-\frac{1}{2}}\left(\lambda(Y_u^{\epsilon,H}) - \widetilde{\lambda}\right)\mathrm{d}u, \\
R_t^{(2)} &:= \widetilde{\mathbb{E}}\left[\int_0^T \left(G'(\widetilde{Y}_s^{\epsilon,H}) - \langle\lambda\lambda'\rangle\right)\int_0^s \rho\left(\frac{1-\gamma}{\gamma}\right)\lambda(Y_u^{\epsilon,H})\mathcal{K}^\epsilon(s-u)\,\mathrm{d}u\,\mathrm{d}s\Big|\mathcal{G}_t\right], \\
R_t^{(3)} &:= \widetilde{\mathbb{E}}\left[\int_0^T \int_0^s \left(\lambda(Y_u^{\epsilon,H}) - \widetilde{\lambda}\right)\mathcal{K}^\epsilon(s-u)\,\mathrm{d}u\,\mathrm{d}s\Big|\mathcal{G}_t\right],
\end{aligned}$$

satisfy, for all $t \in [0,T]$,

$$\lim_{\epsilon\to 0}\epsilon^{H-1}\mathbb{E}\left|R_t^{(j)}\right| = 0, \quad \forall j = 1,2,3, \tag{A.2}$$

Proof. Proof of (A.2) for $j = 3$. It suffices to prove

$$\widetilde{R}_s^{(3)} := \int_0^s \left(\lambda(Y_u^{\epsilon,H}) - \widetilde{\lambda}\right)\mathcal{K}^\epsilon(s-u)\,\mathrm{d}u \sim o(\epsilon^{1-H}) \text{ in } L^2 \text{ uniformly in } s \in [0,T], \tag{A.3}$$

and then use dominated convergence theorem. Noticing that $\mathcal{K}(t) - \frac{t^{H-3/2}}{a\Gamma(H-\frac{1}{2})} \in L^1$, it is equivalent to show that

$$R_s^{(3')} := \int_0^s \left(\lambda(Y_u^{\epsilon,H}) - \widetilde{\lambda}\right)(s-u)^{H-\frac{3}{2}}\,\mathrm{d}u \sim o(1) \text{ in } L^2 \text{ uniformly in } s \in [0,T]. \tag{A.4}$$

To this end, we pick a sequence $c_n \to 0$, denote $s_k = (s-c_n)k/N$, $Z_u^{(3)} = (s-u)^{H-\frac{3}{2}}$, and recall η_u^ϵ defined in (3.6), thus

$$
\begin{aligned}
R_s^{(3')} &= \int_0^s Z_u^{(3)} \frac{\mathrm{d}\eta_u^\epsilon}{\mathrm{d}u}\,\mathrm{d}u = \int_0^{s-c_n} Z_u^{(3)} \frac{\mathrm{d}\eta_u^\epsilon}{\mathrm{d}u}\,\mathrm{d}u + \int_{s-c_n}^s Z_u^{(3)} \frac{\mathrm{d}\eta_u^\epsilon}{\mathrm{d}u}\,\mathrm{d}u \\
&= \sum_{k=0}^{N-1} Z_{s_k}^{(3)} \left(\eta_{s_{k+1}}^\epsilon - \eta_{s_k}^\epsilon\right) + \sum_{k=0}^{N-1} \int_{s_k}^{s_{k+1}} \left(Z_u^{(3)} - Z_{s_k}^{(3)}\right) \frac{\mathrm{d}\eta_u^\epsilon}{\mathrm{d}u}\,\mathrm{d}u + \int_{s-c_n}^s Z_u^{(3)} \frac{\mathrm{d}\eta_u^\epsilon}{\mathrm{d}u}\,\mathrm{d}u \\
&:= R_s^{(3',a)} + R_s^{(3',b)} + R_s^{(3',c)}.
\end{aligned}
$$

The proofs for $R_s^{(3',a)}$ and $R_s^{(3',b)}$ are similar to the ones in [Garnier and Sølna, 2016, Proposition 4.1 Step1]. By Minkowski's inequality,

$$
\begin{aligned}
\left\| R_s^{(3',a)} \right\|_2 &\le 2 \sum_{k=0}^{N} \left\| Z^{(3)} \right\|_\infty \left\| \eta_{s_k}^\epsilon \right\|_2 \le 2(N+1) c_n^{H-\frac{3}{2}} \sup_{u \in [0, s-c_n]} \|\eta_u^\epsilon\|_2 \\
&\le 2(N+1) c_n^{H-\frac{3}{2}} K \epsilon^{1-H}.
\end{aligned}
$$

