

# VARIATIONAL PROBLEMS FOR FÖPPL-VON KÁRMÁN PLATES

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**ABSTRACT.** Some variational problems for a Föppl-von Kármán plate subject to general equilibrated loads are studied. The existence of global minimizers is proved under the assumption that the out-of-plane displacement fulfils homogeneous Dirichlet condition on the whole boundary while the in-plane displacement fulfils nonhomogeneous Neumann condition.

If the Dirichlet condition is prescribed only on a subset of the boundary, then the energy may be unbounded from below over the set of admissible configurations, as shown by several explicit conterexamples: in these cases the analysis of critical points is addressed through an asymptotic development of the energy functional in a neighborhood of the flat configuration. By a  $\Gamma$ -convergence approach we show that critical points of the Föppl-von Kármán energy can be strongly approximated by uniform Palais-Smale sequences of suitable functionals: this property leads to identify relevant features for critical points of approximating functionals, e.g. buckled configurations of the plate.

Eventually we perform further analysis as the plate thickness tends to 0, by assuming that the plate is prestressed and the energy functional depends only on the transverse displacement around the given prestressed state: by this approach, first we identify suitable exponents of plate thickness for load scaling, then we show explicit asymptotic oscillating minimizers as a mechanism to relax compressive states in an annular plate.

## CONTENTS

Introduction	2
1. Minimization of Föppl-von Kármán functional	4
2. Critical points nearby a flat configuration	13
3. Scaling Föppl-von Kármán energy	20
4. Prestressed plates: oscillating versus flat equilibria.	24
References	30

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## INTRODUCTION

The Föppl-von Kármán model is widely used as an effective theoretical tool in the study of the mechanical behavior of thin elastic plates, for its ability to describe the interplay between membrane and bending effects (see [3]). This interplay constitutes the source of a rich phenomenology affecting not only the macroscopic behavior but also the occurrence of local micro-instabilities which are crucial also in the behavior of soft solids, biological tissues, gels ([29]). A relevant problem consists in detecting a precise geometric description of such creased equilibrium configurations in dependence of the geometric and constitutive properties of the plate.

Despite its long and controversial history, a rigorous analysis of the well posedness for variational problems associated to the Föppl-von Kármán functional under general boundary conditions is still far from complete. In particular, the minimization problem under general load conditions is quite subtle. The rigorous derivation of the Föppl-von Kármán plate model from three-dimensional nonlinear elasticity was proved by Friesecke, James and Müller in the seminal paper [22] under the assumption of normal forces, while in [28] the authors carefully analyze the validity of such a theory under in-plane compressive forces and study in detail the instability issue under suitable coercivity hypotheses ([28, Theorem 4]).

In this paper we study the existence of minimizers for the Föppl-von Kármán energy, under general load conditions. In particular, we deal with Dirichlet and Neumann conditions for the out-of-plane displacement on the whole boundary while the in-plane displacement fulfils nonhomogeneous Neumann condition, corresponding to general assumptions on the forces acting on the plate. The existence of minimizers is proved in several cases by exploiting the techniques introduced in [4],[15] to circumvent the lack of coerciveness appearing in related nonconvex minimization problems and by taking advantage of some properties of the Monge-Ampère equation (see [42], [24]).

We exhibit also examples where the energy of admissible configurations is not bounded from below, so that existence of minimizers fails and we turn our attention to the critical points by performing singular perturbation analysis of the functional in a neighborhood of a flat configuration. This analysis leads to detect critical points of the Föppl-von Kármán energy by suitable approximations of Palais-Smale sequences associated to approximating functionals. Our procedure allows to single out global buckling configurations, in cases when the plate has a rectangular shape. As it is well known, wrinkling type phenomena and other micro instabilities (see [17],[20],[21],[41],[23]) manifest themselves in sheets with very small thickness, therefore we focus our analysis on the behavior as thickness tends to 0 and highlight the energetic competition of oscillating configurations versus flat equilibrium configurations.

The detailed outline of the paper is as follows.

In Section 1 we prove existence of minimizers for the Föppl-von Kármán energy (1.11) corresponding to a plate of prescribed thickness  $h > 0$  under the action of balanced loads in three relevant cases:

i) the plate is free at the boundary of a generic Lipschitz open set, while in plane uniform

normal traction or mild uniform normal compression is prescribed on the whole boundary (Theorems 1.1, 1.3);

ii) the plate is simply supported on the whole boundary of a strictly convex set (Theorem 1.6);

iii) the plate is clamped on the whole boundary of a generic Lipschitz open set (Theorem 1.8).

Moreover we focus the analysis on the cases when these conditions at the boundary are loosened, by showing explicit counterexamples where the energy is not bounded from below and minimizers do not exist, even for balanced loads and fixed thickness  $h > 0$ .

Section 2 is devoted to study asymptotic behavior of the energy near a flat configuration; this is achieved by scaling the out-of-plane displacements: Theorem 2.3 shows that critical points of the Föppl-von Kármán energy, say weak solutions of the corresponding Euler-Lagrange equations, can be approximately reconstructed by means of *uniform Palais-Smale sequences* (Definition 2.2) associated to Gamma-converging simpler functionals (concerning Gamma-convergence and critical points we refer also to [26]). This analysis clarifies as some relevant features of critical points, like buckled configurations related to approximating energies, can be recovered by the knowledge of equilibrium configurations related to the flat limit problem (Examples 2.7, 2.8).

In Section 3 we study the limit as  $h \rightarrow 0$  of scaled Föppl-von Kármán energy  $\mathcal{F}_h$  when in-plane forces in (1.11) scale as  $\mathbf{f}_h = h^\alpha \mathbf{f}$ : we show in Theorem 3.1 and Counterexample 3.3 that the natural scaling of the problem (entailing convergence of energies and minimizers) occurs if  $\alpha \geq 2$ : under this restriction, if  $(\mathbf{u}_h, w_h)$  is a minimizer of  $\mathcal{F}_h$  then the scaled pairs  $(h^{-\alpha} \mathbf{u}_h, h^{-\alpha/2} w_h)$  provide a weakly compact sequence in  $H^1 \times H^2$  and the corresponding scaled energy converges to a limit energy (Theorem 3.1 and formula (3.2) therein); on the other hand, if  $\alpha \in [0, 2)$  then the scaled energies may be unbounded from below as  $h \rightarrow 0$  even for free plates or simply supported or clamped ones (Counterexample 3.3 and Remark 3.4).

The results obtained in Sections 1-3 lead us to examine also the case  $\alpha \in [0, 2)$ , by studying the equilibrium configurations of the plate as  $h \rightarrow 0$  through relaxation arguments applied to an energetic functional which takes into account a prestressed state of the plate. Precisely, in Section 4: we perform the analysis of corresponding asymptotic minimizers, show a competition between oscillating and flat equilibria and highlight how this competition is ruled by the mechanical and geometrical parameters: oscillating equilibria act as a mechanism to release compression states in the limit.

Eventually we exhibit a list of creased and non creased equilibrium configurations of an annular plate (Examples 4.5 -4.8), together with a general strategy (Remark 4.9) to build these examples: if both eigenvalues in the stress tensor of the prestressed state are strictly positive almost everywhere, then we can expect only the flat minimizer; whereas possible occurrence of oscillating configurations requires the presence of a compressive state on a region of positive measure (Proposition 4.3, Remark 4.4).

The issues involved in the present article are closely related with a large class of instabilities, according to recent studies ([7], [8], [9], [11], [12],[10], [17], [30], [31], [32], [41]).

*Notation.*  $\text{Sym}_{2,2}(\mathbb{R})$  denotes  $2 \times 2$  real symmetric matrices;  $\mathbf{a} \otimes \mathbf{b}$  denotes the matrix with entries  $a_i b_j$ ,  $\mathbf{a} \odot \mathbf{b} = \frac{1}{2}(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a})$  and  $|\mathbf{a}|^2 = \sum_i a_i^2$  for every  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ ; moreover  $|\mathbb{A}|^2 = \sum_{i,j} A_{ij}^2$  and  $\mathbb{A} : \mathbb{B} = \sum_{i,j} A_{ij} B_{ij}$ , for every  $\mathbb{A}, \mathbb{B} \in \text{Sym}_{2,2}(\mathbb{R})$  with entries respectively  $A_{ij}, B_{ij}$ .

$H^k(\Omega)$  denotes the Sobolev space of functions in the open set  $\Omega \subset \mathbb{R}^2$  whose distributional derivatives up to the order  $k$  belong to  $L^2(\Omega)$ ;  $H_0^k(\Omega)$  denotes the completion of compactly supported functions in the Sobolev  $H^k$  norm;  $H^1(\Omega, \mathbb{R}^2)$  denotes the vector fields with components in  $H^1(\Omega)$ .

$\int_A v \, d\mathbf{x} = |A|^{-1} \int_A v \, d\mathbf{x} \, \forall$  measurable set  $A$  and every integrable function  $v$  defined on  $A$ .  $\mathbf{1}_A(\mathbf{x}) = 1$  if  $\mathbf{x} \in A$ ,  $\mathbf{1}_A(\mathbf{x}) = 0$  if  $\mathbf{x} \notin A$ .  $\chi_U(v) = 0$  if  $v \in U$ ,  $\chi_U(v) = +\infty$  if  $v \notin U$ .

## 1. MINIMIZATION OF FÖPPL-VON KÁRMÁN FUNCTIONAL

Let  $\Omega \subset \mathbb{R}^2$  be a bounded open connected set with Lipschitz boundary  $\partial\Omega$ ,  $\mathbf{x} = (x_1, x_2)$  denotes the coordinates of points in  $\Omega$  referring to the canonical reference frame in  $\mathbb{R}^2$  and  $s > 0$  is the thickness of a thin plate-like region whose reference configuration is  $\Omega \times (-\frac{s}{2}, \frac{s}{2})$ ; moreover set  $s := h s_0$  where  $h$  is a non-dimensional scale factor which remains fixed throughout this Section.

Let  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^2$  and  $w : \Omega \rightarrow \mathbb{R}$  be respectively the in-plane and out-of-plane displacements. In the geometrical linear setting the *stretching tensor*  $\mathbb{D}$  is given by

$$(1.1) \quad \mathbb{D}(\mathbf{u}, w) = \mathbb{E}(\mathbf{u}) + \frac{1}{2} Dw \otimes Dw,$$

where

$$(1.2) \quad \mathbb{E}(\mathbf{u}) = \frac{1}{2}(D\mathbf{u} + D\mathbf{u}^T)$$

denotes the linearized strain tensor.

The kernel of  $\mathbb{E}$ , that is the set of infinitesimal rigid displacements in  $\Omega$ , is denoted by

$$(1.3) \quad \mathcal{R} := \{\mathbf{u} : \mathbb{E}(\mathbf{u}) = \mathbf{0}\}$$

and  $\mathcal{R}(\mathbf{u})$  denotes the projection of  $\mathbf{u} \in H^1(\Omega, \mathbb{R}^2)$  on  $\mathcal{R}$ .

The elastic energy of a plate of thickness  $h s_0 > 0$  is the sum of a membrane energy

$$(1.4) \quad F_h^m(\mathbf{u}, w) = h s_0 \int_{\Omega} J(\mathbb{D}(\mathbf{u}, w)) \, d\mathbf{x}$$

and a bending energy

$$(1.5) \quad F_h^b(w) = \frac{h^3 s_0^3}{12} \int_{\Omega} J(D^2 w) \, d\mathbf{x}.$$

We assume that for every  $\mathbb{A} \in \text{Sym}_{2,2}(\mathbb{R})$  the energy density  $J$  is given by

$$(1.6) \quad J(\mathbb{A}) = \frac{E}{2(1-\nu^2)} (|\text{Tr}(\mathbb{A})|^2 - 2(1-\nu)\det \mathbb{A}) = \frac{E}{2(1+\nu)} |\mathbb{A}|^2 + \frac{E\nu}{2(1-\nu^2)} |\text{Tr} \mathbb{A}|^2$$

where  $E > 0$  is the Young modulus and  $\nu$  is the Poisson ratio,  $-1 < \nu < 1/2$ .

A straightforward consequence of (1.6) which will be exploited in subsequent computations is

$$(1.7) \quad c_\nu \frac{E}{2} |\mathbb{A}|^2 \leq J(\mathbb{A}) \leq C_\nu \frac{E}{2} |\mathbb{A}|^2$$

where  $0 < c_\nu := \min\{(1-\nu)^{-1}, (1+\nu)^{-1}\} \leq C_\nu := \max\{(1-\nu)^{-1}, (1+\nu)^{-1}\} < +\infty$ . By denoting the unit outer normal to  $\partial\Omega$  by  $\mathbf{n}$ , we define

$$(1.8) \quad \begin{aligned} \mathcal{A}^0 &:= \{w \in H^2(\Omega) \mid w = \frac{\partial w}{\partial \mathbf{n}} = 0 \text{ on } \Gamma\} \\ \mathcal{A}^1 &:= \{w \in H^2(\Omega) \mid w = 0 \text{ on } \Gamma\} \\ \mathcal{A}^2 &:= H^2(\Omega) \end{aligned}$$

where the spaces  $\mathcal{A}^0 = \mathcal{A}^0(\Gamma)$ ,  $\mathcal{A}^1 = \mathcal{A}^1(\Gamma)$  actually depend on  $\Gamma$ . We assume in general that

$$(1.9) \quad \Gamma \subset \partial\Omega \quad \text{is a Borel set s.t. } \mathcal{H}^1(\Gamma) > 0.$$

Let

$$(1.10) \quad \mathbf{f}_h \in L^2(\partial\Omega, \mathbb{R}^2), \quad g_h \in L^2(\Omega)$$

respectively be the densities of a given in-plane load distribution and of a given out-of-plane load distribution.

By taking into account the work of external loads and different types of boundary conditions, we define the *Föppl-von Kármán functional*, shortly denoted by  $\mathbf{FvK}$  in the sequel, (1.11)

$$\begin{aligned} \mathcal{F}_h(\mathbf{u}, w) &= \\ &= h s_0 \int_{\Omega} J(\mathbb{D}(\mathbf{u}, w)) \, d\mathbf{x} + \frac{h^3 s_0^3}{12} \int_{\Omega} J(D^2 w) \, d\mathbf{x} - h s_0 \int_{\Omega} g_h w \, d\mathbf{x} - h s_0 \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{u} \, d\mathcal{H}^1. \end{aligned}$$

Throughout the paper we choose units of measurement such that  $s_0 = 1$ .

Equilibrium configurations of the plate under prescribed loads  $\mathbf{f}_h$  and  $g_h$  are obtained by minimizing the functional (1.11) over  $H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^i$ ,  $i = 0, 1, 2$ , corresponding respectively to clamped, simply supported and free plate. The present Section focuses on issues related to existence and non existence of these minimizers: we study in detail existence of such minimizers according to the various choices  $i = 0, 1, 2$  of boundary conditions and loads and we exhibit some counterexamples in which the functional is unbounded from below, hence global minimizers do not exist.

The main obstruction in applying the direct methods of the calculus of variations to this problem relies in the possible lack of coerciveness of the functional (1.11): indeed the kernel of the membrane energy density, which in general is a subset of the set of solutions of the *Monge-Ampère* equation in  $\Omega$  (see Lemma 1.5 below), may be too large to allow balancing of the internal membrane energy versus the effect of external forces, in order to achieve an equilibrium configuration. Notwithstanding this difficulty, an existence theorem can be proved either assuming a sign condition on boundary forces, or an homogeneous Dirichlet condition on the transverse displacement. In the first case the work of the external forces

is bounded away from zero on the kernel of the membrane energy density, thus allowing the global energy to be bounded from below; in the second one a uniqueness result in the theory of *Monge-Ampère* equation implies that the kernel of bending energy reduces to the null transverse displacement (see also [30], [31], [32]). These settings together with a tuning of some techniques introduced in [4] and [15] yield compactness of minimizing sequences, hence existence of minimizers via the direct method.

Assuming  $\mathbf{f}_h = f_h \mathbf{n}$ , we prove existence of minimizers for  $\mathcal{F}_h$  in  $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)$ , first under the assumption that  $f_h$  is a nonnegative constant (Theorem 1.1), second under the assumption that  $f_h$  is a small negative constant (Theorem 1.3).

**Theorem 1.1. (uniform boundary traction of a free plate)**

Assume that  $\Omega \subset \mathbb{R}^2$  is a bounded connected Lipschitz open set and

$$(1.12) \quad \int_{\Omega} g_h \, d\mathbf{x} = \int_{\Omega} x_1 g_h \, d\mathbf{x} = \int_{\Omega} x_2 g_h \, d\mathbf{x} = 0 ,$$

$$(1.13) \quad \mathbf{f}_h = f_h \mathbf{n} \text{ on } \partial\Omega, \quad f_h \geq 0 \text{ is a constant.}$$

Then, for every fixed  $h > 0$ ,  $\mathcal{F}_h$  achieves a minimum over  $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)$ .