The last inequality follows from Lemma A.1(iii), which implies, for any fixed N and c_n, $\left\| R_s^{(3',a)} \right\|_2$ goes to 0 uniformly in s as $\epsilon \to 0$. For the second term

$$
\begin{aligned}
\left\| R_s^{(3',b)} \right\|_2 &\le \|\lambda\|_\infty \sum_{k=0}^{N-1} \int_{s_k}^{s_{k+1}} \left((s-u)^{H-\frac{3}{2}} - (s-s_k)^{H-\frac{3}{2}} \right) \mathrm{d}u \le K \sum_{k=0}^{N-1} c_n^{H-\frac{5}{2}} \frac{1}{N^2} \\
&\le K c_n^{H-\frac{5}{2}} \frac{1}{N},
\end{aligned}
$$

which goes to 0 uniformly in s for any fixed c_n, as $N \to 0$. The last term $R_s^{(3',c)}$ also tends to zero as $c_n \to 0$ uniformly in s by Dini's theorem. Therefore, we get the desired result (A.2) for $j = 3$.

The proof of (A.2) for $j = 1$ follows the same routine as in (A.4) with $Z_u^{(3)}$ replaced by $Z_u^{(1)} = (T-u)^{H-\frac{1}{2}}$.

The proof of (A.2) for $j = 2$ is based on the one of (A.3). To be specific, one has

$$
\begin{aligned}
R_t^{(2)} &= \rho \left(\frac{1-\gamma}{\gamma} \right) \widetilde{\lambda} \widetilde{\mathbb{E}} \left[\int_0^T \left(G'(\widetilde{Y}_s^{\epsilon,H}) - \langle \lambda \lambda' \rangle \right) \int_0^s \mathcal{K}^\epsilon(s-u)\,\mathrm{d}u\,\mathrm{d}s \Big| \mathcal{G}_t \right] \\
&\quad + \rho \left(\frac{1-\gamma}{\gamma} \right) \widetilde{\mathbb{E}} \left[\int_0^T \left(G'(\widetilde{Y}_s^{\epsilon,H}) - \langle \lambda \lambda' \rangle \right) \widetilde{R}_s^{(3)}\,\mathrm{d}s \Big| \mathcal{G}_t \right] \\
&\le K \epsilon^{1-H} \widetilde{\mathbb{E}} \left[\int_0^T \left(G'(\widetilde{Y}_s^{\epsilon,H}) - \langle \lambda \lambda' \rangle \right) s^{H-\frac{1}{2}}\,\mathrm{d}s \Big| \mathcal{G}_t \right] + K' \|G'\|_\infty \widetilde{\mathbb{E}} \left[\int_0^T |\widetilde{R}_s^{(3)}|\,\mathrm{d}s \Big| \mathcal{G}_t \right] \\
&:= K \epsilon^{1-H} \widetilde{\mathbb{E}}[R_T^{(2,a)} | \mathcal{G}_t] + K' R_t^{(2,b)},
\end{aligned}
$$

with

$$
R_T^{(2,a)} = \int_0^T \left(G'(\widetilde{Y}_s^{\epsilon,H}) - \langle \lambda \lambda' \rangle \right) s^{H-\frac{1}{2}}\,\mathrm{d}s, \quad R_t^{(2,b)} = \widetilde{\mathbb{E}} \left[\int_0^T |\widetilde{R}_s^{(3)}|\,\mathrm{d}s \Big| \mathcal{G}_t \right].
$$