*Proof.* In order to achieve the proof it will be enough to show a minimizing sequence equibounded in  $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)$ , since  $\mathcal{F}_h$  is sequentially l.s.c. with respect the weak convergence in such space. Due to  $\inf_{H^1 \times H^2} \mathcal{F}_h \leq \mathcal{F}_h(\mathbf{0}, 0) \leq 0$ , if  $\mathcal{F}_h(\mathbf{u}_n, w_n) \rightarrow \inf_{H^1 \times H^2} \mathcal{F}_h$  we may suppose  $\mathcal{F}_h(\mathbf{u}_n, w_n) \leq 1$  so, by Divergence Theorem, (1.13) and (1.7) we also get

$$(1.14) \quad c_\nu \frac{h^3 E}{24} \int_{\Omega} |D^2 w_n|^2 + c_\nu \frac{h E}{2} \int_{\Omega} |\mathbb{D}(\mathbf{u}_n, w_n)|^2 \leq h f_h \int_{\Omega} \operatorname{div} \mathbf{u}_n + h \int_{\Omega} g_h w_n + 1.$$

Set  $\lambda_n := \|\mathbb{E}(\mathbf{u}_n)\|_{L^2}$  and suppose by contradiction that  $\sup \lambda_n = +\infty$ , hence (up to subsequences without relabeling)  $\lambda_n \rightarrow +\infty$ . Let  $\zeta_n := \lambda_n^{-1/2} w_n$ ,  $\mathbf{v}_n := \lambda_n^{-1} \mathbf{u}_n$  and  $\mathbf{x}_\Omega$  is the center of mass of  $\Omega$ . Possibly different constants denoted by  $C$  actually depend only on  $\Omega$ . Then by substituting in (1.14) and dividing times  $\lambda_n$ , we get via (1.12) and Poincaré inequality

$$(1.15) \quad \begin{aligned} & c_\nu \frac{h^3 E}{24} \int_{\Omega} |D^2 \zeta_n|^2 + \lambda_n c_\nu \frac{h E}{2} \int_{\Omega} |\mathbb{D}(\mathbf{v}_n, \zeta_n)|^2 \leq \\ & \leq h f_h \int_{\Omega} \operatorname{div} \mathbf{v}_n + \lambda_n^{-1/2} h \int_{\Omega} g_h \zeta_n + \lambda_n^{-1} = \\ & = h f_h \int_{\Omega} \operatorname{div} \mathbf{v}_n + \lambda_n^{-1/2} h \int_{\Omega} g_h \left( \zeta_n - \int_{\Omega} \zeta_n - (\mathbf{x} - \mathbf{x}_\Omega) \int_{\Omega} D \zeta_n \right) + \lambda_n^{-1} \leq \\ & \leq h f_h \int_{\Omega} \operatorname{div} \mathbf{v}_n + \lambda_n^{-1/2} h \|g_h\|_{L^2}^2 + \lambda_n^{-1/2} C \int_{\Omega} |D^2 \zeta_n|^2 + \lambda_n^{-1}. \end{aligned}$$

The above inequality together with  $\|\mathbb{E}(\mathbf{v}_n)\|_{L^2} = 1$  entail

$$(1.16) \quad c_\nu \frac{h^3 E}{24} \int_{\Omega} |D^2 \zeta_n|^2 + \lambda_n c_\nu \frac{h E}{2} \int_{\Omega} |\mathbb{D}(\mathbf{v}_n, \zeta_n)|^2 \leq C$$

for large  $n$ . Exploiting  $\|\mathbb{E}(\mathbf{v}_n)\|_{L^2} = 1$ , once more, we get  $D\zeta_n$  are then equibounded in  $H^1(\Omega, \mathbb{R}^2)$ , and, up to subsequences,  $\zeta_n - \int_{\Omega} \zeta_n \rightarrow \zeta$  weakly in  $H^2(\Omega)$ ,  $D\zeta_n \rightarrow D\zeta$  in  $L^4(\Omega, \mathbb{R}^2)$  due to Rellich Theorem and  $\mathbf{v}_n \rightarrow \mathbf{v}$  weakly in  $H^1(\Omega, \mathbb{R}^2)$ .

By taking into account (1.12) we get

$$(1.17) \quad hf_h \int_{\Omega} \operatorname{div} \mathbf{v}_n + \lambda_n^{-1/2} h \int_{\Omega} g_h \zeta_n = hf_h \int_{\Omega} \operatorname{div} \mathbf{v}_n + \lambda_n^{-1/2} h \int_{\Omega} g_h (\zeta_n - \int_{\Omega} \zeta_n) \rightarrow hf_h \int_{\Omega} \operatorname{div} \mathbf{v}.$$

By sequential lower semicontinuity together with (1.17), (1.15) we get

$$(1.18) \quad \begin{aligned} c_\nu \frac{h^3 E}{24} \int_{\Omega} |D^2 \zeta|^2 &\leq \liminf c_\nu \frac{h^3 E}{24} \int_{\Omega} |D^2 \zeta_n|^2 \leq \\ &\leq \liminf \left\{ hf_h \int_{\Omega} \operatorname{div} \mathbf{v}_n + \lambda_n^{-1/2} h \int_{\Omega} g_h (\zeta_n - \int_{\Omega} \zeta_n) + \lambda_n^{-1} \right\} = hf_h \int_{\Omega} \operatorname{div} \mathbf{v}. \end{aligned}$$

Moreover, by taking into account that  $\lambda_n \rightarrow +\infty$ ,

$$(1.19) \quad \lambda_n c_\nu \frac{h E}{2} \int_{\Omega} |\mathbb{D}(\mathbf{v}_n, \zeta_n)|^2 \leq hf_h \int_{\Omega} \operatorname{div} \mathbf{v}_n + \lambda_n^{-1} + \lambda_n^{-1/2} h \int_{\Omega} g_h (\zeta_n - \int_{\Omega} \zeta_n) \leq C$$

and by  $D\zeta_n \rightarrow D\zeta$  in  $L^4(\Omega, \mathbb{R}^2)$ , we have also

$$(1.20) \quad c_\nu \frac{h E}{2} \int_{\Omega} |\mathbb{D}(\mathbf{v}, \zeta)|^2 \leq \liminf c_\nu \frac{h E}{2} \int_{\Omega} |\mathbb{D}(\mathbf{v}_n, \zeta_n)|^2 \leq C \liminf \lambda_n^{-1} = 0.$$

Hence

$\mathbb{D}(\mathbf{v}_n, \zeta_n) \rightarrow \mathbb{D}(\mathbf{v}, \zeta) = 0$ ,  $\mathbb{E}(\mathbf{v}_n) \rightarrow \mathbb{E}(\mathbf{v})$  both in  $L^2(\Omega, \operatorname{Sym}_{2,2}(\mathbb{R}))$  and  $2 \operatorname{div} \mathbf{v} = -|D\zeta|^2$ . Therefore by (1.18)

$$(1.21) \quad c_\nu \frac{h^3 E}{24} \int_{\Omega} |D^2 \zeta|^2 + \frac{1}{2} hf_h \int_{\Omega} |D\zeta|^2 \leq 0$$

and by taking into account that  $\int_{\Omega} \zeta = 0$  we get  $\zeta = 0$  and  $\mathbb{E}(\mathbf{v}) = 0$ , a contradiction since  $\|\mathbb{E}(\mathbf{v}_n)\|_{L^2} = 1$  and  $\mathbb{E}(\mathbf{v}_n) \rightarrow \mathbb{E}(\mathbf{v})$  in  $L^2(\Omega, \operatorname{Sym}_{2,2}(\mathbb{R}))$ . So  $\lambda_n \leq C$  for some  $C > 0$  and  $\mathbf{u}_n - \mathcal{R}(\mathbf{u}_n)$  are equibounded in  $H^1(\Omega, \mathbb{R}^2)$  by Korn inequality, while equiboundedness of  $w_n - \int_{\Omega} w_n$  in  $H^2(\Omega)$  follows from (1.14). Existence of minimizers is then straightforward via direct method.  $\square$

If  $f < 0$  then the analogous of Theorem 1.1 for in-plane compression along the whole boundary cannot be true, as shown by the next particularly telling Counterexample 1.2. Anyway we can deal also with load corresponding to small negative  $f$ , as shown by Theorem 1.3 below.

**Counterexample 1.2. (uniform boundary compression).**

Assume

$$(1.22) \quad \Omega = (-2, 2) \times (-1, 1), \quad \Gamma = \{-2\} \times [-1, 1], \quad g_h \equiv 0$$

$$(1.23) \quad \mathbf{f}_h = f_h \mathbf{n} \text{ on } \partial\Omega, \quad \text{where } f_h \text{ is a given constant s.t. } f_h < -\frac{C_\nu E}{64} h^2.$$

Then  $\inf \mathcal{F}_h = -\infty$  over both  $H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^1$  and  $H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^2$ .

Indeed, let

$$\mathbf{u} = -\frac{(2+x_1)^3}{6} \mathbf{e}_1, \quad \varphi = \frac{(2+x_1)^2}{2},$$

and  $\mathbf{u}_n := n\mathbf{u}$ ,  $\varphi_n := \sqrt{n}\varphi$ ; then  $2\mathbb{E}(\mathbf{u}_n) = -D\varphi_n \otimes D\varphi_n$  and by (1.7)

$$\begin{aligned} \mathcal{F}_h(\mathbf{u}_n, \varphi_n) &\leq \frac{h^3 C_\nu n E}{24} \int_{\Omega} |D^2 \varphi|^2 d\mathbf{x} - nh f_h \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{u} d\mathcal{H}^1 = \\ &= \frac{h^3 C_\nu n E}{24} \int_{\Omega} |D^2 \varphi|^2 d\mathbf{x} - nh f_h \int_{\Omega} \operatorname{div} \mathbf{u} d\mathbf{x} = \\ &= \frac{h^3 C_\nu n E}{24} \int_{\Omega} |D^2 \varphi|^2 d\mathbf{x} + \frac{nh f_h}{2} \int_{\Omega} |D\varphi|^2 d\mathbf{x} = \frac{nh C_\nu}{3} (h^2 E + 64 f_h C_\nu^{-1}) \rightarrow -\infty. \end{aligned}$$

Referring to the bounded connected Lipschitz open set  $\Omega \subset \mathbb{R}^2$ , denote by  $K(\Omega)$  the best constant such that

$$(1.24) \quad \int_{\Omega} \left| \mathbf{v} - \int_{\Omega} \mathbf{v} \right|^2 d\mathbf{x} \leq K(\Omega) \int_{\Omega} |D\mathbf{v}|^2 d\mathbf{x} \quad \forall \mathbf{v} \in H^1(\Omega, \mathbb{R}^2).$$

**Theorem 1.3. (mild uniform boundary compression of a simply supported plate).**

Assume that  $\Omega \subset \mathbb{R}^2$  is a bounded connected Lipschitz open set and

$$(1.25) \quad \mathbf{f}_h = f_h \mathbf{n} \text{ on } \partial\Omega$$

where  $f_h$  is a given constant such that

$$(1.26) \quad f_h > -\frac{h^2 c_\nu E}{12 K(\Omega)}.$$

Then, for every fixed  $h > 0$ ,  $\mathcal{F}_h$  achieves a minimum over  $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega) \cap H_0^1(\Omega)$ .

*Proof.* Here, by setting  $\Gamma = \partial\Omega$ , we have  $\mathcal{A}^1 = H^2(\Omega) \cap H_0^1(\Omega)$ . Let  $\mathcal{F}_h(\mathbf{u}_n, w_n) \rightarrow \inf_{H^1 \times \mathcal{A}^1} \mathcal{F}_h$  and assuming by contradiction that  $\|\mathbb{E}(\mathbf{u}_n)\| \rightarrow +\infty$ . By arguing as in the proof of Theorem 1.1 we can build a sequence  $(\mathbf{v}_n, \zeta_n) \rightarrow (\mathbf{v}, \zeta)$  weakly in  $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)$ ,  $\|\mathbb{E}(\mathbf{v}_n)\| = 1$ ,  $\mathbb{D}(\mathbf{v}_n, \zeta_n) \rightarrow \mathbb{D}(\mathbf{v}, \zeta) = \mathbb{O}$ ,  $\mathbb{E}(\mathbf{v}_n) \rightarrow \mathbb{E}(\mathbf{v})$  both in  $L^2(\Omega, \operatorname{Sym}_{2,2}(\mathbb{R}))$ ,  $2 \operatorname{div} \mathbf{v} = -|D\zeta|^2$  and

$$(1.27) \quad c_\nu \frac{h^3 E}{24} \int_{\Omega} |D^2 \zeta|^2 + \frac{1}{2} h f_h \int_{\Omega} |D\zeta|^2 \leq 0;$$

we emphasize that  $\zeta_n = 0$  at  $\partial\Omega$  entails  $\int_{\Omega} D\zeta_n = 0$ , therefore  $|\int_{\Omega} g_h \zeta_n| \leq C \|g_h\|_{L^2} \|D^2 \zeta_n\|_{L^2}$  for a suitable constant  $C = C(\Omega)$ ; hence (1.27) can be achieved even without assuming

(1.12).

Therefore by taking into account that  $\int_{\Omega} D\zeta = 0$  (due to  $\zeta \in H_0^1$ ), Poincarè inequality (1.24) and assumption (1.26) altogether entail

$$(1.28) \quad c_{\nu} \frac{h^3 E}{24K(\Omega)} \int_{\Omega} |D\zeta|^2 + \frac{1}{2} h f_h \int_{\Omega} |D\zeta|^2 \leq c_{\nu} \frac{h^3 E}{24} \int_{\Omega} |D^2\zeta|^2 + \frac{1}{2} h f_h \int_{\Omega} |D\zeta|^2 \leq 0,$$

So  $D\zeta = 0$  and, by  $\mathbb{D}(\mathbf{v}, \zeta) = \mathbb{O}$ ,  $\mathbb{E}(\mathbf{v}) = \mathbb{O}$ , that is a contradiction since  $\|\mathbb{E}(\mathbf{v}_n)\|_{L^2} = 1$  and  $\mathbb{E}(\mathbf{v}_n) \rightarrow \mathbb{E}(\mathbf{v})$  in  $L^2(\Omega, \text{Sym}_{2,2}(\mathbb{R}))$ . The claim follows by repeating last part of Theorem 1.1 proof: here transverse load balancing (1.12) is not needed, due to boundary condition  $\mathcal{A}^1$ .  $\square$

**Remark 1.4.** By inspection of the proof of Theorem 1.3 we deduce also existence theorems for a plate clamped on a possibly proper subset  $\Gamma$  of the boundary. Precisely, assuming  $\Omega$  bounded, connected, Lipschitz, (1.9), (1.25) with  $f_h > -(h^2 c_{\nu} E)/(12 \tilde{K}(\Omega, \Gamma))$ , where  $\tilde{K}(\Omega, \Gamma)$  is the best constant s.t.  $\int_{\Omega} |\mathbf{v}|^2 dx \leq K(\Omega, \Gamma) \{ \int_{\Omega} |D\mathbf{v}|^2 dx + \int_{\Gamma} |\mathbf{v}|^2 d\mathcal{H}^1 \}$ , then  $\mathcal{F}_h$  achieves a minimum over  $H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^0(\Gamma)$ .

Similar claims in  $H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^1(\Gamma)$  (for plates supported on  $\Gamma$ ) fail, even by adding assumption  $\int_{\Omega} x_1 g_h d\mathbf{x} = \int_{\Omega} x_2 g_h d\mathbf{x} = 0$ . Indeed, if  $\Omega = (0, 1)^2$ ,  $\Gamma = \{0\} \times [0, 1]$ ,  $g_h \equiv 0$ ,  $\mathbf{f}_h = -\lambda^2 h^2 \mathbf{n}$ , then  $\inf \mathcal{F}_h = -\infty$ , as shown by  $\mathbf{u} = -(1/6)(x_1 + m)^3 \mathbf{e}_1$ ,  $w_m = ((x_1 + m)^2 - m^2)/2$ ,  $m \in \mathbb{N}$ .

Concerning existence of minimizers for  $\mathcal{F}_h$  in  $H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^i$  for  $i = 0, 1$ , when  $\Gamma = \partial\Omega$ , that is for clamped and simply supported plates respectively at the whole boundary, in presence of boundary forces which fulfils neither condition (1.13) nor conditions (1.25)-(1.26) we need to state first the following Lemma (see also [22, Proposition 9]) which clarifies the link between  $\ker \mathbb{D}$  and the solutions of the *Monge-Ampère* equation in  $\Omega$ .

**Lemma 1.5.** *Let  $\Omega \subset \mathbb{R}^2$  be an open set and assume that  $\mathbf{u} \in H^1(\Omega, \mathbb{R}^2)$ ,  $\varphi \in H^2(\Omega)$  satisfy*

$$2\mathbb{E}(\mathbf{u}) + D\varphi \otimes D\varphi = 0 \text{ in } \Omega.$$

*Then  $\det D^2\varphi \equiv 0$  in  $\Omega$ , where  $\det D^2\varphi$  is the pointwise hessian of  $\varphi$ .*

*Proof.* Since  $\mathbb{E}(\mathbf{u})$  satisfies the compatibility equation

$$\mathbb{E}_{11,22} + \mathbb{E}_{22,11} = 2\mathbb{E}_{12,12}$$

in the sense of  $\mathcal{D}'(\Omega)$ , we get

$$\int_{\Omega} \psi_{,2} (\mathbb{E}_{11,2} - \mathbb{E}_{12,1}) + \psi_{,1} (\mathbb{E}_{22,1} - \mathbb{E}_{12,2}) d\mathbf{x} = 0, \quad \forall \psi \in C_0^{\infty}(\Omega).$$

Therefore since  $D\varphi \otimes D\varphi = -2\mathbb{E}(\mathbf{u})$  we get

$$\begin{aligned} \mathbb{E}_{22,1} &= -\varphi_{,2} \varphi_{,12} \\ \mathbb{E}_{12,2} &= -\frac{1}{2} \varphi_{,2} \varphi_{,12} - \frac{1}{2} \varphi_{,1} \varphi_{,22} \\ \mathbb{E}_{11,2} &= -\varphi_{,1} \varphi_{,12} \\ \mathbb{E}_{12,1} &= -\frac{1}{2} \varphi_{,2} \varphi_{,11} - \frac{1}{2} \varphi_{,1} \varphi_{,12}. \end{aligned}$$

Summarizing

$$\frac{1}{2} \int_{\Omega} \psi_{,2}(\varphi_{,11}\varphi_{,2} - \varphi_{,1}\varphi_{,21}) + \psi_{,1}(\varphi_{,1}\varphi_{,22} - \varphi_{,2}\varphi_{,21}) d\mathbf{x} = 0, \quad \forall \psi \in C_0^\infty(\Omega).$$

that is  $\text{Det}D^2\varphi = 0$  where  $\text{Det}D^2\varphi$  is the distributional hessian of  $\varphi$ . Since  $\varphi \in H^2(\Omega)$  we have  $\det D^2\varphi = \text{Det}D^2\varphi = 0$  in  $\Omega$ .  $\square$

We are now in a position to state and prove an existence theorem for simply supported plates, whose proof relies on a result by Rauch & Taylor (see [42, Theorem 5.1]) about the Dirichlet problem for the *Monge-Ampère* equation (see also [24]).

**Theorem 1.6. (simply supported plate)**

If  $\Omega \subset \mathbb{R}^2$  is bounded strictly convex and  $\mathbf{f}_h$  is an equilibrated in-plane load distribution, say

$$(1.29) \quad \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{z} d\mathcal{H}^1 = 0 \quad \forall \mathbf{z} \in \mathcal{R}.$$

Then, for every fixed  $h > 0$ , the **FvK** functional  $\mathcal{F}_h$  in (1.11) achieves a minimum over  $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega) \cap H_0^1(\Omega)$ .

*Proof.* Here  $\Gamma \equiv \partial\Omega$  so, referring to (1.8), we look for minimizers of  $\mathcal{F}_h$  over  $H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^1 = H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega) \cap H_0^1(\Omega)$ . The proof will be achieved by showing the existence of a minimizing sequence equibounded in  $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)$ , since  $\mathcal{F}_h$  is sequentially l.s.c. with respect to the weak convergence in this space. Due to  $\inf_{H^1 \times \mathcal{A}^1} \mathcal{F}_h \leq \mathcal{F}_h(\mathbf{0}, 0) \leq 0$ , hence if  $\mathcal{F}_h(\mathbf{u}_n, w_n) \rightarrow \inf_{H^1 \times \mathcal{A}^1} \mathcal{F}_h$  we may suppose  $\mathcal{F}_h(\mathbf{u}_n, w_n) \leq 1$ . So by taking into account (1.29) and (1.7) we get via Korn and Poincarè inequality

$$(1.30) \quad \begin{aligned} c_\nu \frac{h^3 E}{24} \int_{\Omega} |D^2 w_n|^2 + c_\nu \frac{h E}{2} \int_{\Omega} |\mathbb{D}(\mathbf{u}_n, w_n)|^2 &\leq h \int_{\Omega} \mathbf{f}_h \cdot \mathbf{u}_n + h \int_{\Omega} g_h w_n + 1 = \\ &= h \int_{\Omega} \mathbf{f}_h \cdot (\mathbf{u}_n - \mathcal{R}(\mathbf{u}_n)) + h \int_{\Omega} g_h w_n + 1 \leq \|\mathbb{E}(\mathbf{u}_n)\|_{L^2} \|\mathbf{f}_h\|_{L^2} + h \|g_h\|_{L^2} \|Dw_n\|_{L^2} + 1. \end{aligned}$$

Set  $\lambda_n := \|\mathbb{E}(\mathbf{u}_n)\|_{L^2}$ , assume by contradiction  $\lambda_n \rightarrow +\infty$  and set  $\mathbf{v}_n := \lambda_n^{-1} \mathbf{u}_n$   $\zeta_n := \lambda_n^{-1/2} w_n$ . By substituting in (1.30) and dividing times  $\lambda_n$ , via Poincarè inequality in  $H^2 \cap H_0^1$ , we get

$$(1.31) \quad \begin{aligned} c_\nu \frac{h^3 E}{24} \int_{\Omega} |D^2 \zeta_n|^2 + \lambda_n c_\nu \frac{h E}{2} \int_{\Omega} |\mathbb{D}(\mathbf{v}_n, \zeta_n)|^2 &\leq \\ &\leq \|\mathbf{f}_h\|_{L^2} + \lambda_n^{-1/2} h \|g_h\|_{L^2} \|D\zeta_n\|_{L^2} + \lambda_n^{-1} \leq \\ &\leq C + \lambda_n^{-1/2} h \int_{\Omega} |D\zeta_n|^2 \leq C + \lambda_n^{-1/2} h \int_{\Omega} |D^2 \zeta_n|^2 \end{aligned}$$

thus obtaining as in the proof of Theorem 1.1

$$(1.32) \quad c_\nu \frac{h^3 E}{24} \int_{\Omega} |D^2 \zeta_n|^2 + \lambda_n c_\nu \frac{h E}{2} \int_{\Omega} |\mathbb{D}(\mathbf{v}_n, \zeta_n)|^2 \leq C'$$

for a suitable  $C' > 0$ . Since  $\|\mathbb{E}(\mathbf{v}_n)\|_{L^2} = 1$ ,  $D\zeta_n$  are then equibounded in  $H^1(\Omega, \mathbb{R}^2)$  so, up to subsequences,  $\zeta_n \rightarrow \zeta$  weakly in  $H^2(\Omega)$ ,  $D\zeta_n \rightarrow D\zeta$  strongly in  $L^4(\Omega, \mathbb{R}^2)$ ,  $\mathbf{v}_n \rightarrow \mathbf{v}$  weakly in  $H^1(\Omega, \mathbb{R}^2)$  and  $\mathbb{D}(\mathbf{v}_n, \zeta_n) \rightarrow 0$  strongly in  $L^2(\Omega)$ . Hence

$$(1.33) \quad 2\mathbb{E}(\mathbf{v}_n) + D\zeta_n \otimes D\zeta_n \rightarrow 2\mathbb{E}(\mathbf{v}) + D\zeta \otimes D\zeta = \mathbb{O} \quad \text{strongly in } L^2(\Omega, \text{Sym}_{2,2}(\mathbb{R}))$$

and  $\mathbb{E}(\mathbf{v}_n) \rightarrow \mathbb{E}(\mathbf{v})$  strongly in  $L^2(\Omega, \text{Sym}_{2,2}(\mathbb{R}))$ . Then by Lemma 1.5 we have  $\det D^2\zeta = 0$  and by taking into account that  $\Omega$  is strictly convex and  $\zeta = 0$  on the whole  $\partial\Omega$  by uniqueness Theorem 5.1 in [42] we get  $\zeta \equiv 0$  in  $\Omega$ . This implies  $\mathbb{E}(\mathbf{v}) = -\frac{1}{2}D\zeta \otimes D\zeta = \mathbb{O}$ , which is a contradiction since  $\|\mathbb{E}(\mathbf{v}_n)\|_{L^2} = 1$ . Hence  $\lambda_n \leq C$  for suitable  $C > 0$ , so  $\mathbf{u}_n - \mathcal{R}(\mathbf{u}_n)$  are equibounded in  $H^1(\Omega, \mathbb{R}^2)$  and equiboundedness of  $w_n$  in  $H^2(\Omega)$  follows from (1.32). Existence of minimizers is obtained via direct method.  $\square$

Existence of minimizers may fail when  $\Gamma \not\equiv \partial\Omega$  even if the in-plane load  $\mathbf{f}_h$  is equilibrated, as shown by the next Counterexample.

**Counterexample 1.7. (*buckling under in-plane shear*)** Fix  $\gamma > 0$ ,  $\varepsilon > 0$ ,  $h^2 < \gamma/(6EC_\nu)$  and

$$(1.34) \quad \begin{aligned} \Omega_\varepsilon &= \{(x_1, x_2) : |x_1| < 2 + \varepsilon(1 - x_2^2), |x_2| < 1 + \varepsilon(4 - x_1^2)\}, \\ \Gamma_\varepsilon &= \partial\Omega_\varepsilon \cap \{(x_1, x_2) : |x_1 - x_2| \geq 1\}, \end{aligned}$$

$$(1.35) \quad \mathbf{f}_h := \gamma\boldsymbol{\tau}(\mathbf{1}_{\Sigma^{2,\pm}} - \mathbf{1}_{\Sigma^{1,\pm}}),$$

where  $\boldsymbol{\tau}$  denotes the counterclockwise oriented unit vector tangent to  $\partial\Omega_\varepsilon = \Sigma_\varepsilon^{1,\pm} \cup \Sigma_\varepsilon^{2,\pm}$  and

$$\begin{aligned} \Sigma_\varepsilon^{1,\pm} &= \{(x_1, x_2) : |x_1| \leq 2, x_2 = \pm(1 + \varepsilon(4 - x_1^2))\} \\ \Sigma_\varepsilon^{2,\pm} &= \{(x_1, x_2) : |x_2| \leq 1, x_1 = \pm(2 + \varepsilon(1 - x_2^2))\}. \end{aligned}$$

We claim that there exists  $\tilde{\varepsilon}$  such that  $\inf \mathcal{F}_h = -\infty$  over  $H^1(\Omega_{\tilde{\varepsilon}}, \mathbb{R}^2) \times \mathcal{A}^1$  under the assumptions listed above, notwithstanding the strict convexity of  $\Omega_{\tilde{\varepsilon}}$  and the fact that condition (1.29) holds true.

Indeed, let  $\psi \in C^{1,1}(\mathbb{R})$  be an even function, with  $\text{spt } \psi \subset [-1, 1]$ ,  $\psi' = -1$  in  $[1/4, 3/4]$  and  $|\psi''| \leq 4$  in  $\mathbb{R}$ . We set  $\varphi(x_1, x_2) = \psi(x_1 - x_2)$  and define  $w_n := \sqrt{n}\varphi$  and  $\mathbf{u}_n := n\mathbf{u}$ , where

$$u_2(x_1, x_2) = -u_1(x_1, x_2) = \frac{1}{2} \int_{-1}^{x_1 - x_2} |\psi'(\tau)|^2 d\tau.$$

By setting  $\Omega_0 := (-2, 2) \times (-1, 1) \subset \Omega_\varepsilon$ , there is  $C > 0$  such that for every  $0 < \varepsilon \leq 1$

$$\left| \int_{\partial\Omega_\varepsilon} \mathbf{f}_h \cdot \mathbf{u} d\mathcal{H}^1 - \int_{\partial\Omega_0} \mathbf{f}_h \cdot \mathbf{u} d\mathcal{H}^1 \right| \leq C\varepsilon,$$

hence by (1.35) and there exists  $\tilde{\varepsilon} \in (0, 1)$  such that

$$(1.36) \quad \int_{\partial\Omega_{\tilde{\varepsilon}}} \mathbf{f}_h \cdot \mathbf{u} d\mathcal{H}^1 \geq \int_{\partial\Omega_0} \mathbf{f}_h \cdot \mathbf{u} d\mathcal{H}^1 - \frac{\gamma}{2} = \gamma \int_{\Omega_0} 2\mathbb{E}_{12}(\mathbf{u}) d\mathbf{x} - \frac{\gamma}{2}.$$

So

$$u_{1,1}(x_1, x_2) = -\frac{1}{2}|\psi'(x_1 - x_2)|^2 = -\frac{1}{2}\varphi_{,1}^2,$$

$$u_{2,2}(x_1, x_2) = -\frac{1}{2}|\psi'(x_1 - x_2)|^2 = -\frac{1}{2}\varphi_{2,2}^2,$$

$$\frac{u_{1,2} + u_{2,1}}{2} = \frac{1}{2} \left[ \frac{1}{2}|\psi'(x_1 - x_2)|^2 + \frac{1}{2}|\psi'(x_1 - x_2)|^2 \right] = \frac{1}{2}|\psi'(x_1 - x_2)|^2 = -\frac{1}{2}\varphi_{1,1}\varphi_{2,2}$$

that is  $\mathbb{E}(\mathbf{u}_n) = -\frac{1}{2}Dw_n \otimes Dw_n$  and moreover, by (1.7), (1.36) and  $\varphi_{2,2} = -\varphi_{1,1}$  we deduce (1.37)

$$\begin{aligned} \mathcal{F}_h(\mathbf{u}_n, w_n) &\leq C_\nu \frac{h^3 nE}{24} \int_{\Omega_0} |D^2\varphi|^2 dx + C_\nu \frac{h^3 nE}{24} \int_{\Omega_{\tilde{\varepsilon}} \setminus \Omega_0} |D^2\varphi|^2 d\mathbf{x} + \\ &\quad + hn\gamma \int_{\Omega_0} \varphi_{1,1}\varphi_{2,2} d\mathbf{x} + h\frac{\gamma}{2} \leq \\ &\leq C_\nu \frac{8h^3 nE}{3} \left( |\{(x_1, x_2) \in \Omega_0 : 4|x_1 - x_2| \leq 1 \text{ or } 3 \leq 4|x_1 - x_2| \leq 4\}| + |\Omega_{\tilde{\varepsilon}} \setminus \Omega_0| \right) + \\ &\quad - hn\gamma |\{(x_1, x_2) \in \Omega_0 : 1 \leq 4|x_1 - x_2| \leq 3\}| + hn\frac{\gamma}{2} \leq \\ &\leq 3C_\nu E h^3 n - hn\frac{\gamma}{2} \rightarrow -\infty \end{aligned}$$

as  $n \rightarrow +\infty$  whenever  $6EC_\nu h^2 < \gamma$  thus proving the claim.  $\square$

Clearly Theorems 1.1, 1.3, 1.6 hold for the clamped plate too: minimization in  $H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^0$ . Even better, in the case of clamped plate we can drop both convexity assumption on  $\Omega$  and equilibrated out-of-plane load (1.12) as it is shown by the next result.

**Theorem 1.8. (clamped plate)**

If  $\Omega$  is a bounded connected Lipschitz open set and (1.29) holds, then for every fixed  $h > 0$  the functional  $\mathcal{F}_h$  in (1.11) achieves its minimum over  $H^1(\Omega, \mathbb{R}^2) \times H_0^2(\Omega)$ .