Now it reduces to show $\widetilde{\mathbb{E}}[R_T^{(2,a)} | \mathcal{G}_t] \to 0$ and $R_t^{(2,b)} \sim o(\epsilon^{1-H})$ in L^1. Using

$$
\mathbb{E} \left| \widetilde{\mathbb{E}}[R_T^{(2,a)} | \mathcal{G}_t] \right| \le K \left\| R_T^{(2,a)} \right\|_2,
$$

the first one then follows the same line as the proof of (A.4). The second one also holds by

$$
\mathbb{E} \left| R_t^{(2,b)} \right| \le K \left[\mathbb{E} \int_0^T (\widetilde{R}_s^{(3)})^2\,\mathrm{d}s \right]^{1/2} \le K \sup_{s \in [0,T]} \left\| \widetilde{R}_s^{(3)} \right\|_2 \sim o(\epsilon^{1-H})
$$

and the previously proved result (A.3). □

Lemma A.5. *The processes $M_t^{(j)}$, $j = 1, 2, 3$ defined in* (4.20), (4.22) *and* (4.24) *are true $\mathbb{P}$-martingales.*

Proof. By the Burkholder–Davis–Gundy inequality, it suffices to show $\mathbb{E}\left[\left\langle M^{(j)}\right\rangle_T^{1/2}\right] < \infty$, for $j = 1, 2, 3$.

For the case $j = 1$, we compute

$$\mathrm{d}\left\langle M^{(1)}\right\rangle_t = \lambda^2(Y_t^{\epsilon,H})\left(D_1 v^{(0)}(t, X_t^{\pi^{(0)}})\right)^2 \mathrm{d}t \le K\left(v^{(0)}(t, X_t^{\pi^{(0)}})\right)^2 \mathrm{d}t,$$

using Assumption 2.1(i) and the concavity of $v^{(0)}$. Then, under Assumption 4.2

$$\mathbb{E}\left[\left\langle M^{(1)}\right\rangle_T^{1/2}\right] \le \left[\mathbb{E}\int_0^T K(v^{(0)}(t, X_t^{\pi^{(0)}}))^2\,\mathrm{d}t\right]^{1/2} \le K \sup_{t\in[0,T]}\left\|v^{(0)}(t, X_t^{\pi^{(0)}})\right\|_2 < \infty.$$

The martingality of $M_t^{(3)}$ is obtained by a similar derivation with additional estimates from [Fouque and Hu, 2017a, Proposition 3.5]:

$$\left|R^j(t, x; \overline{\lambda})\partial_x^{(j+1)} R(t, x; \overline{\lambda})\right| \le K, \quad 0 \le j \le 3, \quad \forall (t, x) \in [0, T) \times \mathbb{R}^+. \tag{A.5}$$

For the case $j = 2$, similar reasonings lead to

$$\mathrm{d}\left\langle M^{(2)}\right\rangle_t \le K[(\vartheta_t^\epsilon)^2 + (\phi_t^\epsilon)^2]\left(v^{(0)}(t, X_t^{\pi^{(0)}})\right)^2 \mathrm{d}t.$$

Given Assumption 4.2 for $v^{(0)}$ and Lemma A.2(iii) for ϕ_t^ϵ, it suffices to prove

$$\sup_{t\in[0,T]} \|\vartheta_t^\epsilon\|_4 < K\epsilon^{1-H}. \tag{A.6}$$

Recall ϑ_t^ϵ from (3.21) and use Minkowski inequality, one deduces

$$\begin{aligned}\mathbb{E}[(\vartheta_t^\epsilon)^4] &\le \left(\int_t^T \left[\mathbb{E}\left(\mathbb{E}[G'(Y_s^{\epsilon,H})|\mathcal{G}_t]\mathcal{K}^\epsilon(s-t)\right)^4\right]^{1/4} \mathrm{d}s\right)^4 = \left(\int_t^T \mathcal{K}^\epsilon(s-t)\left\|\mathbb{E}[G'(Y_s^{\epsilon,H})|\mathcal{G}_t\right\|_4 \mathrm{d}s\right)^4 \\ &\le \left(\int_t^T \mathcal{K}^\epsilon(s-t)\left\|\left(G'(Y_s^{\epsilon,H})\right)\right\|_4 \mathrm{d}s\right)^4 = \left\langle(\lambda\lambda')^4\right\rangle\left(\int_t^T \mathcal{K}^\epsilon(s-t)\,\mathrm{d}s\right)^4,\end{aligned}$$

and we conclude that $\mathbb{E}[(\vartheta_t^\epsilon)^4]$ is bounded by a constant of order ϵ^{4-4H}, since

$$\left(\int_0^T \mathcal{K}^\epsilon(s)\,\mathrm{d}s\right)^4 \le K\epsilon^{4-4H}$$

using $\mathcal{K}(s) - \frac{s^{H-3/2}}{a\Gamma(H-1/2)} \in L^1$. This completes the proof of (A.6), and we get the desired results for $M_t^{(j)}$, $j = 1, 2, 3$. ◻