*Proof.* Again we need only to exhibit an equibounded minimizing sequence. Indeed, as in the proof of Theorem 1.6 if  $\mathcal{F}_h(\mathbf{u}_n, w_n) \rightarrow \inf_{H^1 \times H_0^2} \mathcal{F}_h$  we may suppose  $\mathcal{F}_h(\mathbf{u}_n, w_n) \leq 1$ . Then, since  $\Gamma = \partial\Omega$  entails  $H_0^2(\Omega) = \mathcal{A}^0 \subset \mathcal{A}^1$ , by setting  $\lambda_n := \|\mathbb{E}(\mathbf{u}_n)\|_{L^2}$ ,  $\mathbf{v}_n := \lambda_n^{-1}\mathbf{u}_n$ ,  $\zeta_n := \lambda_n^{-1/2}w_n$  and assuming  $\lambda_n \rightarrow +\infty$ , arguing as in the previous proofs we achieve the estimates (1.30), (1.31), (1.32). Then the sequence  $D\zeta_n$  is equibounded in  $H^1(\Omega, \mathbb{R}^2)$  so, up to subsequences,  $\zeta_n \rightarrow \zeta$  weakly in  $H^2(\Omega)$ ,  $D\zeta_n \rightarrow D\zeta$  in  $L^4(\Omega, \mathbb{R}^2)$ ,  $\mathbf{v}_n \rightarrow \mathbf{v}$  weakly in  $H^1(\Omega, \mathbb{R}^2)$  and  $\mathbb{D}(\mathbf{v}_n, \zeta_n) \rightarrow \mathbb{O}$  in  $L^2(\Omega, \text{Sym}_{2,2}(\mathbb{R}))$ . Hence

$$(1.38) \quad 2\mathbb{E}(\mathbf{v}_n) + D\zeta_n \otimes D\zeta_n \rightarrow 2\mathbb{E}(\mathbf{v}) + D\zeta \otimes D\zeta = \mathbb{O} \quad \text{strongly in } L^2(\Omega, \text{Sym}_{2,2}(\mathbb{R})),$$

$\mathbb{E}(\mathbf{v}_n) \rightarrow \mathbb{E}(\mathbf{v})$  strongly in  $L^2(\Omega, \text{Sym}_{2,2}(\mathbb{R}))$  and by Lemma 1.5 we have  $\det D^2\zeta = 0$  in the whole  $\Omega$ . Since  $\zeta \equiv \frac{\partial\zeta}{\partial\mathbf{n}} \equiv 0$  on  $\partial\Omega$ , there exists a disk  $\tilde{\Omega}$  (bounded and strictly convex!) such that  $\Omega \subset \tilde{\Omega}$  and the trivial extension  $\tilde{\zeta}$  of  $\zeta$  in  $\tilde{\Omega}$  belongs to  $H_0^2(\tilde{\Omega})$ . Therefore  $\det D^2\tilde{\zeta} = 0$  on  $\tilde{\Omega}$  and still by Theorem 5.1 in [42] we get  $\tilde{\zeta} \equiv 0$  in  $\tilde{\Omega}$  hence  $\zeta \equiv 0$  in  $\Omega$ . Then by (1.38)  $\mathbb{E}(\mathbf{v}) = \mathbb{O}$ , a contradiction since  $\|\mathbb{E}(\mathbf{v}_n)\|_{L^2} = 1$ .  $\square$

## 2. CRITICAL POINTS NEARBY A FLAT CONFIGURATION

When existence of global minimizers fails because the energy is unbounded from below, it is natural to investigate the structure of local minimizers or, more in general of critical points. Since the nonlinearity in the  $\mathbf{FvK}$  functional relies in the interaction between membrane and bending contributions, we will focus in this section on the asymptotic analysis of critical points in the neighborhood of a flat configuration, i.e. we will study the behavior for small out-of-plane displacements. Throughout this section we assume that  $h > 0$  is fixed and

$$(2.1) \quad g_h \equiv 0$$

that is, we restrict our analysis to the case of in-plane load acting on a plate of prescribed thickness. Assume  $\mathbf{f}_h \in L^2(\partial\Omega, \mathbb{R}^2)$  and (1.29) holds true. For every  $(\mathbf{u}, w) \in H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)$ , referring to (1.1)-(1.11), we enclose boundary conditions in the functional, by setting

$$(2.2) \quad \mathcal{F}_h^i(\mathbf{u}, w) = \begin{cases} \mathcal{F}_h(\mathbf{u}, w) & \text{if } \mathbf{u} \in H^1(\Omega, \mathbb{R}^2), w \in \mathcal{A}^i, \\ +\infty & \text{otherwise,} \end{cases}$$

$$(2.3) \quad \mathcal{F}_{h,\varepsilon}^i(\mathbf{u}, w) = \mathcal{F}_h^i(\mathbf{u}, \varepsilon w), \quad \forall \varepsilon > 0.$$

By noticing that  $\mathcal{F}_{h,0} := \mathcal{F}_{h,0}^i$  actually is independent of  $i$ , we also set

$$(2.4) \quad \mathcal{E}_{h,\varepsilon}^i(\mathbf{u}, w) = \varepsilon^{-2} \left( \mathcal{F}_{h,\varepsilon}^i(\mathbf{u}, w) - \min_{H^1(\Omega, \mathbb{R}^2)} \mathcal{F}_{h,0} \right),$$

$$(2.5) \quad \mathcal{E}_h^i(\mathbf{u}, w) = \begin{cases} F_h^b(w) + \frac{h}{2} \int_{\Omega} J'(\mathbb{E}(\mathbf{u})) : Dw \otimes Dw \, dx & \text{if } (\mathbf{u}, w) \in \{\operatorname{argmin} \mathcal{F}_{h,0}\} \times \mathcal{A}^i \\ +\infty & \text{else in } H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega) \end{cases}$$

where

$$(2.6) \quad J'(\mathbb{A}) = \frac{E}{1+\nu} \mathbb{A} + \frac{E\nu}{1-\nu^2} (\operatorname{Tr} \mathbb{A}) \mathbb{I}$$

denotes the derivative of  $J$ .

Functionals  $\mathcal{E}_{h,\varepsilon}^i$  and  $\mathcal{F}_{h,\varepsilon}^i$  are linked via the following result

**Proposition 2.1.**  $\mathcal{E}_h^i = \Gamma \lim_{\varepsilon \rightarrow 0_+} \mathcal{E}_{h,\varepsilon}^i$ .

*Precisely, the following relations hold true:*

*i) for every  $(\mathbf{u}_\varepsilon, w_\varepsilon) \rightharpoonup (\mathbf{u}, w)$  in  $w - H^1 \times H^2$  we have*

$$(2.7) \quad \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_{h,\varepsilon}^i(\mathbf{u}_\varepsilon, w_\varepsilon) \geq \mathcal{E}_h^i(\mathbf{u}, w);$$

ii) for every  $(\mathbf{u}, w) \in H^1 \times H^2$  there exists  $(\tilde{\mathbf{u}}_\varepsilon, \tilde{w}_\varepsilon) \rightarrow (\mathbf{u}, w)$  in  $w - H^1 \times H^2$  such that

$$(2.8) \quad \lim_{\varepsilon \rightarrow 0} \mathcal{E}_{h,\varepsilon}^i(\tilde{\mathbf{u}}_\varepsilon, \tilde{w}_\varepsilon) = \mathcal{E}_h^i(\mathbf{u}, w).$$

*Proof.* Let  $(\mathbf{u}_\varepsilon, w_\varepsilon) \rightarrow (\mathbf{u}, w)$  in  $w - H^1 \times H^2$ : by convexity we have

$$(2.9) \quad \begin{aligned} \mathcal{F}_{h,\varepsilon}^i(\mathbf{u}_\varepsilon, w_\varepsilon) &\geq \varepsilon^2 F_h^b(w) + h \int_\Omega J(\mathbb{E}(\mathbf{u}_\varepsilon)) dx + \\ &+ \frac{h\varepsilon^2}{2} \int_\Omega J'(\mathbb{E}(\mathbf{u}_\varepsilon)) : Dw_\varepsilon \otimes Dw_\varepsilon dx - h \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{u}_\varepsilon d\mathcal{H}^1 \geq \\ &\geq \varepsilon^2 F_h^b(w) + \frac{h\varepsilon^2}{2} \int_\Omega J'(\mathbb{E}(\mathbf{u}_\varepsilon)) : Dw_\varepsilon \otimes Dw_\varepsilon dx + \min \mathcal{F}_{h,0} \end{aligned}$$

and by taking into account that  $Dw_\varepsilon \otimes Dw_\varepsilon \rightarrow Dw \otimes Dw$  strongly in  $L^2(\Omega, \text{Sym}_{2,2}(\mathbb{R}))$  and  $J'(\mathbb{E}(\mathbf{u}_\varepsilon)) \rightarrow J'(\mathbb{E}(\mathbf{u}))$  weakly in  $L^2(\Omega, \text{Sym}_{2,2}(\mathbb{R}))$ , we get

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{E}_{h,\varepsilon}^i(\mathbf{u}_\varepsilon, w_\varepsilon) \geq \mathcal{E}_h^i(\mathbf{u}, w)$$

and i) is proven. The proof of ii) is achieved by taking  $(\tilde{\mathbf{u}}_\varepsilon, \tilde{w}_\varepsilon) \equiv (\mathbf{u}, w)$ .  $\square$

We recall that if  $\mathcal{I} : X \rightarrow \mathbb{R}$  is any  $C^1$  functional defined on a Banach space  $X$  then  $\bar{x} \in X$  is a critical point for  $\mathcal{I}$  if  $\mathcal{I}'(\bar{x}) = 0$  where  $\mathcal{I}' : X \rightarrow X^*$  denotes the Gateaux differential of  $\mathcal{I}$ .

Due to formula (2.10) below,  $\mathcal{F}_{h,\varepsilon}^i$  is a  $C^1$  functional in the Hilbert space  $H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^i$ : precisely, for every  $(\mathbf{u}, w) \in H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^i$  the Gateaux differential of  $\mathcal{F}_{h,\varepsilon}^i$  at  $(\mathbf{u}, w)$  is given by

$$(\mathcal{F}_{h,\varepsilon}^i)'(\mathbf{u}, w)[(\mathbf{z}, \omega)] = \left( \tau_1(\mathbf{u}, w)[\mathbf{z}], \tau_2(\mathbf{u}, w)[\omega] \right), \quad \forall \mathbf{z} \in H^1(\Omega, \mathbb{R}^2), \forall \omega \in \mathcal{A}^i,$$

where

(2.10)

$$\begin{aligned} \tau_1(\mathbf{u}, w)[\mathbf{z}] &:= h \int_\Omega J' \left( \mathbb{E}(\mathbf{u}) + \frac{\varepsilon^2}{2} Dw \otimes Dw \right) : \mathbb{E}(\mathbf{z}) - h \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{z}, \\ \tau_2(\mathbf{u}, w)[\omega] &:= \varepsilon^2 \frac{h^3}{12} \int_\Omega J'(D^2 w) : D^2 \omega + \varepsilon^2 h \int_\Omega J' \left( \mathbb{E}(\mathbf{u}) + \frac{\varepsilon^2}{2} Dw \otimes Dw \right) : Dw \odot D\omega. \end{aligned}$$

$(\tau_1(\mathbf{u}, w)[\mathbf{z}], \tau_2(\mathbf{u}, w)[\omega])$  is replaced by the shorter notation  $(\tau_1[\mathbf{z}], \tau_2[\omega])$ , whenever the dependance on fixed choice for  $(\mathbf{u}, w)$  is understood. Actually (2.10) provides the explicit information that  $(\mathcal{F}_{h,\varepsilon}^i)'(\mathbf{u}, w)$  depends continuously on  $(\mathbf{u}, w)$ .

Hence the Föppl-von Karman plate equations in weak form together with boundary conditions can be written as follows:

$$(2.11) \quad \begin{cases} \mathbf{u}, w \in H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^i, \\ \tau_1(\mathbf{u}, w)[\mathbf{z}] = 0 & \forall \mathbf{z} \in H^1(\Omega, \mathbb{R}^2), \\ \tau_2(\mathbf{u}, w)[\omega] = 0 & \forall \omega \in \mathcal{A}^i. \end{cases}$$

Clearly  $(\mathcal{E}_{h,\varepsilon}^i)'(\mathbf{u}, w) = \varepsilon^{-2}(\mathcal{F}_{h,\varepsilon}^i)'(\mathbf{u}, w)$  hence  $\mathcal{F}_{h,\varepsilon}^i$  and  $\mathcal{E}_{h,\varepsilon}^i$  have the same critical points. Moreover if  $\mathbf{u}_* \in \operatorname{argmin} \mathcal{F}_{h,0}$  then  $\tau_2(\mathbf{u}_*, 0) \equiv 0$  and  $(\mathbf{u}_*, 0)$  is a critical point for  $\mathcal{F}_{h,\varepsilon}^i$ . The next definition tunes the standard notion of *Palais-Smale sequence* to the present context.

**Definition 2.2.** Let  $\mathcal{I}_\varepsilon : X \rightarrow \mathbb{R}$  be a sequence of  $C^1$  functionals and  $X$  be a Banach space  $X$ . A sequence  $\{x_\varepsilon\} \subset X$  is a **uniform Palais-Smale sequence** if there exists  $C > 0$  such that  $\mathcal{I}_\varepsilon(x_\varepsilon) \leq C$  and  $\|\mathcal{I}'_\varepsilon(x_\varepsilon)\|_{X^*} \rightarrow 0$ , as  $\varepsilon \rightarrow 0_+$ .

Notice that the above definition reduces to the usual notion of Palais-Smale sequences when  $\mathcal{I}_\varepsilon \equiv \mathcal{I}$  for every  $\varepsilon > 0$ . Let  $\mathbf{u}_* \in \operatorname{argmin} \mathcal{F}_{h,0}$ , we denote by  $\mathcal{K}_h^i(\mathbf{u}_*)$  the set of critical points in  $\mathcal{A}^i$  of  $\mathcal{E}_h^i(\mathbf{u}_*, \cdot)$  that is

$$(2.12) \quad \mathcal{K}_h^i(\mathbf{u}_*) = \{w \in \mathcal{A}^i : \tau_2(\mathbf{u}_*, w)[\omega] = 0, \forall \omega \in \mathcal{A}^i\} .$$

Next result shows that any critical point of  $\mathcal{E}_h(\mathbf{u}_*, \cdot)$  in  $\mathcal{A}^i$  can be approximated by a uniform Palais-Smale sequence of  $\mathcal{E}_{h,\varepsilon}^i$  whose energy converges to the energy of the critical point itself.

**Theorem 2.3.** Let  $\mathbf{u}_* \in \operatorname{argmin} \mathcal{F}_{h,0}$ ,  $w \in \mathcal{K}_h^i(\mathbf{u}_*)$  and  $\mathbf{z}_w \in \operatorname{argmin} \mathcal{Q}_w(\mathbf{z})$ , where

$$(2.13) \quad \mathcal{Q}_w(\mathbf{z}) := \int_{\Omega} J\left(\mathbb{E}(\mathbf{z}) + \frac{1}{2}Dw \otimes Dw\right) dx$$

Then  $\{(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w)\}_{\varepsilon > 0}$  is a uniform Palais-Smale sequence for  $\mathcal{E}_{h,\varepsilon}^i$  and

$$\lim_{\varepsilon \rightarrow 0_+} \mathcal{E}_{h,\varepsilon}^i(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w) = \mathcal{E}_h^i(\mathbf{u}_*, w) .$$

*Proof.* We have to prove the following conditions

- a)  $\mathcal{E}_{h,\varepsilon}^i(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w) \leq C < +\infty, \quad \forall \varepsilon \in (0, 1],$
- b)  $(\mathcal{E}_{h,\varepsilon}^i)'(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w) \rightarrow 0$  strongly in  $(H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^i)^*$ ,
- c)  $\lim_{\varepsilon \rightarrow 0_+} \mathcal{E}_{h,\varepsilon}^i(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w) = \mathcal{E}_h^i(\mathbf{u}_*, w).$

We first prove c), which implies a) too. Indeed

$$\begin{aligned}
\mathcal{E}_{h,\varepsilon}^i(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w) &= \varepsilon^{-2} [\mathcal{F}_h^1(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, \varepsilon w) - \mathcal{F}_{0,h}(\mathbf{u}_*)] = \\
&= \varepsilon^{-2} \left[ \frac{h^3}{12} \int_{\Omega} J(\varepsilon D^2 w) dx + h \int_{\Omega} J(\mathbb{E}(\mathbf{u}_*) + \varepsilon^2 \mathbb{E}(\mathbf{z}_w) + \frac{\varepsilon^2}{2} Dw \otimes Dw) dx \right] - \\
&\quad - \varepsilon^{-2} \left[ h \int_{\Omega} J(\mathbb{E}(\mathbf{u}_*)) + \varepsilon^2 h \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{z}_w \right] = \\
&= \varepsilon^{-2} \left[ \frac{h^3}{12} \varepsilon^2 \int_{\Omega} J(D^2 w) dx + h \int_{\Omega} J(\mathbb{E}(\mathbf{u}_*) + \varepsilon^4 h \int_{\Omega} J \left( \mathbb{E}(\mathbf{z}_w) + \frac{1}{2} Dw \otimes Dw \right) dx \right] + \\
&\quad + \varepsilon^{-2} \left[ \varepsilon^2 h \int_{\Omega} J'(\mathbb{E}(\mathbf{u}_*)) : \left( \mathbb{E}(\mathbf{z}_w) + \frac{1}{2} Dw \otimes Dw \right) dx - h \int_{\Omega} J(\mathbb{E}(\mathbf{u}_*)) - \varepsilon^2 \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{z}_w \right] = \\
&= \frac{h^3}{12} \int_{\Omega} J(D^2 w) dx + \varepsilon^2 h \int_{\Omega} J \left( \mathbb{E}(\mathbf{z}_w) + \frac{1}{2} Dw \otimes Dw \right) dx + \\
&\quad + h \int_{\Omega} J'(\mathbb{E}(\mathbf{u}_*)) : \left( \mathbb{E}(\mathbf{z}_w) + \frac{1}{2} Dw \otimes Dw \right) dx - h \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{z}_w = \\
&= \frac{h^3}{12} \int_{\Omega} J(D^2 w) dx + \varepsilon^2 h \int_{\Omega} J \left( \mathbb{E}(\mathbf{z}_w) + \frac{1}{2} Dw \otimes Dw \right) dx + \\
&\quad + \frac{h}{2} \int_{\Omega} J'(\mathbb{E}(\mathbf{u}_*)) : Dw \otimes Dw dx
\end{aligned}$$

since, due to minimality of  $\mathbf{u}_*$ ,

$$\int_{\Omega} J'(\mathbb{E}(\mathbf{u}_*)) : \mathbb{E}(\mathbf{z}_w) dx - \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{z}_w = 0.$$

Hence  $\lim_{\varepsilon \rightarrow 0} \mathcal{E}_{h,\varepsilon}^i(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w) = \mathcal{E}_h^i(\mathbf{u}_*, w)$  as claimed.

Eventually we prove b). By recalling (2.4) and (2.10), we get for every  $\mathbf{z} \in H^1(\Omega, \mathbb{R}^2)$  and  $\omega \in \mathcal{A}^i$

$$(\mathcal{E}_{h,\varepsilon}^i)'(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w)[(\mathbf{z}, \omega)] = \varepsilon^{-2} \left( \tau_1(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w)[\mathbf{z}], \tau_2(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w)[\omega] \right).$$

Since  $\mathbf{z}_w \in \operatorname{argmin} \mathcal{Q}(\mathbf{z})$ ,  $\mathbf{u}_* \in \operatorname{argmin} \mathcal{F}_{h,0}$  and  $w \in \mathcal{K}_w^i(\mathbf{u}_*)$  we get:

$$\tau_1(\mathbf{u}_*, 0)[\mathbf{z}] = 0 \quad \forall \mathbf{z} \in H^1(\Omega, \mathbb{R}^2), \quad \tau_2(\mathbf{u}_*, w)[\omega] = 0 \quad \forall \omega \in \mathcal{A}^i,$$

$$\varepsilon^{-2} \tau_2(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w)[\omega] = \varepsilon^2 \int_{\Omega} J' \left( \mathbb{E}(\mathbf{z}_w) + \frac{1}{2} Dw \otimes Dw \right) : Dw \otimes Dw.$$

The above relationships together with (2.12) imply

$$\sup_{\|(\mathbf{z}, \omega)\| \leq 1} |(\mathcal{E}_{h,\varepsilon}^i)'(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w)[(\mathbf{z}, \omega)]| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

where  $\|(\mathbf{z}, \omega)\| = \|\mathbf{z}\|_{H^1} + \|\omega\|_{H^2}$ , thus proving b).  $\square$

**Remark 2.4.** Let  $\mathbf{u}_* \in \operatorname{argmin} \mathcal{F}_{h,0}$ ,  $w \in \mathcal{K}_h^i(\mathbf{u}_*)$  then

$$(2.14) \quad 0 = \mathcal{E}_h^i(\mathbf{u}_*, w)'[(\mathbf{0}, w)] = \frac{h^3}{12} \int_{\Omega} J'(D^2 w) \cdot D^2 w \, d\mathbf{x} + h \int_{\Omega} J'(\mathbb{E}(\mathbf{u}_*)) : Dw \otimes Dw$$

that is  $\mathcal{E}_h^i(\mathbf{u}_*, w) = 0$  and  $\mathcal{E}_{h,\varepsilon}^i(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w) = \varepsilon^2 h \min \mathcal{Q}_w$ .

**Remark 2.5.** In Theorem 2.3 we have shown that every critical point for  $\mathcal{E}_h^i$  of the kind  $(\mathbf{u}_*, w)$ , with  $\mathbf{u}_* \in \operatorname{argmin} \mathcal{F}_{h,0}$  and  $w \in \mathcal{K}_h^i(\mathbf{u}_*)$ , can be approximated (in the strong convergence of  $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)$ ) by uniform Palais-Smale sequences of  $\mathcal{E}_{h,\varepsilon}^i$ . Actually the displacement pair sequence can be chosen explicitly of the kind  $(\mathbf{u}_* + \varepsilon^2 \mathbf{z}_w, w)$ , say with fixed out-of-plane component and in-plane displacement approximated by an infinitesimal correction tuned by the out-of-plane component. Nevertheless we cannot expect that every uniform Palais-Smale sequence of  $\mathcal{E}_{h,\varepsilon}^i$  is equibounded in  $H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^i$ , as we are going to show in the next Counterexample.

**Counterexample 2.6. (*a uniform Palais-Smale sequence lacking compactness*)**

If  $\Omega = (0, a) \times (0, 1)$ ,  $\Gamma \equiv \partial\Omega$  and  $\mathbf{f}_h = \gamma \mathbf{e}_2(\mathbf{1}_{(0,a) \times \{0\}} - \mathbf{1}_{(0,a) \times \{1\}})$ , where  $\gamma$  is a suitable constant to be chosen later, then the unboundedness may develop.

So by Theorem 1.8 (clamped plate),  $\forall h > 0, \forall \varepsilon > 0$  there exists  $(\mathbf{u}_\varepsilon, w_\varepsilon) \in \operatorname{argmin} \mathcal{E}_{h,\varepsilon}^0$ . Hence  $(\mathbf{u}_\varepsilon, w_\varepsilon)$  is a uniform Palais-Smale sequence for  $\mathcal{E}_h^0$ , moreover we show below that such a sequence must lack weak compactness in  $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)$  for big  $\gamma$ . Indeed, if compactness were true, we would obtain (up to subsequences) that  $(\mathbf{u}_\varepsilon, w_\varepsilon) \rightharpoonup (\mathbf{u}, w) \in \operatorname{argmin} \mathcal{E}_h^0$ , due to Proposition 2.1. Eventually we show that  $\inf \mathcal{E}_h^0 = -\infty$ , thus obtaining a contradiction.

Actually, due to Euler equations

$$(2.15) \quad \int_{\Omega} J'(\mathbb{E}(\mathbf{u})) : \mathbb{E}(\mathbf{v}) = \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{v} = -\gamma \int_{\Omega} v_{2,2} \quad \forall \mathbf{v} \in H^1(\Omega, \mathbb{R}^2),$$

so, for every  $\mathbf{u} \in \operatorname{argmin} \mathcal{F}_{h,0}$ ,  $J'(\mathbb{E}(\mathbf{u})) = -\gamma \mathbf{e}_2 \otimes \mathbf{e}_2$ ,  $\mathbf{u} = 2\frac{\gamma\nu}{E} \frac{1+\nu}{1+3\nu} (x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2) + r$ ,  $r \in \mathcal{R}$ , and by (2.5)

$$(2.16) \quad \mathcal{E}_h^0(\mathbf{u}, w) = \begin{cases} \frac{h^3}{12} \int_{\Omega} J(D^2 w) - \frac{h\gamma}{2} \int_{\Omega} |w_{,2}|^2 d\mathbf{x}, & \text{if } \mathbf{u} \in \operatorname{argmin} \mathcal{F}_{h,0}(\cdot, 0), w \in \mathcal{A}^0, \\ +\infty & \text{otherwise in } H^1(\Omega) \times H^2(\Omega). \end{cases}$$

Hence, if  $\mathbf{u} \in \operatorname{argmin} \mathcal{F}_{h,0}$ ,  $w \in \mathcal{A}^0$ , we get

$$\mathcal{E}_h^0(\mathbf{u}, w) \leq \frac{C_\nu E h^3}{24} \int_{\Omega} |D^2 w|^2 d\mathbf{x} - \frac{h}{2} \gamma \int_{\Omega} |w_{,2}|^2 d\mathbf{x}.$$

Set  $w(x_1, x_2) = \alpha(x_1)\beta(x_2)$ , with  $\alpha \in H_0^2(0, a)$  and  $\beta \in H_0^2(0, 1)$ . Then  $w \in H_0^2(\Omega)$  and

$$(2.17) \quad \mathcal{E}_h^2(\mathbf{u}, w) \leq (A_0 C_0 + A_1 C_1 + A_2) C_\nu \frac{E h^3}{24} \int_0^1 |\beta''|^2 dx_2 - \frac{A_2 h \gamma}{2} \int_0^1 |\beta'|^2 dx_2,$$

where

$$A_0 = \int_0^1 |\alpha''|^2 dx_1, \quad A^1 = 2 \int_0^1 |\alpha'|^2 dx_1, \quad A_2 = \int_0^1 \alpha^2 dx_1$$

and  $C_0, C_1$  are the best constants such that

$$\int_0^1 \beta^2 dx_2 \leq C_0 \int_0^1 |\beta''|^2 dx_2, \quad \int_0^1 |\beta'|^2 dx_2 \leq C_1 \int_0^1 |\beta''|^2 dx_2 \quad \forall \beta \in H_0^2(0, 1)$$

If  $\xi \in H_0^2(0, 1)$  is the eigenfunction fulfilling the equality  $\int_0^1 |\xi'|^2 dx_2 = C_1 \int_0^1 |\xi''|^2 dx_2$  and

$$\gamma > \frac{1}{6} (A_0 C_0 + A_1 C_1 + A_2) C_\nu E h^2 / (A_2 C_1).$$

Setting  $\beta_n := n\xi \in H_0^2(0, 1)$  and  $w = \alpha\beta_n$ , the right-hand side of (2.17) goes to  $-\infty$  as  $n \rightarrow \infty$ .

In the previous counterexample we have shown that some uniform Palais-Smale sequence may be not converging to any critical point, while in the next examples we show how Theorem 2.3 can be used to detect buckled configurations of the plate (associated to critical points for **FvK**) by means of uniform Palais-Smale sequences for the approximating functionals.

**Example 2.7. (buckling of a rectangular plate under compressive load)**

Set  $\Omega = (0, a) \times (0, 1)$ ,  $\mathbf{f}_h = \gamma \mathbf{e}_2 (\mathbf{1}_{(0,a) \times \{0\}} - \mathbf{1}_{(0,a) \times \{1\}})$  and  $\Gamma = \Sigma_+ \cup \Sigma_-$ , with  $\Sigma_+ = [0, 1] \times \{1\}$ ,  $\Sigma_- = [0, 1] \times \{0\}$ .

Now  $\Gamma \neq \partial\Omega$ : by arguing as the in previous Counterexample we find noncompact uniform Palais-Smale sequences together with energy of admissible configurations unbounded from below.

In the present case we push forward the analysis: as before we find that if  $\mathbf{u} \in \operatorname{argmin} \mathcal{F}_{h,0}$  and  $w \in \mathcal{A}^i$ ,  $i=0,1,2$ , then  $J'(\mathbb{E}(\mathbf{u})) = -\gamma \mathbf{e}_2 \otimes \mathbf{e}_2$ , so that

$$\mathcal{E}_h^i(\mathbf{u}, w) = \frac{h^3}{12} \int_\Omega J(D^2 w) d\mathbf{x} - \frac{h\gamma}{2} \int_\Omega |w_{,2}|^2 d\mathbf{x} \quad \text{if } \mathbf{u} \in \operatorname{argmin} \mathcal{F}_{h,0}^i(\cdot, 0), \quad w \in \mathcal{A}^i.$$

We look for critical points in the form  $w = w(x_2)$  under the following conditions:

$$\begin{aligned} w(0) = w(1) = w'(0) = w'(1) &= 0, & \text{if } i = 0; \\ w(0) = w(1) &= 0, & \text{if } i = 1; \\ w(0)'' = w''(1) = w'''(0) = w'''(1) &= 0, & \text{if } i = 2. \end{aligned}$$

Since  $J(\mathbf{e}_2 \otimes \mathbf{e}_2) = \frac{E}{2(1-\nu^2)}$ , we have

$$\mathcal{E}_h^i(\mathbf{u}, w) = \frac{E h^3}{24(1-\nu^2)} \int_0^1 |w''(x_2)|^2 dx_2 - \frac{h\gamma a}{2} \int_0^1 |w'(x_2)|^2 dx_2$$

whose non-trivial critical points can be easily computed, via the ODE

$$w'''' + \frac{12\gamma a(1-\nu^2)}{Eh^2}w'' = 0.$$

Theorem 2.3 allows to recover Palais-Smale sequences for  $\mathcal{E}_{h,\varepsilon}^i$ ,  $i = 0, 1, 2$ .

In the clamped case ( $i = 0$ ) the nontrivial buckled solutions occur for discrete choices of  $h$ :

$$h_n = \frac{1}{2n\pi} \sqrt{\frac{12\gamma a(1-\nu^2)}{E}}, \quad w_n(x_2) = 1 + \sin\left(\sqrt{\frac{12\gamma a(1-\nu^2)}{E}} \frac{1}{h}(x_2 - \pi/2)\right), \quad n \in \mathbb{N};$$

else, for any other choice of  $h$ ,  $w \equiv 0$ .

The associated Palais-Smale sequence is  $(2\frac{\gamma\nu}{E}\frac{1+\nu}{1+3\nu}(x_1\mathbf{e}_1+x_2\mathbf{e}_2) + \varepsilon^2\mathbf{z}_{w_n}(x_1, x_2), w_n(x_2))$ , where  $\mathbf{z}_{w_n}(x_1, x_2) = (0, 1/2 \int_0^{x_2} |w'_n(t)|^2 dt)$  and  $w_n$  is given above.

**Example 2.8.** (*buckling of a rectangular plate under shear load*).

Set  $\Omega = (-2, 2) \times (-1, 1)$ ,  $i = 0$ , and  $\Gamma = \Sigma^{1,\pm} \cup \Sigma^{2,\pm}$ , where:

$$\Sigma^{1,+} = [-2, 0] \times \{1\}, \quad \Sigma^{1,-} = [0, 2] \times \{-1\}, \quad \Sigma^{2,+} = \{2\} \times [-1, 1], \quad \Sigma^{2,-} = \{-2\} \times [-1, 1].$$

Assume  $\mathbf{f}_h = \gamma\boldsymbol{\tau}(\mathbf{1}_{S^{2,\pm}} - \mathbf{1}_{S^{1,\pm}})$ , where  $S^{2,\pm} = \Sigma^{2,\pm}$ ,  $S^{1,\pm} = [-2, 2] \times \{\pm 1\}$ ,  $\gamma > 0$ ,  $\boldsymbol{\tau}$  is the counterclockwise oriented tangent unit vector to  $\partial\Omega = S^{1,\pm} \cup S^{2,\pm}$ .

Since  $\mathbf{u} \in \operatorname{argmin} \mathcal{F}_{h,0}$ , by exploiting Euler-Lagrange equations as before, we obtain  $J'(\mathbb{E}(\mathbf{u})) = \gamma(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$  and by (2.5)

$$\mathcal{E}_h^0(\mathbf{u}, w) = \frac{h^3}{12} \int_{\Omega} J(D^2w) dx + h\gamma \int_{\Omega} w_{,1} w_{,2} dx.$$

We look for critical points in the form

$$(2.18) \quad w = \begin{cases} \psi(x_1 - x_2) & \text{if } (x_1, x_2) \in \Omega, \quad |x_1 - x_2| \leq 1 \\ 0 & \text{else in } \Omega, \end{cases}$$

and satisfying  $\psi(\pm 1) = \psi'(\pm 1) = 0$ .

By  $J(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1) = \frac{2E}{1-\nu^2}$  we obtain

$$\mathcal{E}_h^0(\mathbf{u}, w) = \frac{h^3 E}{3(1-\nu^2)} \int_{-1}^1 |\psi''(t)|^2 dt - 2h\gamma \int_{-1}^1 |\psi'(t)|^2 dt$$

whose nontrivial critical points can be easily computed, via the ODE

$$\psi'''' + \frac{6\gamma(1-\nu^2)}{Eh^2} \psi'' = 0, \quad \psi(\pm 1) = \psi'(\pm 1) = 0.$$

Therefore even now the nontrivial buckled solutions occur for (different) discrete choices of  $h$ :

$$w = w_n(x_1, x_2) = \psi_n(x_1 - x_2) := 1 + \sin\left(\sqrt{\frac{12\gamma a(1-\nu^2)}{E}} \frac{1}{h_n}(x_1 - x_2 + 1/2)\right)$$

$$\text{if } h_n = \frac{1}{n\pi} \sqrt{\frac{12\gamma a(1-\nu^2)}{E}}, \quad \text{with } n \in \mathbb{N};$$

else, we have the flat solution  $w \equiv 0$  for any other choice of  $h$ .

The associated Palais-Smale sequence is  $(\mathbf{u}(x_1, x_2) + \varepsilon^2 \mathbf{z}_{w_n}(x_1, x_2), w_n(x_1, x_2))$ , where  $\mathbf{u}(x_1, x_2) = \gamma \frac{1+\nu}{E}(x_2, x_1)$ ,  $\mathbf{z}_{w_n}(x_1, x_2) = \left( - (1/2) \int_{-1}^{x_1-x_2} |w'_n(t)|^2 dt, (1/2) \int_{-1}^{x_1-x_2} |w'_n(t)|^2 dt \right)$ .

**Remark 2.9.** In Examples 2.7, 2.8, when nontrivial solutions exist the period of the oscillations has order  $h$ . By scaling loads, that is by taking  $\mathbf{f}_h = h^\alpha \mathbf{f}$ , we get  $J'(\mathbb{E}(\mathbf{u})) = -h^\alpha \gamma (\mathbf{e}_2 \otimes \mathbf{e}_2)$  and  $J'(\mathbb{E}(\mathbf{u})) = h^\alpha \gamma (\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$  respectively, while related limit functionals become respectively

$$\mathcal{E}_h^i(\mathbf{u}, w) = \frac{Eh^3}{24(1-\nu^2)} \int_0^1 |w''(x_2)|^2 dx_2 - \frac{h^{\alpha+1}\gamma a}{2} \int_0^1 |w'(x_2)| dx_2, \quad i = 0, 1, 2,$$

$$\mathcal{E}_h^0(\mathbf{u}, w) = \frac{h^3 E}{3(1-\nu^2)} \int_{-1}^1 |w''(t)|^2 dt - 2h^{\alpha+1}\gamma \int_{-1}^1 |w'(t)|^2 dt,$$

whose nontrivial critical points obviously exhibit oscillation period of order  $h^{1-\alpha/2}$ .

Computations in Remark 2.9 proves useful in the next Section when studying asymptotics of the problem as the thickness tends to  $0_+$ .