Lemma A.6. *The random variable $R_{t,T}^{(j)}$, $j = 1, 2, 3, 4$ defined in* (4.25)-(4.28)

$$\begin{aligned}R_{t,T}^{(1)} &:= \int_t^T \phi_s^\epsilon\left[\frac{1}{2}(\lambda^2(Y_s^{\epsilon,H}) - \overline{\lambda}^2)(D_2 + 2D_1)D_1 v^{(0)}(s, X_s^{\pi^{(0)}})\right]\mathrm{d}s, \\ R_{t,T}^{(2)} &:= \int_t^T \epsilon^{1-H}\rho\left(\lambda(Y_s^{\epsilon,H}) - \widetilde{\lambda}\right)D_1^2 v^{(0)}(s, X_s^{\pi^{(0)}})\theta_s\,\mathrm{d}s, \\ R_{t,T}^{(3)} &:= \int_t^T \rho\lambda(Y_s^{\epsilon,H})D_1^2 v^{(0)}(s, X_s^{\pi^{(0)}})\widetilde{\theta}_s^\epsilon\,\mathrm{d}s, \\ R_{t,T}^{(4)} &:= \int_t^T \frac{1}{2}\epsilon^{1-H}\rho\widetilde{\lambda}(\lambda^2(Y_s^{\epsilon,H}) - \overline{\lambda}^2)(D_2 + 2D_1)v^{(1)}(s, X_s^{\pi^{(0)}})\,\mathrm{d}s,\end{aligned}$$

are of order $o(\epsilon^{1-H})$:

$$\lim_{\epsilon\to 0}\epsilon^{H-1}\,\mathbb{E}\left|R_{t,T}^{(j)}\right| = 0, \quad \forall j = 1, 2, 3, 4. \tag{A.7}$$

Proof. The proofs here are similar to the ones in Lemma A.4.

To prove (A.7) with $j = 1$, we denote $t_k = t + (T-t)k/N$, $Z_s^{(1)} = (D_2 + 2D_1)D_1 v^{(0)}(s, X_s^{\pi^{(0)}})$ and recall φ_t^ϵ defined in (A.1), thus $R_{t,T}^{(1)}$ can be written as

$$\begin{aligned} R_{t,T}^{(1)} &= \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} Z_s^{(1)} \frac{\mathrm{d}\varphi_s^\epsilon}{\mathrm{d}s} \,\mathrm{d}s = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} Z_{t_k}^{(1)} \frac{\mathrm{d}\varphi_s^\epsilon}{\mathrm{d}s} \,\mathrm{d}s + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (Z_s^{(1)} - Z_{t_k}^{(1)}) \frac{\mathrm{d}\varphi_s^\epsilon}{\mathrm{d}s} \,\mathrm{d}s \\ &= \sum_{k=0}^{N-1} Z_{t_k}^{(1)} (\varphi_{t_{k+1}}^\epsilon - \varphi_{t_k}^\epsilon) + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (Z_s^{(1)} - Z_{t_k}^{(1)}) \frac{\mathrm{d}\varphi_s^\epsilon}{\mathrm{d}s} \,\mathrm{d}s \\ &:= R_{t,T}^{(1,a)} + R_{t,T}^{(1,b)}. \end{aligned}$$