### 3. SCALING FÖPPL-VON KÁRMÁN ENERGY

Here we focus on the asymptotic analysis of the mechanical problems for **Fvk** plate as  $h \rightarrow 0_+$ . To highlight properties of the limit solution we examine the behavior of suitably scaled energy: all along this Section we assume that there is no transverse load, say  $g_h \equiv 0$ , while we refer to a parameter  $\alpha$  characterizing different asymptotic regimes of in-plane load  $\mathbf{f}_h$ , say

$$(3.1) \quad \mathbf{f}_h = h^\alpha \mathbf{f} \quad \text{where } \alpha \geq 0 \quad \text{and } \mathbf{f} \in L^2(\partial\Omega, \mathbb{R}^2).$$

The next result and the subsequent counterexample show how the choice of  $\alpha$  may influence the asymptotic behavior of functionals  $\mathcal{F}_h$  when  $h \rightarrow 0_+$ .

**Theorem 3.1.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded connected Lipschitz open set,  $\alpha \geq 2$  and  $i = 0, 1, 2$ . If  $i = 0$  (clamped plate) assume (1.29) and  $\Gamma = \partial\Omega$  (as in Theorem 1.8).*

*If  $i = 1$  (simply supported plate) assume (1.29),  $\Omega$  strictly convex,  $\Gamma = \partial\Omega$  (as in Theorem 1.6).*

*If  $i = 2$  (free plate) assume (1.12) and (1.13) (as in Theorem 1.1).*

Set

$$(3.2) \quad \mathcal{F}^{i,\alpha}(\mathbf{v}, \zeta) = \begin{cases} \mathcal{F}_1^i(\mathbf{v}, \zeta) & \text{if } \alpha = 2 \\ \mathcal{F}_1^i(\mathbf{v}, \zeta) + \chi_{\{D^2\zeta=0\}}(\zeta) & \text{if } \alpha > 2, \end{cases}$$

where  $\chi_{\{D^2\zeta \equiv 0\}}(\zeta) = 0$  if  $D^2\zeta \equiv 0$ ,  $= +\infty$  else.

Fix  $i \in \{0, 1, 2\}$  and a sequence  $(\mathbf{u}_h, w_h)$  in  $\operatorname{argmin} \mathcal{F}_h^i$ .

Then there exists  $(\mathbf{v}, \zeta) \in \operatorname{argmin} \mathcal{F}^{i,\alpha}$  such that, up to subsequences,

$$(3.3) \quad (h^{-\alpha}\mathbf{u}_h, h^{-\alpha/2}w_h) \rightarrow (\mathbf{v}, \zeta) \quad \text{weakly in } H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega), \text{ as } h \rightarrow 0_+.$$

Moreover

$$(3.4) \quad h^{-2\alpha-1}\mathcal{F}_h^i(\mathbf{u}_h, w_h) \rightarrow \mathcal{F}^{i,\alpha}(\mathbf{v}, \zeta), \text{ as } h \rightarrow 0_+.$$

*Proof.* The case  $\alpha = 2$  is trivial since  $(\mathbf{u}_h, w_h) \in \operatorname{argmin} \mathcal{F}_h^i$  if and only if  $(h^{-2}\mathbf{u}_h, h^{-1}w_h) \in \operatorname{argmin} \mathcal{F}_1^i$  for every  $h$ .

If  $\alpha > 2$ ,  $i = 0, 1$  and  $(\mathbf{u}_h, w_h) \in \operatorname{argmin} \mathcal{F}_h^i$ , set  $\mathbf{v}_h := h^{-\alpha}\mathbf{u}_h$ ,  $\zeta_h := h^{-\alpha/2}w_h$ ,  $\lambda_h = \|\mathbb{E}(\mathbf{v}_h)\|_{L^2}$  and assume by contradiction  $\lambda_h \rightarrow +\infty$ . Then by taking into account minimality of  $(\mathbf{u}_h, w_h)$ , (1.7), (1.29) and setting  $\varphi_h = \lambda_h^{-1/2}\zeta_h$ ,  $\mathbf{z}_h = \lambda_h^{-1}\mathbf{v}_h$  we get

$$(3.5) \quad c_\nu \frac{h^{2-\alpha} E}{24} \int_{\Omega} |D^2\varphi_h|^2 + \lambda_h c_\nu \frac{E}{2} \int_{\Omega} |\mathbb{D}(\mathbf{z}_h, \varphi_h)|^2 \leq \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{z}_h \leq C.$$

Hence  $|D^2\varphi_h| \rightarrow 0$  in  $L^2(\Omega, \operatorname{Sym}_{2,2}(\mathbb{R}))$  and by taking into account that  $\varphi_h = 0$  on  $\partial\Omega$  we get  $\varphi_h \rightarrow 0$  in  $H^2(\Omega)$ ; therefore  $\mathbb{E}(\mathbf{z}_h) \rightarrow \mathbb{O}$  in  $L^2(\Omega, \mathbb{R}^2)$ , a contradiction since  $\|\mathbb{E}(\mathbf{z}_h)\|_{L^2} = 1$ . Then  $\lambda_h$  is bounded from above and by taking into account minimality of  $(\mathbf{u}_h, w_h)$ , (1.7), (1.29) we get

$$(3.6) \quad c_\nu \frac{h^{2-\alpha} E}{24} \int_{\Omega} |D^2\zeta_h|^2 + c_\nu \frac{E}{2} \int_{\Omega} |\mathbb{D}(\mathbf{v}_h, \zeta_h)|^2 \leq \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v}_h \leq \|\mathbf{f}\| \lambda_h \leq C$$

which entails  $D^2\zeta_h \rightarrow 0$  in  $L^2(\Omega)$  and equiboundedness of  $D\zeta_h$  in  $L^4(\Omega, \mathbb{R}^2)$ .

When  $i = 2$  we take again  $\lambda_h = \|\mathbb{E}(\mathbf{v}_h)\|_{L^2}$  and assume by contradiction  $\lambda_h \rightarrow +\infty$ . Then estimate (3.5) continues to hold and as before  $|D^2\varphi_h| \rightarrow 0$  in  $L^2(\Omega)$  which entails  $\varphi_h - \int_{\Omega} \varphi_h \rightarrow 0$  in  $L^2(\Omega)$ ,  $D\varphi_h \rightarrow \mathbf{c}$  in  $L^4$  and  $2\mathbb{E}(\mathbf{z}_h) \rightarrow -\mathbf{c} \otimes \mathbf{c}$  strongly in  $L^2(\Omega, \operatorname{Sym}_{2,2}(\mathbb{R}))$  for a suitable  $\mathbf{c} \in \mathbb{R}^2$ . Therefore (1.13), (3.5) yield

$$(3.7) \quad 0 \leq \lim_{h \rightarrow 0_+} \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{z}_h = \lim_{h \rightarrow 0_+} f \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{z}_h = \lim_{h \rightarrow 0_+} f \int_{\Omega} \operatorname{div} \mathbf{z}_h = -\frac{f}{2} |\Omega| |\mathbf{c}|^2$$

that is  $\mathbf{c} = \mathbf{0}$  so  $\mathbb{E}(\mathbf{z}_h) \rightarrow \mathbb{O}$  in  $L^2(\Omega, \operatorname{Sym}_{2,2}(\mathbb{R}))$  as in the previous cases, again a contradiction. Thus equiboundedness holds in this case too. Since, for  $0 < h \leq 1$ , the w.l.s.c. functionals  $\mathcal{F}^{i,\alpha}$  fulfil  $\mathcal{F}^{i,\alpha} \leq h^{-2\alpha-1}\mathcal{F}_h^i$ , the proof can be completed by a standard argument in  $\Gamma$  convergence.  $\square$

**Remark 3.2.** It is worth noticing that when  $D^2w \equiv 0$  then

$$(3.8) \quad \mathcal{F}_1(\mathbf{v}, w) = \mathcal{F}_1(\mathbf{v}, 0), \quad \text{if } i = 0, 1,$$

$$(3.9) \quad \mathcal{F}_1(\mathbf{v}, w) = \mathcal{F}_1(\mathbf{v}, \boldsymbol{\xi} \cdot \mathbf{x}) = \int_{\Omega} J(\mathbb{E}(\mathbf{v})) + \frac{1}{2} \boldsymbol{\xi} \otimes \boldsymbol{\xi} - \int_{\partial\Omega} \mathbf{f}_h \cdot \mathbf{v} \quad \text{for } w = \boldsymbol{\xi} \cdot \mathbf{x}, \text{ if } i = 2.$$

Theorem 3.1 is optimal in the sense that if  $\alpha < 2$  we cannot expect neither that  $h^{-2\alpha-1} \min_{\mathcal{A}^i} \mathcal{F}_h$  are bounded from below nor that minimizers are equibounded in  $H^1(\Omega, \mathbb{R}^2) \times H^{1,4}(\Omega)$  when we let  $h \rightarrow 0_+$ . This phenomenon may take place even if  $\Omega$  is a rectangle as shown by the next Counterexample, where we consider a plate with the same geometry and load of Counterexample 2.6, nevertheless here we push further the analysis of this case.

**Counterexample 3.3.** Let  $a > EC_\nu$ ,  $\alpha \in [0, 2)$ ,  $\mathbf{f}_h = h^\alpha \mathbf{f}$  with

$$(3.10) \quad \Omega = (0, a) \times (0, 1), \quad \Gamma = \partial\Omega, \quad g_h \equiv 0, \quad \mathbf{f} = (\mathbf{1}_{\{y=0\}} - \mathbf{1}_{\{y=1\}}) \mathbf{e}_2.$$

Then for any sequence  $(\mathbf{u}_h, w_h) \in \arg \min \mathcal{F}_h^0$  (such sequences do exist due to Theorem 1.8), the scaled sequence  $(h^{-\alpha} \mathbf{u}_h, h^{-\alpha/2} w_h)$  is not equibounded in  $H^1(\Omega, \mathbb{R}^2) \times H^{1,4}(\Omega)$ . Moreover,  $\inf h^{-2\alpha-1} \mathcal{F}_h^0 \rightarrow -\infty$  as  $h \rightarrow 0_+$ .

Indeed we can set:  $\mathbf{v}_h := h^{-\alpha} \mathbf{u}_h$ ,  $\zeta_h := h^{-\alpha/2} w_h$ , and

$$(3.11) \quad \mathcal{W}_h(\mathbf{v}_h, \zeta_h) := h^{-1-2\alpha} \mathcal{F}_h(\mathbf{u}_h, w_h) = \frac{h^{2-\alpha}}{12} \int_{\Omega} J(D^2 \zeta_h) + \int_{\Omega} J(\mathbb{D}(\mathbf{v}_h, \zeta_h)) - \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v}_h,$$

$$(3.12) \quad \mathcal{I}^+(\mathbf{v}, \zeta) := \inf \left\{ \limsup_{h \rightarrow 0_+} \mathcal{W}_h(\mathbf{v}_h, \zeta_h) : \mathbf{v}_h \xrightarrow{w-H^1} \mathbf{v}, \zeta_h \xrightarrow{w-H^{1,4}} \zeta \right\},$$

$$(3.13) \quad \mathcal{I}^-(\mathbf{v}, \zeta) := \inf \left\{ \liminf_{h \rightarrow 0_+} \mathcal{W}_h(\mathbf{v}_h, \zeta_h) : \mathbf{v}_h \xrightarrow{w-H^1} \mathbf{v}, \zeta_h \xrightarrow{w-H^{1,4}} \zeta \right\},$$

$$(3.14) \quad \mathcal{J}(\mathbb{B}, \boldsymbol{\eta}) = \frac{E}{8(1+\nu)} |\mathbb{B} + \mathbb{B}^T + \boldsymbol{\eta} \otimes \boldsymbol{\eta}|^2 + \frac{E\nu}{8(1-\nu^2)} |\text{Tr}(\mathbb{B} + \mathbb{B}^T + \boldsymbol{\eta} \otimes \boldsymbol{\eta})|^2.$$

Then by arguing as in Lemma 4.1 of [14] we get

$$(3.15) \quad \mathcal{I}^+(\mathbf{v}, \zeta) \leq \Lambda(\mathbf{v}, \zeta) := \int_{\Omega} \mathcal{J}(\mathbb{D}(\mathbf{v}, \zeta)) dx - \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v} d\mathcal{H}^1.$$

Then by denoting with  $Q\mathcal{J}$  the quasiconvex envelope of  $\mathcal{J}$ , since  $\mathcal{I}^+$  is sequentially lower semicontinuous in  $w-H^1 \times w-H^{1,4}$ , we obtain

$$(3.16) \quad \mathcal{I}^+(\mathbf{v}, \zeta) \leq \int_{\Omega} Q\mathcal{J}(D\mathbf{v}, D\zeta) dx - \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v} dx.$$

On the other hand for every  $\mathbf{v}_h \xrightarrow{w-H^1} \mathbf{v}$ ,  $\zeta_h \xrightarrow{w-H^{1,4}} \zeta$  we get

$$\liminf_{h \rightarrow 0_+} h^{-1-2\alpha} \mathcal{F}_h(h^\alpha \mathbf{v}_h, h^{\alpha/2} \zeta_h) \geq \int_{\Omega} Q\mathcal{J}(D\mathbf{v}, D\zeta) - \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v}$$

that is

$$\mathcal{I}^-(\mathbf{v}, \zeta) \geq \int_{\Omega} Q\mathcal{J}(D\mathbf{v}, D\zeta) - \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v}.$$

By

$$(3.17) \quad \mathcal{I}(\mathbf{v}, \zeta) := \int_{\Omega} Q\mathcal{J}(D\mathbf{v}, D\zeta) - \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v} \geq \mathcal{I}^+(\mathbf{v}, \zeta) \geq \mathcal{I}^-(\mathbf{u}, w) \geq \mathcal{I}(\mathbf{v}, \zeta)$$

we get

$$(3.18) \quad \Gamma \lim_{h \rightarrow 0_+} \mathcal{W}_h = \mathcal{I}.$$

Therefore, if  $(h^{-\alpha} \mathbf{u}_h^*, h^{-\alpha/2} w_h^*)$  were equibounded in  $H^1(\Omega, \mathbb{R}^2) \times H^{1,4}(\Omega)$  then

$$h^{-1-2\alpha} \mathcal{F}_h(\mathbf{u}_h^*, w_h^*) \rightarrow \min \mathcal{I} = \inf \Lambda$$

since  $\Lambda$  is the relaxed functional of  $\mathcal{I}$ , and we will show that this leads to a contradiction. Indeed, we choose

$$(3.19) \quad \zeta_n(x, y) = \frac{1}{\sqrt{n}} \varphi(ny) \psi_n(x), \quad \mathbf{v}_n(x, y) = (0, \frac{-n}{2}y),$$

with

$$(3.20) \quad \varphi : \mathbb{R} \rightarrow \mathbb{R}, \quad 1\text{-periodic}, \quad \varphi(y) = \frac{1}{2}(1 - |1 - 2y|) \quad \forall y \in (0, 1)$$

$$(3.21) \quad \psi_n(x) = nx \mathbf{1}_{\{[0, 1/n]\}} + \mathbf{1}_{\{[1/n, a-1/n]\}} - n(x-a) \mathbf{1}_{\{[a-1/n, a]\}}.$$

We get

$$\mathbb{E}(\mathbf{v}_n) = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{n}{2} \end{bmatrix}, \quad \mathbb{D}(\mathbf{v}_n, \zeta_n) = \begin{bmatrix} \frac{1}{2n} (\psi'_n(x))^2 (\varphi(ny))^2 & \frac{1}{2} \psi_n(x) \psi'_n(x) \varphi(ny) \varphi'(ny) \\ \frac{1}{2} \psi_n(x) \psi'_n(x) \varphi(ny) \varphi'(ny) & \frac{n}{2} (\psi_n^2(x) |\varphi'(ny)|^2 - 1) \end{bmatrix}$$

and by taking into account (1.6), (1.7) and that  $2|\varphi| \leq 1$ ,  $|\varphi'| = 1$ ,  $|\psi| \leq 1$ ,  $|\psi'_n| \leq n$ ,  $\text{spt } \psi'_n \subset [0, 1/n] \cup [a-1/n, a]$ ,  $|\psi_n| = 1$  on  $[1/n, a-1/n]$ ,  $a > EC_\nu$ ,

$$\begin{aligned} \Lambda(\mathbf{v}_n, \zeta_n) &= \int_0^a \int_0^1 J(\mathbb{D}(\mathbf{v}_n, \zeta_n)) \, dx \, dy - \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v}_n \, dx \, dy \leq \\ &\leq \int_0^a \int_0^1 \frac{EC_\nu}{8} \left( n^{-2} |\psi'_n(x)|^4 |\varphi(ny)|^4 + 2 |\psi_n(x)|^2 |\psi'_n(x)|^2 |\varphi(ny)|^2 |\varphi'(ny)|^2 + n^2 (\psi_n^2(x) |\varphi'(ny)|^2 - 1)^2 \right) - \frac{na}{2} \\ &\leq \int_0^a \int_0^1 \frac{EC_\nu}{8} \left( n^2 \mathbf{1}_{[0, 1/n] \cup [a-1/n, a]} + n^2 (\psi_n^2(x) - 1)^2 \right) - \frac{na}{2} \leq \frac{nEC_\nu}{2} - \frac{na}{2} \rightarrow -\infty. \end{aligned}$$

leads to a contradiction.

So  $(h^{-\alpha} \mathbf{u}_h^*, h^{-\alpha/2} w_h^*)$  are not equibounded in  $H^1(\Omega, \mathbb{R}^2) \times H^{1,4}(\Omega)$  and the first claim follows. Eventually we prove the second claim. By (3.15) there exists  $(\mathbf{v}_{n,h}, \zeta_{n,h}) \rightarrow (\mathbf{v}_n, \zeta_n)$  weakly in  $H^1(\Omega, \mathbb{R}^2) \times H^2(\Omega)$  such that  $\limsup \mathcal{W}_h(\mathbf{v}_{n,h}, \zeta_{n,h}) \leq \mathcal{I}(\mathbf{v}_n, \zeta_n) \leq -Kn$  for suitable  $K > 0$ , hence by using a diagonal argument we achieve the claim.

**Remark 3.4.** If  $a > EC_\nu$ ,  $\alpha \in [0, 2)$ ,  $\mathbf{f}_h = h^\alpha \mathbf{f}$  with

$$(3.22) \quad \Omega = (0, a) \times (0, 1), \quad g_h \equiv 0, \quad \mathbf{f} = (\mathbf{1}_{\{y=0\}} - \mathbf{1}_{\{y=1\}}) \mathbf{e}_2, \quad \Gamma = \partial\Omega,$$

Then  $h^{-1-2\alpha} \inf \mathcal{F}_h^i \rightarrow -\infty$  as  $h \rightarrow 0_+$  holds true also for  $i = 1, 2$ .

Indeed, though existence of minimizers of  $\mathcal{F}_h^i$ , ( $i = 1, 2$ ) may fail, nevertheless  $\inf \mathcal{F}_h^i \leq \inf \mathcal{F}_h^0$  for  $i = 1, 2$ ; hence the claim follows by previous Counterexample.

#### 4. PRESTRESSED PLATES: OSCILLATING VERSUS FLAT EQUILIBRIA.

Counterexample 3.3 and Remark 3.4 show that the Föppl Von Karman functional might not be suitable for studying equilibria of plates when thickness  $h \rightarrow 0_+$ , at least in presence of in-plane loads scaling as  $h^\alpha$ , when  $\alpha \in [0, 2)$  and  $h$  is the scale factor for the plate thickness.

To circumvent this difficulty, as in the case of many practical engineering applications, we assume that our plate-like structure is initially prestressed and undergoes a transverse displacement about the prestressed state.

Precisely, in this Section we fix  $g_h \equiv 0$ ,  $\mathbf{f} \in L^2(\partial\Omega, \mathbb{R}^2)$ ,  $\alpha \in [0, 2)$  and we assume that the prestressed state is caused by the (scaled) force field  $\mathbf{f}_h = h^\alpha \mathbf{f}$  and is given by every  $\mathbf{u}^* \in H^1(\Omega, \mathbb{R}^2)$ ,  $\mathbf{u}^* = h^\alpha \mathbf{v}^*$  where  $\mathbf{v}^*$  is a minimizer of the functional

$$(4.1) \quad \mathcal{F}(\mathbf{v}) := \int_{\Omega} J(\mathbb{E}(\mathbf{v})) - \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v}$$

The transverse displacement  $w$  is chosen such that the pair  $(\mathbf{u}^*, w)$  minimizes the functional  $\mathcal{G}_h$  over  $H^1(\Omega, \mathbb{R}^2) \times \mathcal{A}^i$ , defined by

$$\mathcal{G}_h(\mathbf{u}, w) = \begin{cases} \mathcal{F}_h(\mathbf{u}, w) & \text{if } \mathbf{u} = \mathbf{u}^* \text{ and } w \in \mathcal{A}^i, \\ +\infty & \text{else.} \end{cases}$$

Moreover we have  $\mathcal{G}_h(\mathbf{u}, w) = \tilde{\mathcal{G}}_h(\mathbf{v}, \zeta)$  when setting  $\mathbf{v} := h^{-\alpha} \mathbf{u}$ ,  $\zeta := h^{-\alpha/2} w$  and

$$\tilde{\mathcal{G}}_h(\mathbf{v}, \zeta) = \begin{cases} h^\alpha F_h^b(\zeta) + h^{2\alpha+1} \int_{\Omega} J(\mathbb{D}(\mathbf{v}, \zeta)) - h^{2\alpha+1} \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v}, & \text{if } \mathbf{v} \in \operatorname{argmin} \mathcal{F}, \zeta \in \mathcal{A}^i, \\ +\infty & \text{else in } H^1(\Omega) \times \mathcal{A}^i. \end{cases}$$

We aim to capture the nature of the transverse minimizer through a detailed study of the asymptotic behavior of minimizers of  $\tilde{\mathcal{G}}_h$  as  $h \rightarrow 0_+$ . A first hint in this perspective is the next result.

**Theorem 4.1.** For every  $\mathbf{v} \in \operatorname{argmin} \mathcal{F}$ , let  $I_{\mathbf{v}}^{**}(\mathbf{x}, \cdot)$  be the convex envelope of  $I_{\mathbf{v}}(\mathbf{x}, \cdot)$  where  $I_{\mathbf{v}}(\mathbf{x}, \boldsymbol{\xi}) := J\left(\mathbb{E}(\mathbf{v})(\mathbf{x}) + \frac{1}{2} \boldsymbol{\xi} \otimes \boldsymbol{\xi}\right)$ , and

$$(4.2) \quad \mathcal{G}^{**}(\mathbf{v}, \zeta) := \int_{\Omega} I_{\mathbf{v}}^{**}(\mathbf{x}, D\zeta) dx - \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v} d\mathcal{H}^1 \quad \forall \zeta \in H^{1,4}(\Omega).$$

Then, for every  $\alpha \in [0, 2)$ ,

$$(4.3) \quad h^{-2\alpha-1} \min_{\mathcal{A}^i} \tilde{\mathcal{G}}_h \rightarrow \begin{cases} \min \{ \mathcal{G}^{**}(\mathbf{v}, \zeta) : \zeta \in H^{1,4}(\Omega), \zeta = 0 \text{ on } \Gamma \} & \text{if } i = 0, 1 \\ \min \{ \mathcal{G}^{**}(\mathbf{v}, \zeta) : \zeta \in H^{1,4}(\Omega) \} & \text{if } i = 2. \end{cases}$$

Moreover if  $(\mathbf{v}, \zeta_h) \in \arg \min_{\mathcal{A}^i} \tilde{\mathcal{G}}_h$  then  $\zeta_h \rightarrow \zeta$  weakly in  $H^{1,4}(\Omega)$ , up to subsequences, with  $(\mathbf{v}, \zeta) \in \arg \min \mathcal{G}^{**}$ .

*Proof.* The claim is a straightforward consequence of techniques developed in Lemma 4.1 of [14] and standard relaxation of integral functionals.  $\square$

In order to characterize equilibrium configurations of  $\tilde{\mathcal{G}}_h$ , additional information about minimizers of functional  $\mathcal{G}^{**}$  are needed: actually a careful use of Theorem 4.1 allows to show explicit examples capturing the qualitative behavior of minimizers and their dependance on the thickness  $h$ .

To this aim, if  $\mathbb{A} \in \text{Sym}_{2,2}(\mathbb{R})$  we denote its ordered eigenvalues by  $\lambda_1(\mathbb{A}) \leq \lambda_2(\mathbb{A})$  and by  $\mathbf{v}_1(\mathbb{A}), \mathbf{v}_2(\mathbb{A})$  their corresponding normalized eigenvectors, which afterwards will be denoted shortly with  $\lambda_1, \lambda_2, \mathbf{v}_1, \mathbf{v}_2$  whenever there is no risk of confusion.

For every  $\nu \neq 1$ ,  $\boldsymbol{\xi} \in \mathbb{R}^2$  and  $\mathbb{A} \in \text{Sym}_{2,2}(\mathbb{R})$  we set

$$(4.4) \quad g_{\mathbb{A}}(\boldsymbol{\xi}) = |\mathbb{A} + \boldsymbol{\xi} \otimes \boldsymbol{\xi}|^2 + \frac{\nu}{(1-\nu)} (\text{Tr } \mathbb{A} + |\boldsymbol{\xi}|^2)^2.$$

**Lemma 4.2.** *If  $\nu \in (-1, 1/2)$ , then*

$$(4.5) \quad \min_{\boldsymbol{\xi} \in \mathbb{R}^2} g_{\mathbb{A}}(\boldsymbol{\xi}) = \begin{cases} g_{\mathbb{A}}(\mathbf{0}) & \text{if } \nu\lambda_2 + \lambda_1 \geq 0 \\ (1+\nu)(\lambda_2(\mathbb{A}))^2 & \text{if } \nu\lambda_2 + \lambda_1 < 0. \end{cases}$$

*Proof.* It is worth noticing that minimum in (4.5) is achieved since  $g_{\mathbb{A}} \in C(\mathbb{R}^2)$  and  $g_{\mathbb{A}}(\boldsymbol{\xi}) \rightarrow +\infty$  as  $|\boldsymbol{\xi}| \rightarrow +\infty$ . Let  $\mathbb{M} \in O(2)$  be such that  $\mathbb{M}^T \mathbb{A} \mathbb{M} = \text{diag}(\lambda_1, \lambda_2)$ . Then it is readily seen that by setting  $x := \boldsymbol{\xi} \cdot \mathbf{v}_1$ ,  $y := \boldsymbol{\xi} \cdot \mathbf{v}_2$  we have

$$\tilde{g}_{\mathbb{A}}(x, y) := g_{\mathbb{A}}(\boldsymbol{\xi}) = (x^2 + \lambda_1)^2 + (y^2 + \lambda_2)^2 + 2x^2y^2 + \frac{\nu}{1-\nu} (\lambda_1 + \lambda_2 + x^2 + y^2)^2$$

and an easy computation shows that if  $\nu\lambda_2 + \lambda_1 \geq 0$  then minimum is attained at  $(x, y) = (0, 0)$ . Else, if  $\nu\lambda_2 + \lambda_1 < 0$  then either  $\nu\lambda_1 + \lambda_2 \geq 0$  or  $\nu\lambda_2 + \lambda_1 \leq \nu\lambda_1 + \lambda_2 < 0$ .

In the first case  $D\tilde{g}_{\mathbb{A}}(x, y) = (0, 0)$  if and only if  $(x, y) \in \{(\pm\sqrt{-\nu\lambda_2 - \lambda_1}, 0), (0, 0)\}$  and  $\tilde{g}_{\mathbb{A}}(x, y) = (1+\nu)\lambda_2^2$  or  $g_{\mathbb{A}}(x, y) = g_{\mathbb{A}}(0, 0) > (1+\nu)\lambda_2^2$ ; in the latter one  $D\tilde{g}_{\mathbb{A}}(x, y) = (0, 0)$  also at  $(x_*, \pm y_*) = (0, \pm\sqrt{-\nu\lambda_2 - \lambda_1})$  with  $\tilde{g}_{\mathbb{A}}(x_*, \pm y_*) = (1+\nu)\lambda_1^2$ . Hence

$$\min_{\boldsymbol{\xi} \in \mathbb{R}^2} g_{\mathbb{A}}(\boldsymbol{\xi}) = (1+\nu)\lambda_2^2$$

if  $\nu\lambda_2 + \lambda_1 < 0 \leq \nu\lambda_1 + \lambda_2$  and

$$\min_{\boldsymbol{\xi} \in \mathbb{R}^2} g_{\mathbb{A}}(\boldsymbol{\xi}) = (1+\nu) \min\{\lambda_2^2, \lambda_1^2\}$$

if  $\nu\lambda_2 + \lambda_1 \leq \nu\lambda_1 + \lambda_2 < 0$ . In the latter case if  $\nu \in (-1, 0)$  then  $\lambda_1 \leq \lambda_2 \leq -\nu\lambda_1$ , hence  $\lambda_1 \leq \lambda_2 \leq 0$  and  $|\lambda_1| \geq |\lambda_2|$ . If  $\nu \in [0, 1/2)$  then  $\lambda_1 < 0$  and either  $\lambda_2 > |\lambda_1| > 0$  or  $\lambda_1 \leq \lambda_2 \leq 0$ . In the first case we get necessarily  $\nu > 0$  and  $|\lambda_1| > \nu^{-1}(1 - \nu)\lambda_2 > \lambda_2$ , a contradiction. Therefore  $|\lambda_2| \leq |\lambda_1|$  and

$$\min_{\boldsymbol{\xi} \in \mathbb{R}^2} g_{\mathbb{A}}(\boldsymbol{\xi}) = (1 + \nu)\lambda_2^2$$

whenever  $\nu\lambda_2 + \lambda_1 < 0$  thus proving the thesis.  $\square$

Lemma 4.2 proves quite useful in the perspective of the next Proposition and the subsequent Examples, since the two alternatives in the right-hand side of (4.5) correspond respectively to locally flat or oscillating equilibrium configurations.

**Proposition 4.3.** *If  $\mathbf{v}_* \in \operatorname{argmin} \mathcal{F}$  and the ordered eigenvalues  $\lambda_1 \leq \lambda_2$  of  $\mathbb{E}(\mathbf{v}_*)$  fulfil  $\nu\lambda_2 + \lambda_1 \geq 0$  in the whole set  $\Omega$ , then*

$$(4.6) \quad \tilde{\mathcal{G}}_h(\mathbf{v}_*, \zeta) \geq \tilde{\mathcal{G}}_h(\mathbf{v}_*, 0).$$

*If in addition  $\nu\lambda_2 + \lambda_1 > 0$  in a set of positive measure, then the inequality in (4.6) is strict for every  $\zeta \neq 0$ .*

*Proof.* Due to (4.5) in Lemma 4.2:  $\nu\lambda_2 + \lambda_1 \geq 0$  entails  $g_{2\mathbb{E}(\mathbf{u}_*)}(\boldsymbol{\xi}) \geq g_{2\mathbb{E}(\mathbf{u}_*)}(\mathbf{0})$ , moreover  $\nu\lambda_2 + \lambda_1 > 0$  entails  $g_{2\mathbb{E}(\mathbf{u}_*)}(\boldsymbol{\xi}) > g_{2\mathbb{E}(\mathbf{u}_*)}(\mathbf{0})$ . Hence

$$J(\mathbb{D}(\mathbf{v}_*, \zeta)) = \frac{E}{8(1 + \nu)} g_{2\mathbb{E}(\mathbf{v}_*)}(Dw) \geq J(\mathbb{E}(\mathbf{v}_*))$$

and, for  $\zeta \in \mathcal{A}^i$ ,

$$\begin{aligned} \tilde{\mathcal{G}}_h(\mathbf{v}_*, \zeta) &= h^\alpha F_h^b(\zeta) + h^{2\alpha+1} \int_{\Omega} J(\mathbb{D}(\mathbf{v}_*, \zeta)) - h^{2\alpha+1} \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v}_* \geq \\ &\geq h^\alpha F_h^b(\zeta) + h^{2\alpha+1} \int_{\Omega} J(\mathbb{E}(\mathbf{v}_*)) - h^{2\alpha+1} \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v}_* \\ &\geq \tilde{\mathcal{G}}_h(\mathbf{v}_*, 0). \end{aligned}$$

Moreover the first inequality in the last computation is strict whenever  $\nu\lambda_2 + \lambda_1 > 0$  in a set of positive measure.  $\square$

**Remark 4.4.** Notice that  $s_1 := \frac{E}{1-\nu^2}(\nu\lambda_2 + \lambda_1)$  is the smallest eigenvalue of the stress tensor  $\mathbb{T}(\mathbf{v}) = J'(\mathbb{E}(\mathbf{v}))$ . Therefore Proposition 4.3 shows that, if the eigenvalues of the stress tensor are both strictly positive almost everywhere, then we can expect only one flat minimizer ( $\zeta \equiv 0$ ). On the other hand, the possible occurrence of oscillating configurations requires the presence of a compressive state on a region of positive measure: that is to say the stress tensor must have at least one negative eigenvalue on set of positive measure.

We show some examples clarifying how the asymptotic behavior of functionals  $\tilde{\mathcal{G}}_h$  provides useful information about minimizers when  $\Omega$  is an annular set.

Set  $0 < R_1 < R_2$ ,  $p_1, p_2 \in \mathbb{R}$ ,  $\Omega := B_{R_2} \setminus B_{R_1}$ , and consider uniform in-plane normal traction/compression at each component of the boundary.

$$\mathbf{f} = -p_1 \frac{\mathbf{x}}{R_1} \mathbf{1}_{\{|\mathbf{x}|=R_1\}} + p_2 \frac{\mathbf{x}}{R_2} \mathbf{1}_{\{|\mathbf{x}|=R_2\}}.$$

Therefore  $\mathbf{v} \in \arg \min \mathcal{F}_{1,0}$  entails

$$(4.7) \quad \mathbf{v}(\mathbf{x}) = (a + b|\mathbf{x}|^{-2})\mathbf{x},$$

and exploiting polar coordinates  $\mathbf{x} = (r \cos \theta, r \sin \theta)$  we obtain

$$\mathbb{E}(\mathbf{v}) = \begin{bmatrix} a - \frac{b}{r^2} \cos 2\theta & -\frac{b}{r^2} \sin 2\theta \\ -\frac{b}{r^2} \sin 2\theta & a + \frac{b}{r^2} \cos 2\theta \end{bmatrix}.$$

By using Neumann boundary condition  $J'(\mathbb{E}(\mathbf{v}))\mathbf{n} = \mathbf{f}$  on  $\partial\Omega$ , we get :

$$(4.8) \quad p_i = E(1 + \nu)^{-1}(a(1 + \nu)(1 - \nu)^{-1} - bR_i^{-2}), \quad i = 1, 2$$

that is

$$a = \frac{(1 - \nu)(p_2 R_2^2 - p_1 R_1^2)}{E(R_2^2 - R_1^2)}; \quad b = \frac{(1 + \nu)(p_2 - p_1)R_1^2 R_2^2}{E(R_2^2 - R_1^2)}.$$

It is worth noticing that  $a - br^{-2}$ ,  $a + br^{-2}$  are the eigenvalues of  $\mathbb{E}(\mathbf{v})$  and  $(\cos \theta, \sin \theta)$ ,  $(-\sin \theta, \cos \theta)$  the corresponding normalized eigenvectors  $\forall r \in [R_1, R_2]$ ; order may change according to  $\text{sign}(b)$ .