To proceed the analysis of $R_{t,T}^{(1,a)}$ and $R_{t,T}^{(1,b)}$, we first state two properties of $Z_s^{(1)}$: (a) it has a finite second moment uniformly in ϵ and $s \in [0,T]$

$$\mathbb{E}[(Z_s^{(1)})^2] \leq K\mathbb{E}[(v^{(0)}(s, X_s^{\pi^{(0)}}))^2] \leq K \sup_{s\in[0,T]} \mathbb{E}[(v^{(0)}(s, X_s^{\pi^{(0)}}))^2] < \infty \tag{A.8}$$

using the concavity of $v^{(0)}$, the estimates (A.5) and Assumption 4.2; and (b) its increments are bounded in L^2 by

$$\mathbb{E}[(Z_u^{(1)} - Z_v^{(1)})^2] \leq K\,|u-v|\,. \tag{A.9}$$

Part (b) is obtained by firstly using Itô formula

$$Z_u^{(1)} - Z_v^{(1)} = \int_v^u \mathcal{L}_{t,x}(\lambda(Y_s^{\epsilon,H}))Z_s^{(1)} \,\mathrm{d}s + \int_v^u \lambda(Y_s^{\epsilon,H}) D_1 Z_s^{(1)} \,\mathrm{d}W_s,$$

then, squaring both sides, together with the boundedness of λ and the estimates (A.5)

$$\mathbb{E}[(Z_u^{(1)} - Z_v^{(1)})^2] \leq K \left(\int_v^u \left\| v^{(0)}(s, X_s^{\pi^{(0)}}) \right\|_2 \mathrm{d}s \right)^2 + K' \int_v^u \mathbb{E}[(v^{(0)}(s, X_s^{\pi^{(0)}}))^2] \,\mathrm{d}s,$$

and Assumption 4.2.

Now we proceed to the proof (A.7) with $j = 1$.

$$\mathbb{E}\left|R_{t,T}^{(1,a)}\right| \leq \sqrt{2} \sum_{k=0}^{N-1} \left\| Z_{t_k}^{(1)} \right\|_2 [\mathbb{E}(\varphi_{t_k}^\epsilon)^2 + \mathbb{E}(\varphi_{t_{k+1}}^\epsilon)^2]^{1/2} \leq 2N \sup_{s\in[t,T]} \left\| (Z_s^{(1)} \right\|_2 \sup_{s\in[t,T]} \|\varphi_s^\epsilon\|_2$$

and is of order $o(\epsilon^{1-H})$ for any fixed N by Lemma A.2(ii). For the second term, using (A.9) gives

$$\begin{aligned} \mathbb{E}\left|R_{t,T}^{(1,b)}\right| &\leq \|\lambda\|_\infty \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \left\| Z_s^{(1)} - Z_{t_k}^{(1)} \right\|_2 \|\phi_s^\epsilon\|_2 \,\mathrm{d}s \\ &\leq \|\lambda\|_\infty K\epsilon^{1-H} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (s-t_k)^{1/2} \,\mathrm{d}s = K\epsilon^{1-H} \frac{1}{\sqrt{N}}, \end{aligned}$$

and

$$\lim_{\epsilon\to 0} \epsilon^{H-1}\, \mathbb{E}\left|R_{t,T}^{(1,b)}\right| \leq \frac{K}{\sqrt{N}}$$

holds for any N. Thus, we have the desired result, by letting $N \to \infty$.

Proof of (A.7) for $j = 2$ (*resp.* $j = 4$) is done by applying essentially the same argument to $Z_s^{(2)} = D_1^2 v^{(0)}(s, X_s^{\pi^{(0)}})\theta_s$ (*resp.* $Z_s^{(4)} = (D_2 + 2D_1)v^{(1)}(s, X_s^{\pi^{(0)}})$) which also satisfies (A.8) and (A.9), and $\epsilon^{1-H}\eta_t^\epsilon$ (*resp.* $\epsilon^{1-H} I_t^\epsilon$).

Proof of (A.7) for $j = 3$ is given by

$$\mathbb{E}\left|R_{t,T}^{(3)}\right| \leq \rho \int_t^T \mathbb{E}\left|\lambda(Y_s^{\epsilon,H}) D_1^2 v^{(0)}(s, X_s^{\pi^{(0)}})\widetilde{\theta}_s^\epsilon\right| \mathrm{d}s \leq K \int_t^T \left\|v^{(0)}(s, X_s^{\pi^{(0)}})\right\|_2 \left\|\widetilde{\theta}_s^\epsilon\right\|_2 \mathrm{d}s$$
$$\leq K \sup_{s\in[t,T]} \left\|v^{(0)}(s, X_s^{\pi^{(0)}})\right\|_2 \sup_{s\in[t,T]} \left\|\widetilde{\theta}_s^\epsilon\right\|_2$$

and Lemma A.1(i). □

B Assumptions for Theorem 4.5

This set of assumptions is used to establish the approximation accuracy (4.34) (*resp.* (4.35)) to $V_t^{\pi,\epsilon}$ defined in (4.30). To be specific, these assumptions will ensure that $\widetilde{M}_t^\epsilon$ (*resp.* $\widehat{M}_t^\epsilon$) is a true martingale and that $\widetilde{R}_t^\epsilon$ (*resp.* $\widehat{R}_t^\epsilon$) is of order $o(\epsilon^{1-H})$ (*resp.* $\mathcal{O}(\epsilon^{(1-H)\wedge\alpha})$).