We examine several different cases which may occur.

**Example 4.5. Radially oscillating minimizers.** Set  $\Gamma = \partial\Omega$ ,  $\nu \in (-1, 1/2)$ ,  $i = 0$  and either  $p_1 \leq p_2 < 0$  or  $p_2 \leq p_1 < 0$ . In the first case we get  $b \geq 0$  in the second one  $b \leq 0$ . However in both cases  $\nu\lambda_2 + \lambda_1 < 0$  in the whole annular set.

Set also  $\mathbf{v}(\mathbf{x}) = (a + b|\mathbf{x}|^{-2})\mathbf{x} \in \arg \min \mathcal{F}_{0,1}$ .

Choose  $\sigma_h \rightarrow 0_+$ ,  $\beta_h \rightarrow +\infty$ ,  $\psi_h : \mathbb{R} \rightarrow \mathbb{R}$   $(R_2 - R_1)$ -periodic such that

$$(4.9) \quad \psi_h(t) = \max \{0, \min \{t - R_1 - \sigma_h, R_2 - \sigma_h - t\}\}$$

and set  $\psi_h^* := \psi_h * \rho_h$  being  $\rho_h$  mollifiers such that  $\text{spt } \rho_h \subset [-\sigma_h, \sigma_h]$ . Then by denoting the floor of a real number (maximum integer not exceeding the number) with  $[\cdot]$  and setting  $r = |\mathbf{x}|$ ,

$$\zeta_h(r) = \begin{cases} [\beta_h]^{-1} \sqrt{2(1 - \nu)br^{-2} - 2a(\nu + 1)} \psi_h^*(R_1 + (r - R_1)[\beta_h]) & \text{if } p_1 \leq p_2 < 0, \\ [\beta_h]^{-1} \sqrt{2(\nu - 1)br^{-2} - 2a(\nu + 1)} \psi_h^*(R_1 + (r - R_1)[\beta_h]) & \text{if } p_2 \leq p_1 < 0, \end{cases}$$

$\zeta'_h := \partial\zeta/\partial r$ ,  $D\zeta_h = (\zeta_{h,1}, \zeta_{h,2}) = (x_1/r, x_2/r) \zeta'_h$  and

$$\mathbb{M}(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta. \end{bmatrix} \quad \mathbb{S}(\theta) = \begin{bmatrix} \cos^2 \theta & \sin \theta \cos \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix} = (\zeta'_h)^{-2} D\zeta_h \otimes D\zeta_h.$$

So  $\mathbb{M}^T \mathbb{S} \mathbb{M} = \mathbf{e}_1 \otimes \mathbf{e}_1$  and there exists  $\Omega_h \subset \Omega$  with  $|\Omega_h| \sim \sigma_h$  such that,  $|(\psi_h^*)'| = 1$  on  $\Omega \setminus \Omega_h$ . Then referring to (4.4) and (4.7), for every  $x \in \Omega \setminus \Omega_h$  we have

$$\begin{aligned} g_{2\mathbb{E}(\mathbf{v})}(D\zeta_h) &= |2\mathbb{E}(\mathbf{v}) + D\zeta_h \otimes D\zeta_h|^2 + \frac{\nu}{1-\nu} |2 \operatorname{div} \mathbf{v} + |D\zeta_h|^2|^2 = \\ &|2\mathbb{M}^T E(\mathbf{v})\mathbb{M} + \mathbb{M}^T D\zeta_h \otimes D\zeta_h \mathbb{M}|^2 + \frac{\nu}{1-\nu} |4a + |D\zeta_h|^2|^2 = \\ &|2(a - br^{-2})\mathbf{e}_1 \otimes \mathbf{e}_1 + 2(a + br^{-2})\mathbf{e}_2 \otimes \mathbf{e}_2 + |\zeta_h'|^2 \mathbb{M}^T \mathbb{S} \mathbb{M}|^2 + \frac{\nu}{1-\nu} |4a + |\zeta_h'|^2|^2 = \\ &(2a - 2br^{-2} + |\zeta_h'|^2)^2 + 4(a + br^{-2})^2 + \frac{\nu}{1-\nu} |4a + |\zeta_h'|^2|^2. \end{aligned}$$

If  $p_1 \leq p_2 < 0$ , we have  $b \geq 0$ ,  $|\zeta_h'|^2 = 2(1-\nu)br^{-1} - 2a(\nu+1) + O([\beta_h]^{-2})$  on  $\Omega \setminus \Omega_h$ , hence

$$\begin{aligned} g_{2\mathbb{E}(\mathbf{v})}(D\zeta_h) &= 4(1+\nu)(a + br^{-2})^2 + O([\beta_h]^{-1}), \\ \int_{\Omega} I_{\mathbf{v}}(\mathbf{x}, D\zeta_h) d\mathbf{x} &= \int_{\Omega \setminus \Omega_h} I_{\mathbf{v}}(\mathbf{x}, D\zeta_h) d\mathbf{x} + \int_{\Omega_h} I_{\mathbf{v}}(\mathbf{x}, D\zeta_h) d\mathbf{x} = \\ &= \frac{E}{2(1-\nu)} \int_{\Omega \setminus \Omega_h} \{(a + b|\mathbf{x}|^{-2})^2 + O(\beta_h^{-1})\} d\mathbf{x} + O(\sigma_h) \rightarrow \frac{E}{2(1-\nu)} \int_{\Omega} (a + b|\mathbf{x}|^{-2})^2 dx \end{aligned}$$

Analogously, if  $p_2 \leq p_1 < 0$ , then  $b \leq 0$  and  $|\zeta_h'|^2 = 2(\nu-1)br^{-2} - 2a(\nu+1) + O([\beta_h]^{-1})$  on  $\Omega \setminus \Omega_h$ , hence

$$\begin{aligned} g_{2\mathbb{E}(\mathbf{v})}(D\zeta_h) &= 4(1+\nu)(a - br^{-2})^2 + O([\beta_h]^{-1}), \\ \int_{\Omega} I_{\mathbf{v}}(\mathbf{x}, D\zeta_h) d\mathbf{x} &\rightarrow \frac{E}{2(1-\nu)} \int_{\Omega} (a - b|\mathbf{x}|^{-2})^2 d\mathbf{x}. \end{aligned}$$

By Lemma 4.2 we know that

$$\min_{\boldsymbol{\xi} \in \mathbb{R}^2} I_{\mathbf{v}}(x, \boldsymbol{\xi}) = \begin{cases} \frac{E}{2(1-\nu)} (a + b|\mathbf{x}|^{-2})^2 & \text{if } p_2 \leq p_1 < 0, \\ \frac{E}{2(1-\nu)} (a - b|\mathbf{x}|^{-2})^2 & \text{if } p_1 \leq p_2 < 0, \end{cases}$$

therefore in both cases we have proved that

$$\int_{\Omega} I_{\mathbf{v}}(\mathbf{x}, D\zeta_h) dx \rightarrow \min\{\mathcal{G}^{**}(\mathbf{v}, \zeta) : \zeta \in H^{1,4}(\Omega), \zeta = 0 \text{ in } \partial\Omega\}.$$

Moreover

$$\begin{aligned} h^{-2\alpha-1} \tilde{\mathcal{G}}(\mathbf{v}, \zeta_h) &= h^{-\alpha-1} F_h^b(\zeta_h) + \int_{\Omega} I_{\mathbf{v}}(x, D\zeta_h) d\mathbf{x} - \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v} d\mathcal{H}^1, \\ h^{-\alpha-1} F_h^b(\zeta_h) &\sim h^{2-\alpha} \beta_h \sigma_h^{-1}. \end{aligned}$$

Therefore by Theorem 4.1 for every choice of  $\beta_h$ ,  $\sigma_h$  satisfying the conditions detailed before,  $(\mathbf{v}, \zeta_h)$  can be viewed as an asymptotically minimizing sequence of  $\tilde{\mathcal{G}}_h$  whose out-of-plane component exhibits periodic oscillations (period:  $\beta_h^{-1}$ ; asymptotic amplitude:

$\sqrt{2(1-\nu)br^{-2} - 2a(\nu+1)}$  if  $p_1 \leq p_2 < 0$  and  $\sqrt{2(\nu-1)br^{-2} - 2a(\nu+1)}$  if  $p_2 \leq p_1 < 0$ ) in the radial direction in the whole annular set. The optimal choice of  $\beta_h$  can be determined heuristically as follows: previous estimates show that

$$h^{-2\alpha-1}\tilde{\mathcal{G}}(\mathbf{v}, \zeta_h) - \min \mathcal{G}^{**} = R_h$$

where  $R_h \sim h^{2-\alpha}\beta_h\sigma_h^{-1} + \beta_h^{-1} + \sigma_h$ . So, approximatively, we have to minimize the last term. A direct calculation shows that the best choice corresponds to  $\beta_h^{-1} \sim h^{2/3-\alpha/3}$ ,  $\sigma_h \sim h^{5/3(2-\alpha)}$ .

**Example 4.6. Flat minimizer.** Let  $\Gamma = \partial\Omega$ ,  $\nu \in [0, 1/2)$ ,  $i = 0$  or  $i = 1$ ,  $p_1 \geq 0$ , so that  $R_1^2 a \geq (1-2\nu)b$  and by Lemma 4.2 we get

$$\min\{\mathcal{G}^{**}(\mathbf{v}, \zeta) : \zeta \in H^{1,4}(\Omega), \zeta = 0 \text{ in } \partial\Omega\} = \int_{\Omega} I_{\mathbf{v}}(\mathbf{x}, 0) dx.$$

Obviously the minimum is attained at  $\zeta \equiv 0$  that is we have a flat minimizer.

**Remark 4.7.** Let  $\Gamma = \partial\Omega$ ,  $\nu \in (-1, 1/2)$ ,  $i=0$ ,  $p_1 < 0 \leq p_2$ . Hence  $a > 0$ ,  $b > 0$ ,  $\nu\lambda_2 + \lambda_1 = a - br^{-2} + \nu(a + br^{-2}) \geq 0$  in the annular set  $A_1 = \{\bar{R} := \sqrt{(1-\nu)(1+\nu)^{-1}ba^{-1}} \leq r \leq R_2\}$  and  $< 0$  in the annular set  $A_2 = \{R_1 \leq r < \bar{R}\}$ . Then by the same computations performed in previous examples we can build minimizers which are flat in  $A_1$  and oscillating in  $A_2$ .

**Example 4.8. Tangentially oscillating minimizers.** Let  $\Gamma = \partial B_{R_1}$ ,  $\nu \in (-1, 1/2)$ ,  $i = 1$  and choose  $p_1 > 0$ ,  $p_2 > 0$  such that  $p_2 R_2^2 = p_1 R_1^2$ . If  $\mathbf{v} \in \arg \min \mathcal{F}_{1,0}$  we find again  $\mathbf{v}(\mathbf{x}) = (a + b|\mathbf{x}|^{-2})\mathbf{x}$  with

$$(4.10) \quad a = 0, \quad b = -(1+\nu)E^{-1}p_1 R_1^2 < 0.$$

Hence  $\lambda_1 = br^{-2} < 0 < -br^{-2} = \lambda_2$  are the eigenvalues of  $\mathbb{E}(\mathbf{v})$  and  $\mathbf{v}_1 = (-\sin \theta, \cos \theta)$ ,  $\mathbf{v}_2 = (\cos \theta, \sin \theta)$  the corresponding normalized eigenvectors.

Choose  $\sigma_h \rightarrow 0_+$ ,  $\beta_h \rightarrow +\infty$ ,  $\phi_h : \mathbb{R} \rightarrow \mathbb{R}$ ,  $2\pi$ -periodic defined by

$$(4.11) \quad \phi_h(t) = \max\{0, \min\{t - \sigma_h, 2\pi - \sigma_h - t\}\}$$

and set  $\phi_h^* := \phi_h * \rho_h$  being  $\rho_h$  mollifiers such that  $\text{spt } \rho_h \subset [-\sigma_h, \sigma_h]$ . Let

$$\zeta_h(r, \theta) = \sqrt{-2b(1-\nu)} [\beta_h]^{-1} \phi_h^*([\beta_h]\theta) (\delta_h^{-1}(r - R_1)\mathbf{1}_{[R_1, R_1+\delta_h]}(r) + \mathbf{1}_{[R_1+\delta_h, R_2]}(r))$$

with  $\delta_h \rightarrow 0_+$ ,  $\beta_h^{-1}\delta_h^{-1} \rightarrow 0$ . Then there exists  $\Omega_h \subset \Omega$  with  $|\Omega_h| \sim \sigma_h$  such that for every  $x \in \Omega \setminus \Omega_h$  we have  $|(\phi_h^*)'| = 1$  on  $\Omega \setminus \Omega_h$ . Therefore referring to (4.4) and (4.7) and by setting

$$R_*(\theta) = \begin{bmatrix} -\sin \theta & \cos \theta \\ \cos \theta & \sin \theta \end{bmatrix}$$

we get

$$\begin{aligned}
& \int_{\Omega \setminus \Omega_h} \left( |2E(\mathbf{v}) + D\zeta_h \otimes D\zeta_h|^2 + \frac{\nu}{1-\nu} |D\zeta_h|^4 \right) d\mathbf{x} = \\
& = \int_{\Omega \setminus \Omega_h} \left( |2R_*^T E(\mathbf{v}) R_* + R_*^T D\zeta_h \otimes D\zeta_h R_*|^2 + \frac{\nu}{1-\nu} |D\zeta_h|^4 \right) d\mathbf{x} = \\
& = \int_{\Omega \setminus \Omega_h} 4(1+\nu)b^2 |\mathbf{x}|^{-4} d\mathbf{x} + O([\beta_h]^{-1} \delta_h^{-1}) + O(\sigma_h) + O(\delta_h).
\end{aligned}$$

By using now Lemma 4.2 and by arguing as in Example 4.5 we get

$$\int_{\Omega} I_{\mathbf{v}}(\mathbf{x}, D\zeta_h) dx - \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v} d\mathcal{H}^1 \rightarrow \min \{ \mathcal{G}^{**}(\mathbf{v}, \zeta) : \zeta \in H^{1,4}(\Omega), \zeta = 0 \text{ in } \partial\Omega \} .$$

$$h^{-2\alpha-1} \tilde{\mathcal{G}}(\mathbf{v}, \zeta_h) \rightarrow \min \mathcal{G}^{**} = \frac{Eb^2}{2(1+\nu)} \int_{\Omega} |\mathbf{x}|^{-4} d\mathbf{x} - \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v} d\mathcal{H}^1 .$$

Moreover, since  $h^{-\alpha-1} F_h^b(\zeta_h) \sim h^{2-\alpha} \beta_h \sigma_h^{-1}$ , we get

$$\begin{aligned}
h^{-2\alpha-1} \tilde{\mathcal{G}}(\mathbf{v}, \zeta_h) &= h^{-\alpha-1} F_h^b(\zeta_h) + \int_{\Omega} I_{\mathbf{v}}(x, D\zeta_h) dx - \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v} d\mathcal{H}^1 = \\
&= h^{2-\alpha} \beta_h \sigma_h^{-1} + O([\beta_h]^{-1} \delta_h^{-1}) + O(\sigma_h) + O(\delta_h) .
\end{aligned}$$

Hence, here the optimal choice is  $\beta_h^{-1} \sim h^{1-\alpha/2}$ ,  $\delta_h \sim \beta_h^{-1/2}$ ,  $\sigma_h \sim h^{1-\alpha/2} \beta_h^{1/2}$ .

**Remark 4.9.** Thanks to Lemma 4.2 and Proposition 4.3, Examples 4.5, 4.6, 4.8 constitute a paradigm for the construction of oscillating versus flat approximated minimizers.

Moreover we sketch another technique to devise new ones, by this procedure: first take a boundary force field, construct the corresponding prestressed state (in 2D there are a lot of significant classical examples, see for instance those of Examples 2.7, 2.8) and look at the eigenvalues of the strain matrix: it is not difficult to obtain examples according to either  $\nu\lambda_2 + \lambda_1 \geq 0$  or  $\nu\lambda_2 + \lambda_1 < 0$  in the whole plate.

In the first case through Lemma 4.2 and Proposition 4.3 we argue that there is only a flat minimizer, in the second one a careful use of Lemma 4.2 on the pattern of Examples 4.5, 4.8 allows an easy construction.

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