Assumption B.1. *Let $\widetilde{\mathcal{A}}_t^\epsilon\left[\widetilde{\pi}^0, \widetilde{\pi}^1, \alpha\right]$ be the family of trading strategies defined in* (4.1). *Recall that X^π is the wealth process generated by the strategy $\pi = \widetilde{\pi}^0 + \epsilon^\alpha \widetilde{\pi}^1$ as defined in* (4.31). *In order to condense the notation, we systematically omit the arguments (s, X_s^π) of $v^{(0)}$ and $v^{(1)}$ and the argument $Y_s^{\epsilon,H}$ of μ and σ in what follows. According to the different cases, we further require:*

(i) If $\widetilde{\pi}^0 \equiv \pi^{(0)}$, the quantities below, for any $t \in [0,T]$, are of order ϵ^{1-H} in L^1 sense:
$\int_t^T \phi_s^\epsilon \mu \widetilde{\pi}_t^1 (\partial_x + R(s, X_s^\pi; \overline{\lambda})\partial_{xx}) D_1 v^{(0)} \,\mathrm{d}s$, $\int_t^T \phi_s^\epsilon \sigma^2 (\widetilde{\pi}_t^1)^2 \partial_{xx} D_1 v^{(0)} \,\mathrm{d}s$,
and the following quantities are uniformly bounded in ϵ:
$\mathbb{E}\int_0^T \left(\sigma\widetilde{\pi}_t^1 v_x^{(0)}\right)^2 \mathrm{d}s$, $\mathbb{E}\left|\int_0^T \mu\widetilde{\pi}_t^1 v_x^{(1)} \,\mathrm{d}s\right|$, $\mathbb{E}\left|\int_0^T \mu\widetilde{\pi}_t^1 R(t, X_t^\pi; \overline{\lambda}) v_{xx}^{(1)} \,\mathrm{d}s\right|$, $\mathbb{E}\left|\int_0^T \sigma^2 \left(\widetilde{\pi}_t^1\right)^2 v_{xx}^{(1)} \,\mathrm{d}s\right|$,
$\mathbb{E}\left(\int_0^T \left(\sigma\widetilde{\pi}_t^1 v_x^{(0)} \phi_s^\epsilon\right)^2 \mathrm{d}s\right)^{\frac{1}{2}}$, $\mathbb{E}\left(\int_0^T \left(\sigma\widetilde{\pi}_t^1 v_x^{(1)}\right)^2 \mathrm{d}s\right)^{\frac{1}{2}}$,

(ii) If $\widetilde{\pi}^0 \not\equiv \pi^{(0)}$, we require the uniformly boundedness (in ϵ) of the following:
$\mathbb{E}\left|\int_0^T \mu\widetilde{\pi}_t^1 v_x^{(0)} \,\mathrm{d}s\right|$, $\mathbb{E}\left|\int_0^T \sigma^2 \widetilde{\pi}_t^0 \widetilde{\pi}_t^1 v_{xx}^{(0)} \,\mathrm{d}s\right|$, $\mathbb{E}\left|\int_0^T \sigma^2 \left(\widetilde{\pi}_t^1\right)^2 v_{xx}^{(0)} \,\mathrm{d}s\right|$, $\mathbb{E}\left(\int_0^T \left(\sigma\widetilde{\pi}_t^0 v_x^{(0)}\right)^2 \mathrm{d}s\right)^{\frac{1}{2}}$,
$\mathbb{E}\left(\int_0^T \left(\sigma\widetilde{\pi}_t^1 v_x^{(0)}\right)^2 \mathrm{d}s\right)^{\frac{1}{2}}$.

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